Blow up for $u_t - \triangle u = g(u)$ revisited.

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1. Introduction. In this paper we are concerned with the relations between the existence of global, classical solutions of the evolution equation

$$\begin{cases} u_t - \triangle u = g(u) \quad \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \\ u(0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1}$$

and the existence of weak solutions of the stationary problem

$$\begin{cases} -\triangle u = g(u) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Here, and throughout the paper, $\Omega \subset \mathbf{R}^N$ is a smooth, bounded domain and $g : [0,\infty) \to [0,\infty)$ is a C^1 convex, nondecreasing function. For **some** results, we will also assume that there exists $x_0 \geq 0$ such that $g(x_0) > 0$ and

$$\int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty. \tag{3}$$

Solutions u of (1) and (2) are always assumed to be nonnegative. The initial condition u_0 is always assumed to be in $L^\infty(\Omega)$ and $u_0 \geq 0$, so that a classical solution of (1) exists on a maximal interval $(0, T_m)$.

By a weak solution of (2), we mean the following.

Definition 1. *A weak solution of (2) is a function $u \in L^1(\Omega)$, $u \geq 0$ such that*

$$g(u)\delta \in L^1(\Omega), \tag{4}$$

where δ denotes the function distance to the boundary,

$$\delta(x) = \mathrm{dist}(x, \partial\Omega), \tag{5}$$

and

$$-\int_\Omega u \triangle\zeta = \int_\Omega g(u)\zeta, \tag{6}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. (Note that the second integral makes sense since $|\zeta(x)| \le C\delta(x)$ for all $x \in \Omega$.)

Our first result is the following.

Theorem 1. *Assume (3). If there exists a global, classical solution of (1) for some $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, then there exists a weak solution of (2).*

Remark 1. Theorem 1 is quite surprising since we do not assume any bound (as $t \to \infty$) for the global solution u.

Remark 2. The existence of a global solution of (1) does not in general imply the existence of a **classical** solution of (2). In many examples, the existence of a weak solution of (2) implies the existence of a classical solution of (2). However, there exist situations where there is no classical solution for the stationary problem, and still there exists a global, classical solution of the evolution equation. See Theorem 2 and Remark 5.

An obvious consequence of Theorem 1 is the following:

Corollary 1. *Assume (3). If there is no weak solution of (2), then for any initial value $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, the solution of (1) blows up in finite time.*

Remark 3. There are very sharp results concerning the existence or nonexistence of weak solutions of (2). See properties a) and d) below and Corollary 2.

There is a converse of Theorem 1, which does not require assumption (3).

Theorem 2. *If there exists a weak solution w of (2), then for any $u_0 \in L^\infty(\Omega)$ with $0 \le u_0 \le w$, the solution u of (1) with $u(0) = u_0$ is global.*

Remark 4. If w is a classical solution of (2), then the existence of a global solution of (1) follows immediately from the maximum principle. On the other hand, if $w \notin L^\infty(\Omega)$, then the conclusion is far from obvious. Indeed, suppose that the solution blows up in finite time T_m. Clearly $u(t,x) \le w(x)$ on $(0,T_m)\times\Omega$, but this estimate in itself does not prevent $\|u(t)\|_{L^\infty}$ from blowing up in finite time. It is well known that $u(t,x)$ can converge to a blow up profile $u(T_m,x)$, which may be finite everywhere except at one point (see e.g. Weissler [16]).

A basic ingredient in the proof of Theorem 2 consists in proving that some "perturbations" of (2) have classical solutions if (2) has a weak solution. A typical result in that direction is the following:

Theorem 3. *If there exists a* **weak** *solution w of (2), then, for every $\varepsilon \in (0,1)$, there exists a* **classical** *solution w_ε of*

$$\begin{cases} -\triangle w_\varepsilon = (1-\varepsilon)g(w_\varepsilon) \quad \text{in} \quad \Omega, \\ w_\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

Theorem 3 allows us to sharpen some well known results concerning the problem

$$\begin{cases} -\triangle u = \lambda g(u) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

Here we assume in addition that

$$g(0) > 0 \quad \text{and} \quad g \not\equiv g(0). \tag{9}$$

We recall that there exists $0 < \lambda^* < \infty$ such that:

a) For every $0 < \lambda < \lambda^*$ equation (8) has a minimal, positive classical solution $u(\lambda)$, which is the unique stable solution of (8); stability means that

$$\lambda_1(-\triangle - \lambda g'(u(\lambda))) > 0.$$

(There may exist, for some values of $\lambda \in (0, \lambda^*)$, one or many other solutions, which are all unstable.)

b) The map $\lambda \mapsto u(\lambda)$ is increasing.

c) For $\lambda > \lambda^*$, there is **no classical solution** of (2).

d) For $\lambda = \lambda^*$, and if

$$\frac{g(u)}{u} \underset{u\to\infty}{\longrightarrow} \infty, \tag{10}$$

then there is a weak solution $u^* = \lim_{\lambda \uparrow \lambda^*} u(\lambda)$ of (8).

For all these results, we refer to I.M. Gel'fand [7], H.B. Keller and D.S. Cohen [10], H.B. Keller and J. Keener [11], M.G. Crandall and P.H. Rabinowitz [3], H. Brezis and L. Nirenberg [2].

Property d) is not absolutely standard, see Lemma 5.

Remark 5. The solution u^* is sometimes a classical solution. For example when $g(u) = e^u$ and $N \leq 9$ or when $g(u) = (1+u)^p$ and $N \leq 10$ (see F. Mignot and J.-P. Puel [14]). However, there are important cases where there is **no classical solution** at $\lambda = \lambda^*$. For example when Ω is the unit ball of $\mathbf{R}^N$ with $N \geq 10$ and $g(u) = e^u$; in this case $\lambda^* = 2(N-2)$ and $u^*(x) = \log\left(\frac{1}{|x|^2}\right)$ (see D.D. Joseph and T.S. Lundgren [8]).

The main novelty is:

Corollary 2. *Assume (9). If $\lambda > \lambda^*$, then there is* **no weak solution** *of (8).*

This is an obvious consequence of Theorem 3 applied to the function λg, and the characterization of λ^*.

Remark 6. A result similar to Corollary 2 was obtained by Gallouët, Mignot and Puel [6] in the case $g(u) = e^u$ (and for a stronger notion of weak solution).

Putting together Theorems 1, 2 and 3, we can now state the following.

Corollary 3. *Assume (3) and (9), and consider the (classical) solution u of*

$$\begin{cases} u_t - \triangle u = \lambda g(u) \quad \textit{in} \quad \Omega, \\ u = 0 \quad \textit{on} \quad \partial\Omega, \\ u(0) = 0 \quad \textit{in} \quad \Omega. \end{cases} \tag{11}$$

If $\lambda \le \lambda^*$, then u is global. If $\lambda > \lambda^*$, then u blows up in finite time.

Remark 7. It is somewhat surprising that one finds the same dividing line λ^* in the stationary problem and the evolution problem.

Starting with the celebrated papers of H. Fujita [4,5], dealing with the case $g(u) = e^u$, a number of authors have investigated the question of blow up in finite time or the existence of a global solution for (11). A. Lacey [12] had established that the solution of (11) blows up in finite time for $\lambda > \lambda^*$ under some additional assumption: either $u^* \in L^\infty(\Omega)$ or Ω is a ball. H. Bellout [1] had reached the same conclusion, with the additional assumption that $\left(\dfrac{g}{g'}\right)'' \le 0$. On the other hand, A Lacey and D. Tzanetis [13] proved that for $\lambda = \lambda^*$ the solution of (11) is global when Ω is a ball and $u_0 \le u^*$, $u_0 \in L^\infty(\Omega)$ and u_0 is spherically symmetric (and also for general domains but under various restrictive conditions).

2. Proof of Theorem 3.

We begin with a lemma concerning the linear Laplace equation.

Lemma 1. *Given $f \in L^1(\Omega, \delta(x)dx)$, there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} -\triangle v = f \quad \text{in} \quad \Omega, \\ v_{|\partial\Omega} = 0, \end{cases} \tag{12}$$

in the sense that

$$-\int_\Omega v\triangle\zeta = \int_\Omega f\zeta, \tag{13}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Moreover,

$$\|v\|_{L^1} \le C\|f\|_{L^1(\Omega,\delta(x)dx)}, \tag{14}$$

for some constant C independent of f. In addition, if $f \ge 0$ a.e. in Ω, then $v \ge 0$ a.e. in Ω.

Proof. The uniqueness is clear. Indeed, let v_1 and v_2 be two solutions of (12). Then $v = v_1 - v_2$ satisfies

$$\int_\Omega v\triangle\zeta = 0,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Given any $\varphi \in \mathcal{D}(\Omega)$ let ζ be the solution of

$$\begin{cases} \triangle\zeta = \varphi \quad \text{in } \Omega, \\ \zeta_{|\partial\Omega} = 0. \end{cases}$$

It follows that

$$\int_\Omega v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.

For the existence, we may assume that $f \ge 0$ (otherwise we write $f = f_+ - f_-$). Given an integer $k \ge 0$ set $f_k(x) = \min\{f(x), k\}$, so that $f_k \underset{k\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let v_k be the solution of

$$\begin{cases} -\triangle v_k = f_k \quad \text{in } \Omega, \\ v_k = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{15}$$

The sequence $(v_k)_{k\geq 0}$ is clearly monotone nondecreasing. It is also a Cauchy sequence in $L^1(\Omega)$ since

$$\int_\Omega (v_k - v_\ell) = \int_\Omega (f_k - f_\ell)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} -\triangle \zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Hence

$$\int_\Omega |v_k - v_\ell| \leq C \int_\Omega |f_k - f_\ell| \delta(x)\, dx.$$

Passing to the limit in (15) (after multiplication by ζ), we obtain (13). Finally, taking $\zeta = \zeta_0$ in (13), we obtain

$$\|v\|_{L^1} = \int_\Omega v = \int_\Omega f\zeta_0 \leq C\|f\|_{L^1(\Omega,\delta(x)dx)},$$

and (14) follows. □

Our next lemma is a variant of Kato's inequality (see [9]).

Lemma 2. *Let $f \in L^1(\Omega, \delta(x)dx)$, and let $u \in L^1(\Omega)$ be the weak solution of (12). Let $\Phi \in C^2(\mathbf{R})$ be concave, with Φ' bounded and $\Phi(0) = 0$. Then*

$$-\triangle \Phi(u) \geq \Phi'(u) f,$$

in the sense that

$$-\int_\Omega \Phi(u)\triangle\zeta \geq \int_\Omega \Phi'(u) f\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ such that $\zeta = 0$ on $\partial\Omega$.

Proof. Consider $(f_n)_{n\geq 0} \subset \mathcal{D}(\Omega)$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let u_n be the solution of

$$\begin{cases} -\triangle u_n = f_n & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It follows from Lemma 1 that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^1(\Omega)$. On the other hand we have

$$\triangle\Phi(u_n) = \Phi'(u_n)\triangle u_n + \Phi''(u_n)|\nabla u_n|^2 \leq \Phi'(u_n)\triangle u_n = -\Phi'(u_n) f_n.$$

Therefore,

$$-\int_\Omega \Phi(u_n)\triangle\zeta \geq \int_\Omega \Phi'(u_n) f_n\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ such that $\zeta = 0$ on $\partial\Omega$; and so the result follows easily by letting $n \to \infty$. □

Lemma 3. *Let $\overline{w}$ be a weak super-solution of (2), in the sense that $\overline{w} \in L^1(\Omega)$, $\overline{w} \geq 0$, $g(\overline{w})\delta \in L^1(\Omega)$, where δ is given by (5), and*

$$-\int_\Omega \overline{w}\triangle\zeta \geq \int_\Omega g(\overline{w})\zeta, \tag{17}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Then there exists a weak solution w of (2) with $0 \leq w \leq \overline{w}$.

Proof. We use a standard monotone iteration argument: define the sequence $(w_n)_{n\geq 1}$ by

$$\begin{cases} -\triangle w_{n+1} = g(w_n) \quad \text{in} \quad \Omega, \\ w_{n+1} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

for $n \geq 1$, starting with $w_1 = \overline{w}$. It is easy to check that $\overline{w} = w_1 \geq w_2 \geq \cdots \geq 0$. Indeed, it suffices to prove that $w_1 \geq w_2 \geq 0$, and the rest follows by induction, using Lemma 1. We have

$$\int_\Omega (w_1 - w_2)(-\triangle\zeta) \geq 0, \tag{18}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Given $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, let ζ_φ be the solution of

$$\begin{cases} -\triangle\zeta_\varphi = \varphi \quad \text{in} \quad \Omega, \\ \zeta_\varphi = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Taking $\zeta = \zeta_\varphi$ in (18), we obtain

$$\int_\Omega (w_1 - w_2)\varphi \geq 0.$$

Since $\varphi \geq 0$ is arbitrary, we deduce that $w_2 \leq w_1$ a.e. in Ω. On the other hand, it follows from Lemma 1 that $w_2 \geq 0$.

Since the sequence $(w_n)_{n\geq 1}$ is nonincreasing, it converges to a limit $u \in L^1(\Omega)$, which is clearly a weak solution of (2). $\square$

An essential ingredient in the proof of Theorem 3 is the following.

Lemma 4. *Assume $g(0) > 0$ and set*

$$h(u) = \int_0^u \frac{ds}{g(s)},$$

for all $u \geq 0$. Let $\widetilde{g}$ be a C^1 positive function on $[0, \infty)$ such that $\widetilde{g} \leq g$ and $\widetilde{g}' \leq g'$. Set

$$\widetilde{h}(u) = \int_0^u \frac{ds}{\widetilde{g}(s)},$$

and

$$\Phi(u) = \widetilde{h}^{-1}(h(u)),$$

for all $u \geq 0$. Then

(i) *$\Phi(0) = 0$ and $0 \leq \Phi(u) \leq u$.*

(ii) *Φ is increasing, concave and $\Phi'(u) \leq 1$ for all $u \geq 0$.*

(iii) *If $h(\infty) < \infty$ and $\widetilde{g} \not\equiv g$, then $\Phi(\infty) < \infty$.*

Proof. Properties (i) and (iii) are clear. We have

$$\Phi'(u) = \frac{\widetilde{g}(\Phi(u))}{g(u)} > 0,$$

and

$$\Phi''(u) = \frac{g(u)\widetilde{g}'(\Phi(u))\Phi'(u) - \widetilde{g}(\Phi(u))g'(u)}{g(u)^2} = \frac{\widetilde{g}(\Phi(u))(\widetilde{g}'(\Phi(u)) - g'(u))}{g(u)^2}.$$

Since $\widetilde{g}'(\Phi(u)) \leq g'(\Phi(u)) \leq g'(u)$, it follows that Φ is concave. Hence (ii). $\square$

Proof of Theorem 3. If $g(0) = 0$, then 0 is a weak solution of (7), so we assume $g(0) > 0$. We consider two cases.

Case 1. Suppose $\displaystyle\int_0^\infty \frac{ds}{g(s)} < \infty$. Let $v = \Phi(w)$, with the notation of Lemma 4, where $\widetilde{g} = (1-\varepsilon)g$. It follows from Lemmas 2 and 4 that $v \in L^\infty(\Omega)$ is a super-solution of (7). The result follows from Lemma 3.

Case 2. Suppose $\displaystyle\int_0^\infty \frac{ds}{g(s)} = \infty$. Let $\widetilde{g} = (1-\varepsilon)g$, and consider the function Φ introduced in Lemma 4. Set

$$v_1 = \Phi(w).$$

We have $0 \leq v_1 \leq w$. We observe that by concavity of the function $h(u) = \displaystyle\int_0^u \frac{ds}{g(s)}$,

$$h(w) \leq h(v_1) + (w - v_1)h'(v_1) = h(v_1) + \frac{w - v_1}{g(v_1)}.$$

Since $h(v_1) = (1-\varepsilon)h(w)$, we deduce that

$$\varepsilon g(v_1) \leq \frac{w - v_1}{h(w)} \leq \frac{w}{h(w)} \leq C(1 + w),$$

so that in particular, $g(v_1) \in L^1(\Omega)$. Now, we observe that by Lemma 2, v_1 is a weak super-solution of the equation

$$\begin{cases} -\triangle u_1 = (1-\varepsilon)g(u_1) \quad \text{in} \quad \Omega, \\ u_1 = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{19}$$

Therefore, it follows from Lemma 3 that there exists a weak solution u_1 of (19) such that $0 \leq u_1 \leq v_1$. In particular, we have $0 \leq g(u_1) \leq g(v_1) \in L^1(\Omega)$, so that $u_1 \in L^p(\Omega)$, for all $p \geq 1$ such that (see e.g. Stampacchia [15])

$$p < \frac{N}{N-2} \quad (p \leq \infty \text{ if } N = 1, p < \infty \text{ if } N = 2). \tag{20}$$

By the same construction, we find a solution u_2 of the equation

$$\begin{cases} -\triangle u_2 = (1-\varepsilon)^2 g(u_2) \quad \text{in} \quad \Omega, \\ u_2 = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

such that $0 \leq u_2 \leq u_1$ and $g(u_2) \leq C(1 + u_1)$. In particular, $g(u_2) \in L^p(\Omega)$, for all $p \geq 1$ verifying (20). This implies that $u_2 \in L^r(\Omega)$, for all $r \geq 1$ such that $r < \dfrac{N}{N-4}$ ($r \leq \infty$ if $N = 1, 2, 3$, $r < \infty$ if $N = 4$). By iteration, we find that if $k(N) = [N/2] + 1$, then the solution u_k of the equation

$$\begin{cases} -\triangle u_k = (1-\varepsilon)^k g(u_k) \quad \text{in} \quad \Omega, \\ u_k = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

belongs to $L^\infty(\Omega)$. Since $\varepsilon \in (0,1)$ is arbitrary, this completes the proof. $\square$

3. Proof of Theorem 1. We assume $g(0) > 0$, for otherwise $w \equiv 0$ is a weak solution of (2). Furthermore, we may also assume that $u_0 = 0$, so that $u \geq 0$ and $u_t \geq 0$ for all $t \geq 0$.

Next, observe that $g'(u) \underset{u\to\infty}{\longrightarrow} +\infty$ by (3), so that there exists a constant $M > 0$ such that

$$g(s) - \lambda_1 s \geq \frac{1}{2} g(s) \quad \text{for} \quad s \geq M, \tag{21}$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. Let $\varphi \in C^2(\overline{\Omega})$ with $\varphi_{|\partial\Omega} = 0$. It follows from (1) that

$$\frac{d}{dt}\int_\Omega u(t)\varphi + \int_\Omega u(t)(-\triangle\varphi) = \int_\Omega g(u(t))\varphi. \tag{22}$$

We first claim that

$$\sup_{t\geq 0} \int_\Omega g(u)\varphi_1 \leq (1+\lambda_1)M, \tag{23}$$

where M is as in (21) and φ_1 is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$ such that $\displaystyle\int_\Omega \varphi_1 = 1$. Indeed, taking $\varphi = \varphi_1$ in (22), we find

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 + \lambda_1 \int_\Omega u(t)\varphi_1 = \int_\Omega g(u(t))\varphi_1 \geq g\left(\int_\Omega u(t)\varphi_1\right), \tag{24}$$

by Jensen's inequality. If there exists $t_0 \geq 0$ such that $\displaystyle\int_\Omega u(t_0)\varphi_1 > M$, then it follows from (24) and (21) that

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \geq \frac{1}{2} g\left(\int_\Omega u(t)\varphi_1\right),$$

for $t \geq t_0$, which is absurd by (3); and so

$$\int_\Omega u(t)\varphi_1 \leq M,$$

for all $t \geq 0$. Integrating (24) on $(t, t+1)$ and since $u_t \geq 0$, we find

$$\int_\Omega g(u(t))\varphi_1 \leq \int_t^{t+1}\int_\Omega g(u)\varphi_1 \leq \int_\Omega u(t+1)\varphi_1 + \lambda_1 \int_t^{t+1}\int_\Omega u\varphi_1 \leq (1+\lambda_1)M,$$

hence (23).

We next claim that there exists K such that

$$\sup_{t\geq 0} \|u(t)\|_{L^1} \leq K. \tag{25}$$

Indeed, let ζ_0 be the solution of (16). Taking $\varphi = \zeta_0$ in (22) and integrating on $(t, t+1)$, we find

$$\int_\Omega u(t) \leq \int_t^{t+1}\int_\Omega u = \int_\Omega u(t)\zeta_0 - \int_\Omega u(t+1)\zeta_0 + \int_t^{t+1}\int_\Omega g(u)\zeta_0,$$

and (25) follows by applying (23).

By monotone convergence, it follows from (25) and (23) that $u(t)$ has a limit w in $L^1(\Omega)$ and that $g(u)$ converges to $g(w)$ in $L^1(\Omega, \delta(x)dx)$, as $t \to \infty$. Let $\varphi \in C^2(\overline{\Omega})$, $\varphi_{|\partial\Omega} = 0$. Integrating (22) on $(t, t+1)$, we obtain

$$\left[\int_\Omega u\varphi\right]_t^{t+1} + \int_t^{t+1}\int_\Omega u(t)(-\triangle\varphi) = \int_t^{t+1}\int_\Omega g(u(t))\varphi.$$

Letting $t \to \infty$, we find

$$\int_\Omega w(-\triangle\varphi) = \int_\Omega g(w)\varphi.$$

Therefore, w is a weak solution of (2). $\square$

We now give an alternative proof of Theorem 1 in the spirit of the proof of Theorem 3. It makes use of the following lemma.

Lemma 5. *Assume (9), and let λ^* be the supremum of all $\lambda > 0$ such that (8) has a minimal, positive, classical solution $u(\lambda)$. Then $\lambda^* < \infty$. If furthermore (10) holds, then $\lim_{\lambda \uparrow \lambda^*} u(\lambda) = u^*$ is a weak solution of (8) with $\lambda = \lambda^*$.*

Proof. We first observe that by (9) and convexity of g, there exists $\varepsilon > 0$ such that

$$g(u) \geq \varepsilon u,$$

for all $u \geq 0$; and so

$$-\triangle u(\lambda) \geq \lambda \varepsilon u(\lambda). \tag{26}$$

Let λ_1 be the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, and let φ_1 be a corresponding eigenvector. Multiplying (26) by φ_1, we see that $\lambda\varepsilon \leq \lambda_1$; and so,

$$\lambda^* \leq \frac{\lambda_1}{\varepsilon}.$$

If (10) holds, then there exists C such that

$$g(u) \geq \frac{2\lambda_1}{\lambda^*} u - C,$$

for all $u \geq 0$. Multiplying (8) by φ_1, we obtain

$$\lambda \int_\Omega g(u(\lambda))\varphi_1 = \lambda_1 \int_\Omega u(\lambda)\varphi_1 \leq \frac{\lambda^*}{2} \int_\Omega (g(u(\lambda)) + C)\varphi_1.$$

Letting $\lambda \uparrow \lambda^*$, we deduce that

$$\lim_{\lambda \uparrow \lambda^*} \int_\Omega g(u(\lambda))\varphi_1 < \infty. \tag{27}$$

Multiplying now (8) by the solution ζ_0 of (16), we obtain

$$\int_\Omega u(\lambda) = \lambda \int_\Omega g(u(\lambda))\zeta_0 \leq C\lambda \int_\Omega g(u(\lambda))\varphi_1,$$

so that $u(\lambda)$ is bounded in $L^1(\Omega)$ by (27). Since $u(\lambda)$ is increasing in λ, it follows that $u(\lambda)$ has a limit $u^* \in L^1(\Omega)$ and that $g(u(\lambda))$ converges to $g(u^*)$ in $L^1(\Omega, \delta(x)\,dx)$. It follows easily that u^* is a weak solution of (8) with $\lambda = \lambda^*$. □

Alternative proof of Theorem 1. We may assume as above that $g(0) > 0$ and $u_0 = 0$. Given $0 < \varepsilon < 1$, let $\widetilde{g} = (1 - \varepsilon)g$ and let Φ be as in Lemma 4. Set

$$v_\varepsilon(t) = \Phi(u(t)),$$

for all $t \geq 0$. It follows from Lemma 4 that there exists $M_\varepsilon < \infty$ such that

$$0 \leq v_\varepsilon \leq M_\varepsilon. \tag{28}$$

Furthermore, it follows from Lemmas 2 and 4 that

$$\begin{aligned}
-\triangle v_\varepsilon &\geq \Phi'(u)(-\triangle u) \\
&= \Phi'(u)(g(u) - u_t) \\
&= (1-\varepsilon)g(v_\varepsilon) - (v_\varepsilon)_t,
\end{aligned}$$

so that v_ε is a super-solution of the equation

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \triangle u_\varepsilon = (1-\varepsilon)g(u_\varepsilon), \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \\ u_\varepsilon(0) = 0. \end{cases} \tag{29}$$

It now follows from (28) that the solution u_ε of (29) is global and bounded by M_ε. As above, we deduce that $w_\varepsilon = \lim\limits_{t\to\infty} u_\varepsilon(t)$ a (classical) solution of the equation

$$\begin{cases} -\triangle w_\varepsilon = (1-\varepsilon)g(w_\varepsilon) \quad \text{in} \quad \Omega, \\ w_\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

It follows (see property c) in the introduction) that $\lambda^* \geq 1$. By Lemma 5, (2) has a weak solution. □

4. Proof of Theorem 2.

Since the hypotheses of Theorem 2 allow g to vanish at the origin, we need a variant of Lemma 4 that applies to the case $g(0) = 0$.

Lemma 6. *Assume (3). There exist constants $K \geq 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there is a function $\Phi_\varepsilon \in C^2([0,\infty))$, concave, increasing, with*

$$\Phi_\varepsilon(0) = 0, \tag{30}$$

$$0 < \Phi_\varepsilon(x) \leq x \quad \textit{for} \quad x > 0, \tag{31}$$

$$1 \geq \Phi_\varepsilon'(x) \geq \frac{(g(\Phi_\varepsilon(x)) - \varepsilon K)^+}{g(x)} \quad \textit{for} \quad x \geq 0. \tag{32}$$

Moreover, $\sup\limits_{x\geq 0} \Phi_\varepsilon(x) < \infty$.

Proof. If $g(0) > 0$ we apply Lemma 4 with $\widetilde{g}(u) = g(u) - \varepsilon$ and the conclusions follow with $\varepsilon_0 = g(0)$ and $K = 1$.

Therefore we may assume that $g(0) = 0$. Let $a > 0$ be the unique solution of $g(a) = 1$. Set

$$H(x) = a + \int_a^x \frac{ds}{g(s)} \text{for} \quad x \geq a.$$

Since $g(a) = 1$, there exists $0 < \varepsilon_0 < 1$ such that

$$0 < \varepsilon < g((1-\varepsilon)a),$$

for $0 < \varepsilon < \varepsilon_0$. For such an ε, let

$$H_\varepsilon(x) = a + \int_{(1-\varepsilon)a}^x \frac{ds}{g(s) - \varepsilon} \quad \text{for} \quad x \geq (1-\varepsilon)a.$$

Note that $H_\varepsilon((1-\varepsilon)a) = a = H(a)$. Moreover,

$$\lim_{x\to\infty} H_\varepsilon(x) > a + \int_{(1-\varepsilon)a}^{\infty} \frac{ds}{g(s)} \geq \lim_{x\to\infty} H(x).$$

Thus

$$\Psi_\varepsilon(x) = H_\varepsilon^{-1}(H(x)),$$

is well-defined for $x \geq a$, $\Psi_\varepsilon(a) = (1-\varepsilon)a$ and $\sup_{x\geq a} \Psi_\varepsilon(x) < \infty$. Furthermore, for $x \geq a$,

$$\Psi_\varepsilon'(x) = \frac{g(\Psi_\varepsilon(x)) - \varepsilon}{g(x)}. \tag{33}$$

In addition, for $x \geq a$ we have

$$\Psi_\varepsilon''(x) = \frac{g(x)g'(\Psi_\varepsilon(x))\Psi_\varepsilon'(x) - (g(\Psi_\varepsilon(x)) - \varepsilon)g'(x)}{g(x)^2} = \frac{(g(\Psi_\varepsilon(x)) - \varepsilon)(g'(\Psi_\varepsilon(x)) - g'(x))}{g(x)^2} \leq 0,$$

since $\Psi_\varepsilon(x) \leq x$ thus $g'(\Psi_\varepsilon(x)) \leq g'(x)$. We finally consider a concave function $\Phi_\varepsilon \in C^2([0,\infty))$ such that $\Phi_\varepsilon(x) = \Psi_\varepsilon(x)$ for $x \geq a$, $\Phi_\varepsilon(0) = 0$, and $\Phi_\varepsilon'(x) \leq 1$ for all $x \geq 0$. Such a function exists since

$$\Psi_\varepsilon'(a) \leq \frac{\Psi_\varepsilon(a)}{a} \leq 1.$$

Clearly Φ_ε verifies (30) and (31). We claim that (32) holds with $K = 1 + ag'(a)$. Indeed, it follows from (33) that for $x \geq a$

$$\Phi_\varepsilon'(x) \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)},$$

so that (32) holds for $x \geq a$ (since $\Phi_\varepsilon' \geq 0$). For $x \leq a$, we have

$$\Phi_\varepsilon'(x) \geq \Phi_\varepsilon'(a) = g((1-\varepsilon)a) - \varepsilon.$$

Furthermore, by convexity,

$$g((1-\varepsilon)a) \geq g(a) - \varepsilon a g'(a) = 1 - \varepsilon(K-1);$$

and so, for $x \leq a$,

$$\Phi_\varepsilon'(x) \geq 1 - \varepsilon K = 1 - \frac{\varepsilon K}{g(a)} \geq 1 - \frac{\varepsilon K}{g(x)} = \frac{g(x) - \varepsilon K}{g(x)} \geq \frac{g(\Phi_\varepsilon(x)) - \varepsilon K}{g(x)}.$$

It follows that (32) is verified for $x \leq a$, which completes the proof. □

Lemma 7. *Let δ be given by (5). For every $0 < T < \infty$, there exists $\varepsilon_1(T) > 0$ such that if $0 < \varepsilon \leq \varepsilon_1$, then the solution Z of the equation*

$$\begin{cases} Z_t - \triangle Z = -\varepsilon \quad \text{in} \quad (0,\infty)\times\Omega, \\ Z = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \\ Z(0) = \delta, \end{cases}$$

verifies $Z \geq 0$ on $[0,T]\times\overline{\Omega}$.

Proof. Let $(T(t)))_{t\geq 0}$ be the heat semigroup with Dirichlet boundary condition, and consider the solution ζ_0 of (16). We have

$$\zeta_0 = T(t)\zeta_0 + \int_0^t T(s)1_\Omega \, ds,$$

for all $t \geq 0$. Since $T(t)\zeta_0 \geq 0$, it follows that

$$\int_0^t T(s)1_\Omega \, ds \leq \zeta_0 \leq C\delta, \tag{35}$$

for all $t \geq 0$. On the other hand, we have

$$Z(t) = T(t)\delta - \varepsilon \int_0^t T(s)1_\Omega \, ds;$$

and so,

$$Z(t) \geq T(t)\delta - \varepsilon C\delta.$$

Consider now $c_0, c_1 > 0$ such that $c_0\varphi_1 \leq \delta \leq c_1\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$, associated to the eigenvalue λ_1. We have

$$T(t)\delta \geq c_0 T(t)\varphi_1 = c_0 e^{-\lambda_1 t}\varphi_1 \geq \frac{c_0}{c_1} e^{-\lambda_1 t}\delta.$$

Therefore,

$$Z(t) \geq \left(\frac{c_0}{c_1} e^{-\lambda_1 t} - \varepsilon C\right)\delta.$$

It follows that $Z(t) \geq 0$ on $[0, T]$, provided $\varepsilon \leq \dfrac{c_0}{c_1 C} e^{-\lambda_1 T}$. $\square$

Proof of Theorem 2. If (3) fails, then the solution of

$$\begin{cases} \theta' = g(\theta), \\ \theta(0) = \|u_0\|_{L^\infty}, \end{cases}$$

is global. Since $\theta(t)$ is a super-solution of (1) and 0 is a sub-solution, it follows that all the solutions of (1) are global.

We now assume that (3) holds. Furthermore, we may assume

$$w \notin L^\infty(\Omega), \tag{36}$$

since otherwise $u(t) \leq w$ by the maximum principle, and so u is global. We denote by $[0, T_m)$ the maximal interval of existence of u, and we now proceed in five steps.

Step 1. We have $u(t) \leq w$ for all $t \in [0, T_m)$. (Note that if w were a smooth solution of (2), this would follow from the maximum principle.) Fix $T < T_m$. Let $h(t,x) \in \mathcal{D}((0,T) \times \Omega)$, $h \geq 0$, and let ζ be the solution of

$$\begin{cases} -\zeta_t - \triangle\zeta - g'(u)\zeta = h, \\ \zeta_{|\partial\Omega} = 0, \\ \zeta(T) = 0. \end{cases}$$

Since $g'(u) \in L^\infty((0,T)\times\Omega)$ we have in particular $\zeta \in C([0,T], C^2(\overline{\Omega}) \cap C_0(\Omega))$ and $\zeta \geq 0$. Multiplying (1) by ζ and integrating on $(0,T)\times\Omega$, we find

$$-\int_\Omega u_0\zeta(0) + \int_0^T\int_\Omega u[h + g'(u)\zeta] = \int_0^T\int_\Omega g(u)\zeta.$$

On the other hand,

$$-\int_0^T\int_\Omega w\zeta_t - \int_\Omega w\zeta(0) = 0,$$

and

$$-\int_0^T\int_\Omega w\triangle\zeta = \int_0^T\int_\Omega g(w)\zeta.$$

Therefore,

$$-\int_\Omega (u_0 - w)\zeta(0) + \int_0^T\int_\Omega (u-w)h = \int_0^T\int_\Omega (g(u) - g(w) - g'(u)(u-w))\zeta.$$

By convexity,

$$g(u) - g(w) - g'(u)(u-w).$$

Since $\zeta \geq 0$ and $u_0 - w \leq 0$, this yields

$$\int_0^T\int_\Omega (u-w)h \leq 0.$$

Since h is arbitrary, we conclude that $u - w \leq 0$.

Step 2. There exist $0 < \tau < T_m$ and $C_0, c_0 > 0$ such that

$$u(\tau) \leq C_0\delta, \tag{37}$$

and

$$u(\tau) \leq w - c_0\delta. \tag{38}$$

Set $v_0 = \min\{w, 1 + u_0\}$. We have $v_0 \geq u_0$ and $v_0 \not\equiv u_0$ by (36). In particular, there exists a function $\gamma : [0,\infty) \to \mathbf{R}$ such that $\gamma(t) > 0$ for $t > 0$ and

$$T(t)(v_0 - u_0) \geq \gamma(t)\delta, \tag{39}$$

where δ is defined by (5) and $(T(t))_{t\geq 0}$ is the heat semigroup with Dirichlet boundary condition. Let v be the solution of (1) with the initial value $v(0) = v_0$, and let $[0,\overline{T})$ be the maximal interval of existence of v. We have $v \geq 0$, and by Step 1, $v \leq w$. Let $z(t) = u(t) + T(t)(v_0 - u_0)$ for $0 \leq t < \overline{T}$. We have

$$\begin{cases} z_t - \triangle z = g(u) \leq g(z) & \text{in} \quad (0,\overline{T})\times\Omega, \\ z = 0 & \text{on} \quad \partial\Omega, \\ z(0) = v_0 & \text{in} \quad \Omega, \end{cases}$$

so that $z \leq v$ by the maximum principle. Therefore,

$$u(t) \leq v(t) - T(t)(v_0 - u_0) \leq w - T(t)(v_0 - u_0) \leq w - \gamma(t)\delta, \tag{40}$$

for $0 \le t < \overline{T}$ by (39). Fix $0 < T < \min\{\overline{T}, T_m\}$. u is bounded by some constant M on $[0,T] \times \overline{\Omega}$, so that

$$u(t) \le MT(t)1_\Omega + g(M) \int_0^t T(s)1_\Omega \, ds.$$

There exists a function $\overline{C} : (0,\infty) \to \mathbf{R}$ such that $T(t)1_\Omega \le C(t)\delta$ for $t > 0$, so that we deduce from (35) that

$$u(t) \le MC(t)\delta + g(M)C\delta, \tag{41}$$

for $0 < t \le T$. (37) and (38) now follow from (40) and (41).

Step 3. We may assume without loss of generality that

$$u_0 \le C_0\delta, \tag{42}$$

and

$$u_0 \le w - c_0\delta, \tag{43}$$

where C_0, c_0 are as in Step 2. Indeed, we need only consider $u(\cdot + \tau)$ instead of $u(\cdot)$.

Step 4. Let ε_0 and Φ_ε be as in Lemma 6, and set $w_\varepsilon = \Phi_\varepsilon(w)$ for $0 < \varepsilon < \varepsilon_0$. Then

$$w_\varepsilon \in L^\infty(\Omega), \tag{44}$$

and

$$\int_\Omega \zeta(-\triangle w_\varepsilon) \ge \int_\Omega (g(w_\varepsilon) - \varepsilon K)\zeta, \tag{45}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \ge 0$ on Ω and $\zeta_{|\partial\Omega} = 0$. Moreover, there exists $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$u_0 \le w_\varepsilon - \frac{c_0}{2}\delta, \tag{46}$$

for $0 < \varepsilon < \varepsilon_1$, where c_0 is as in (43). Indeed, (44) and (45) follow from Lemmas 2 and 6. In order to prove (46), set

$$\eta = \min\{w, (C_0 + c_0)\delta\},$$

and

$$\eta_\varepsilon = \Phi_\varepsilon(\eta).$$

Here, δ is given by (5) and C_0 is as in (42). It follows from (42) and (43) that

$$u_0 \le \eta - c_0\delta. \tag{47}$$

We claim that

$$\eta \le \eta_\varepsilon + \frac{c_0}{2}\delta, \tag{48}$$

for $\varepsilon > 0$ small enough. Note that it follows from (47) and (48) that

$$u_0 \le \eta_\varepsilon - \frac{c_0}{2}\delta,$$

and (46) follows since $\eta_\varepsilon \le w_\varepsilon$ (since Φ_ε is nondecreasing). Thus we need only prove (48). Note that

$$\eta_\varepsilon \le \eta \le M,$$

where $M = (C_0 + c_0)\|\delta\|_{L^\infty}$, and that

$$\Phi'_\varepsilon(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} 1,$$

uniformly on $[0, M]$ by Lemma 6. Therefore,

$$\eta - \eta_\varepsilon \le \eta \sup_{0\le x\le M} (1 - \Phi'_\varepsilon(x)) \le (C_0 + c_0)\delta \sup_{0\le x\le M} (1 - \Phi'_\varepsilon(x)) \le \frac{c_0}{2}\delta,$$

for ε small enough, and (48) follows.

Step 5. Conclusion. Assume by contradiction that $T_m < \infty$. Let $\varepsilon > 0$ be small enough so that

$$u_0 \le w_\varepsilon - \frac{c_0}{2}\delta,$$

(see Step 4), and so that the solution Z of the equation

$$\begin{cases} Z_t - \triangle Z = -\varepsilon K \quad \text{in} \quad (0, T_m) \times \Omega, \\ Z = 0 \quad \text{on} \quad \partial\Omega, \\ Z(0) = \dfrac{c_0}{2}\delta \quad \text{in} \quad \Omega, \end{cases}$$

is nonnegative on $[0, T_m] \times \overline{\Omega}$ (see Lemma 7. Here, K is given by Lemma 6). Let v be the solution of

$$\begin{cases} v_t - \triangle v = g(|v|) - \varepsilon K \quad \text{in} \quad (0, T) \times \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega, \\ v(0) = w_\varepsilon \quad \text{in} \quad \Omega. \end{cases}$$

Let $[0, S_m)$ be the maximal interval of existence of v. Set $z(t) = Z(t) + u(t)$ for $0 \le t < T_m$. We have $z \ge u \ge 0$ and

$$\begin{cases} z_t - \triangle z = g(u) - \varepsilon K \le g(z) - \varepsilon K \quad \text{on} \quad (0, T_m) \times \Omega, \\ z_{|\partial\Omega} = 0, \\ z(0) = u_0 + \dfrac{c_0}{2}\delta \le w_\varepsilon \quad \text{in} \quad \Omega. \end{cases}$$

By the maximum principle, we have $z \le v$ on $[0, \min\{T_m, S_m\})$. In particular, $v \ge 0$ on $[0, \min\{T_m, S_m\})$; by the maximum principle and (45), $v \le w_\varepsilon$. Since $w_\varepsilon \in L^\infty(\Omega)$, this implies that $T_m < S_m = +\infty$. Therefore, $u \le z \le v \le w_\varepsilon$ on $[0, T_m)$, which is absurd. $\square$

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