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Cálculo III - 2^a Prova

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Nome: ________________________________ Matrícula: ________________

Questão 1 *(3.0 pts) Seja* $\vec{F} = P\vec{i} + Q\vec{j}$, *onde* $P = \cos^2(x) + x^2y^3 - \frac{y}{2-x}$ *e* $Q = x^3y^2 + \ln(2-x)$.

(a) *Mostre que* $\vec{F}$ *é um campo conservativo. Justifique;*

(b) *Encontre um potencial para* $\vec{F}$;

(c) *Calcule* $\int_C \vec{F}\cdot d\vec{r}$, *onde* C *é a curva parametrizada por*

$$\alpha : [0,\, 2\pi] \to \mathbb{R}^2,\ \alpha(t) = \left(\frac{1}{2} + \frac{1}{2}\cos(t),\ \frac{1}{2}sen(t)\right)$$

Questão 2 *(2.0 pts.) Calcule o trabalho realizado pelo campo*

$$\vec{F} = x(x^2+y^2)\vec{i} + y(x^2+y^2)\vec{j}$$

para mover uma partícula ao longo da curva dada por $9x^2 + 4y^2 = 36$; *do ponto* $(2,\,0)$ *até o ponto* $(0,3)$.

Questão 3 *(2.0 pts) Seja* $\vec{F}$ *um campo definido em* $\mathbb{R}^2 \setminus \{(-2,\,0), (2,\,0)\}$, *tal que* $\nabla \times \vec{F} = \vec{0}$ *em todos os pontos do domínio. Suponha que:*

$$\oint_{C_1} \vec{F}\cdot d\vec{r} = 4 \qquad e \qquad \oint_{C_2} \vec{F}\cdot d\vec{r} = 2,$$

onde C_1 *é o círculo de raio* 1 *e centro* $(-2,0)$ *e* C_2 *é o círculo de raio* 1 *e centro* $(2,\,0)$, *ambos orientados no sentido anti-horário. Calcule* $\oint_C \vec{F}\cdot d\vec{r}$, *onde* C *é o círculo de raio* 4 *e centro* $(0,\,0)$, *orientado no sentido anti-horário.*

Questão 4 *(3.0 pts) Calcule:*

(a) $\int_C xdx+ydy+zdz$, *C é a curva interseção das superfícies* $z = x^2+y^2$ *e* $2x+2y-z = 1$, *orientada de maneira que sua projeção no plano XY seja percorrida no sentido anti-horário;*

(b) $\int_C \frac{1}{x^2+y^2+z^2}ds$, *$C$ é a curva parametrizada por*

$$\alpha(t) = \left(e^t\cos(t),\, e^t\sin(t),\, e^t\right), \quad 0 \le t \le 1.$$

Boa Prova !!

C3 P2 2024.1

① $\vec{F} = P\vec{i} + Q\vec{j}$, $P = \cos^2 x + x^2 y^3 - \dfrac{y}{2-x}$

$Q = x^3 y^2 + \ln(2-x)$

(a) Mostrar que $\vec{F}$ é conservativo;

(b) Encontrar um potencial para $\vec{F}$;

(c) Calcular $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, C é a curva parametriza

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \quad \alpha(t) = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos t, \ \tfrac{1}{2}\operatorname{sen} t\right)$$

Solução

(a) $\vec{F}$ está definido em todo $\mathbb{R}^2$ (que é simplesmente conexo). Além disso,

$$\frac{\partial Q}{\partial x} = 3x^2y^2 - \frac{1}{2-x} \quad \text{e} \quad \frac{\partial P}{\partial y} = 3x^2y^2 - \frac{1}{2-x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad \therefore \quad \operatorname{rot}(\vec{F}) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \vec{0}$$

Daí, $\vec{F}$ é conservativo.

(b) Queremos encontra uma função $f : \mathbb{R}^2 \to \mathbb{R}$ (de classe C^1) t.q. $\nabla f = \vec{F}$.

Ou seja, $f_x = P$ e $f_y = Q$

$$f_x = P \Rightarrow f_x = \cos^2 x + x^2y^3 - \frac{y}{2-x}$$

$$\Rightarrow f = \int \cos^2 x\, dx + \frac{1}{3}x^3y^3 + y\ln(2-x)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \quad \therefore \int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\operatorname{sen} 2x$$

Daí,

$$f = \frac{1}{2}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{3}x^3y^3 + y\ln(2-x) + g(y)$$

Derivando c/r a y :

$$f_y = x^3 y^2 - \ln(2-x) + g'(y)$$

$$f_y = Q \Rightarrow \quad x^3y^2 + \ln(2-x) + g'(y) = x^3y^2 + \ln(2-x)$$

$$\Rightarrow \quad g'(y) = 0 \quad \therefore \quad g(y) = C = cte$$

Daí

$$f = \frac{1}{2}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{3}x^3y^3 + y\ln(x-2) + C.$$

(c) $$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \nabla f \cdot d\vec{r}$$

$$= f(\alpha(2\pi)) - f(\alpha(0)) = 0$$

$$\left(\begin{array}{l} \alpha(2\pi) = (1,0) \qquad \alpha(0) = (1,0) \\ (\text{a curva é fechada}) \end{array} \right)$$

$$\left(\begin{array}{l} \begin{cases} x = \frac{1}{2} + \frac{1}{2}\cos t \\ y = \frac{1}{2}\operatorname{sen} t \end{cases} \Rightarrow \begin{array}{r} 2(x - 1/2) = \cos t \\ 2y = \operatorname{sen} t \end{array} \\ \Rightarrow 4(x-1/2)^2 + 4y^2 = 1 \\ \Rightarrow \underline{(x-1/2)^2 + y^2 = 1/4} \end{array} \right)$$

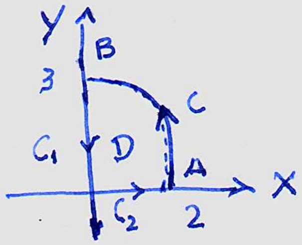
a curva é uma circunf. de raio $1/2$
e centro $(1/2, 0)$

(2) $\vec{F} = x(x^2+y^2)\vec{i} + y(x^2+y^2)\vec{j}$

C: $9x^2+4y^2=36$ ... do pto $A=(2,0)$ até o pto $B=(0,3)$

calcular o trabalho realizado por $\vec{F}$ para mover uma partícula ao longo de C do pto A até o pto B

<u>Solução</u>



$9x^2+4y^2=16$

$\frac{x^2}{4}+\frac{y^2}{9}=1$

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$

$C: \begin{cases} x = 2\cos t \\ y = 3\,\mathrm{sen}\,t \end{cases} \quad 0 \le t \le \pi/2 \qquad \begin{cases} dx = -2\,\mathrm{sen}\,t\,dt \\ dy = 3\cos t\,dt \end{cases}$

$W = \int_C \vec{F}\cdot d\vec{r}$

$x(x^2+y^2)dx + y(x^2+y^2)dy = (x^2+y^2)(x\,dx + y\,dy)$

$\begin{aligned} x\,dx &= -4\,\mathrm{sen}\,t\cos t\,dt \\ y\,dy &= 9\,\mathrm{sen}\,t\cos t\,dt \end{aligned} \quad \Big| \quad \underline{x\,dx + y\,dy = 5\,\mathrm{sen}\,t\cos t\,dt}$

$\underline{x^2+y^2 = 4\cos^2 t + 9\,\mathrm{sen}^2 t}$

$(x^2+y^2)(x\,dx+y\,dy) = (20\cos^3 t\,\mathrm{sen}\,t + 45\,\mathrm{sen}^3 t\cos t)\,dt.$

$W = \int_0^{\pi/2}(20\cos^3 t\,\mathrm{sen}\,t + 45\,\mathrm{sen}^3 t\cos t)\,dt$

$= -5\cos^4 t\Big|_0^{\pi/2} + \frac{45}{4}\,\mathrm{sen}^4 t\Big|_0^{\pi/2}$

$= -5(0-1) + \frac{45}{4}(1-0) = 5 + \frac{45}{4} = \frac{65}{4}$

Outra solução:

Seja $D \subseteq \mathbb{R}^2$ a região t.q. $\partial D = C \cup C_1 \cup C_2$

$$C_1: \begin{cases} x = 0 \\ y = 3 - t \end{cases} \quad t \in [0,3] \qquad \begin{cases} dx = 0 \\ dy = -dt \end{cases}$$

$$C_2: \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in [0,2] \qquad \begin{matrix} dx = dt \\ dy = 0 \end{matrix}$$

Green:

$$\int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dA$$

$$\left.\begin{matrix} P = x(x^2+y^2) \Rightarrow P_y = 2xy \\ Q = y(x^2+y^2) \Rightarrow Q_x = 2xy \end{matrix}\right\} \Rightarrow Q_x - P_y = 0$$

$$\therefore \int_D (Q_x - P_y)\,dA = 0$$

$$\int_{\partial D} P\,dx + Q\,dy = \int_C P\,dx + \dots + \int_{C_1} P\,dx + \dots + \int_{C_2} P\,dx + \dots = 0$$

$$\therefore \int_C P\,dx + Q\,dy = -\int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy$$

$$\int_{C_1} P\,dx + Q\,dy = \int_0^3 -(3-t)^3\,dt = \left.\frac{(3-t)^4}{4}\right|_0^3 = \frac{-3^4}{4}$$

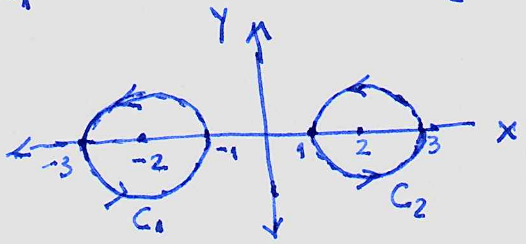
$$\int_{C_2} P\,dx + Q\,dy = \int_0^2 t^3\,dt = \left.\frac{1}{4}t^4\right|_0^2 = 4$$

$$\therefore \int_C P\,dx + Q\,dy = \frac{3^4}{4} - 4 = \frac{81-16}{4} = \frac{65}{4}$$

③ $\vec{F}: \mathbb{R}^2 \setminus \{(-2,0),(2,0)\} \longrightarrow \mathbb{R}^2$

$\nabla \times \vec{F} = \vec{0}$ (i.e. $rot(\vec{F}) = \vec{0}$)

$\oint_{C_1} \vec{F} \cdot d\vec{r} = 4$, $\oint_{C_2} \vec{F} \cdot d\vec{r} = 2$

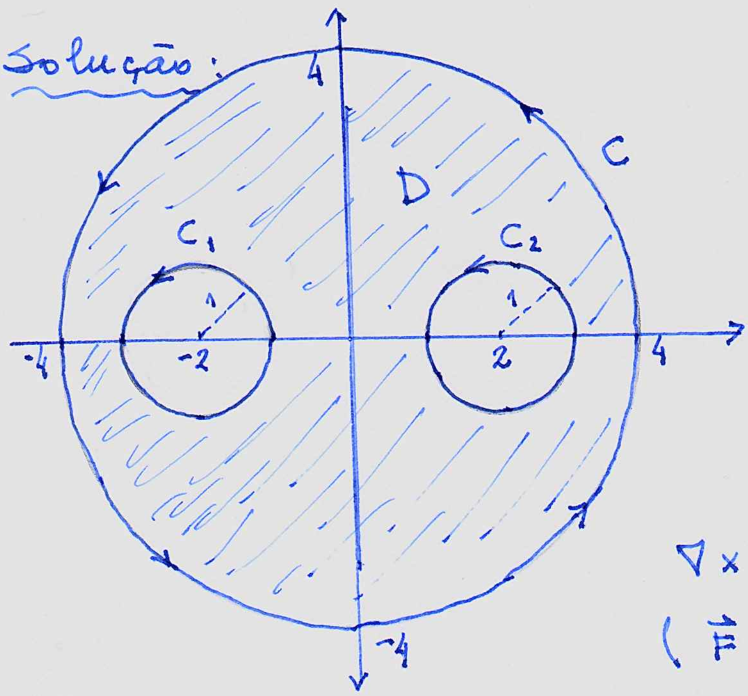


C_1: circunf. c/ centro $(-2,0)$ e raio 1

C_2: circunf. c/ centro $(2,0)$ e raio 1

Calcular $\oint_C \vec{F} \cdot d\vec{r}$ C circunf. de centro $(0,0)$ e raio 4

Solução:



Seja D a região cuja fronteira ∂D é $C \cup C_1 \cup C_2$ orientada positivamente

$\boxed{\partial D = C \cup C_1^- \cup C_2^-}$

$\nabla \times \vec{F} = rot(\vec{F}) = (Q_x - P_y)\vec{k}$

$(\vec{F} = P\vec{i} + Q\vec{j} + 0\vec{k})$

Pelo Teor. de Green

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_{\partial D} P\,dx + Q\,dy = \int_D (Q_x - P_y)\,dA = 0$$

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_{C_1^-} \vec{F} \cdot d\vec{r} + \int_{C_2^-} \vec{F} \cdot d\vec{r} = 0$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = -\int_{C_1^-} \vec{F} \cdot d\vec{r} - \int_{C_2^-} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= 4 + 2 = \underline{\underline{6}}$$

④ Calcule:

(a) $\displaystyle\int_C x\,dx + y\,dy + z\,dz$ $\qquad C: \begin{cases} \text{Interseção de } z = x^2+y^2 \\ \text{com } 2x+2y-z=1 \end{cases}$

Solução:

$$\left.\begin{array}{l} z = 2x+2y-1 \\ z = x^2+y^2 \end{array}\right\} \Rightarrow \begin{array}{l} x^2+y^2 = 2x+2y-1 \\ x^2-2x+y^2-2y = -1 \\ (x-1)^2+(y-1)^2 = 1 \end{array}$$

$$\begin{cases} x = 1+\cos t \\ y = 1+\operatorname{sen} t \\ z = 2(1+\cos t)+2(1+\operatorname{sen} t)-1 = 3+2\cos t+2\operatorname{sen} t \end{cases} \qquad 0 \le t \le 2\pi$$

$dx = -\operatorname{sen} t\,dt$

$dy = \cos t\,dt$

$dz = (-2\operatorname{sen} t + 2\cos t)\,dt$

$$x\,dx + y\,dy + z\,dz = \Big(-(1+\cos t)\operatorname{sen} t + (1+\operatorname{sen} t)\cos t + (3+2\cos t+2\operatorname{sen} t)(-2\operatorname{sen} t + 2\cos t)\Big)dt$$

$$= \left(\cos t - \operatorname{sen} t - 6\operatorname{sen} t + 6\cos t \cancel{-4\operatorname{sen} t\cos t} + 4\cos^2 t - 4\operatorname{sen}^2 t \cancel{+4\operatorname{sen} t\cos t}\right)dt$$

$$= \left(7\cos t - 7\operatorname{sen} t + 4\cos^2 t - 4\operatorname{sen}^2 t\right)dt$$

$$= \left(7(\cos t - \operatorname{sen} t) + 4\cos 2t\right)dt$$

$$\int_C x\,dx + y\,dy + z\,dz = \int_0^{2\pi}\left(7(\cos t - \operatorname{sen} t) + 4\cos 2t\right)dt$$

$$= -7(\cos t + \operatorname{sen} t)\Big|_0^{2\pi} + 2\operatorname{sen} 2t\Big|_0^{2\pi}$$

$$= \underline{\underline{0}}$$

(b) $\displaystyle\int_C \frac{ds}{x^2+y^2+z^2}$ C: parametrizada por

$\alpha(t) = (e^t\cos t, e^t\operatorname{sen} t, e^t)$

$0 \leq t \leq 1$

Solução

$$\alpha'(t) = (e^t\cos t - e^t\operatorname{sen} t,\ e^t\operatorname{sen} t + e^t\cos t,\ e^t)$$

$$\|\alpha'(t)\|^2 = e^{2t}\left((\cos t - \operatorname{sen} t)^2 + (\operatorname{sen} t + \cos t)^2 + 1\right)$$

$$= e^{2t}\Big(\underbrace{\cos^2 t + \operatorname{sen}^2 t}_{1} - \cancel{2\cos t\operatorname{sen} t} + \underbrace{\operatorname{sen}^2 t + \cos^2 t}_{1} + \cancel{2\operatorname{sen} t\cos t} + 1\Big)$$

$$= e^{2t} \cdot 3 = 3e^{2t}$$

$$\therefore \quad \|\alpha'\| = \sqrt{3}\, e^t$$

$$x^2+y^2+z^2 = e^{2t}(\cos^2 t + \operatorname{sen}^2 t + 1) = 2e^{2t}$$

$$\int_C \frac{ds}{x^2+y^2+z^2} = \int_0^1 \frac{1}{2e^{2t}} \cdot \sqrt{3}\, e^t\, dt$$

$$= \int_0^1 \frac{\sqrt{3}}{2} \cdot \frac{1}{e^t}\, dt = \frac{\sqrt{3}}{2}\int_0^1 e^{-t}\, dt$$

$$= -\frac{\sqrt{3}}{2} e^{-t}\Big|_0^1 = -\frac{\sqrt{3}}{2}\left(e^{-1} - 1\right)$$

$$= \frac{\sqrt{3}}{2}\left(1 - \frac{1}{e}\right)$$