UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
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Cálculo III - Tarde - 1^a Prova
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Nome: ______________________________ Matrícula: ________________

Questão 1 *(2.0 pts) Calcule a massa do sólido W limitado dentro do cone $z = \sqrt{3(x^2+y^2)}$ e limitado por cima pela esfera $x^2+y^2+(z-1)^2 = 1$, sendo sua densidade igual ao quadrado da distância do ponto $(x,y,z) \in W$ ao plano $z = 0$.*

Questão 2 *(2.0 pts.) Calcule $\int_D (x^4-y^4)e^{xy}dA$, onde D é a região no primeiro quadrante limitada pelas hipérboles $xy = 1$, $xy = 3$, $x^2-y^2 = 3$ e $x^2-y^2 = 4$.*
Sugestão: *Tente a mudança de variáveis $u = xy$ $v = x^2-y^2$.*

Questão 3 *(3.0 pts) Dada a integral dupla: $\int_D f dA = \int_{-1}^{1}\int_{-1-\sqrt{1-x^2}}^{-1+\sqrt{1-x^2}} f(x,y)dydx$*

(a) Faça um esboço da região D;

(b) inverta a ordem de integração;

(b) Calcule a integral no caso em que $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.

Questão 4 *(3.0 pts.) Calcule as integrais abaixo:*

(a) $\int_W \sqrt{1+(x^2+y^2+z^2)^{\frac{3}{2}}}dV, \qquad W : x^2+y^2+z^2 \leq 1;$

(b) $\int_W (x^2+y^2)dV, \qquad W : \sqrt{x^2+y^2} \leq z \leq 1.$

Boa Prova !!

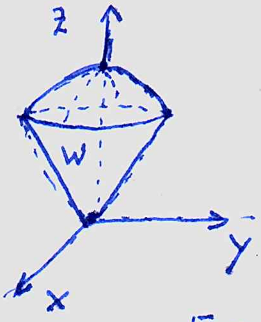
C3 P1: 2024.1 (Tarde)

(1) Massa de W = ?

W $\begin{cases} \text{limitado dento do cone } z = \sqrt{3(x^2+y^2)} \text{ e} \\ \text{limitado por cima pela esfera } x^2+y^2+(z-1)^2 = 1 \end{cases}$

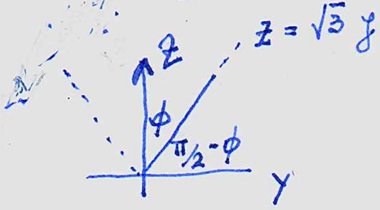
densidade = quadrado da distância ao plano $z = 0$

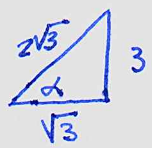
$(\ f(x,y,z) = z^2\ )$



Coord. Esféricas : $\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$

$W_{\rho\phi\theta}: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/6 \\ 0 \leq \rho \leq 2\cos\phi \end{cases}$





$\alpha = \pi/2 - \phi$

$tg(\pi/2 - \phi) = \sqrt{3}$

$\cos\alpha = \frac{\sqrt{3}}{2\sqrt{3}} = 1/2$

$\Rightarrow \alpha = 60°$

$\phi = 30° = \pi/6$

$x^2+y^2+(z-1)^2 = 1$

$\Rightarrow x^2+y^2+z^2-2z+\cancel{1} = \cancel{1}$

$\Rightarrow \rho^2 - 2\rho\cos\phi = 0$

$\Rightarrow \rho = 2\cos\phi$

$$M(W) = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\cos\phi} \rho^2\cos^2\phi \cdot \rho^2 \operatorname{sen}\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/6} \left(\frac{1}{5}\rho^5 \Big|_0^{2\cos\phi} \right) \cos^2\phi \operatorname{sen}\phi \, d\phi$$

$$= \frac{2\pi}{5} \cdot 2^5 \int_0^{\pi/6} \cos^7\phi \operatorname{sen}\phi \, d\phi \qquad \left(\cos \pi/6 = \sqrt{3}/2 \right)$$

$$= \frac{2^6}{5}\pi \left(-\frac{1}{8}\cos^8\phi \Big|_0^{\pi/6} \right)$$

$$= \frac{2^6}{5}\pi \left(\frac{1}{8} - \left(\frac{\sqrt{3}}{2}\right)^8 \cdot \frac{1}{8} \right) = \frac{2^3}{5}\pi \left(1 - \frac{3^4}{2^8} \right)$$

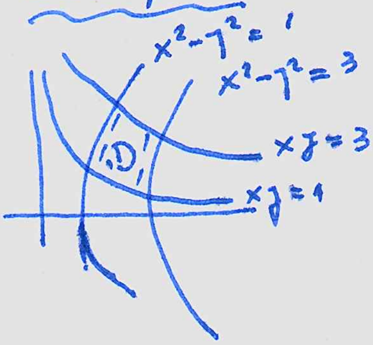
$$= \dots = \frac{35}{32}\pi$$

② Calcular $\int_D (x^4-y^4)\,e^{xy}\,dA$

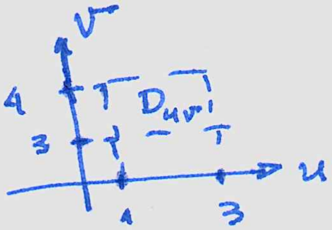
D região no 1º quadrante limitada por

$xy=1$, $xy=3$, $x^2-y^2=3$ e $x^2-y^2=4$

Solução



$$\begin{cases} u = xy \\ v = x^2-y^2 \end{cases}$$



$$D_{uv}: \begin{cases} 1 \leq u \leq 3 \\ 3 \leq v \leq 4 \end{cases}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} = -2y^2-2x^2 = -2(x^2+y^2)$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{-1}{2(x^2+y^2)}$$

$$(x^4-y^4)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = (x^4-y^2)\,\frac{1}{2(x^2+y^2)} = \frac{1}{2}(x^2-y^2) = \frac{1}{2}v$$

$$\int_D (x^4-y^4)e^{xy}\,dA = \int_1^3\int_3^4 \frac{1}{2}\,v\,e^{u}\,dv\,du$$

$$= \frac{1}{4}\int_1^3 \left(v^2\Big|_3^4\right)e^u\,du = \frac{1}{4}(16-9)\int_1^3 e^u\,du$$

$$= 7/4\; e^u\Big|_1^3 = \frac{7}{4}(e^3-e)$$

(3) $$I = \int_D f\,dA = \int_{-1}^{1} \int_{-1-\sqrt{1-x^2}}^{-1+\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

a) Esboço da região D

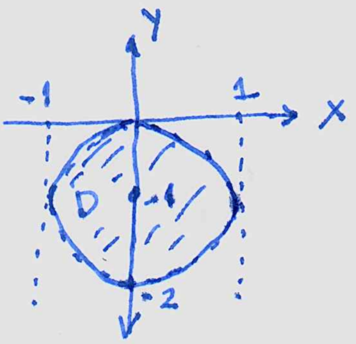
b) Inverter a ordem de integração

c) calcular I no caso $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$

Solução

a) $$D: \begin{cases} -1 \leq x \leq 1 \\ -1-\sqrt{1-x^2} \leq y \leq -1+\sqrt{1-y^2} \end{cases}$$

$$y = -1+\sqrt{1-x^2}$$
$$\Rightarrow (y+1)^2 = 1-x^2$$
$$\Rightarrow x^2+(y+1)^2 = 1$$



b) $$x^2+(y+1)^2 = 1 \Rightarrow x^2 = 1-(y+1)^2$$
$$\Rightarrow x = \pm\sqrt{1-(y+1)^2}$$

$$D: \begin{cases} -2 \leq y \leq 0 \\ -\sqrt{1-(y+1)^2} \leq x \leq \sqrt{1-(y+1)^2} \end{cases}$$

$$I = \int_{-2}^{0} \int_{-\sqrt{1-(y+1)^2}}^{\sqrt{1-(y+1)^2}} f(x,y)\,dx\,dy$$

c) $$\int_{-1}^{1} \int_{-1-\sqrt{1-x^2}}^{-1+\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$$

coord. Polares
$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ dy\,dx = r\,dr\,d\theta \end{cases}$$

$$D_{r\theta}: \begin{cases} \pi \leq \theta \leq 2\pi \\ 0 \leq r \leq -2\operatorname{sen}\theta \end{cases}$$

$$y = -1 \pm \sqrt{1-x^2}$$
$$(y+1)^2 = 1 - x^2$$
$$x^2 + (y+1)^2 = 1$$
$$x^2 + y^2 + 2y + 1 = 1$$
$$r^2 + 2r\operatorname{sen}\theta = 0$$
$$\{ r = -2\operatorname{sen}\theta$$

$$I = \int_{\pi}^{2\pi} \int_{0}^{-2\operatorname{sen}\theta} \frac{1}{\sqrt{r^2}} \cdot r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi} -2\operatorname{sen}\theta\, d\theta$$

$$= 2\cos\theta \Big|_{\pi}^{2\pi} = 2(1-(-1))$$

$$= 4$$

(4) Calcular

(a) $\displaystyle\int_{W} \sqrt{1 + (x^2+y^2+z^2)^{3/2}}\, dV \qquad W: x^2+y^2+z^2 \leq 1$

solução

coord. Esféricas.

$$W_{\rho\phi\theta}: \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} \sqrt{1+\rho^3} \cdot \rho^2 \operatorname{sen}\phi\, d\phi\, d\theta\, d\rho$$

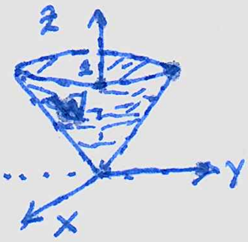
$$= 2\pi \int_0^1 \sqrt{1+\rho^3}\, \rho^2 \left(-\cos\phi \Big|_0^{\pi}\right) d\rho$$

$$= 4\pi \int_0^1 \sqrt{1+\rho^3}\, \rho^2\, d\rho = \frac{4\pi}{3} \int_0^1 \sqrt{1+\rho^3} \cdot 3\rho^2\, d\rho$$

$$= \frac{4\pi}{3} \cdot \frac{2}{3} (1+\rho^3)^{3/2} \Big|_0^1 = \frac{8\pi}{9}\left(2^{3/2} - 1\right)$$

(b) $\int_W (x^2+y^2)\,dV \qquad W: \sqrt{x^2+y^2} \leq z \leq 1$

Solução



coord cilíndricas

$$W_{r\theta z}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 1 \end{cases} \qquad \left(\begin{array}{l} (x,y) \in D \\ \quad D: x^2+y^2 \leq 1 \end{array} \right)$$

$$\int_0^1 \int_0^{2\pi} \int_r^1 r^2 \cdot r \, dz\, d\theta\, dr$$

$$= 2\pi \int_0^1 r^3 (1-r)\, dr = 2\pi \int_0^1 (r^3 - r^4)\, dr$$

$$= 2\pi \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{2\pi}{20} = \pi/10$$