UNIVERSIDADE FEDERAL DA PARAÍBA
CCEN - Departamento de Matemática
http://www.mat.ufpb.br

Cálculo III - 1^a Prova - 2024.2

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Nome: ______________________________ Matrícula: ____________________

Obs: Esta Prova vale 5 pontos.

Questão 1 *(3.0 pts) Em cada caso, calcule a integral:* $\int_D f(x,y)dA$.

(a) $f(x,y) = x,\qquad D : \begin{cases} x^2+y^2 \leq 4 \\ x^2+y^2 \geq 2x \\ x \geq 0,\ y \geq 0 \end{cases}$

(b) $f(x,y) = \frac{x}{x^2+y^2}\qquad D : \begin{cases} x^2+y^2 \leq 4 \\ x \geq 1 \end{cases}$

Questão 2 *(2.0 pts.) O volume do cilindro sobre o domínio D limitado superiormente pelo gráfico da função $f : D \subset \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = e^{x^2+y^2}$ é dado por:*

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}dydx - \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}dydx$$

(a) Faça um esboço da região D e expresse V numa única integral dupla sobre D;

(b) Calcule o volume V.

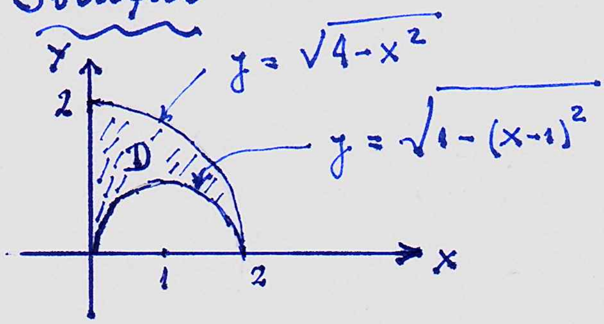
Boa Prova !!

C3 - Prova 1 - 2024.2

① Calcular $\displaystyle\iint_D f\,dA$.

(a) $f(x,y) = x$ $\quad D: \begin{cases} x^2+y^2 \leq 4 \\ x^2+y^2 \geq 2x \\ x \geq 0,\ y \geq 0 \end{cases}$

Solução



$x^2+y^2 = 2x$

$\Rightarrow x^2-2x+y^2 = 0$

$\Rightarrow (x-1)^2+y^2 = 1$

Coord. Polares

$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$ $\quad x^2+y^2 = r^2$

$\quad dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta$

$\Rightarrow r = 2\cos\theta$.

$D_{r\theta}: \begin{cases} 2\cos\theta \leq r \leq 2 \\ 0 \leq \theta \leq \pi/2 \end{cases}$

$$\iint_D f\,dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{1}{3}r^3\Big|_{2\cos\theta}^{2}\right)\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}(8-8\cos^3\theta)\cos\theta\,d\theta$$

$$\int_D f\,dA = \frac{8}{3}\int_0^{\pi/2}(\cos\theta - \cos^4\theta)\,d\theta$$

$$= \frac{8}{3}\,\text{sen}\,\theta\Big|_0^{\pi/2} - \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{8}{3} - \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^4\theta = \frac{1}{4}(1+\cos 2\theta)^2$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cdot\frac{1}{2}(1+\cos 4\theta)$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

$$\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{3}{8}\cdot\pi/2 + \underbrace{\frac{1}{4}\,\text{sen}\,2\theta\Big|_0^{\pi/2}}_{0} + \underbrace{\frac{1}{32}\,\text{sen}\,4\theta\Big|_0^{\pi/2}}_{0}$$

$$= \frac{3}{16}\pi$$

Daí

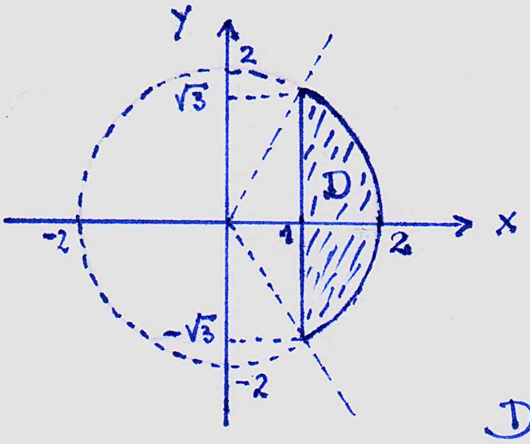
$$\int_D f\,dA = \frac{8}{3} - \frac{8}{3}\cdot\frac{3}{16}\pi$$

$$= \frac{8}{3}\left(1 - \frac{3\pi}{16}\right) = \frac{8}{3\cdot 16}(16 - 3\pi)$$

$$= \frac{16-3\pi}{6}$$

(b) $f(x,y) = \dfrac{x}{x^2+y^2}$ $\quad D: \begin{cases} x^2+y^2 \leq 4 \\ x \geq 1 \end{cases}$

Solução

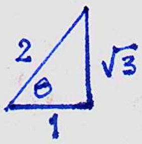


$$\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

$$x = 1 \Rightarrow r\cos\theta = 1$$
$$\Rightarrow r = \frac{1}{\cos\theta} = \sec\theta$$

$$D_{r\theta}: \begin{cases} \frac{1}{\cos\theta} \leq r \leq 2 \\ -\pi/3 \leq \theta \leq \pi/3 \end{cases}$$

$$\left.\begin{matrix} x^2+y^2 = 4 \\ x = 1 \end{matrix}\right\} \Rightarrow y^2 = 3$$
$$\Rightarrow y = \pm\sqrt{3}$$



$$\left.\begin{matrix} \cos\theta = 1/2 \\ \operatorname{sen}\theta = \frac{\sqrt{3}}{2} \end{matrix}\right\} \Rightarrow \theta = 60^\circ = \pi/3$$

$$\int_D f\,dA = \int_{-\pi/3}^{\pi/3} \int_{1/\cos\theta}^{2} \frac{r\cos\theta}{r^2} \cdot r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left(2 - \frac{1}{\cos\theta}\right)\cos\theta\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (2\cos\theta - 1)\,d\theta$$

$$= 2\operatorname{sen}\theta \Big|_{-\pi/3}^{\pi/3} - 2\pi/3$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - \frac{2\pi}{3} = 2\sqrt{3} - \frac{2\pi}{3}$$

(2) $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{x^2+y^2}$

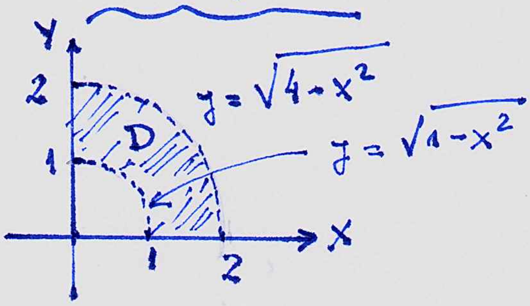
$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy\, dx - \int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$$

(a) Esboçar D

Escrever V numa única integral

(b) Calcular V

Solução



$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

$$D_{r\theta} : \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \pi/2 \end{cases}$$

$$V = \int_1^2 \int_0^{\pi/2} e^{r^2} \cdot r\, d\theta\, dr$$

$$V = \int_1^2 (\pi/2 - 0)\, r e^{r^2}\, dr$$

$$= \pi/2 \cdot \frac{1}{2} e^{r^2} \Big|_1^2 = \frac{\pi}{4}\left(e^4 - e\right)$$