**VI Workshop in Nonlinear PDE's and Geometric Analysis of UFPB**

# Hénon type equations with jumping nolinearities involving critical growth

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December 27, 2017

## Abstract

In this work, we search for two non-trivial radially symmetric solutions of the Dirichlet problem involving a Hénon-type equation of the form

$$\begin{cases} -\Delta u &= \lambda u + |x|^{\alpha} k(u_+) + f(x) \quad \text{in} \quad B_1, \\ u &= 0 \quad \text{on} \quad \partial B_1, \end{cases} \tag{1}$$

where $\lambda > 0$, $\alpha \geq 0$, $B_1$ is a unity ball centered at the origin of $\mathbf{R}^N$ $(N \geq 3)$ and $k(s) = s^{2^*_\alpha - 1} + g(s)$ with $2^*_\alpha = 2(N+\alpha)/(N-2)$ and $g(s)$ is a $C^1$ function in $[0, +\infty)$ which is assumed to be in the subcritical growth range.

The proofs are based on variational methods and to ensure that the considered minimax levels lie in a suitable range, special classes of approximating functions which have disjoint support with Talenti functions (Hénon version) are constructed.

### 0.1 Hypotheses

Before stating our main results, we shall introduce the following assumptions on the nonlinearity $g$:

$(g_0)$ $g \in C(\mathbb{R}, \mathbb{R}^+)$, $g(s) = o(s)$ when $s \to 0_+$ and $g(s) = 0$ for all $s \leq 0$.

$(g_1)$ There exist positive constants $c_1, D$ and $s_0$ and $2 < p+1 < 2^*_\alpha$ such that $g(s) \leq c_1 s^p + D$ for all $s \geq s_0$.

$(g_2)$ There exists $c_2 > 0$ and $q$ such that $g(s) \geq c_2 s^q$ for all $s \in \mathbb{R}^+$, where

$$
\begin{cases}
2^*_\alpha - \dfrac{2N-8}{3N-8} & < \quad q+1 \quad < \quad 2^*_\alpha & & \text{for} \quad N \geq 5; \\[2ex]
(4+\alpha) - \dfrac{2}{5} & < \quad q+1 \quad < \quad 4+\alpha & = \quad 2^*_\alpha & \text{for} \quad N = 4; \\[2ex]
(6+2\alpha) - \dfrac{2}{5} & < \quad q+1 \quad < \quad 6+2\alpha & = \quad 2^*_\alpha & \text{for} \quad N = 3.
\end{cases} \tag{2}
$$

Let us consider $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots$ the sequence of eigenvalues of $(-\Delta, H^1_0(B_1))$ and $e_j$ is a $j^{th}$ eigenfunction of $(-\Delta, H^1_0(B_1))$. Assuming $(g_0)$ and that $\lambda \neq \lambda_j$ for all $j$, one can prove that $\psi$ is a nonpositive solution of 1 if and only if it is a nonpositive solution for the linear problem

$$
\begin{cases}
-\Delta\psi & = \lambda\psi + f(x) \quad \text{in} \quad B_1, \\
\psi & = 0 \qquad\qquad\quad \text{on} \quad \partial B_1.
\end{cases} \tag{3}
$$

In order to obtain such solutions for 3, we assume that

$(f_1)$ $f(x) = h(x) + te_1(x)$,

where $h \in L^\mu(B_1)$, $\mu > N$ and

$$
\int_{B_1} he_1 \quad dx = 0. \tag{4}
$$

The parameter $t$ will be used in the proof of the first Theorem of this work.

## 0.2 Statement of main results

We divide our results in two theorems. The first one deals with the first solution of the problem, which is nonpositive and is obtained by a simple remark about a linear problem related to our equation. The other theorem concerns the second solution and we need to consider the dimension which we are working. On condition $(f_1)$, for $N \geq 5$, we only need to assume $\mu > N$ in order to recover the compactness of the functional associated to Problem 1. In dimensions $N = 4$ and 3, we should consider $\mu \geq 8$ and $\mu \geq 12$, respectively, for this purpose.

**Theorem 1 (The linear problem)** *Assume $(f_1)$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then there exists a constant $T = T(h) > 0$ such that:*

**(i)** *If $\lambda < \lambda_1$, there exists $\psi_t < 0$, a solution for 3 and, consequently, for 1, for all $t < -T$.*

**(ii)** *If $\lambda > \lambda_1$, there exists $\psi_t < 0$, a solution for 3 and, consequently, for 1, for all $t > T$.*

*Furthermore, if $f$ is radially symmetric, then $\psi_t$ is radially symmetric as well.*

**Theorem 2** *Assume the existence of nonpositive radial solution $\psi$ of 1, conditions $(g_0) - (g_2)$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then, 1 possesses a second radial solution provided that $f \in L^\mu(B_1)$ with $\mu \geq 12$ if $N = 3$, $\mu \geq 8$ if $N = 4$ and $\mu > N$ if $N \geq 5$.*