Existence of bound and ground states for a class of Kirchhoff-Schrödinger equations involving critical Trudinger-Moser growth *

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Abstract

In this work we discuss the existence of bound and ground state solutions for a class of fractional Kirchhoff equations defined on the whole real line. The equation involves a nonlinear term with critical exponential growth in the Trudinger-Moser sense. We deal with periodic and asymptotically periodic potential which may change sign. We handle with the lack of compactness due to the unboundedness of the domain and the critical behavior of the nonlinearity. The main theorems are stated without the well known Ambrosetti-Rabinowitz condition at infinity.

1 Introduction

This work deals with the following class of fractional Kirchhoff equations

$$\left(a+b[u]_{1/2}^2\right)(-\Delta)^{1/2}u+V(x)u=f(x,u), \quad x\in\mathbb{R}, \tag{P}$$

where $a>0$, $b\geq 0$, $(-\Delta)^{1/2}$ denotes the *square root of the Laplacian* and the term

$$[u]_{1/2}=\left(\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y\right)^{1/2}$$

is the so-called *Gagliardo semi-norm* of the function u. We study the existence of bound and ground solutions for equation (P). It is worthwhile to mention that a solution u of finite energy for (P) is called a *bound state solution*. It is well known that $u\neq 0$ is called *ground state solution* if admits the smallest energy among the all nontrivial bound states of (P). This class of equations imposes several difficulties. The first one is the "lack of compactness" inherited by the nature of the problem by reason of equation (P) is defined in the whole space $\mathbb{R}$ and involves critical nonlinear terms. Furthermore, we have the presence of the term $b[u]_{1/2}^2$ that causes some mathematical difficulties and makes the study of such class of equations very interesting, since it is a nonlocal function, that is, it takes care of the behavior of the solution in the whole space. Moreover, we deal with periodic functions and perturbations of periodic functions. We consider a class of bounded potentials which may change the sign. The nonlinear terms have critical exponential growth in the sense of Trudinger-Moser and are assumed without the well known Ambrosetti-Rabinowitz condition at infinity. In order to overcome these difficulties, we use a variational approach based on minimax theorems. To the best of our knowledge, there are few papers in the literature on fractional Kirchhoff equations involving critical exponential growth. Our results may be considered as the extension of the ones obtained in [1], see Remark 1.1.

Remark 1.1. *If $a=1$ and $b=0$, then Problem (P) reduces to the fractional Schrödinger equation*

$$(-\Delta)^s u+V(x)u=f(x,u), \quad x\in\mathbb{R}^N,$$

which had been studied by many authors under many different assumptions on the potential $V(x)$ and nonlinearity $f(x,u)$. Here we improve the results obtained in [1], since we obtain ground state solutions and we are not assuming the Ambrosetti-Rabinowitz condition at infinity.

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First, we are interested to study the following class of fractional Kirchhoff equations

$$\left(a+b[u]_{1/2}^2\right)(-\Delta)^{1/2}u+V_0(x)u=f_0(x,u),\quad x\in\mathbb{R}, \tag{P_0}$$

where $V_0(x)$ and $f_0(x,u)$ are periodic functions on x and $V_0(x)$ satisfies the following assumptions:

(V_1) There exists a positive constant v_0 such that $V_0(x)\geq -v_0$ for all $x\in\mathbb{R}$;

(V_2) The infimum

$$\inf_{\substack{u\in E_0\\ \|u\|_2=1}}\left(\frac{a}{2\pi}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}}V_0(x)u^2\,\mathrm{d}x\right)$$

is positive.

We are interested in the case that the nonlinear term has *critical exponential growth* and in addition, we suppose that $f_0(x,t)$ is a continuous 1-periodic function in x and satisfies the following hypotheses:

(f_1) $0\leq \lim\limits_{t\to 0}\dfrac{f_0(x,t)}{t}<\lambda_1$;

(f_2) $f_0:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is locally bounded in t, that is, for any bounded interval $J\subset\mathbb{R}$, there exists $C>0$ such that $|f_0(x,t)|\leq C$, for all $(x,t)\in\mathbb{R}\times J$;

(f_3) $f_0(x,t)/|t|^3$ is strictly increasing for $t\in\mathbb{R}$, for all $x\in\mathbb{R}$;

(f_4) There exist constants $p>4$ and $C_p>0$ such that

$$f_0(x,t)\geq C_p t^{p-1},\quad \text{for all }\ (x,t)\in\mathbb{R}\times\mathbb{R}.$$

2 Main Results

Theorem 2.1. *Assume that (V_1), (V_2), (f_1)-(f_4) hold and f_0 has critical exponential growth. Then problem (P_0) has a nontrivial bound state solution for some $C_p>0$ large enough.*

We are also concerned with the existence of solutions to the class of fractional Kirchhoff equations given in (P). In this case in order to deal with the difficulties imposed by the lack of periodicity, we require assumptions which compare the periodic terms with the asymptotically periodic terms. On the potential $V(x)$ we assume that:

(V_3) $V_0-V\in\mathcal{F}$ and $V_0(x)\geq V(x)\geq -v_0$, for all $x\in\mathbb{R}$;

(V_4) The infimum

$$\inf_{\substack{u\in E\\ \|u\|_2=1}}\left(\frac{a}{2\pi}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}}V(x)u^2\,\mathrm{d}x\right)$$

is positive.

On the nonlinear term, we assume the following assumptions:

(f_5) For any $\varepsilon>0$, there exists $\eta>0$ such that for $t\in\mathbb{R}$ and $|x|\geq\eta$ we have

$$|f(x,t)-f_0(x,t)|\leq\varepsilon \mathrm{e}^{\alpha t^2}.$$

Moreover, $f_0(x,t)\leq f(x,t)$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}$.

Theorem 2.2. *Assume that the potential $V(x)$ satisfies (V_1)-(V_4). If the nonlinearity f satisfies (f_1)-(f_5) and has critical exponential growth then Problem (P) has a ground state solution for some $C_p>0$ large enough.*

References

[1] Araújo Y. L., Souza M. de - On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth. *Math. Nachr.* **289**, 610–625, 2016.