

SWITCHING CONTROLS FOR ANALYTIC SEMIGROUPS AND APPLICATIONS TO PARABOLIC SYSTEMS*

FELIPE WALLISON CHAVES-SILVA[†], SYLVAIN ERVEDOZA[‡], AND DIEGO A. SOUZA[§]

Abstract. In this work, we extend the analysis of the problem of switching controls proposed in [E. Zuazua, *J. Eur. Math. Soc. (JEMS)*, 13 (2011), pp. 85–117]. The problem asks the following question: Assuming that one can control a system using two or more actuators, does there exist a control strategy such that at all times, only one actuator is active? We answer positively when the controlled system corresponds to an analytic semigroup spanned by a positive self-adjoint operator which is null-controllable in arbitrary small times. Similarly to [E. Zuazua, *J. Eur. Math. Soc. (JEMS)*, 13 (2011), pp. 85–117], our proof relies on analyticity arguments and will also work in finite dimensional settings and under some further spectral assumptions when the operator spans an analytic semigroup but is not necessarily self-adjoint.

Key words. switching control, parabolic systems, controllability, analyticity

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1. Introduction.

Setting and main results. In this article, we are interested in the following system:

$$(1.1) \quad y' + Ay = Bu, \quad t \in (0, T), \quad y(0) = y_0 \in H.$$

Here, y is the state variable, assumed to belong to a Hilbert space H , $'$ denotes the time derivative, A describes the free dynamics, and $-A$ generates a C^0 semigroup. The function u is the control, acting on the system through the control operator B , which is assumed to be in $\mathcal{L}(U, H)$, where U is a Hilbert space, and u will be searched in the space $L^2(0, T; U)$, with $T > 0$.

Controllability of systems of the form (1.1) has been analyzed thoroughly in many works. We do not intend to give an exhaustive account of the theory, and we simply refer the reader to the textbook [30].

Here, we focus on the case where U can be identified with $U_1 \times U_2$ through an isomorphism, i.e.,

$$(1.2) \quad \text{there exists a linear isomorphism } \pi : U_1 \times U_2 \rightarrow U,$$

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[†]Department of Mathematics, Federal University of Paraíba, UFPB, CEP 58050-085, João Pessoa-PB, Brazil (fchaves@mat.ufpb.br).

[‡]Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux, Bordeaux INP, CNRS, F-33400 Talence, France (sylvain.ervedoza@math.u-bordeaux.fr).

[§]Dpto. E.D.A.N., University of Seville, Aptdo 1160, 41080 Seville, Spain (desouza@us.es).

so that we can associate to $B \in \mathcal{L}(U, H)$ two operators $B_1 \in \mathcal{L}(U_1, H)$ and $B_2 \in \mathcal{L}(U_2, H)$ such that

$$(1.3) \quad \forall (u_1, u_2) \in U_1 \times U_2, \quad B\pi(u_1, u_2) = B_1 u_1 + B_2 u_2.$$

The control problem (1.1) can then be rewritten as

$$(1.4) \quad y' + Ay = B_1 u_1 + B_2 u_2, \quad t \in (0, T), \quad y(0) = y_0,$$

with $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$.

The question we are interested in is the possibility of constructing *switching controls*, that is, controls $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$ such that

$$(1.5) \quad \text{a.e. in } t \in (0, T), \quad \|u_1(t)\|_{U_1} \|u_2(t)\|_{U_2} = 0.$$

Informally, this means that at each time t , only one control is active.

Of course, under condition (1.5) one cannot expect to have better controllability properties for (1.4) than for the general case (1.1). We thus assume some controllability properties for (1.1) and discuss which properties can be obtained for control problem (1.4) under condition (1.5).

More precisely, we will assume that system (1.1) is null-controllable in arbitrary small times; i.e., for all $T > 0$, there exists a constant C_T such that for all $y_0 \in H$, there exists $u \in L^2(0, T; U)$ such that the solution y of (1.1) satisfies

$$(1.6) \quad y(T) = 0,$$

and the control u verifies the inequality

$$(1.7) \quad \|u\|_{L^2(0, T; U)} \leq C_T \|y_0\|_H.$$

In fact, we would rather use the following equivalent observability property (see, e.g., [30, Theorem 11.2.1]): For all $T > 0$, there exists C_T such that for all $z_T \in H$, the solution z of

$$(1.8) \quad -z' + A^* z = 0, \quad t \in (0, T), \quad z(T) = z_T \in H$$

satisfies

$$(1.9) \quad \|z(0)\|_H \leq C_T \|B^* z\|_{L^2(0, T; U)}.$$

Our goal then is to show the following result.

THEOREM 1.1. *Assume that system (1.1) is null-controllable in arbitrary small times and that one of the following two conditions holds:*

- $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint positive definite operator with compact resolvent, H being a Hilbert space;
- H is a finite dimensional vector space.

Let $B \in \mathcal{L}(U, H)$, where U is a Hilbert space, and assume that U is isomorphic to $U_1 \times U_2$ for some Hilbert spaces U_1 and U_2 , and define B_1 and B_2 as in (1.3).

Then system (1.4) is null-controllable in arbitrary small times with switching controls, i.e., controls satisfying (1.5). More precisely, given any $T > 0$ and any $y_0 \in H$, there exist control functions $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$ such that the solution y of (1.4) satisfies (1.6), while the control functions satisfy the switching condition (1.5).

The proof of Theorem 1.1 is given in section 2. It is strongly inspired by the work [31] and revisits two ideas which are already presented there but that we exploit further. Indeed, to construct controls u_1 and u_2 , for $z_T \in H$ we minimize the functional

$$(1.10) \quad J(z_T) = \frac{1}{2} \int_0^T \max\{\|B_1^* z(t)\|_{U_1}^2, \alpha(t)\|B_2^* z(t)\|_{U_2}^2\} dt + \langle y_0, z(0) \rangle_H,$$

where z is the solution of the adjoint problem (1.8), and $\alpha = \alpha(t)$ is given by

$$(1.11) \quad \alpha(t) = 1 + \frac{1}{2} \sin(\omega t), \quad t \in \mathbb{R},$$

where $\omega \in \mathbb{R}^*$ is suitably chosen.

Similarly to [31], the main difficulty is guaranteeing that for any minimizer Z_T of J (in a suitable class to be defined later), the set $\{t \in (0, T), \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t)\|B_2^* Z(t)\|_{U_2}^2\}$ is of measure zero and thus either guarantees the switching structure of the controls provided or corresponds to the straightforward case $Z_T = 0$; see section 2 for more details.

As it turns out, this property depends on the analyticity of the semigroup of generator $-A^*$. Moreover, we shall use the fact that α is analytic and oscillates at infinity, and therefore no resonances effect preventing from the switching structure (1.5) can arise.

Before going further, let us remark that the work [31] proposed a similar strategy, see [31, pp. 94–95 and Theorem 2.2], but did not manage to conclude that the set $\{t \in (0, T), \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t)\|B_2^* Z(t)\|_{U_2}^2\}$ either is of zero measure or corresponds to the trivial case $Z_T = 0$ in the general setup we propose; there, only the finite dimensional case was considered, and it was assumed that B_1 and B_2 were scalar (i.e., $U_1 = U_2 = \mathbb{R}$) and that $(A, B_1 - \alpha_- B_2)$ and $(A, B_1 + \alpha_+ B_2)$ satisfy Kalman rank conditions for some α_- and α_+ in the accumulation sets of α at $-\infty$ and $+\infty$, respectively. Note in particular that these conditions are not satisfied for the 2×2 control system,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Some extensions were given in some particular infinite dimensional settings and for nonscalar control operators but under strong spectral assumptions. Namely, only the case of the heat equation has been discussed when the following assumptions are satisfied:

- The set of eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ satisfies the fact that for all $\Lambda \in \mathbb{R}$, there is at most one pair (k, ℓ) such that $\lambda_k + \lambda_\ell = \Lambda$,
- eigenvectors $(\varphi_k)_{k \in \mathbb{N}}$ of the Laplace operator satisfy $\|B_1^* \varphi_k\|_{U_1} \neq \|B_2^* \varphi_k\|_{U_2}$ for all $k \in \mathbb{N}$.

Here, our arguments avoid these strong spectral requirements by using the analytic function $\alpha = \alpha(t)$ in (1.10) and the fact that for α of the form (1.11), the set of accumulation points at $-\infty$ is a nontrivial interval. We emphasize that our work differs from [31] in the analysis of the set $\{t \in (0, T), \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t)\|B_2^* Z(t)\|_{U_2}^2\}$ and the sufficient conditions required to prove that it is of zero measure, allowing us to state the existence of switching controls under the minimal assumption that system (1.1) is null-controllable.

Remark 1.2. Let us also point out that this result is easy to obtain in finite dimensional settings, as was mentioned to us by Marius Tucsnak. Indeed, when H is of finite dimension, it is easy to check that for all $T > 0$, for any $i \in \{1, 2\}$, considering any nonempty open time interval I_i , the set $R_i(I_i)$ defined by

$$R_i(I_i) = \left\{ \int_0^T e^{-(T-s)A} B_i \mathbf{1}_{I_i}(s) u_i(s) ds, \text{ with } u_i \in L^2(0, T) \right\},$$

i.e., the reachable set for (1.4) at time T starting from $y_0 = 0$ and with control u_i acting only in the time interval I_i ($\mathbf{1}_{I_i}$ is the indicator function of the interval I_i), the other control being null, equals the set $\mathcal{R}_i$ defined by

$$\mathcal{R}_i = \text{Ran}(B_i, AB_i, \dots, A^{d-1}B_i),$$

where d is the dimension of the space H . In particular, $R_i(I_i)$ is independent of the choice of the time interval I_i .

Recall that if H is a finite dimensional space of dimension d and system (1.1) is controllable, the Kalman rank condition is satisfied, i.e., $\text{Ran}(B, AB, \dots, A^{n-1}B) = \mathbb{R}^d$, so that by construction (recall (1.3)) $\mathcal{R}_1 + \mathcal{R}_2 = \mathbb{R}^d$.

Therefore, using the above comments, given any initial datum $y_0 \in H$ and nonempty open time subintervals I_1 and I_2 of $(0, T)$, there exist controls $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$ such that the solution y of (1.4) satisfies (1.6), while u_1 is supported in I_1 and u_2 is supported in I_2 .

Even if this is a stronger statement than Theorem 1.1 in the case of finite dimension, our approach has the advantage of building a strategy which naturally constructs switching controls and optimizes the choice of switching times, while the above result gives switching structures through a priori choices of supports of controls u_1 and u_2 .

In fact, our proofs can be adapted to the case of more than two control operators and to unbounded control operators $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$. Assume that U is isomorphic to $U_1 \times \dots \times U_n$ for some $n \in \mathbb{N}^*$ satisfying $n \geq 2$, i.e.,

$$(1.12) \quad \text{there exists a linear isomorphism } \pi : U_1 \times \dots \times U_n \rightarrow U,$$

so that we can associate to $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$ n operators $B_i \in \mathcal{L}(U_i, \mathcal{D}(A^*)')$, $i \in \{1, \dots, n\}$, by the formula

$$(1.13) \quad \forall (u_1, \dots, u_n) \in U_1 \times \dots \times U_n, \quad B\pi(u_1, \dots, u_n) = \sum_{i=1}^n B_i u_i.$$

When we have n controls $u_i \in L^2(0, T; U_i)$, the interesting notion of *switching control* is the following:

$$(1.14) \quad \text{a.e. in } t \in (0, T), \quad \prod_{i=1}^n \left(\sum_{j \neq i} \|u_j(t)\|_{U_j} \right) = 0.$$

In other words, we say that controls $(u_1, \dots, u_n) \in L^2(0, T; U_1 \times \dots \times U_n)$ are switching if almost everywhere (a.e.) in $t \in (0, T)$, at most one control is active.

We then claim that Theorem 1.1 can be generalized to this case as follows.

THEOREM 1.3. *Assume that system (1.1) is null-controllable in arbitrary small times and that one of the following two conditions holds:*

- $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint positive definite operator with compact resolvent, with H being a Hilbert space;
- H is a finite dimensional vector space.

Let $B \in \mathcal{L}(U, \mathcal{D}(A^*))'$, where U is a Hilbert space, let $n \in \mathbb{N}^*$ with $n \geq 2$, assume that U is isomorphic to $U_1 \times \cdots \times U_n$ for some Hilbert spaces U_i , $i \in \{1, \dots, n\}$, and define B_i for $i \in \{1, \dots, n\}$ as in (1.13).

Then the system

$$(1.15) \quad y' + Ay = \sum_{i=1}^n B_i u_i, \quad t \in (0, T), \quad y(0) = y_0,$$

is null-controllable in arbitrary small times with switching controls, i.e., controls satisfying (1.14). More precisely, given any $T > 0$ and any $y_0 \in H$, there exist n control functions $u_i \in L^2(0, T; U_i)$, $i \in \{1, \dots, n\}$, such that solution y of (1.15) satisfies (1.6), while the control functions satisfy switching condition (1.14).

The proof of Theorem 1.3 is given in section 3 and follows the same steps as those in the proof of Theorem 1.1.

In section 4 we will give several examples of applications, in particular regarding general parabolic systems and the Stokes problem. We also explain under which assumptions Theorems 1.1 and 1.3 can be extended to non-self-adjoint operators A with a compact resolvent which generates an analytic semigroup; see section 5 and Theorem 5.1. However, it is important to note immediately that the assumptions required to deal with non-self-adjoint operators seem quite delicate to check in practice (as we will explain in two examples) due to the possible complexity of the spectrum in those cases.

Related results. As stated above, this work is strongly related to the work [31], which triggered our analysis. But more generally, it is related to the common idea that minimizing ℓ^1 norms enforces sparsity. This idea has been developed thoroughly in the context of optimal control; see, e.g., [1, 21, 22, 23] and references therein.

As we will see later in the examples in section 4, when considering parabolic systems or the Stokes problem, Theorem 1.3 will easily provide controllability results with controls having at each time at most one active component. This is in sharp contrast to the questions addressed for parabolic systems or Stokes models when the control can act on only one component, in which the controllability properties can be strongly modified depending on the geometry of the domains or the time of controllability (see, e.g., [2, 3, 14] and the references therein), while the use of nonlinear terms may help reestablish control properties; see, e.g., the works [7, 9, 10]. In other words, the notion we are analyzing in this context truly lies in between the notions of controllability with controls acting on all components and controllability with controls acting on only one component.

2. Proof of Theorem 1.1. The structure of the proof of Theorem 1.1 is exactly the same whether A is a self-adjoint operator or H is a finite dimensional space, and it closely follows the proof presented in [31].

Let $y_0 \in H$ and $T > 0$ be fixed, and then introduce the functional J defined in (1.10) for $z_T \in H$ and z solving (1.8).

Since $\inf \alpha = 1/2 > 0$ and $\sup \alpha = 3/2 < \infty$, it is clear that the observability property (1.9) implies that for all $T > 0$, there exists a constant C_T such that for all

$z_T \in H$,

$$(2.1) \quad \|z(0)\|_H^2 \leq C_T^2 \int_0^T \max\{\|B_1^* z(t)\|_{U_1}^2, \alpha(t)\|B_2^* z(t)\|_{U_2}^2\} dt.$$

Although the functional J in (1.10) is convex, the functional J is, in general, not coercive with respect to the norm of H (this is, for instance, the case when considering the heat equation). We thus introduce the space

$$(2.2) \quad X = \overline{H}^{\|\cdot\|_{obs}},$$

i.e., the completion of the space H with respect to the norm $\|\cdot\|_{obs}$ given by

$$(2.3) \quad \|z_T\|_{obs}^2 = \int_0^T \max\{\|B_1^* z(t)\|_{U_1}^2, \alpha(t)\|B_2^* z(t)\|_{U_2}^2\} dt.$$

One then easily checks that, since this norm is equivalent to

$$\int_0^T \|B^* z(t)\|_U^2 dt,$$

for α of the form (1.11), the space X does not depend on the choice of the parameter ω in (1.11).

Using (2.1), it is clear that the functional J in (1.10) admits a unique extension (still denoted the same way) as a continuous functional in X , is coercive in X , and stays convex.

The functional J has therefore a minimizer $Z_T \in X$. To derive the Euler–Lagrange equation satisfied by Z_T , it is convenient to first analyze when the set

$$(2.4) \quad I = \{t \in (0, T), \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t)\|B_2^* Z(t)\|_{U_2}^2\}$$

is of nonzero measure.

Note that, when H is of finite dimension, $X = H$, and thus, for $Z_T \in H$, the function $t \mapsto \|B_1^* Z(t)\|_{U_1}^2 - \alpha(t)\|B_2^* Z(t)\|_{U_2}^2$ is in fact continuous on $[0, T]$. When H is of infinite dimension, the set X might be more intricate than H ; still, as we will see in the proof of Lemma 2.1, for $Z_T \in X$, the function $t \mapsto \|B_1^* Z(t)\|_{U_1}^2 - \alpha(t)\|B_2^* Z(t)\|_{U_2}^2$ is in fact continuous on any interval of the form $(0, T')$ with $T' < T$ (see (2.11)), and thus the set I is properly defined.

For the two cases we are interested in, we claim that the set I can be of nonzero measure only in the straightforward case $Z_T = 0$. This is precisely given in the following lemmas.

LEMMA 2.1. *When A is a self-adjoint positive definite operator with compact resolvent and α is as in (1.11) with $\omega \in \mathbb{R} \setminus \{0\}$, the set I is necessarily of zero measure, except in the case $\|B_1^* Z\|_{L^2(0,T;U_1)} = \|B_2^* Z\|_{L^2(0,T;U_2)} = 0$ where $I = (0, T)$.*

LEMMA 2.2. *Let H be a finite dimensional space. Let $(\lambda_k)_{k \in \{1, \dots, K\}}$ be the eigenvalues of the matrix A^* ordered so that $\Re(\lambda_k) \leq \Re(\lambda_{k+1})$ for all k , and define the set W as follows:*

$$(2.5) \quad W = \{0\} \cup \left\{ \Im(\lambda_k) - \Im(\lambda_{k_1}), \frac{1}{2}(\Im(\lambda_k) - \Im(\lambda_{k_1})), \right. \\ \left. \forall (k, k_1) \text{ such that } \Re(\lambda_k) = \Re(\lambda_{k_1}) \right\}.$$

Then, for α as in (1.11) with $\omega \in \mathbb{R} \setminus W$, the set I is necessarily of zero measure, except in the trivial case $\|B_1^*Z\|_{L^2(0,T;U_1)} = \|B_2^*Z\|_{L^2(0,T;U_2)} = 0$ where $I = (0, T)$.

The proofs of Lemmas 2.1 and 2.2 are postponed to sections 2.1 and 2.2, respectively.

Remark 2.3. We point out that Lemmas 2.1 and 2.2 do not use the unique continuation property $\|B_1^*Z\|_{L^2(0,T;U_1)} = \|B_2^*Z\|_{L^2(0,T;U_2)} = 0$, which implies that $Z = 0$ in $(0, T)$, but only the analyticity of the semigroup and the clear structure of the spectrum of the operator A when A is a matrix or a self-adjoint operator. This will be of interest when extending Theorem 1.1 to n operators; see section 3.

Based on the above results, using the observability property (1.9), we deduce that the set I is of zero measure except in the trivial case $Z_T = 0$. Therefore, when $Z_T \neq 0$, setting

$$(2.6) \quad I_1 = \{t \in (0, T), \|B_1^*Z(t)\|_{U_1}^2 > \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\},$$

$$(2.7) \quad I_2 = \{t \in (0, T), \|B_1^*Z(t)\|_{U_1}^2 < \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\},$$

we see that the Euler–Lagrange equation satisfied by Z easily yields the fact that for all $z_T \in H$,

$$(2.8) \quad 0 = \int_{I_1} \langle B_1^*Z(t), B_1^*z(t) \rangle_{U_1} dt + \int_{I_2} \alpha(t) \langle B_2^*Z(t), B_2^*z(t) \rangle_{U_2} dt + \langle y_0, z(0) \rangle_H;$$

see [31, pp. 91–93] for the careful justification of this identity, which we briefly recall in the appendix for completeness.

It is then easy to check that, setting

$$(2.9) \quad u_1(t) = \begin{cases} B_1^*Z(t) & \text{for } t \in I_1, \\ 0 & \text{for } t \in I_2, \end{cases} \quad u_2(t) = \begin{cases} 0 & \text{for } t \in I_1, \\ \alpha(t)B_2^*Z(t) & \text{for } t \in I_2, \end{cases}$$

the corresponding solution y of (1.4) satisfies (1.6), while u_1 and u_2 satisfy the switching condition (1.5).

On the other hand, it is easy to check that if $Z_T = 0$, then $y_0 = 0$, and the controls $u_1 = 0$ and $u_2 = 0$ are also suitable for controlling the trajectory (1.4) to zero at time T (i.e., (1.6)), and they obviously satisfy the switching condition (1.5).

It therefore remains to show Lemmas 2.1 and 2.2, whose proofs are given in the next sections.

2.1. Proof of Lemma 2.1: The case of a self-adjoint positive definite operator A with compact resolvent. In order to prove that the set I is of zero measure except when $\|B_1^*Z\|_{L^2(0,T;U_1)} = \|B_2^*Z\|_{L^2(0,T;U_2)} = 0$, we will consider a strictly positive and strictly increasing sequence T_n going to T as $n \rightarrow \infty$ and will show that for all $n \in \mathbb{N}$, the set

$$(2.10) \quad I_n = I \cap (0, T_n)$$

is of zero measure except in the trivial case where B_1^*Z and B_2^*Z vanish identically on $(0, T_n)$. This will entail as well that I is of zero measure except in the trivial case where B_1^*Z and B_2^*Z vanish identically.

Let $n \in \mathbb{N}$ and consider the corresponding T_n . From (1.9) applied between T_n and T , there exists C_n such that for all $z_T \in H$, the solution z of (1.8) satisfies

$$\sup_{t \in (0, T_n)} \|z(t)\|_H \leq C_n \|B^* z\|_{L^2(0, T; U)} \leq \sqrt{2} C_n \|z_T\|_{obs}.$$

Therefore, the map $z_T \in H \mapsto z(t) \in C^0([0, T_n]; H)$ extends by continuity to X , and in particular, for $Z_T \in X$,

$$(2.11) \quad \begin{aligned} &Z \text{ is well defined and continuous on } (0, T_n) \text{ with values in } H, \text{ and} \\ &-Z' + A^*Z = 0, \quad t \in (0, T_n), \quad Z(T_n) = Z_n \in H, \end{aligned}$$

and the set I_n can be equivalently defined as

$$I_n = \{t \in (0, T_n), \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t) \|B_2^* Z(t)\|_{U_2}^2\}.$$

Now, since Z satisfies (2.11), Z is an analytic function on $(0, T_n)$ because $-A^* = -A$ is the generator of an analytic semigroup, and it can thus be extended uniquely as an analytic function on $(-\infty, T_n)$ as the solution of

$$(2.12) \quad -Z' + A^*Z = 0, \quad t \in (-\infty, T_n), \quad Z(T_n) = Z_n \in H.$$

Therefore, since α also is an analytic function, if I_n is of positive measure, then

$$(2.13) \quad \forall t \in (-\infty, T_n), \quad \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t) \|B_2^* Z(t)\|_{U_2}^2.$$

Our next goal is to prove that (2.13) cannot be satisfied except in the trivial case $\|B_1^* Z\|_{L^2(0, T_n; U_1)} = \|B_2^* Z\|_{L^2(0, T_n; U_2)} = 0$. We thus assume (2.13).

Now, since A is a positive definite self-adjoint operator with compact resolvent, its spectrum is given by a positive strictly increasing sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \lambda_{k+1} \rightarrow \infty$ and of corresponding eigenspace $H_k = \text{Kernel}(A - \lambda_k I)$, which are two by two orthogonal.

We expand $Z_n \in H$ using this basis,

$$(2.14) \quad Z_n = \sum_{k \in \mathbb{N}} w_k, \quad \text{with } w_k \in H_k, \quad \text{and} \quad \|Z_n\|_H^2 = \sum_k \|w_k\|_H^2,$$

so that

$$(2.15) \quad \forall t < T_n, \quad Z(t) = \sum_{k \in \mathbb{N}} w_k e^{\lambda_k(t-T_n)}.$$

Now, let

$$(2.16) \quad k_0 = \inf\{k \in \mathbb{N}, \|B_1^* w_k\|_{U_1} + \|B_2^* w_k\|_{U_2} \neq 0\}.$$

Our goal is thus to check that k_0 cannot be finite. If k_0 is finite, then we should have

$$(2.17) \quad \|B_1^* w_{k_0}\|_{U_1} + \|B_2^* w_{k_0}\|_{U_2} \neq 0.$$

Therefore, setting

$$Z_r(t) = \sum_{k \neq k_0} w_k e^{\lambda_k(t-T_n)} \quad (t < T_n),$$

the identity (2.13) implies that for all $t < T_n$,

$$\begin{aligned} & \|B_1^* w_{k_0}\|_{U_1}^2 - \alpha(t) \|B_2^* w_{k_0}\|_{U_2}^2 \\ &= -2e^{-\lambda_{k_0}(t-T_n)} \Re(\langle B_1^* w_{k_0}, B_1^* Z_r(t) \rangle_{U_1} - \alpha(t) \langle B_2^* w_{k_0}, B_2^* Z_r(t) \rangle_{U_2}) \\ &\quad - e^{-2\lambda_{k_0}(t-T_n)} (\|B_1^* Z_r(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z_r(t)\|_{U_2}^2). \end{aligned}$$

Since

$$\exists C > 0 \forall t < T_n, \quad \|B_1^* Z_r(t)\|_{U_1} + \|B_2^* Z_r(t)\|_{U_2} \leq Ce^{\lambda_{k_0+1}(t-T_n)},$$

the last identity yields for all $t < T_n$,

$$|\|B_1^* w_{k_0}\|_{U_1}^2 - \alpha(t) \|B_2^* w_{k_0}\|_{U_2}^2| \leq Ce^{(\lambda_{k_0+1}-\lambda_{k_0})(t-T_n)} + Ce^{2(\lambda_{k_0+1}-\lambda_{k_0})(t-T_n)}.$$

Since $\lambda_{k_0+1} > \lambda_{k_0}$, making $t \rightarrow -\infty$, we obtain that

$$(2.18) \quad \forall \alpha_\infty \in \left[\liminf_{t \rightarrow -\infty} \alpha, \limsup_{t \rightarrow -\infty} \alpha \right], \quad \|B_1^* w_{k_0}\|_{U_1}^2 - \alpha_\infty \|B_2^* w_{k_0}\|_{U_2}^2 = 0.$$

Since $\liminf_{t \rightarrow -\infty} \alpha < \limsup_{t \rightarrow -\infty} \alpha$, we easily get that this implies

$$B_1^* w_{k_0} = 0 \quad \text{and} \quad B_2^* w_{k_0} = 0.$$

This contradicts (2.17), so that k_0 is infinite, and thus $B_1^* Z = 0$ and $B_2^* Z = 0$ on $(-\infty, T_n)$. This shows that, except when $B_1^* Z$ and $B_2^* Z$ vanish identically on $(0, T_n)$, I_n is of zero measure. In particular, passing to the limit $n \rightarrow \infty$, we easily get that I is of zero measure except if $B_1^* Z$ and $B_2^* Z$ vanish identically on $(0, T)$.

Remark 2.4. In the above proof, we did not use the specific form of α . In fact, as one can check, the proof of Lemma 2.1 works for any function α satisfying

$$(2.19) \quad \begin{cases} \alpha \text{ is an analytic function on } \mathbb{R}, \\ 0 < \inf_{\mathbb{R}} \alpha < \sup_{\mathbb{R}} \alpha < \infty, \\ \liminf_{t \rightarrow -\infty} \alpha < \limsup_{t \rightarrow -\infty} \alpha. \end{cases}$$

2.2. Proof of Lemma 2.2: The case of a finite dimensional space H . In order to prove Lemma 2.2, we will use the following result.

LEMMA 2.5. *Let J be a finite set, and let $(\mu_j)_{j \in J}$ be a finite sequence of two by two distinct real numbers.*

Then, for any finite sequence $(a_j)_{j \in J}$ of elements of $\mathbb{C}$ such that

$$(2.20) \quad \lim_{t \rightarrow -\infty} \left(\sum_{j \in J} a_j e^{i\mu_j t} \right) = 0,$$

we have

$$(2.21) \quad \forall j \in J, \quad a_j = 0.$$

Proof. To prove Lemma 2.5, we use the fact that since there is a finite number of μ_j ,

$$\int_0^1 \left| \sum_{j \in J} b_j e^{i\mu_j t} \right|^2 dt$$

is a norm on $\{b = (b_j)_{j \in J}, b_j \in \mathbb{C}\}$ and is thus equivalent to the quantity

$$\sum_{j \in J} |b_j|^2.$$

Now, for $(a_j)_{j \in J}$ as in (2.20), we have for any $T \in \mathbb{R}$,

$$\sum_{j \in J} |a_j|^2 = \sum_{j \in J} |a_j e^{-i\mu_j T}|^2 \leq C \int_0^1 \left| \sum_{j \in J} a_j e^{i\mu_j(t-T)} \right|^2 dt \leq C \int_{-T}^{-T+1} \left| \sum_{j \in J} a_j e^{i\mu_j t} \right|^2 dt.$$

Thus, choosing T going to $+\infty$, the assumption (2.20) and the above estimates give Lemma 2.5. $\square$

Let us now come back to the proof of Lemma 2.2. To begin, we put the matrix A^* into its Jordan form and call $(\lambda_k)_{k \in \{1, \dots, K\}}$ its eigenvalues ordered so that $\Re(\lambda_k) \leq \Re(\lambda_{k+1})$ for all k , and we call H_k the corresponding generalized eigenspaces.

We then prove that when $\omega \in \mathbb{R} \setminus W$ (recall the definition of W in (2.5)), with the choice of α as in (1.11), I necessarily is of zero measure except in the trivial case $\|B_1^* Z\|_{L^2(0,T;U_1)} = \|B_2^* Z\|_{L^2(0,T;U_2)} = 0$.

We thus assume that I is of nonzero measure and we let Z be the solution of (1.8) with initial datum $Z_T \in X$. Here, since H is finite dimensional, $X = H$ and $Z_T \in H$. Then the solution Z of (1.8) can be defined on $\mathbb{R}$ and is an analytic function of time, and we write it under the form

$$(2.22) \quad Z(t) = \sum_k e^{\lambda_k(t-T)} \left(\sum_{\ell=0}^{m_k} (T-t)^\ell w_{k,\ell} \right) \quad (t \in \mathbb{R}),$$

where m_k is the size of the maximal Jordan block corresponding to λ_k (or, equivalently, its algebraic multiplicity), and each $w_{k,\ell}$ belongs to H_k . Besides, since we assume that I is of nonzero measure and since Z in (2.22) is analytic with respect to time, we should have $I = (0, T)$, and it follows that

$$(2.23) \quad \forall t \in \mathbb{R}, \quad \|B_1^* Z(t)\|_{U_1}^2 = \alpha(t) \|B_2^* Z(t)\|_{U_2}^2.$$

Now, let

$$k_0 = \inf \{k \in \{1, \dots, K\} : \exists \ell \in \{0, \dots, m_k\} \text{ such that } \|B_1^* w_{k,\ell}\|_{U_1} + \|B_2^* w_{k,\ell}\|_{U_2} \neq 0\}.$$

If $k_0 < \infty$, we define ℓ_1 by

$$\ell_1 = \sup \{\ell : \exists k \text{ with } \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_1^* w_{k,\ell}\|_{U_1} + \|B_2^* w_{k,\ell}\|_{U_2} \neq 0\},$$

and consider the set

$$D = \{k : \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_1^* w_{k,\ell_1}\|_{U_1} + \|B_2^* w_{k,\ell_1}\|_{U_2} \neq 0\},$$

which describes the indices giving the dominant terms in $\|B_1^* Z(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z(t)\|_{U_2}^2$ as $t \rightarrow -\infty$. Indeed, setting

$$(2.24) \quad Z_d(t) = \sum_{k \in D} w_{k,\ell_1} e^{i\Im(\lambda_k)(t-T)} \quad \text{and} \quad Z_r(t) = Z(t) - e^{\Re(\lambda_{k_0})(t-T)} (T-t)^{\ell_1} Z_d(t),$$

we have, for some C independent of time,

$$(2.25) \quad \begin{aligned} \forall t \in (-\infty, T), \quad & \|B_1^* Z_r(t)\|_{U_1} + \|B_2^* Z_r(t)\|_{U_2} \\ & \leq \begin{cases} C e^{\Re(\lambda_{k_0})t} (1 + (T-t)^{\ell_1-1}) & \text{if } \ell_1 \geq 1, \\ C e^{(\Re(\lambda_{k_0+1}) + \Re(\lambda_{k_0}))t/2} & \text{if } \ell_1 = 0, \end{cases} \end{aligned}$$

and thus, possibly changing the constant,

$$(2.26) \quad \forall t \in (-\infty, T-1), \quad \|B_1^* Z_r(t)\|_{U_1} + \|B_2^* Z_r(t)\|_{U_2} \leq C e^{\Re(\lambda_{k_0})t} (T-t)^{\ell_1-1}.$$

Therefore, using (2.23), we easily get that

$$(2.27) \quad \forall t \in (-\infty, T-1), \quad \left| \|B_1^* Z_d(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z_d(t)\|_{U_2}^2 \right| \leq \frac{C}{T-t}.$$

Now, we expand $\|B_1^* Z_d(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z_d(t)\|_{U_2}^2$ as follows:

$$(2.28) \quad \begin{aligned} & \|B_1^* Z_d(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z_d(t)\|_{U_2}^2 \\ &= \sum_{k \in D} \|B_1^* w_{k, \ell_1}\|_{U_1}^2 - \left(1 + \frac{\sin(\omega t)}{2}\right) \sum_{k \in D} \|B_2^* w_{k, \ell_1}\|_{U_2}^2 \\ & \quad + 2 \sum_{k \in D} \sum_{k_1 \in D, k_1 > k} \Re \left(e^{i(\Im(\lambda_k) - \Im(\lambda_{k_1}))t} \langle B_1^* w_{k, \ell_1}, B_1^* w_{k_1, \ell_1} \rangle_{U_1} \right) \\ & \quad - 2 \left(1 + \frac{\sin(\omega t)}{2}\right) \sum_{k \in D} \sum_{k_1 \in D, k_1 > k} \Re \left(e^{i(\Im(\lambda_k) - \Im(\lambda_{k_1}))t} \langle B_2^* w_{k, \ell_1}, B_2^* w_{k_1, \ell_1} \rangle_{U_2} \right). \end{aligned}$$

From this, we deduce that the function $\|B_1^* Z_d(t)\|_{U_1}^2 - \alpha(t) \|B_2^* Z_d(t)\|_{U_2}^2$ is of the form $\sum_j a_j e^{i\mu_j t}$, where

$$\{\mu_j\} = \{0, \pm\omega, (\Im(\lambda_k) - \Im(\lambda_{k_1})), \pm\omega + (\Im(\lambda_k) - \Im(\lambda_{k_1})) \text{ for } k, k_1 \in D\}.$$

This set is finite, but there might be some nondistinct values in the set given on the right-hand side. We shall thus rely on the choice $\omega \notin W$ (recall that W is defined in (2.5)), which guarantees that 0 and ω appear only once in the above list. Therefore, using (2.27), Lemma 2.5 guarantees at least that the numbers in front of the constant term (corresponding to $\mu = 0$) and of $e^{i\omega t}$ in (2.2) vanish, i.e.,

$$\begin{aligned} 0 &= \sum_{k \in D} \|B_1^* w_{k, \ell_1}\|_{U_1}^2 - \sum_{k \in D} \|B_2^* w_{k, \ell_1}\|_{U_2}^2, \\ 0 &= \sum_{k \in D} \|B_2^* w_{k, \ell_1}\|_{U_2}^2. \end{aligned}$$

Combining the above two identities, we easily deduce that

$$\forall k \in D, \quad \|B_1^* w_{k, \ell_1}\|_{U_1} + \|B_2^* w_{k, \ell_1}\|_{U_2} = 0.$$

From its definition, it follows that the set D is necessarily empty. This contradicts the definition of k_0 and ℓ_1 . Hence, $k_0 = \infty$, $B_1^* Z(t) = 0$, and $B_2^* Z(t) = 0$ for all $t \in \mathbb{R}$.

3. Proof of Theorem 1.3. Of course, the proof of Theorem 1.3 follows the proof of Theorem 1.1. We point out only the main differences that are needed in the proof of Theorem 1.1 to conclude Theorem 1.3.

To fix ideas, we consider only the case $n = 3$, as the case of $n \geq 4$ control operators can be treated in the same way as the price of adding some notation.

Given $y_0 \in H$, we consider the functional

$$(3.1) \quad J(z_T) = \frac{1}{2} \int_0^T \max\{\alpha_1(t) \|B_1^* z(t)\|_{U_1}^2, \alpha_2(t) \|B_2^* z(t)\|_{U_2}^2, \alpha_3(t) \|B_3^* z(t)\|_{U_3}^2\} dt \\ + \langle y_0, z(0) \rangle_H,$$

defined for $z_T \in \mathcal{D}(A^*)$, where z is the solution of the adjoint problem (1.8), and $\alpha_i = \alpha_i(t)$ is given by

$$(3.2) \quad \alpha_i(t) = 1 + \frac{1}{2} \sin(\omega_i t), \quad t \in \mathbb{R}, \quad i \in \{1, 2, 3\},$$

where the frequencies ω_i are suitably chosen.

Similarly to the proof of Theorem 1.1, the functional J can be extended by continuity on the space

$$X = \overline{\mathcal{D}(A^*)}^{\|\cdot\|_{obs}},$$

where the norm $\|\cdot\|_{obs}$ is the one defined by

$$\|z_T\|_{obs}^2 = \int_0^T \max\{\alpha_1(t) \|B_1^* z(t)\|_{U_1}^2, \alpha_2(t) \|B_2^* z(t)\|_{U_2}^2, \alpha_3(t) \|B_3^* z(t)\|_{U_3}^2\} dt$$

and is coercive on that space X . Therefore, J has a minimizer $Z_T \in X$. Next, to properly derive the Euler–Lagrange equation satisfied by Z_T , we study the sets

$$(3.3) \quad \forall (i, j) \in \{1, 2, 3\}^2 \text{ with } i < j, \\ I_{i,j} = \left\{ t \in (0, T), \alpha_i(t) \|B_i^* Z(t)\|_{U_i}^2 = \alpha_j(t) \|B_j^* Z(t)\|_{U_j}^2 \right\}.$$

The case when A is a self-adjoint positive definite operator with compact resolvent. In this case, Lemma 2.1 can be easily adapted to show the following result.

LEMMA 3.1. *When A is a self-adjoint positive definite operator with compact resolvent, and $(\alpha_i)_{i \in \{1, 2, 3\}}$ are as in (3.2) with $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}_+^3$ two by two distinct, for all $i, j \in \{1, \dots, 3\}$ with $i \neq j$, the set $I_{i,j}$ is necessarily of zero measure, except in the trivial case $\|B_i^* Z\|_{L^2(0, T; U_i)} = \|B_j^* Z\|_{L^2(0, T; U_j)} = 0$.*

Since the proof of Lemma 3.1 is the same as the proof of Lemma 2.1 and relies on the fact that α_i/α_j admits a set of accumulation points at $-\infty$ which contains a nontrivial interval, we skip it and leave it to the reader.

The case when H is a finite dimensional vector space. In this case, we choose the parameters ω_i successively; for instance, we can take

$$(3.4) \quad \omega_1 = 0, \quad \omega_2 \in \mathbb{R} \setminus W,$$

where W is defined as in (2.5), and

$$(3.5) \quad \omega_3 \in \mathbb{R} \setminus W_3,$$

where W_3 is defined by

$$(3.6) \quad W_3 = W \cup \{\pm\omega_2, \pm\omega_2 + \Im(\lambda_k) - \Im(\lambda_{k_1}) \mid \forall (k, k_1) \text{ such that } \Re(\lambda_k) = \Re(\lambda_{k_1})\}.$$

We then prove the following result.

LEMMA 3.2. *When H is a finite dimensional space, setting $\omega_1 = 0$ and choosing $\omega_2 \in \mathbb{R} \setminus W$ (defined in (2.5)) and $\omega_3 \in \mathbb{R} \setminus W_3$ (defined in (3.6)) and taking α_i as in (1.11) corresponding to ω_i , we have that for all $(i, j) \in \{1, 2, 3\}$ with $i < j$, the set $I_{i,j}$ is necessarily of zero measure, except in the trivial case $\|B_i^* Z\|_{L^2(0,T;U_i)} = \|B_j^* Z\|_{L^2(0,T;U_j)} = 0$.*

We briefly sketch the proof of Lemma 3.2 below.

Sketch of the proof of Lemma 3.2. Clearly, when $i = 1$, the proof of Lemma 3.2 reduces to the proof of Lemma 2.2.

We thus focus on the case when $i = 2$ and $j = 3$. Similarly to the proof of Lemma 2.2, we assume that $I_{2,3}$ is of positive measure. By analyticity, this implies that $I_{2,3} = (0, T)$ and, by extending Z on $\mathbb{R}$ by analyticity, that for all $t \in \mathbb{R}$, $\alpha_2(t)\|B_2^* Z(t)\|_{U_2}^2 = \alpha_3(t)\|B_3^* Z(t)\|_{U_3}^2$. We then expand Z as in (2.22) and define, as in the proof of Lemma 2.2,

$$k_0 = \inf \{k \in \{1, \dots, K\} : \exists \ell \in \{0, \dots, m_k\} \text{ such that } \|B_2^* w_{k,\ell}\|_{U_2} + \|B_3^* w_{k,\ell}\|_{U_3} \neq 0\},$$

and, if $k_0 < \infty$,

$$\begin{aligned} \ell_1 &= \sup \{\ell : \exists k \text{ with } \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_2^* w_{k,\ell}\|_{U_2} + \|B_3^* w_{k,\ell}\|_{U_3} \neq 0\}, \\ D &= \{k : \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_2^* w_{k,\ell_1}\|_{U_2} + \|B_3^* w_{k,\ell_1}\|_{U_3} \neq 0\}, \\ Z_d(t) &= \sum_{k \in D} w_{k,\ell_1} e^{i\Im(\lambda_k)(t-T)} \quad (t \in \mathbb{R}). \end{aligned}$$

With the above choices, similarly to (2.2), for all $t \in \mathbb{R}$ we have the formula

$$\begin{aligned} (3.7) \quad & \alpha_2(t)\|B_2^* Z_d(t)\|_{U_2}^2 - \alpha_3(t)\|B_3^* Z_d(t)\|_{U_3}^2 \\ &= \left(1 + \frac{\sin(\omega_2 t)}{2}\right) \sum_{k \in D} \|B_2^* w_{k,\ell_1}\|_{U_2}^2 - \left(1 + \frac{\sin(\omega_3 t)}{2}\right) \sum_{k \in D} \|B_3^* w_{k,\ell_1}\|_{U_3}^2 \\ &+ 2 \left(1 + \frac{\sin(\omega_2 t)}{2}\right) \sum_{k \in D} \sum_{k_1 \in D, k_1 > k} \Re \left(e^{i(\Im(\lambda_k) - \Im(\lambda_{k_1}))t} \langle B_2^* w_{k,\ell_1}, B_2^* w_{k_1,\ell_1} \rangle_{U_2} \right) \\ &- 2 \left(1 + \frac{\sin(\omega_3 t)}{2}\right) \sum_{k \in D} \sum_{k_1 \in D, k_1 > k} \Re \left(e^{i(\Im(\lambda_k) - \Im(\lambda_{k_1}))t} \langle B_3^* w_{k,\ell_1}, B_3^* w_{k_1,\ell_1} \rangle_{U_3} \right), \end{aligned}$$

which holds instead of (2.2). Additionally, since for all $t \in \mathbb{R}$ we have $\alpha_2(t)\|B_2^* Z(t)\|_{U_2}^2 = \alpha_3(t)\|B_3^* Z(t)\|_{U_3}^2 = 0$, we can also deduce, as in (2.26), that

$$(3.8) \quad \forall t \in (-\infty, T-1), \quad \left| \alpha_2(t)\|B_2^* Z_d(t)\|_{U_2}^2 - \alpha_3(t)\|B_3^* Z_d(t)\|_{U_3}^2 \right| \leq \frac{C}{T-t}.$$

Accordingly, using Lemma 2.5 on function $t \mapsto \alpha_2(t)\|B_2^* Z_d(t)\|_{U_2}^2 - \alpha_3(t)\|B_3^* Z_d(t)\|_{U_3}^2$, which goes to 0 as $t \rightarrow -\infty$, and considering the coefficients in front of the constant

term and in front of $e^{i\omega_3 t}$ in (2), which appear only once in the expansion (2) since $\omega_3 \notin W_3$, we deduce

$$\begin{aligned} 0 &= \sum_{k \in D} \|B_2^* w_{k, \ell_1}\|_{U_2}^2 - \sum_{k \in D} \|B_3^* w_{k, \ell_1}\|_{U_3}^2, \\ 0 &= \sum_{k \in D} \|B_3^* w_{k, \ell_1}\|_{U_3}^2. \end{aligned}$$

This easily yields that $k_0 = \infty$ and, consequently, that $\|B_2^* Z(t)\|_{U_2} + \|B_3^* Z(t)\|_{U_3} = 0$ for all $t \in \mathbb{R}$, and concludes the proof of Lemma 3.2. $\square$

End of the proof of Theorem 1.3. We choose the coefficients $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ such that either the assumptions of Lemma 3.1 are satisfied in the case of a self-adjoint operator or the assumptions of Lemma 3.2 are satisfied when considering the case of H of finite dimension. According to Lemmas 3.1 and 3.2, if $I_{i,j}$ is of positive measure for some $i, j \in \{1, 2, 3\}$ with $i \neq j$, taking $\ell \in \{1, 2, 3\} \setminus \{i, j\}$, only two cases arise as follows:

- If $t \mapsto \|B_\ell^* Z(t)\|_{U_\ell}^2$ is identically zero, then the observability property (1.9) implies that $Z = 0$ identically, which corresponds to a minimizer for J only in the case $y_0 = 0$, which can be steered to 0 by keeping all the controls equal to 0 at all times.
- If $t \mapsto \|B_\ell^* Z(t)\|_{U_\ell}^2$ is not identically zero, since it is an analytic function, its zero set has no accumulation point, and thus

$$\text{a.e. } t \in (0, T), \quad \alpha_\ell(t) \|B_\ell^* Z(t)\|_{U_\ell}^2 > \max\{\alpha_i(t) \|B_i^* Z(t)\|_{U_i}^2, \alpha_j(t) \|B_j^* Z(t)\|_{U_j}^2\}.$$

Accordingly, except in the trivial case $Z_T = 0$, we have the following:

$$(3.9) \quad \text{a.e. } t \in (0, T), \exists! \ell \in \{1, 2, 3\}, \text{ such that } \alpha_\ell(t) \|B_\ell^* Z(t)\|_{U_\ell}^2 > \max_{i \neq \ell} \{\alpha_i(t) \|B_i^* Z(t)\|_{U_i}^2\}.$$

We can then write the Euler–Lagrange equation satisfied by a minimizer Z_T of J and obtain, after setting for each $i \in \{1, 2, 3\}$,

$$u_i(t) = \begin{cases} \alpha_i(t) B_i^* Z(t) & \text{when } \alpha_i(t) \|B_i^* Z(t)\|_{U_i}^2 > \max_{j \neq i} \{\alpha_j(t) \|B_j^* Z(t)\|_{U_j}^2\}, \\ 0 & \text{otherwise,} \end{cases}$$

that the corresponding solution y of (1.15) satisfies $y(T) = 0$, while the controls u_1, u_2, u_3 satisfy the switching condition (1.14).

4. Examples.

4.1. Examples in finite dimension. Theorems 1.1 and 1.3 have many interesting consequences—even for finite dimensional systems. Below we give some examples.

Example 1: General matrix A . Let us fix $H = \mathbb{R}^d$ for $d \in \mathbb{N}^*$, and let A be a $d \times d$ matrix. Then it is clear that the control system

$$(4.1) \quad y' + Ay = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix}, \quad t \in (0, T), \quad y(0) = y_0 \in \mathbb{R}^d,$$

is exactly controllable at any time T . Indeed, controllability can be achieved as follows: given y_0 and y_1 in $\mathbb{R}^d$, we take y , a smooth function of time with values in $\mathbb{R}^d$ such that $y(0) = y_0$ and $y(T) = y_1$, and simply set $u = y' + Ay$.

Therefore, it is clear that Theorem 1.3 applies when considering the operators $B_i u_i = u_i e_i$ for $i \in \{1, \dots, d\}$, where e_i is the vector of $\mathbb{R}^d$ whose i th component equals 1 and all other components vanish. We thus get the following result.

THEOREM 4.1. *Let $d \in \mathbb{N}^*$, $H = \mathbb{R}^d$, and A be a $d \times d$ matrix. Then for any $y_0 \in \mathbb{R}^d$, there exist d control functions $u_i \in L^2(0, T; \mathbb{R})$ such that the controlled trajectory of (4.1) satisfies $y(T) = 0$ and with control functions satisfying condition (1.14), i.e., such that a.e. in $(0, T)$, at most one of the controls $u_i(t)$ for $i \in \{1, \dots, d\}$ is nonzero.*

This result can be applied, for instance, to the following case, which corresponds to the space semidiscretization of the 1-dimensional heat equation on $(0, L)$ with homogeneous Dirichlet boundary conditions at $x = 0$ and $x = L$:

$$(4.2) \quad \begin{cases} y'_j - \frac{1}{h^2}(y_{j+1} - 2y_j + y_{j-1}) = u_j, & t \in (0, T), j \in \{1, \dots, d\}, \\ y_0(t) = y_{d+1}(t) = 0, & t \in (0, T), \\ y_j(0) = y_j^0, & j \in \{1, \dots, d\}, \end{cases}$$

where $h > 0$ is a (small) parameter. Indeed, (4.2) can be seen as the finite difference approximation of the heat equation

$$(4.3) \quad \begin{cases} \partial_t y - \partial_{xx} y = u, & t \in (0, T), x \in (0, L), \\ y(t, 0) = y(t, L) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, L), \end{cases}$$

choosing the parameter h in (4.2) of the form $h = L/(d+1)$. Theorem 4.1 then yields that (4.2) can be controlled to zero with controls $u_i \in L^2(0, T; \mathbb{R})$ for each $i \in \{1, \dots, d\}$ such that at any time, only one of the controls u_i is active.

It is not clear how that process can pass to the limit as $d \rightarrow \infty$, and this is an interesting open question.

Example 2: General matrices (A, B) satisfying Kalman condition. If A is a $d \times d$ matrix and B is a $d \times n$ matrix, it is well known (see, e.g., [30]) that system (1.1) is controllable if and only if the following Kalman condition is satisfied:

$$(4.4) \quad \text{Rank}(B, AB, A^2B, \dots, A^{d-1}B) = d.$$

Now, we have chosen B under the form of a $d \times n$ matrix, which means that the control function u belongs to $u \in L^2(0, T; \mathbb{R}^n)$. As before, when $n \geq 2$, it is interesting to write

$$(4.5) \quad Bu = \sum_{i=1}^n B_i u_i, \quad \text{where } B_i \text{ is the } i\text{th column of } B.$$

Applying Theorem 1.3, we get the following result.

THEOREM 4.2. *Let A be a $d \times d$ matrix, let B be a $d \times n$ matrix such that the Kalman rank condition (4.4) holds, and let B_i denote the i th column of the matrix B . Then for any $y_0 \in \mathbb{R}^d$, there exist n control functions $u_i \in L^2(0, T; \mathbb{R})$ such that the controlled trajectory of (1.15) satisfies $y(T) = 0$ and with control functions satisfying condition (1.14), i.e., such that a.e. in $(0, T)$, at most one of the controls $u_i(t)$ for $i \in \{1, \dots, d\}$ is nonzero.*

Again, a nice application is given by the space semidiscretization of some PDE, for instance, of the wave equation. Indeed, if we consider the wave equation

$$(4.6) \quad \begin{cases} \partial_{tt}y - \partial_{xx}y = u, & t \in (0, T), x \in (0, L), \\ y(t, 0) = y(t, L) = 0, & t \in (0, T), \\ (y(0, x), \partial_t y(0, x)) = (y_0(x), y_1(x)), & x \in (0, L), \end{cases}$$

its finite difference semidiscretization is given by

$$(4.7) \quad \begin{cases} y_j'' - \frac{1}{h^2}(y_{j+1} - 2y_j + y_{j-1}) = u_j, & t \in (0, T), j \in \{1, \dots, d\}, \\ y_0(t) = y_{d+1}(t) = 0, & t \in (0, T), \\ (y_j(0), y_j'(0)) = (y_j^0, y_j^1), & j \in \{1, \dots, d\}, \end{cases}$$

where $h = L/(d+1)$. It is clear that system (4.7) is controllable in any arbitrary time, so that Theorem 4.2 applies immediately and provides controls $u_i \in L^2(0, T)$ for all $i \in \{1, \dots, d\}$ such that at all times only one of the controls is active.

Here again, it is completely unclear how this process can pass to the limit as $d \rightarrow \infty$. It is probably more difficult to analyze here than in the previous example since the limit equation (4.6) does not correspond to an analytic semigroup. Still, recent works on sparse optimal controls for the wave equation (see, in particular, [24]) may yield some insight into this problem.

4.2. Distributed control of parabolic systems. To give a nontrivial PDE example, we consider a smooth bounded domain Ω of $\mathbb{R}^N$ ($N \geq 1$), an open subset $\mathcal{O} \subset \Omega$, and the parabolic system

$$(4.8) \quad \begin{cases} \partial_t y - D\Delta y + Py = 1_{\mathcal{O}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \text{ with } d_1, d_2 > 0,$$

with $P = P(x) \in L^\infty(\Omega; S_2^+(\mathbb{R}))$, where $S_2^+(\mathbb{R})$ denotes the set of symmetric positive definite 2×2 matrices with real coefficients. Here, the control

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

acts on system (4.8) on $\mathcal{O}$ through multiplication by the indicator function $1_{\mathcal{O}}$ of the subset $\mathcal{O}$.

System (4.8) fits into the framework of Theorem 1.1 by setting

$$(4.9) \quad A = -D\Delta_x + P, \quad \text{in } H = (L^2(\Omega))^2 \text{ with domain } \mathcal{D}(A) = (H^2 \cap H_0^1(\Omega))^2,$$

and

$$Bu = 1_{\mathcal{O}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ for } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad U = (L^2(\mathcal{O}))^2.$$

Indeed, the operator A in (4.9) is obviously self-adjoint with compact resolvent. Additionally, the following result is a straightforward consequence of the Carleman estimates in [18].

PROPOSITION 4.3. *System (4.8) is null-controllable in arbitrary small times with control functions u in $L^2(0, T; (L^2(\mathcal{O}))^2)$.*

Thus, to apply Theorem 1.1, a natural example consists of choosing

$$(4.10) \quad B_1 u_1 = 1_{\mathcal{O}} \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad U_1 = L^2(\mathcal{O}), \quad \text{and} \quad B_2 u_2 = 1_{\mathcal{O}} \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \quad U_2 = L^2(\mathcal{O}).$$

Theorem 1.1 then readily implies the following.

THEOREM 4.4. *System (4.8) is null-controllable in arbitrary small times, with controls u_1 and u_2 in $L^2(0, T; L^2(\mathcal{O}))$ satisfying the additional switching constraints (1.5).*

Remark 4.5. By a shifting argument, Theorem 4.4 remains true if we only consider P as a bounded symmetric matrix.

Here, we emphasize that our results are different from the ones in which the controls may act on only one component. Indeed, in such a case, it is clear that more conditions are needed, since when $P = 0$ and acting on only one component, the second component will be free of control.

Of course, when $P = 0$, it is easy to check that one can control system (4.8) with controls having a switching structure, since one can control the first component y_1 to 0 at time $T/2$ by keeping the control $u_2 = 0$ in $(0, T/2)$ and can then control the second component y_2 to 0 on $(T/2, T)$ by keeping the control $u_1 = 0$ in $(T/2, T)$. However, when $P \neq 0$, this strategy does not seem to be directly applicable.

On the other hand, when one wants to control a system through one component only, it is clear that the coupling terms should play an important role; see, for instance, [14].

Therefore, our results fall between the questions of controllability of parabolic systems when the controls act on all the components of the state and when the controls may act on only one (or some of) the components of the state.

4.3. Distributed controls of 3D Stokes equations. Let Ω be a smooth bounded domain of $\mathbb{R}^3$, and consider the following Stokes equation:

$$(4.11) \quad \begin{cases} \partial_t y - \Delta y + \nabla p = 1_{\mathcal{O}} u & \text{in } (0, T) \times \Omega, \\ \operatorname{div} y = 0 & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases}$$

Here, $y = y(t, x) \in \mathbb{R}^3$ denotes the velocity field of an incompressible fluid, p is the pressure, and the control u acts through the nonempty open subset $\mathcal{O}$ of Ω .

This example fits the setting of Theorem 1.3 by choosing the state space

$$(4.12) \quad H = V_n^0(\Omega) = \{y \in L^2(\Omega; \mathbb{R}^3), \operatorname{div} y = 0 \text{ in } \Omega \text{ and } y \cdot \mathbf{n}_x = 0 \text{ on } \partial\Omega\},$$

the operator A as

$$(4.13) \quad A = -\mathbb{P}\Delta, \text{ with } \mathcal{D}(A) = \{y \in H^2 \cap H_0^1(\Omega; \mathbb{R}^3), \operatorname{div} y = 0 \text{ in } \Omega\} \text{ in } H,$$

where $\mathbf{n}_x$ is the outward normal to $x \in \partial\Omega$, $\mathbb{P}$ is the orthogonal projection on $V_n^0(\Omega)$ in $L^2(\Omega; \mathbb{R}^3)$, and the control operator

$$Bu = \mathbb{P}1_{\mathcal{O}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \text{with } U = (L^2(\mathcal{O}))^3.$$

It is then natural to define the operators B_1 , B_2 , and B_3 as follows:

$$(4.14) \quad B_1 u_1 = \mathbb{P}1_{\mathcal{O}} \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 u_2 = \mathbb{P}1_{\mathcal{O}} \begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix}, \quad B_3 u_3 = \mathbb{P}1_{\mathcal{O}} \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix},$$

with $U_1 = U_2 = U_3 = L^2(\mathcal{O})$.

Indeed, we have the following results:

- The operator A is self-adjoint on $V_n^0(\Omega)$; see, e.g., [6, Lemma IV.5.4].
- The Stokes problem (4.11) is null-controllable in arbitrary small times; see [20].

We can therefore readily apply Theorem 1.3 as follows.

THEOREM 4.6. *Given any $y_0 \in V_n^0(\Omega)$, there exist control functions u_1 , u_2 , and u_3 in $L^2(0, T; L^2(\mathcal{O}))$ such that the controlled trajectory y of (4.11) satisfies $y(T) = 0$ in Ω and with control functions u_1 , u_2 and u_3 satisfying condition (1.14), i.e., such that a.e. in $(0, T)$, at most one of the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ is nonzero.*

It is interesting to consider this case, since the controllability of the Stokes equation (4.11) with controls having one or two vanishing components has been studied in the literature. In particular, it has been shown in [9] that given $\ell \in \{1, 2, 3\}$, system (4.11) is null-controllable in arbitrary small times with controls $u \in L^2(0, T; (L^2(\mathcal{O}))^3)$ satisfying $u_\ell \equiv 0$. Additionally, the result in [26] shows that system (4.11) may not be null-controllable (in fact, not even approximate controllable) in some specific geometric settings with controls having two vanishing components.

Note that the result in [10] about the null-controllability of the 3D incompressible Navier–Stokes equation with controls having two vanishing components depends on the nonlinear term in the Navier–Stokes equation in the spirit of the celebrated *Coron’s return method* and thus does not apply to the linear problem (4.11).

4.4. Boundary control of a system of coupled heat equations. This example is closely related to the one in section 4.2. Let us consider a smooth bounded domain Ω and the following parabolic system:

$$(4.15) \quad \begin{cases} \partial_t y - D\Delta y + Py = 0 & \text{in } (0, T) \times \Omega, \\ y = u1_\Gamma & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad D = \text{diag } (d_1, \dots, d_n), \quad \text{with } d_i > 0 \ \forall i \in \{1, \dots, n\},$$

and $P = P(x) \in L^\infty(\Omega; S_n^+(\mathbb{R}))$, where $S_n^+(\mathbb{R})$ denotes the set of symmetric positive definite $n \times n$ matrices with real coefficients. Here, the control

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

acts on the system (4.15) on a nonempty open subset Γ of the boundary $\partial\Omega$ through the multiplication by the indicator function 1_Γ .

System (4.15) fits into the framework of Theorem 1.3 by setting

$$(4.16) \quad A = -D\Delta_x + P, \quad \text{in } H = (L^2(\Omega))^n \text{ with domain } \mathcal{D}(A) = (H^2 \cap H_0^1(\Omega))^n,$$

and the control operator B as

$$Bu = \tilde{A}\text{Dir}_\Gamma(u) \quad \text{for } u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad U = (L^2(\Gamma))^n,$$

where $\text{Dir}_\Gamma : (L^2(\Gamma))^n \mapsto (L^2(\Omega))^n$ is the Dirichlet operator given by

$$\text{Dir}_\Gamma u = z, \quad \text{where } z \text{ solves } \begin{cases} -D\Delta z + Pz = 0 & \text{in } \Omega, \\ z = u1_\Gamma & \text{on } \partial\Omega, \end{cases}$$

and $\tilde{A}$ denotes the extension of A of domain $(L^2(\Omega))^n$ on $((H^2 \cap H_0^1(\Omega))^n)'$; see [30, Proposition 3.4.5 and section 10.7].

Similarly to Proposition 4.3, one can show the following using classical Carleman estimates (see [18]).

PROPOSITION 4.7. *System (4.15) is null-controllable in arbitrary small times with control functions $u = (u_1, \dots, u_n)$ in $L^2(0, T; (L^2(\Gamma))^n)$.*

One can then readily apply Theorem 1.3 as follows.

THEOREM 4.8. *System (4.15) is null-controllable in arbitrary small times with controls $u = (u_1, \dots, u_n)$ in $L^2(0, T; (L^2(\Gamma))^n)$ satisfying the additional switching constraints (1.14).*

Again, we emphasize that our results complement those where the controls act on only one component of the system, in which the situation is much more intricate since controllability results will depend on delicate coupling conditions; see, for instance, [3] and references therein.

4.5. Boundary control of 3D Stokes equations. Again, one can consider Stokes equations but now controlled from the boundary. Using [20] (see also [16]), we find that in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, the 3D Stokes equations are null-controllable in any time T through any nonempty open subset of its boundary. More precisely, we let Ω be a smooth bounded domain of $\mathbb{R}^3$ and let Γ be a nonempty open subset of $\partial\Omega$, and we consider the Stokes equation,

$$(4.17) \quad \begin{cases} \partial_t y - \Delta y + \nabla p = 0 & \text{in } (0, T) \times \Omega, \\ \text{div } y = 0 & \text{in } (0, T) \times \Omega, \\ y = 1_\Gamma(x)u & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases}$$

where 1_Γ is the indicator function of the set Γ , and u is assumed to belong to $L^2(0, T; L^2(\Gamma; \mathbb{R}^3))$ and satisfy

$$(4.18) \quad \forall t \in (0, T), \quad \int_\Gamma u(t, x) \cdot \mathbf{n}_x \, d\sigma = 0,$$

where $\mathbf{n}_x$ is the outward normal to $\partial\Omega$ at $x \in \partial\Omega$. Condition (4.18) can be seen as a compatibility condition with the divergence free condition $\operatorname{div} y = 0$ and can be obtained immediately by integrating it in Ω .

Properly speaking, [20] does not deal with boundary controls, but the following result can be easily obtained from [20] using the classical extension/restriction argument to get controllability results with controls on the boundary.

THEOREM 4.9 ([20]). *System (4.17) is null-controllable in any time T . More precisely, for all $T > 0$, for any $y_0 \in V_n^0(\Omega)$, there exists a control function $u \in L^2(0, T; L^2(\Gamma; \mathbb{R}^3))$ satisfying (4.18) such that the controlled trajectory y of (4.17) satisfies $y(T) = 0$ in Ω .*

Because of condition (4.18), it is natural to decompose the space $\{u \in L^2(\Gamma; \mathbb{R}^3) : \int_{\Gamma} u(x) \cdot \mathbf{n}_x d\sigma = 0\}$ using tangential and normal components of u . Therefore, we choose a family of triplets $(e_1(x), e_2(x), \mathbf{n}_x)$ indexed by $x \in \Gamma$ such that for all $x \in \Gamma$, $(e_1(x), e_2(x), \mathbf{n}_x)$ is an orthonormal basis of $\mathbb{R}^3$, and we define $U_1 = U_2 = L^2(\Gamma; \mathbb{R})$ and $U_3 = \{u_3 \in L^2(\Gamma; \mathbb{R}) \text{ with } \int_{\Gamma} u_3(x) d\sigma = 0\}$, also denoted by $L_0^2(\Gamma; \mathbb{R})$, and the isomorphism π in (1.12) is then given by

$$(4.19) \quad \pi : (u_1, u_2, u_3) \in U_1 \times U_2 \times U_3 \mapsto (x \mapsto (u_1(x)e_1(x) + u_2(x)e_2(x) + u_3(x)\mathbf{n}_x)).$$

Now, as before (see, e.g., [29]), to properly define the operator B in this case, we need to introduce the Dirichlet operator D_{Γ} defined by

$$D_{\Gamma}u = z, \text{ where } z \text{ solves } \begin{cases} -\Delta z + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} z = 0 & \text{in } \Omega, \\ z = 1_{\Gamma}u & \text{on } \partial\Omega, \end{cases}$$

and the operator B is defined by

$$Bu = \tilde{A}\mathbb{P}D_{\Gamma}u,$$

where $\tilde{A}$ denotes the extension of the Stokes operator (defined in (4.12)–(4.13)) from $V_n^0(\Omega)$ to $\mathcal{D}(A)'$, and $\mathbb{P}$ denotes the Leray projection, that is, the orthogonal projection on $V_n^0(\Omega)$ in $L^2(\Omega; \mathbb{R}^3)$. The full system (4.17) can then be written as

$$(4.20) \quad \begin{cases} \mathbb{P}y' + \tilde{A}\mathbb{P}y = Bu, & t \in (0, T), \\ \mathbb{P}y(0) = \mathbb{P}y_0, \\ (I - \mathbb{P})y = (I - \mathbb{P})D_{\Gamma}u, & t \in (0, T). \end{cases}$$

Accordingly, the quantities $\mathbb{P}y$ and $(I - \mathbb{P})y$ should be handled separately. In particular (see [29, Theorems 2.3 and 3.1]), for $u \in L^2(0, T; L^2(\Gamma; \mathbb{R}^3))$ satisfying (4.18), the solution y of (4.20) with initial datum $\mathbb{P}y_0 \in V_n^0(\Omega)$ satisfies $\mathbb{P}y \in L^2(0, T; V_n^0(\Omega)) \cap_{\varepsilon > 0} L^2(0, T; V^{1/2-\varepsilon}(\Omega)) \cap H^{1/4}(0, T; V^0(\Omega)) \cap C^0([0, T]; V^{-1}(\Omega))$ and $(I - \mathbb{P})y \in L^2(0, T; V^{1/2}(\Omega))$. Here, $V_n^0(\Omega)$ is the space defined in (4.12), and the other spaces are

$$\begin{aligned} V^s(\Omega) &= \{y \in H^s(\Omega; \mathbb{R}^3), \operatorname{div} y = 0 \text{ in } \Omega, \text{ with } \langle y \cdot n, 1 \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0\} \quad (s \geq 0), \\ V_0^1(\Omega) &= \{y \in H_0^1(\Omega; \mathbb{R}^3), \operatorname{div} y = 0 \text{ in } \Omega\}, \end{aligned}$$

and $V^{-1}(\Omega)$ is the dual of $V_0^1(\Omega)$, with $V_n^0(\Omega)$ as the pivot space.

Theorem 1.3 then yields the following result.

THEOREM 4.10. *Let Ω be a smooth bounded domain of $\mathbb{R}^3$, and let Γ be a nonempty open subset of $\partial\Omega$. Given a family of orthonormal triplets $(e_1(x), e_2(x), \mathbf{n}_x)$ for $x \in \Gamma$ which defines the control operators B_1 , B_2 , and B_3 according to (1.13) through the isomorphism π in (4.19), the control system (4.17) is null-controllable in arbitrary small times with controls $(u_1, u_2, u_3) \in L^2(0, T; L^2(\Gamma; \mathbb{R})^2 \times L_0^2(\Gamma; \mathbb{R}))$ which satisfy the switching condition (1.14) in the following sense: for any $T > 0$, for any $y_0 \in V_n^0(\Omega)$, there exist control functions u_1, u_2 in $L^2(0, T; L^2(\Gamma; \mathbb{R}))$, and $u_3 \in L^2(0, T; L_0^2(\Gamma; \mathbb{R}))$ satisfying the switching condition (1.14) such that the solution y of (4.20) satisfies $\mathbb{P}y(T) = 0$.*

Remark 4.11. Although Theorem 4.10 states only the control of $\mathbb{P}y$ at time T , extending the controls (u_1, u_2, u_3) by 0 for $t \geq T$, one easily checks that $\mathbb{P}y$ and $(I - \mathbb{P})y$ vanish for $t \geq T$. The difficulty is that $(I - \mathbb{P})y$ does not a priori make sense at time T since it only belongs to $L^2(0, T; V^{1/2}(\Omega))$.

To the best of our knowledge, there are almost no results regarding the controllability of Stokes system with controls acting on only normal or tangential components. We are only aware of [17] for the case of tangential controls on the whole boundary and of the results in [8] for the Stokes equation in a channel when the control is localized on the whole boundary of one side of the channel.

5. Extensions. Theorem 1.3 focuses on the case of operators A which are either positive self-adjoint with compact resolvent or matrices. Thus, it is natural also to consider the case of general operators A , which generate an analytic semigroup and are possibly non-self-adjoint. The goal of this section is precisely to discuss this case. Our arguments will require the introduction of several spectral assumptions which are hard to check in practice.

THEOREM 5.1. *Let A be an operator on the Hilbert space H having compact resolvent and such that $-A$ generates an analytic semigroup.*

Assume that the Hilbert space H can be decomposed as

$$(5.1) \quad H = \oplus_{k \in \mathbb{N}} H_k, \quad \text{where } H_k \text{ are finite dimensional vector spaces}$$

such that for all $k \in \mathbb{N}$,

$$(5.2) \quad A^*(H_k) \subset H_k, \text{ and } A^*|_{H_k} = A_k^*,$$

where A_k^ is of the form $\lambda_k I + N_k$, with $\lambda_k \in \mathbb{C}$ and N_k nilpotent.*

Also assume for simplicity that $\Re(\lambda_0) \leq \Re(\lambda_1) \leq \dots \leq \Re(\lambda_k) \leq \dots \rightarrow \infty$.

Furthermore, denoting by $\mathbb{P}_k$ the projection on H_k parallel to $\oplus_{j \neq k} H_j$, we assume that there exists $T_0 > 0$ large enough so that

$$(5.3) \quad \forall t \geq T_0, \quad e^{-tA^*} = \sum_k e^{-tA_k^*} \mathbb{P}_k,$$

i.e., the right-hand side is norm convergent for $t \geq T_0$.

Let $B \in \mathcal{L}(U, \mathcal{D}(A^))$, where U is a Hilbert space, let $n \in \mathbb{N}$ with $n \geq 2$, and assume that U is isomorphic to $U_1 \times \dots \times U_n$ for some Hilbert spaces U_i , $i \in \{1, \dots, n\}$, and define B_i for $i \in \{1, \dots, n\}$ as in (1.13).*

We assume that system (1.1) is null-controllable in arbitrary small times.

Then the system (1.15) is null-controllable in arbitrary small times with switching controls, i.e., satisfying (1.14). More precisely, given any $T > 0$ and any $y_0 \in H$,

there exist n control functions $u_i \in L^2(0, T; U_i)$, $i \in \{1, \dots, n\}$, such that the solution y of (1.15) satisfies (1.6), while the control functions satisfy the switching condition (1.14).

Before giving the proof of Theorem 5.1, let us emphasize that the assumptions on A^* may be delicate to prove for general operators A generating an analytic semigroup.

Of course, each H_k corresponds to the generalized eigenspaces corresponding to the eigenvalues λ_k , and the projections $\mathbb{P}_k$ correspond to the spectral projections. However, condition (5.3) is difficult to check in practice; see, e.g., [19] for an introduction to spectral theory for non-self-adjoint operators.

To better illustrate that fact, we present two examples of interest. The first is borrowed from [5].

Let us take A_0 as a positive self-adjoint operator with compact resolvent defined on a Hilbert space H_0 with domain $\mathcal{D}(A_0)$, which we will assume for simplicity to have only simple eigenvalues. Then, for $f \in \mathcal{C}^\infty(\mathbb{R}_+^*; \mathbb{R}_+^*)$ bounded at infinity, define

$$(5.4) \quad \hat{A} = \begin{pmatrix} A_0 & Id \\ 0 & A_0 + f(A_0) \end{pmatrix} \text{ in } H = (H_0)^2, \quad \text{with } \mathcal{D}(\hat{A}) = (\mathcal{D}(A_0))^2.$$

It is easy to check that such an $\hat{A}$ generates an analytic semigroup in H , since it is a bounded perturbation of the operator $\text{Diag}(A_0, A_0)$. Additionally, its spectrum can be expressed easily in terms of those of A_0 . If $(\lambda_{k,0})_{k \in \mathbb{N}}$ is the set of eigenvalues of A_0 , corresponding to a family of normalized eigenvectors $(\varphi_{k,0})_{k \in \mathbb{N}}$, then it is easy to check that the eigenvalues of $\hat{A}$ are given by the family $(\lambda_{k,1}, \lambda_{k,2})_{k \in \mathbb{N}}$ with $\lambda_{k,1} = \lambda_{k,0}$ and $\lambda_{k,2} = \lambda_{k,0} + f(\lambda_{k,0})$. The corresponding eigenvectors are given for $k \in \mathbb{N}$ by

$$\varphi_{k,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_{k,0}, \quad \varphi_{k,2} = \frac{1}{\sqrt{1 + f(\lambda_{k,0})^2}} \begin{pmatrix} 1 \\ f(\lambda_{k,0}) \end{pmatrix} \varphi_{k,0}.$$

It is then easy to check that

$$\begin{aligned} \mathbb{P}_{k,1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \varphi_{k,1} \left\langle \begin{pmatrix} 1 \\ -\frac{1}{f(\lambda_k)} \end{pmatrix} \varphi_{k,0}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle_H, \\ \mathbb{P}_{k,2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \varphi_{k,2} \left\langle \begin{pmatrix} 0 \\ \frac{\sqrt{1+f(\lambda_k)^2}}{f(\lambda_k)} \end{pmatrix} \varphi_{k,0}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle_H. \end{aligned}$$

When f goes to zero at infinity, the norms of these projections behave like $1/f(\lambda_k)$. In particular, if for $T_0 > 0$ there exists C such that $f(s) \leq Ce^{-T_0 s}$ for s large enough, we see that the right-hand side of (5.3) is not norm convergent for $t \in (0, T_0)$. Of course, this also means that when considering $f(s) = \exp(-s^2)$, condition (5.3) is not satisfied no matter what $T_0 > 0$ is.

This example shows that even for rather gentle perturbations of self-adjoint operators, condition (5.3) should be analyzed with caution.

We also present another example in this direction, based on the works [11, 12] discussing the operator A_α defined for complex number $\alpha \in \mathbb{C} \setminus \{0\}$ with $\text{Arg}(\alpha) < \pi/4$ on $L^2(\mathbb{R})$ by

$$A_\alpha y = -\alpha^{-2} y'' + \alpha^2 x^2 y.$$

In fact, to be perfectly rigorous, the operator A_α has to be defined as the closed densely defined operator associated to the quadratic form

$$\int_{\mathbb{R}} (\alpha^{-2} |y'(x)|^2 + \alpha^2 x^2 y(x)^2) dx,$$

originally defined on $\mathcal{C}_c^\infty(\mathbb{R})$.

According to [11], the eigenvalues of the operator A_α do not depend on α for $\alpha \in \mathbb{C} \setminus \{0\}$ with $|\arg(\alpha)| < \pi/4$ and thus coincide with the usual ones for the harmonic operator (which are $2\mathbb{N} + 1$); however, except in the case when $\alpha \in \mathbb{R}_+^*$, the spectrum of A_α is wild [11, Theorem 9], meaning that, denoting by $\mathbb{P}_k$ the spectral projector on the k th eigenvector, $\|\mathbb{P}_k\|$ cannot be bounded by a polynomial in k .

In fact, the situation is even worse, and for $\alpha \notin \mathbb{R}$, the formula

$$e^{-t\alpha^2 A_\alpha} = \sum_{k \in \mathbb{N}} e^{-t\alpha^2 \lambda_k} \mathbb{P}_k$$

holds only for t large enough (see [12, Corollary 4]) due to the fact that $\|\mathbb{P}_k\|$ behaves like $\exp(c\Re(\lambda_k))$ for some strictly positive c as $k \rightarrow \infty$.

To sum up, we see that condition (5.3) is rather delicate to deal with. Although it is automatically satisfied in finite dimensional contexts or when A is self-adjoint, when considering general operators A generating an analytic semigroup, condition (5.3) should be carefully analyzed.

Proof. The proof of Theorem 5.1 closely follows the proofs of Theorems 1.1 and 1.3.

For the sake of simplicity, we will only focus on the case where $n = 2$ and $B \in \mathcal{L}(U, H)$, similarly to Theorem 1.1, since the general case where $n \geq 3$ and $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$ can be handled similarly as in section 3 by minor adaptations of the case where $n = 2$.

In fact, it is easy to check that the only point which needs further analysis is the counterpart of Lemmas 2.1 and 2.2.

We thus take X as in (2.2) and let $Z_T \in X$ be a minimizer of the functional J in (1.10), and we study the set I defined in (2.4).

LEMMA 5.2. *Assume that A is an operator on the Hilbert space H having compact resolvent and such that $-A$ generates an analytic semigroup. Also assume that the Hilbert space H can be decomposed as in (5.1) such that A^* satisfies (5.2) for all $k \in \mathbb{N}$, where the corresponding eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ are ordered such that $\Re(\lambda_0) \leq \Re(\lambda_1) \leq \dots \leq \Re(\lambda_k) \leq \dots \rightarrow \infty$. Further assume that, denoting by $\mathbb{P}_k$ the projection on H_k parallel to $\oplus_{j \neq k} H_j$, there exists $T_0 > 0$ large enough such that (5.3) holds.*

Define the set W as in (2.5).

Let $B \in \mathcal{L}(U, H)$, and assume that system (1.1) is null-controllable in arbitrary small times.

Then, for α as in (1.11) with $\omega \in \mathbb{R} \setminus W$, the set I is necessarily of zero measure, except in the trivial case $\|B_1^ Z\|_{L^2(0,T;U_1)} = \|B_2^* Z\|_{L^2(0,T;U_2)} = 0$.*

After Lemma 5.2 is proved, the end of the proof of Theorem 5.1 will follow line by line the proof of Theorem 1.1 by showing that the Euler–Lagrange equation satisfied by Z_T is given by (2.8) when $Z_T \neq 0$, entailing that the controls u_1 and u_2 given by (2.9) are of switching forms and indeed control (1.4). As before, the case $Z_T = 0$ corresponds to the case $y_0 = 0$, and then taking the controls u_1 and u_2 to be identically zero solves the problem. $\square$

Proof of Lemma 5.2. In order to prove that the set I is of zero measure except when $\|B_1^* Z\|_{L^2(0,T;U_1)} = \|B_2^* Z\|_{L^2(0,T;U_2)} = 0$, we consider a strictly positive and strictly increasing sequence T_n going to T as $n \rightarrow \infty$, and we show that for all $n \in \mathbb{N}$, the set $I_n = I \cap (0, T_n)$ is of zero measure except in the trivial case in which both $B_1^* Z$ and $B_2^* Z$ vanish identically on $(0, T_n)$.

As in the proof of Lemma 2.1, the small time null-controllability implies that since $Z_T \in X$, the trajectory $Z|_{(0, T_n)}$ is well defined and in fact solves (2.11) with some initial datum $Z_n \in H$.

Accordingly, since $-A^*$ generates an analytic semigroup, the function $t \mapsto Z(t)$ is in fact analytic on $(0, T_n)$ with values in H and can be extended analytically to $(-\infty, T_n)$.

We now assume that I_n is not of zero measure. According to the analyticity properties above, this implies that the identity (2.13) holds.

To conclude as in the proof of Lemma 2.1 or Lemma 2.2, we would like to write formula (2.15). This cannot be done for all $t < T_n$ as before, but according to (5.3), it is still true for $t \leq T_n - T_0$:

$$(5.5) \quad \forall t \leq T_n - T_0, \quad Z(t) = \sum_{k \in \mathbb{N}} e^{A_k^*(t-T_n)} \mathbb{P}_k Z_n.$$

Each H_k is a finite dimensional vector space. Therefore, writing the Jordan decomposition of $A^*|_{H_k}$ for each $k \in \mathbb{N}$, denoting by m_k the size of the maximal Jordan block corresponding to λ_k ,

$$e^{A_k^*(t-T_n)} = e^{\lambda_k(t-T_n)} \sum_{\ell \in \{0, \dots, m_k\}} \frac{(t-T_n)^\ell}{\ell!} N_k^\ell.$$

We then follow the proof of Lemma 2.2, introducing

$$k_0 = \inf \{k \in \mathbb{N} : \exists \ell \in \{0, \dots, m_k\} \text{ such that } \|B_1^* N_k^\ell \mathbb{P}_k Z_n\|_{U_1} + \|B_2^* N_k^\ell \mathbb{P}_k Z_n\|_{U_2} \neq 0\}.$$

Our goal is to show that k_0 is necessarily infinite. Indeed, if k_0 is infinite, then for all k , $B_1^* e^{A_k^*(t-T_n)} \mathbb{P}_k Z_n$ and $B_2^* e^{A_k^*(t-T_n)} \mathbb{P}_k Z_n$ identically vanish, so that using formula (5.5), we see that $B_1^* Z$ and $B_2^* Z$ identically vanish on $(-\infty, T_n - T_0)$ and by analyticity on $(0, T_n)$ as well. We prove that k_0 is necessarily infinite by contradiction, assuming that k_0 is finite.

Next, we define ℓ_1 by

$$\ell_1 = \sup \{\ell : \exists k \text{ with } \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_1^* N_k^\ell \mathbb{P}_k Z_n\|_{U_1} + \|B_2^* N_k^\ell \mathbb{P}_k Z_n\|_{U_2} \neq 0\}$$

and define the set

$$D = \left\{k : \Re(\lambda_k) = \Re(\lambda_{k_0}) \text{ and } \|B_1^* N_k^{\ell_1} \mathbb{P}_k Z_n\|_{U_1} + \|B_2^* N_k^{\ell_1} \mathbb{P}_k Z_n\|_{U_2} \neq 0\right\}.$$

According to the above definition, we can decompose Z as

$$\begin{aligned} Z_d(t) &= e^{\Re(\lambda_{k_0})t} \frac{(T_n - t)^{\ell_1}}{\ell_1!} \sum_{k \in D} N_k^{\ell_1} \mathbb{P}_k Z_n e^{i\Im(\lambda_k)(t-T_n)} \quad (t \in (-\infty, T_n)), \\ Z_{d,2}(t) &= \sum_{k \text{ with } \Re(\lambda_k) = \Re(\lambda_{k_0})} e^{\lambda_k(t-T_n)} \left(\sum_{\ell \in \{0, \dots, \ell_1-1\}} \frac{(T_n - t)^\ell}{\ell!} N_k^\ell \mathbb{P}_k Z_n \right) \quad (t \in (-\infty, T_n)), \\ Z_{d,3}(t) &= \sum_{k \text{ with } \Re(\lambda_k) = \Re(\lambda_{k_0})} e^{\lambda_k(t-T_n)} \left(\sum_{\ell \geq \ell_1+1} \frac{(T_n - t)^\ell}{\ell!} N_k^\ell \mathbb{P}_k Z_n \right) \quad (t \in (-\infty, T_n)), \\ Z_0(t) &= \sum_{k \text{ with } \Re(\lambda_k) < \Re(\lambda_{k_0})} e^{A_k^*(t-T_n)} \mathbb{P}_k Z_n \quad (t \in (-\infty, T_n)), \\ Z_r(t) &= \sum_{k \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})} e^{A_k^*(t-T_n)} \mathbb{P}_k Z_n \quad (t \in (-\infty, T_n - T_0]). \end{aligned}$$

By the definitions of k_0 and ℓ_1 , we easily see that

$$(5.6) \quad \forall t \in (-\infty, T_n), \quad \|B_1^* Z_{d,3}(t)\|_{U_1} + \|B_1^* Z_0(t)\|_{U_1} + \|B_2^* Z_{d,3}(t)\|_{U_2} + \|B_2^* Z_0(t)\|_{U_2} = 0.$$

It is also easy to check, since the sum defining $Z_{d,2}$ is finite, that there exists a constant C such that $Z_{d,2}$ satisfies

$$(5.7) \quad \forall t \leq T_n - 1, \quad \|B_1^* Z_{d,2}(t)\|_{U_1} + \|B_2^* Z_{d,2}(t)\|_{U_2} \leq e^{\Re(\lambda_{k_0})t} C (T_n - t)^{\ell-1}.$$

We claim that there exist constants C and $\mu > \Re(\lambda_{k_0})$ such that

$$(5.8) \quad \forall t \leq T_n - T_0 - 1, \quad \|Z_r(t)\|_H \leq C e^{\mu t}.$$

Indeed, denoting $A_r^* = A^*|_{\oplus_{k \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})} H_k}$, Z_r solves

$$-Z_r' + A_r^* Z_r = 0, \quad t \in (-\infty, T_n - T_0), \quad Z_r|_{t=T_n-T_0} = \sum_{k \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})} e^{-A_k^* T_0} \mathbb{P}_k Z_n.$$

Since A^* generates an analytic semigroup on H , it is easy to check that $A_r^* = A^*|_{\oplus_{k \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})} H_k}$ also generates an analytic semigroup on $\oplus_{k \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})} H_k$ and that its spectral abscissa is given by $\inf\{\Re(\lambda_k), \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})\}$. According to [28, Theorem 4.3], Z_r thus decays exponentially at any rate smaller than

$$\inf\{\Re(\lambda_k), \text{ with } \Re(\lambda_k) > \Re(\lambda_{k_0})\}.$$

Since this quantity is strictly larger than $\Re(\lambda_{k_0})$, we have proved (5.8).

Estimate (5.8) in turns imply that

$$(5.9) \quad \forall t \leq T_n - T_0 - 1, \quad \|B_1^* Z_r(t)\|_{U_1} + \|B_2^* Z_r(t)\|_{U_2} \leq C e^{\mu t}$$

for some $\mu > \Re(\lambda_{k_0})$.

Using the identity (2.13) and the decay estimates (5.6), (5.7), and (5.9), we easily obtain the counterpart of (2.27), that is, the existence of positive constants C_1, C_2 such that for all $t \leq T_n - T_0 - 1$,

$$(5.10) \quad \left\| \left\| B_1^* \left(\sum_{k \in D} N_k^{\ell_1} \mathbb{P}_k Z_n e^{i\Im(\lambda_k)(t-T_n)} \right) \right\|_{U_1}^2 - \alpha(t) \left\| B_2^* \left(\sum_{k \in D} N_k^{\ell_1} \mathbb{P}_k Z_n e^{i\Im(\lambda_k)(t-T_n)} \right) \right\|_{U_2}^2 \right\| \leq \frac{C}{T_n - t}.$$

As in the proof of Lemma 2.2, we then easily get that, if α is as in (1.11) with $\omega \notin W$, for all $k \in D$,

$$\left\| B_1^* N_k^{\ell_1} \mathbb{P}_k Z_n \right\|_{U_1} + \left\| B_2^* N_k^{\ell_1} \mathbb{P}_k Z_n \right\|_{U_2} = 0.$$

This contradicts the definition of k_0 when $k_0 < \infty$ and concludes the proof of Lemma 5.2. $\square$

6. Further comments and open problems.

6.1. Further comments.

Approximate controllability. In this article, we focused on the null-controllability property, but several other notions can be used and developed similarly. For instance, we could consider the approximate controllability property at time T , which reads as follows for system (1.1): for any $y_0 \in H$ and $\varepsilon > 0$, there exists $u \in L^2(0, T)$ such that the solution y of (1.1) satisfies $\|y(T)\|_H \leq \varepsilon$.

It is classical (see, for instance, [25]) that this is equivalent to the following unique continuation property for the adjoint equation: if $z_T \in H$ is such that the solution z of (1.8) satisfies $B^*z = 0$ in $L^2(0, T; U)$, then $z_T = 0$.

In this context, following the same strategy as before, we can prove the following counterpart of Theorem 1.1.

THEOREM 6.1. *Assume that system (1.1) is approximately controllable at time T and that one of the following two conditions holds:*

- $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint positive definite operator with compact resolvent, with H being a Hilbert space;
- H is a finite dimensional vector space.

Let $B \in \mathcal{L}(U, H)$, where U is a Hilbert space, and assume that U is isomorphic to $U_1 \times U_2$ for some Hilbert spaces U_1 and U_2 , and define B_1 and B_2 as in (1.3).

Then system (1.4) is approximately controllable at time T with switching controls, i.e., satisfying (1.5). More precisely, given any $\varepsilon > 0$ and any $y_0 \in H$, there exist control functions $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$ such that the solution y of (1.4) satisfies $\|y(T)\|_H \leq \varepsilon$, while the control functions satisfy the switching condition (1.5).

The proof of Theorem 6.1 can be performed the same way as the proof of Theorem 1.1 by minimizing, instead of J in (1.10), the functional J_ε given by

$$(6.1) \quad J_\varepsilon(z_T) = \frac{1}{2} \int_0^T \max\{\|B_1^*z(t)\|_{U_1}^2, \alpha(t)\|B_2^*z(t)\|_{U_2}^2\} dt + \varepsilon\|z_T\|_H + \langle y_0, z(0) \rangle_H,$$

where z is the solution of the adjoint problem (1.8), and $\alpha = \alpha(t)$ is as in (1.11) for a suitable choice of $\omega \in \mathbb{R}^*$.

Details of the proof are left to the reader.

Similarly, counterparts of Theorems 1.3 and 5.1 can also be proved in the context of approximate controllability by penalizing the functional under consideration by the additional term $\varepsilon\|z_T\|_H$ as in (6.1); the rest of the proof is the same. Precise statements and proofs are left to the reader.

Handling source terms. In the proofs of Theorems 1.1, 1.3, and 5.1, we assume that system (1.1) is null-controllable in arbitrary small times. As we said earlier, this is equivalent to saying that for all $T > 0$, any solution z of (1.8) with initial datum $z_T \in H$ satisfies (1.9). It is then easy to check that this property implies that for all $z_T \in H$, the solution z of (1.8) satisfies

$$\frac{1}{T} \int_0^T \frac{1}{C_{T-t}^2} \|z(t)\|_H^2 dt \leq \sup_{(0,T)} \left\{ \frac{1}{C_{T-t}^2} \|z(t)\|_H^2 \right\} \leq \|B^*z\|_{L^2(0,T;U)}^2$$

and thus entails the existence of a positive function $\rho_T \in L_{loc}^1([0, T])$ such that

$$(6.2) \quad \int_0^T \rho_T(t)^2 \|z(t)\|_H^2 dt \leq \|B^*z\|_{L^2(0,T;U)}^2.$$

Additionally, easy considerations allow us to show that ρ_T can be chosen as a strictly positive function which may degenerate to zero only as $t \rightarrow T$.

This allows us to handle source terms in the control problems corresponding to (1.1). For simplicity, as before we only focus on the counterpart of Theorem 1.1, since the counterparts of Theorems 1.3 and 5.1 can be performed similarly.

THEOREM 6.2. *Let us assume $A : \mathcal{D}(A) \subset H \rightarrow H$ is a self-adjoint positive definite operator with compact resolvent, with H being a Hilbert space.*

Let $B \in \mathcal{L}(U, H)$, where U is a Hilbert space, assume that U is isomorphic to $U_1 \times U_2$ for some Hilbert spaces U_1 and U_2 , and define B_1 and B_2 as in (1.3).

Assume that system (1.1) is null-controllable in arbitrary small times and satisfies the observability inequality (6.2) for some functions $(\rho_T)_{T>0}$ a.e. strictly positive with $\rho_T \in L^1_{loc}(0, T)$.

Then given any $T > 0$, any $y_0 \in H$, and $f \in L^2(0, T; H)$ satisfying

$$(6.3) \quad \int_0^T \frac{1}{\rho_T(t)^2} \|f(t)\|_H^2 dt < \infty,$$

there exist control functions $u_1 \in L^2(0, T; U_1)$ and $u_2 \in L^2(0, T; U_2)$ such that the solution y of

$$(6.4) \quad y' + Ay = B_1 u_1 + B_2 u_2 + f, \quad t \in (0, T), \quad y(0) = y_0,$$

satisfies (1.6), while the control functions satisfy the switching condition (1.5).

Again, the proof of Theorem 6.2 can be easily adapted from the proof of Theorem 1.1 by minimizing, instead of the functional J in (1.10), the functional J_s defined for $z_T \in H$ by

$$(6.5) \quad J_s(z_T) = \frac{1}{2} \int_0^T \max\{\|B_1^* z(t)\|_{U_1}^2, \alpha(t) \|B_2^* z(t)\|_{U_2}^2\} dt + \int_0^T \langle f(t), z(t) \rangle_H dt + \langle y_0, z(0) \rangle_H,$$

where z is the solution of the adjoint problem (1.8), and $\alpha = \alpha(t)$ is as in (1.11) for a suitable choice of $\omega \in \mathbb{R}^*$.

The condition (6.3) is there to guarantee that the term

$$\int_0^T \langle f(t), z(t) \rangle_H dt$$

is well defined in the space X in (2.2) and to preserve the coercivity of the functional J_s . Again, the rest of the proof of Theorem 6.2 follows verbatim that of Theorem 1.1 and is left to the reader.

The interest of Theorem 6.2 is that it allows one to handle source terms and therefore paves the way for proving local null-controllability results with switching controls for semilinear equations in the presence of superlinear nonlinearities.

To do so, one should add suitable weights in the design of the controls. These weights can depend only on time, as in the work [27] based on the knowledge of the cost of controllability in small times, or to more general weights depending on time and space variables as it occurs naturally when using Carleman estimates; see, e.g., [15, 18].

6.2. Open problems.

Time-dependent coefficients. One of the important restrictions of our approach is that it is based on spectral decompositions of the space, and therefore seems to be strongly limited to operators which are independent of time. It is natural to discuss this property more closely. In fact, looking at our proof, it seems that the only relevant assumption should be an analytic dependence of the operators with respect to the time t . However, so far this problem seems to be out of reach.

Positive time of controllability. Our arguments are limited to the case of analytic semigroups which are null-controllable in arbitrary small times, but several recent results have shown that there are analytic semigroups which are null-controllable only after some strictly positive critical time. This is the case, for instance, for the one dimensional heat equation controlled from one well-chosen point (see [13]) or when considering Grushin operators (see [4] and references therein).

Our proofs fail to handle these cases, since we do not know how to prove that for $Z_T \in X$ (defined in (2.2)), the function $t \mapsto B^*Z(t)$ (also $t \mapsto B_1^*Z(t)$, $t \mapsto B_2^*Z(t)$) is analytic in time on strict subintervals of $(0, T)$, which is an essential element of our analysis in the study of the set I in (2.4).

Appendix A. Proof of (2.8). The goal of this appendix is to present the proof of the derivation of the Euler–Lagrange equation (2.8) satisfied by a minimizer Z of the functional J in (1.10) when the set I in (2.4) is of zero measure.

Here, we follow the arguments in [31, pp. 91–93].

We keep the notation of section 2: $Z_T \neq 0$ is assumed to be the minimizer of the functional J in (1.10) on X (defined in (2.2)), the set I in (2.4) is of zero measure, and I_1 and I_2 are defined as in (2.6)–(2.7).

For $z_T \in H$, and a.e. in $t \in (0, T)$, we clearly have

$$(A.1) \quad \begin{aligned} & \frac{1}{h} \left(\max\{\|B_1^*(Z + hz)(t)\|_{U_1}^2, \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2\} \right. \\ & \quad \left. - \max\{\|B_1^*Z(t)\|_{U_1}^2, \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\} \right) \\ & \xrightarrow{h \rightarrow 0} \begin{cases} 2\langle B_1^*Z(t), B_1^*z(t) \rangle_{U_1} & \text{if } t \in I_1, \\ 2\alpha(t)\langle B_2^*Z(t), B_2^*z(t) \rangle_{U_2} & \text{if } t \in I_2, \end{cases} \end{aligned}$$

i.e., pointwise convergence in the set I , which is of full measure. Thus, to establish that the Gateaux derivative of J in Z_T is given by (2.8), we only have to prove that this convergence also holds in $L^1(0, T)$.

Using Lebesgue's dominated convergence, we only have to find an $L^1(0, T)$ majorant to the aforementioned ratio as $h \rightarrow 0$.

Fix $t \in (0, T)$, and denote by $i \in \{1, 2\}$ the index in which the maximum of the expression $\max\{\|B_1^*(Z + hz)(t)\|_{U_1}^2, \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2\}$ is achieved and by $j \in \{1, 2\}$ the index in which the maximum of the expression $\max\{\|B_1^*Z(t)\|_{U_1}^2, \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\}$ is achieved (i and j depend on t , but this dependence is omitted for simplicity).

Of course, if $i = j$, it is easy to derive the following bounds:

$$\begin{aligned} & \frac{1}{h} \left(\max\{\|B_1^*(Z + hz)(t)\|_{U_1}^2, \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2\} \right. \\ & \quad \left. - \max\{\|B_1^*Z(t)\|_{U_1}^2, \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\} \right) \\ & = \begin{cases} 2\langle B_1^*Z(t), B_1^*z(t) \rangle_{U_1} + h\|B_1^*z(t)\|_{U_1}^2 & \text{if } i = j = 1, \\ 2\alpha(t)\langle B_2^*Z(t), B_2^*z(t) \rangle_{U_2} + h\alpha(t)\|B_2^*z(t)\|_{U_2}^2 & \text{if } i = j = 2. \end{cases} \end{aligned}$$

When $i \neq j$, this is slightly more delicate. Let us assume, for instance, that $(i, j) = (1, 2)$; the case $(i, j) = (2, 1)$ is the same. Then we have

$$\|B_1^*(Z + hz)(t)\|_{U_1}^2 > \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2 \quad \text{and} \quad \|B_1^*Z(t)\|_{U_1}^2 < \alpha(t)\|B_2^*Z(t)\|_{U_2}^2.$$

Accordingly,

$$\begin{aligned} & \frac{1}{h} \left(\max\{\|B_1^*(Z + hz)(t)\|_{U_1}^2, \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2\} \right. \\ & \quad \left. - \max\{\|B_1^*Z(t)\|_{U_1}^2, \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\} \right) \\ &= \frac{1}{h} \left(\|B_1^*(Z + hz)(t)\|_{U_1}^2 - \alpha(t)\|B_2^*Z(t)\|_{U_2}^2 \right) \\ & \begin{cases} \leq \frac{1}{h} \left(\|B_1^*(Z + hz)(t)\|_{U_1}^2 - \|B_1^*Z(t)\|_{U_1}^2 \right) \\ \geq \frac{1}{h} \left(\alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2 - \alpha(t)\|B_2^*Z(t)\|_{U_2}^2 \right), \end{cases} \end{aligned}$$

which have been estimated in the cases $i = j$.

Thus, we get that for all $(i, j) \in \{1, 2\}^2$,

$$\begin{aligned} & \frac{1}{h} \left(\max\{\|B_1^*(Z + hz)(t)\|_{U_1}^2, \alpha(t)\|B_2^*(Z + hz)(t)\|_{U_2}^2\} \right. \\ & \quad \left. - \max\{\|B_1^*Z(t)\|_{U_1}^2, \alpha(t)\|B_2^*Z(t)\|_{U_2}^2\} \right) \\ & \leq \max\{2\langle B_1^*Z(t), B_1^*z(t) \rangle_{U_1} + h\|B_1^*z(t)\|_{U_1}^2, 2\alpha(t)|\langle B_2^*Z(t), B_2^*z(t) \rangle_{U_2}| \\ & \quad + h\alpha(t)\|B_2^*z(t)\|_{U_2}^2\}. \end{aligned}$$

The right-hand side of this estimate is clearly in $L^1(0, T)$ since B_i^*Z and B_i^*z belong to $L^2(0, T; U_i)$ for $i \in \{1, 2\}$ and $\alpha \in L^\infty(0, T)$. Combining these results with the pointwise convergence (A), we can use Lebesgue's dominated convergence theorem to deduce the Euler–Lagrange equation (2.8).

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