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On the controllability of parabolic equations with large parameters in small time

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ABSTRACT

In this work, we deal with the heat equation perturbed by an anomalous diffusion term scaled by a large parameter. More precisely, we study the cost of null controllability for the system as the parameter goes to infinity. We show that is possible to construct null controls that exhibit exponential decay with respect to the parameter. Our study gives a positive answer to a conjecture raised by J.-L. Lions and E. Zuazua [Rev. Mat. Univ. Complut. Madrid, 10 (2)(1997), 481-524]. Additionally, using the so-called source term method and an inverse mapping theorem, we prove that it is possible to achieve controllability results for certain nonlinear systems with a control with a cost decaying to zero as well.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded connected open set, whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and let ω be a nonempty subset of Ω which will usually be referred to as *control domain*. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$.

In this paper, we deal with the heat equation with an anomalous diffusion term given by

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u = f_k 1_\omega & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $0 \leq \theta < 1$, $k \in \mathbb{R}$, and the operator $(-\Delta)^\theta$ stands for the spectral fractional Laplacian, i.e.

$$(-\Delta)^\theta u(x) \;:=\; \sum_{j=1}^{\infty} \mu_j^\theta \left\langle u, e_j \right\rangle_{L^2(\Omega)} e_j(x) \tag{1.2}$$

with (μ_j, e_j) denoting the eigenvalues and corresponding eigenfunctions of the Laplace operator $-\Delta$ with homogeneous Dirichlet boundary conditions. For more details on this operator, we refer the reader to the article [1].

It is not difficult to prove that for every $T > 0$, each initial condition $u_0 \in L^2(\Omega)$, and any $k \in \mathbb{R}$, there exists a control $f_k \in L^2(\omega \times (0,T))$ such that the associated solution u of system (1.1) satisfies

$$u(T) = 0 \quad \text{in } \Omega. \tag{1.3}$$

Moreover, there exists a constant $C_k(T) > 0$, such that

$$\|f_k\|_{L^2(\omega\times(0,T))} \;\leq\; C_k(T)\, \|u_0\|_{L^2(\Omega)}\,. \tag{1.4}$$

The problem of finding a control function f_k such that the solution of (1.1) satisfies (1.3) is known as the null controllability problem for Eq. (1.1).

The constant $C_k(T)$ in (1.4) is referred to as the *cost of controllability* at time T. This constant measures the difficulty of driving the initial state u_0 to zero at time T. Understanding the behavior of this constant is important in Control Theory and has been a subject of intensive research in recent years. Notably, in [2], Fernández-Cara and Zuazua investigated the cost of approximate controllability for the heat equation with Dirichlet boundary conditions. We also refer to [3] for recent developments in the case of dynamic boundary conditions. In the null controllability case, the works [4,5] by T. Seidman provide optimal estimates on the cost with respect to time for the heat equation. These results were later revisited and further formalized by L. Miller in [6], who used the spectral approach of G. Lebeau and L. Robbiano in [7] to recover and refine Seidman's estimates. It is also worth mentioning results for fluid models: in [8], the optimal cost for the Stokes system is obtained, and in [9], the authors study the cost for the Oseen system in the two-dimensional case.

For equations depending on parameters, several results have also been established. In [10], S. Guerrero and G. Lebeau analyze the null controllability cost for a transport-diffusion equation in the vanishing

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viscosity limit, showing that the cost may either blow up or become exponentially small depending on the time and the associated vector field. See also [11] for recent advances on compact manifolds, and [12–15] for results in one-dimensional settings. We also mention [16,17], where the behavior of the null controllability cost with respect to the viscosity for the heat and the Stokes system is investigated. Finally, we cite [18], where the controllability of the heat equation with vanishing viscosity in networks is studied.

Our interest in studying Eq. (1.1) is motivated by the work [19] of J.-L. Lions and E. Zuazua. In their work, they consider the problem in the case $k \to -\infty$, and showed that, due to the instability of the system, the cost of controllability, i.e. constant $C_k(T)$ in (1.4), blows up. That is to say, when $k \to -\infty$, it costs more and more to drive the solution of (1.1) to rest at a given final state. Conversely, the following conjecture was proposed in [19]:

Conjecture 1 (*Lions-Zuazua*). *As $k \to +\infty$, the cost of controllability for system (1.1) goes to zero, i.e. system (1.1) becomes cheaper and cheaper to be controlled.*

The main purpose of this work is to give a positive answer to this conjecture. To achieve this, we will analyze the behavior of the constant $C_k(T)$ as $k \to +\infty$. Specifically, we prove the following result.

Theorem 1.1. *Let $0 < T_0 \leq 1$ and let $u_0 \in L^2(\Omega)$. Then, for every $T \in (0, T_0)$, there exists a control function $f_k \in L^2(\omega \times (0,T))$ such that the solution to (1.1) satisfies*

$$u(T) = 0 \quad \text{in } \Omega. \tag{1.5}$$

Moreover, for any $k > 0$, we have the estimate

$$\|f_k\|_{L^2(\omega\times(0,T))} \leq C_1 e^{-C_2 kT} e^{\frac{C_3}{T}} \|u_0\|_{L^2(\Omega)},$$

where the constants C_1, C_2, C_3 are positive and do not depend on k.

In particular, Theorem 1.1 gives a positive answer to the conjecture posed by Lions and Zuazua. Moreover, our result extend beyond the original conjecture by showing that the cost of controllability actually decreases exponentially to zero as $k \to +\infty$.

Remark 1.2. Using the classical Lebeau–Robbiano strategy (see [7]), it is not difficult to see that for any $T > 0$, any initial condition $u_0 \in L^2(\Omega)$, and every $k > 0$ there exists a control f_k such that the solution of (1.1) satisfies (1.5), and the control f_k has the estimate (1.4) with $C_k(T)$ bounded with respect to k. Thus, the importance of Theorem (1.1) is the precise estimate for the decay of the norm of the control.

To prove Theorem 1.1, we use the well-known fact that null controllability for a system is equivalent to a suitable observability inequality for its adjoint system. More precisely, Theorem 1.1 is a consequence of the Proposition below.

Proposition 1.3. *Let $0 < T_0 \leq 1$. There exists three positive constants C_1, C_2, C_3 such that, for every $T \in (0, T_0)$, the observability inequality*

$$\|z(T)\|^2_{L^2(\Omega)} \leq C_1 e^{-C_2 kT} e^{\frac{C_3}{T}} \int_0^T \int_\omega |z|^2 \, dx dt \tag{1.6}$$

holds for any $k > 0$ and for all solution z of the adjoint system

$$\begin{cases} z_t - \Delta z + k(-\Delta)^\theta z = 0 & \text{in} \quad Q, \\ z = 0 & \text{on} \quad \Sigma, \\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{1.7}$$

with $z_0 \in L^2(\Omega)$.

The constants C_1, C_2, C_3 in (1.6) are independent of k.

We prove Proposition 1.3 using some arguments of T. Seidman in [5].

Remark 1.4. Observe that when $\theta = 0$, system (1.1) reduces to the standard heat equation with a zero-order term that becomes unbounded as $k \to +\infty$. In this case, the combination of the Hilbert Uniqueness Method (HUM) and the intrinsic dissipation of the system allows us to derive the result stated in Theorem 1.1.

Indeed, if u is a solution of system (1.1) with $\theta = 0$, then $v(x,t) = e^{kt}u(x,t)$ satisfies the following system:

$$\begin{cases} v_t - \Delta v = p 1_\omega & \text{in} \quad Q, \\ v = 0 & \text{on} \quad \Sigma, \\ v(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{1.8}$$

It is straightforward to verify that if p is a control for (1.8), then $f = e^{-kt}p$ is a control for system (1.1) with $\theta = 0$. Moreover, for any fixed $T > 0$ and every $u_0 \in L^2(\Omega)$, there exists a control function $p \in L^2(\omega \times (0,T))$ such that

$$v(T) = 0 \quad \text{in } \Omega,$$

with

$$\|p\|_{L^2(\omega\times(0,T))} \leq Ce^{C/T} \|u_0\|_{L^2(\Omega)}.$$

In particular, this estimate implies that the associated control f satisfies the bound

$$\|f\|_{L^2(\omega\times(0,T))} \leq Ce^{C/T} \|u_0\|_{L^2(\Omega)},$$

which is independent of k, but does not yield any decay with respect to k. To obtain an estimate that reflects exponential decay in k, it suffices to fix $\varepsilon > 0$ and define the control for system (1.1) with $\theta = 0$ as

$$f = \begin{cases} 0, & \text{if} \quad t \in (0, \varepsilon), \\ e^{-kt}p, & \text{if} \quad t \in (\varepsilon, T), \end{cases}$$

and then one has

$$\|f\|_{L^2(\omega\times(0,T))} \leqslant Ce^{-2k\varepsilon} e^{\frac{C}{T-\epsilon}} \|u_0\|_{L^2(\Omega)},$$

with $C = C(\Omega, \omega) > 0$. This estimate gives an explicit exponential decay with respect to the parameter k.

In this paper, we also analyze some nonlinear versions of system (1.1). The motivation for this arises from realizing that the strategy used to deal with the linear system allow us go further and handle certain nonlinear problems as well. To do this, first we must study the null controllability problem for the linear system (1.1) with an additional term on the right-hand side, namely:

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u = g + f 1_\omega & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.9}$$

with g being a right-hand side which belongs to an appropriate weighted space.

The approach of using the controllability of a linear system without external forces to derive the controllability of the same system with appropriate external forces was introduced by Y. Liu, T. Takahashi, and M. Tucsnak in [20] (see also [21]). This technique, known as the *source term method*, relies on obtaining a precise estimate of the cost of controllability for the linear system.

The source term method has been extensively used in recent years. For instance, combining the source term method with Banach's fixed-point theorem, in [22], a local null controllability result is established for a nonlinear stochastic parabolic equation. In [23], this method was used to achieve local controllability to trajectories for a nonlocal viscous Burgers equation. Similarly, [24] applied it to obtain a local exact boundary controllability result for constant trajectories of the nonlinear phase-field system. Here, we combine the source term method with an inverse mapping theorem as done, for instance, by M. Duprez and P. Lissy in [25] to achieve a bilinear local controllability to trajectories for the Fokker–Planck equation.

As mentioned earlier, in the final part of this paper, we consider nonlinear versions of (1.1). More precisely, we show that it is possible to control nonlinear systems with large parameter uniformly with respect to the parameter. The results are stated as follows:

Theorem 1.5. *Let $1 \le N \le 4$, and let $0 < T_0 \le 1$, $T \in (0, T_0)$. For each $k > 0$, consider the system*

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u + |u|^{l-1}u = f 1_\omega & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.10}$$

with

$$\begin{cases} 1 < l < \infty & \text{if} \quad N = 1, 2, \\ 1 < l < 5 & \text{if} \quad N = 3, \\ 1 < l < 3 & \text{if} \quad N = 4. \end{cases} \tag{1.11}$$

Then, there exists $\gamma > 0$ such that, for any $u_0 \in L^2(\Omega)$ satisfying $\|u_0\|_{L^2(\Omega)} \le \gamma$, there exists a control $f_k \in L^2(\omega \times (0, T))$, and constants $C_1, C_2 > 0$, satisfying

$$\|f_k\|^2_{L^2(\omega\times(0,T))} \le C_1 e^{-C_2 kT},$$

such that the corresponding solution to (1.10) satisfies

$$u(T) = 0 \quad \text{in } \Omega.$$

Here, the constants γ, C_1 and C_2 do not depend on k.

The next result addresses nonlinearities that involve the gradient of the solution.

Theorem 1.6. *Let $1 \le N \le 3$, and let $0 < T_0 \le 1$, $T \in (0, T_0)$ and let $\vec{A} \in \mathbb{R}^N$ be a constant vector. For each $k > 0$, consider the following system*

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u + u(\vec{A} \cdot \nabla u) = f 1_\omega & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{1.12}$$

Then, there exists $\gamma > 0$ such that, for any $u_0 \in L^2(\Omega)$ satisfying $\|u_0\|_{L^2(\Omega)} \le \gamma$, there exists a control $f_k \in L^2(\omega \times (0, T))$, and constants $C_1, C_2 > 0$, satisfying

$$\|f_k\|^2_{L^2(\omega\times(0,T))} \le C_1 e^{-C_2 kT},$$

such that the corresponding solution to (1.12) satisfies

$$u(T) = 0 \quad \text{in } \Omega.$$

Here, the constants γ, C_1 and C_2 do not depend on k.

We prove both Theorems 1.5 and 1.6 in Section 4.

Remark 1.7. For the case $N = 4$, following the proof, one can show that if $u_0 \in H^1_0(\Omega)$ with $\|u_0\|_{H^1_0(\Omega)}$ small enough, then there exists a control $f_k \in L^2(\omega \times (0, T))$, uniformly bounded with respect to k, i.e $\|f_k\|^2_{L^2(\omega\times(0,T))} \le C$, that drives the solution of (1.12) to zero at time T. However, in this case, we are unable to obtain an exponential decay for the norm of the control because we do not have a regularity result as Proposition A.2.

Notice that both Theorems 1.5 and 1.6 establish local uniform controllability results, and their proofs rely on the application of the Graves inverse mapping theorem (see [26]). This approach has been extensively used in the literature for systems without parameters (see, for instance, [27–31]). The case of equation which depends on a parameter is more delicate and has been less studied. In this regard, we are only aware of the works [32,33], in the first they show local uniform controllability for the Keller–Segel system around a constant solution, in the second they prove local-controllability to positive constant trajectories of a nonlinear system of two coupled ODE equations modeled for the nonlinear reaction–diffusion Gray–Scott model.

This article is organized as follows. In Section 2, we focus on the proof of Theorem 1.1, which also provides a positive answer to the conjecture proposed by Lions and Zuazua. Section 3 addresses the null controllability of system (1.1) in the presence of a right-hand side term exhibiting suitable exponential decay. In Section 4, we extend our analysis to the nonlinear setting, tackling the corresponding controllability issues. Finally, in Section 5, we outline some open problems and possible directions for future research.

2. Cost of controllability for the linear system

This section is devoted to give the proof of Proposition 1.3. As we said before, this result answers positively for the Lions–Zuazua's Conjecture in [19]. The main difficulty we face is the obtainment of the optimal cost of controllability with respect to k, i.e., estimate (1.6). Our approach is based on some ideas of T. Seidman in [5] (see also [6,8]) to prove the estimate in a direct way. The most important ingredient in the proof is the following spectral inequality.

Proposition 2.1. *Let $\omega \subset\subset \Omega$ be a nonempty open set. There exist constants $M > 0$, $K > 0$ such that, for every sequence of complex numbers a_j and every real $\Lambda > 0$, we have*

$$\sum_{\mu_j \le \Lambda} \left|a_j\right|^2 = \int_\Omega \left|\sum_{\mu_j \le \Lambda} a_j e_j(x)\right|^2 dx \le M e^{K\sqrt{\Lambda}} \int_\omega \left|\sum_{\mu_j \le \Lambda} a_j e_j(x)\right|^2 dx,$$

where $\{e_j\}_{j=1}^{+\infty}$ is the sequence of eigenfunctions for the Laplace operator with Dirichlet boundary condition and the sequence $\{\mu_j\}_{j=1}^{+\infty}$ the respective eigenvectors.

A proof of Proposition 2.1 in the case of opens sets can be found in [7], and in [34] for the case of measurable sets when the domain is star-shaped.

Before turning to the proof of Proposition 1.3, we give two auxiliary lemmata that play a fundamental role in the forthcoming estimates. These results are central to the argument and will be used at key steps of the proof. Throughout this section, the constant $C > 0$ may change from line to line.

The first lemma provides a decay estimate for solutions of system (1.7), while the second concerns an estimate that is valid in the short-time regime.

Lemma 2.2. *Let $z_0 \in L^2(\Omega)$. For any $0 \le t_1 < t_2$, the solution z of (1.7) satisfies*

$$\|z(t_2)\|^2_{L^2(\Omega)} \le e^{-Ck(t_2-t_1)}\|z(t_1)\|^2_{L^2(\Omega)},$$

where $C > 0$ does not depend on k.

Proof. This follows directly by writing $z(t)$ in terms of the eigenfunctions of the Laplace operator with Dirichlet boundary condition. □

Lemma 2.3. *Let $T > 0$, $z_0 \in L^2(\Omega)$ and z the solution of (1.7). Then, for any $0 < \epsilon < 1$, it holds*

$$\|z(T)\|^2_{L^2(\Omega)} \le \frac{2}{\epsilon T} e^{-Ck\epsilon T} \int_{(1-\epsilon)T}^{T} \|z(t)\|^2_{L^2(\Omega)} dt,$$

where $C > 0$ does not depend on k.

Proof. From Lemma 2.2, one has

$$\|z(T)\|^2_{L^2(\Omega)} \le e^{-Ck(T-t)}\|z(t)\|^2_{L^2(\Omega)}.$$

Integrating this inequality from $(1-\epsilon)T$ to $\frac{T+(1-\epsilon)T}{2}$, the result follows. □

Let us now prove Proposition 1.3.

Proof of Proposition 1.3. Let $T \in (0, T_0)$, for some $T_0 > 0$ to be chosen later, and $z_0 \in L^2(\Omega)$. For any $\Lambda > 0$, define the low-frequency space

$$H_\Lambda = \operatorname{span}\{e_j : \mu_j \le \Lambda\},$$

where μ_j and e_j are the eigenvalues and corresponding eigenfunctions of the Laplace operator with Dirichlet boundary conditions.

We decompose the initial datum as $z_0 = u_0 + v_0$, with $u_0 \in H_\Lambda$ and $v_0 \in H_\Lambda^\perp$. Accordingly, we write the solution z of (1.7) as $z = u + v$, where u and v solve (1.7) with initial data $u_0 \in H_\Lambda$ and $v_0 \in H_\Lambda^\perp$, respectively.

It is not difficult to see that the high-frequency component, whose spectral support lies in the region where $\mu_j > \Lambda$, decays exponentially. More precisely, we have the estimate

$$\|v(t)\|_{L^2(\Omega)} \le e^{-(\Lambda+k\Lambda^\theta)t}\|v_0\|_{L^2(\Omega)}, \quad \forall t > 0. \tag{2.1}$$

On the other hand, since $u(t) \in H_\Lambda$ for all $t > 0$, from Lemma 2.3 and Proposition 2.1, it follows that

$$\|u(T)\|^2_{L^2(\Omega)} \le \frac{2M}{M_1} e^{-Ck\epsilon T} e^{\frac{M_1}{\epsilon T}} e^{K\sqrt{\Lambda}} \int_{(1-\epsilon)T}^{T}\int_\omega |u(t)|^2\,dxdt, \tag{2.2}$$

for some $M_1 > 0$ to be chosen, and any $0 < \epsilon < 1$.

By taking the $L^2(\omega)$-norm of $z = u + v$ and then integrating over $((1-\epsilon)T, T)$, we obtain

$$\int_{(1-\epsilon)T}^{T}\int_\omega |u(t)|^2\,dxdt \le 2\left(\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt + \int_{(1-\epsilon)T}^{T}\int_\omega |v(t)|^2\,dxdt\right). \tag{2.3}$$

To simplify notation, we define

$$f(T) := \frac{M_1}{2M} e^{Ck\epsilon T} e^{-\frac{M_1+K}{\epsilon T}}. \tag{2.4}$$

Now, setting the frequency $\sqrt{\Lambda} = \frac{1}{\epsilon T}$, from (2.2)–(2.3), we obtain that

$$f(T)\|u(T)\|^2_{L^2(\Omega)} \le 2\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt \;+\; 2\int_{(1-\epsilon)T}^{T}\int_\omega |v(t)|^2\,dxdt. \tag{2.5}$$

We estimate the last integral on the right-hand side of (2.5) using the following claim.

Claim 2.1. *For any $0 < \epsilon < 1$, we have the estimate*

$$\int_{(1-\epsilon)T}^{T} \|v(t)\|^2_{L^2(\Omega)}dt \;\le\; \epsilon T e^{\left(-Ck-2\Lambda-2k\Lambda^\theta\right)\frac{(1-\epsilon)T}{2}}\|v_0\|^2_{L^2(\Omega)}.$$

Proof of the Claim 2.1. Let $t > 0$. From Lemma 2.2, we have the estimate

$$\|v(t)\|^2_{L^2(\Omega)} \;\le\; e^{-Ck\frac{t}{2}}\left\|v\left(\tfrac{t}{2}\right)\right\|^2_{L^2(\Omega)}.$$

Moreover, from (2.1) we also have that

$$\left\|v\left(\tfrac{t}{2}\right)\right\|_{L^2(\Omega)} \;\le\; e^{-(\Lambda+k\Lambda^\theta)\frac{t}{2}}\|v_0\|_{L^2(\Omega)}$$

Therefore, Claim 2.1 follows by combining these two estimates and then integrating from $(1-\epsilon)T$ to T. □

From (2.5) and Claim 2.1, we readily see that

$$f(T)\|z(T)\|^2_{L^2(\Omega)} \le 2\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt + 2\epsilon T e^{\left(-Ck-2\Lambda-2k\Lambda^\theta\right)\frac{(1-\epsilon)T}{2}}\|v_0\|^2_{L^2(\Omega)} + f(T)\|v(T)\|^2_{L^2(\Omega)}.$$

Using that $\|v_0\|_{L^2(\Omega)} \le \|z_0\|_{L^2(\Omega)}$, and applying estimate (2.1) at $t = T/2$, we get

$$f(T)\|z(T)\|^2_{L^2(\Omega)} - \left(2\epsilon T e^{\left(-Ck-2\Lambda-2k\Lambda^\theta\right)\frac{(1-\epsilon)T}{2}} + f(T)e^{\left(-Ck-2\Lambda-2k\Lambda^\theta\right)\frac{T}{2}}\right)\|z_0\|^2_{L^2(\Omega)} \le 2\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt.$$

In particular, this gives

$$f(T)\|z(T)\|^2_{L^2(\Omega)} - e^{\left(-Ck-2\Lambda\right)\frac{(1-\epsilon)T}{2}}\left(2\epsilon T + f(T)\right)\|z_0\|^2_{L^2(\Omega)} \le 2\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt.$$

Since $\Lambda = \frac{1}{\epsilon^2 T^2}$, we can use the estimate

$$2\epsilon T + f(T) \le 2\epsilon T + \frac{M_1}{2M} e^{Ck\epsilon T} e^{-\frac{M_1+K}{\epsilon T_0}}, \quad \forall T < T_0,$$

and the definition of $f(T)$ in (2.4) to see that

$$f(T)\|z(T)\|^2_{L^2(\Omega)} - e^{-Ck\frac{(1-\epsilon)T}{2}} e^{-\frac{(1-\epsilon)}{\epsilon^2 T}}\left(2\epsilon T + \frac{M_1}{2M}e^{Ck\epsilon T}e^{-\frac{M_1+K}{\epsilon T_0}}\right)\|z_0\|^2_{L^2(\Omega)} \le 2\int_{(1-\epsilon)T}^{T}\int_\omega |z(t)|^2\,dxdt. \tag{2.6}$$

To finish, we need the following lemma, in which we choose both M_1 and T_0.

Lemma 2.4. *Let $T_0 \le 1$. There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, there exists a constant $m \in (0,1)$, independent of T_0 and k, such that*

$$e^{-Ck\frac{(1-\epsilon)T}{2}} e^{-\frac{(1-\epsilon)}{\epsilon^2 T}}\left(2\epsilon T + \frac{M_1}{2M}e^{Ck\epsilon T}e^{-\frac{M_1+K}{\epsilon T_0}}\right) \;\le\; f(mT), \tag{2.7}$$

for all $T \in (0, T_0)$ and all $k > 0$.

Proof of Lemma 2.4. We begin noticing that, for any $k > 0$ and every $m \in (0,1)$, we have

$$\frac{M_1}{2M}e^{-\frac{M_1+K}{\epsilon T m}} \le f(mT). \tag{2.8}$$

We divide the proof into two cases:

Case 1: $0 < k \le 1$.

In this case, it is not difficult to see that

$$2\epsilon T + \frac{M_1}{2M}e^{Ck\epsilon T}e^{-\frac{M_1+K}{\epsilon T_0}} \;\le\; e^{C\epsilon T}\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right). \tag{2.9}$$

Therefore, from estimates (2.8) and (2.9), it is sufficient to find $m \in (0,1)$ such that

$$e^{-Ck\frac{(1-\epsilon)T}{2}} e^{-\frac{(1-\epsilon)}{\epsilon^2 T}} e^{C\epsilon T}\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right) \;\le\; \frac{M_1}{2M}e^{-\frac{M_1+K}{\epsilon T m}}, \tag{2.10}$$

or equivalently,

$$\frac{M_1+K}{\epsilon T m} \le \frac{(1-\epsilon)}{\epsilon^2 T} + \ln\left(\frac{M_1}{2M\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right)}\right) + Ck\frac{(1-\epsilon)T}{2} - C\epsilon T. \tag{2.11}$$

The result follows from the fact that there exist $\epsilon_0 > 0$ and $\overline{M}_1 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$ and $M_1 \ge \overline{M}_1$, one has

$$\frac{(1-\epsilon)}{\epsilon^2 T} - C\epsilon T \ge \frac{1}{2}\frac{(1-\epsilon)}{\epsilon^2 T} \quad \text{and} \quad \frac{1}{2(M_1+K)}\frac{(1-\epsilon)}{\epsilon} > 1,$$

and

$$\frac{M_1}{2M\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right)} \ge 1.$$

In this case, we may take

$$1 < m^{-1} \le \frac{1}{2(M_1+K)}\frac{(1-\epsilon)}{\epsilon}.$$

Case 2: $k > 1$.

Here, using the estimate

$$2\epsilon T + \frac{M_1}{2M}e^{Ck\epsilon T}e^{-\frac{M_1+K}{\epsilon T_0}} \;\le\; e^{Ck\epsilon T}\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right), \tag{2.12}$$

we seek $m \in (0,1)$ such that

$$e^{-Ck\frac{(1-\epsilon)T}{2}} e^{-\frac{(1-\epsilon)}{\epsilon^2 T}} e^{Ck\epsilon T}\left(\frac{2}{C} + \frac{M_1}{2M}e^{-\frac{K}{\epsilon}}\right) \le \frac{M_1}{2M}e^{-\frac{M_1+K}{\epsilon T m}}. \tag{2.13}$$

Proceeding as in the first case, we may take

$$1 < m^{-1} \le \frac{1}{M_1+K}\frac{(1-\epsilon)}{\epsilon},$$

where ϵ is sufficiently small and M_1 is large enough. □

Let us now finish the proof of Proposition 1.3. From (2.6) and Lemma 2.4, we have

$$f(t)\|z(t)\|^2_{L^2(\Omega)} - f(mt)\|z_0\|^2_{L^2(\Omega)} \le 2\int_0^t\int_\omega |z(s)|^2\,dxds. \tag{2.14}$$

Next, we take $\tau_j = m^j(1-m)T = m^{j+1}T$, $j \in \mathbb{N}$, and consider a disjoint partition $\cup(T_{j+1}, T_j]$ of $(0,T)$ with $T_{j+1} = T_j - \tau_j$, $T_0 = T$.

We apply (2.14) to $z_0 = z(T_{j+1})$ and $t = \tau_j$, obtaining

$$f(\tau_j)\|z(T_j)\|^2 - f(\tau_{j+1})\|z(T_{j+1})\|^2 \leq 2\int_{T_{j+1}}^{T_j}\int_\omega |z|^2\,dxdt,\ \ j \in \mathbb{N}.$$

We add these inequalities, to get

$$f(\tau_0)\|z(T)\|^2 - f(\tau_j)\|z(T_j)\|^2 \leq 2\int_{T_j}^{T}\int_\omega |z|^2\,dxdt,\ \ j \in \mathbb{N}.$$

Finally, taking $j \longrightarrow \infty$, and using that the function $t \mapsto \|z(t)\|$ is bounded in $[0,T]$ and that, for each k, $f(\tau_j) \longrightarrow 0$, we obtain

$$\|z(T)\|^2 \leq \frac{2}{f((1-m)T)}\int_0^T\int_\omega |z|^2\,dxdt,$$

for any $T \in (0,T_0)$, T_0 small, and k large. This finishes the proof of Proposition 1.3. $\square$

3. Linear case with a right-hand side

In this section, we show that one can obtain null controllability for the linear system (1.1) with a right-hand side in an appropriate weighted space. A crucial step here is the construction of the weighted spaces where the external forces may live. The construction of such spaces has been formalized by Y. Liu, T. Takahashi and M. Tucsnak in [20]. Establishing this result will be crucial to derive the local null controllability of (1.1) in the following section.

Given g in a space to be defined (see Theorem 3.2 below), we would like to find a control function $f_k \in L^2(\omega\times(0,T))$ such that the solution of

$$\begin{cases} L_k(u) = g + f_k 1_\omega & \text{in} \quad Q,\\ u = 0 & \text{on} \quad \Sigma,\\ u(0) = u_0 & \text{in} \quad \Omega,\end{cases} \tag{3.1}$$

where

$$L_k(u) = u_t - \Delta u + k(-\Delta)^\theta u$$

verifies

$$u(T) = 0 \quad \text{in } \Omega. \tag{3.2}$$

As said before, to deal with the nonlinear problems, we must solve the previous problem in an appropriate weighted space. Before introducing the spaces where we solve (3.1)–(3.2), we need to define several functions that will be used in the sequel.

For $T > 0$, consider the function $\gamma : (0,T] \longrightarrow \mathbb{R}$, given by

$$\gamma(t) = C_1 e^{\frac{C_3}{t}}, \tag{3.3}$$

where C_1, C_3 are the same constants given in Theorem 1.1.

Let $q > 1$ and $s_0 > 0$ such that

$$C_3 q^2 - s_0(q-1) < 0. \tag{3.4}$$

We introduce the basic weight $\rho_0 : [0,T] \longrightarrow (0,\infty)$ given by

$$\rho_0(t) = e^{-\frac{s_0}{T-t}} e^{s_1 k(T-t)}, \tag{3.5}$$

with $s_1 > 0$.

Let us define the function $\rho_k : [T(1-\frac{1}{q^2}), T] \longrightarrow (0,\infty)$ by

$$\rho_k(t) \ := \ \rho_0\big(q^2(t-T)+T\big)\gamma\big((q-1)(T-t)\big). \tag{3.6}$$

In particular, using the definition of γ and ρ_0 in (3.3) and (3.5), respectively, ρ_k can be expressed as:

$$\rho_k(t) \ = \ C_1 e^{s_1 k q^2 (T-t)} e^{\frac{C_3 q^2 - s_0(q-1)}{q^2(q-1)(T-t)}},$$

with C_1, C_3 the same constants in Theorem 1.1.

Remark 3.1. Note that both functions ρ_0 and ρ_k are non-increasing and satisfy

$\rho_0(T) = \rho_k(T) = 0.$

We extend ρ_k to $[0,T]$, i.e. $\rho_k : [0,T] \longrightarrow (0,\infty)$, by the formula

$$\rho_k(t) = \begin{cases} \rho_k(t), & [T(1-\frac{1}{q^2}), T]\\ \rho_k(T(1-\frac{1}{q^2})), & [0, T(1-\frac{1}{q^2})].\end{cases}$$

Now, we proceed to the definition of the spaces where (3.1)–(3.2) will be solved. We introduce:

$$E_0 = \Big\{ (u,f) \ : \ \rho_0^{-1}\big(L_k(u) - f 1_\omega\big) \in L^2(Q), \quad u = 0 \text{ on } \Sigma,$$
$$\rho_k^{-1} u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \text{and } \rho_k^{-1} f 1_\omega \in L^2(Q) \Big\}.$$

Notice that E_0 is a Banach space with norm

$$\|(u,f)\|_{E_0} \ := \ \|\rho_k^{-1}u\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))} \ + \ \|\rho_k^{-1} f 1_\omega\|_{L^2(Q)}.$$

The first result of this section is the following.

Theorem 3.2. *Let $T_0 > 0$ be the same as in Theorem 1.1 and let $T \in (0,T_0)$. Then, for every $u_0 \in L^2(\Omega)$ and g such that*

$\rho_0^{-1} g \in L^2(Q),$

there exists $(u,f) \in E_0$ satisfying

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u = g + f 1_\omega & \text{in} \quad Q,\\ u = 0 & \text{on} \quad \Sigma,\\ u(0) = u_0 & \text{in} \quad \Omega.\end{cases} \tag{3.7}$$

Moreover, there exists constants $\mathfrak{C}_1, \mathfrak{C}_2 > 0$, not depending on k, such that the following estimate holds

$$\|\rho_k^{-1}u\|^2_{C([0,T],L^2(\Omega))\cap L^2((0,T);H_0^1(\Omega))} \ + \ \|\rho_k^{-1} f 1_\omega\|^2_{L^2(Q)} \leq \ \mathfrak{C}_1\Big(e^{-2s_1 kT} e^{\frac{\mathfrak{C}_2}{T}}\|u_0\|^2_{L^2(\Omega)} \ + \ \|\rho_0^{-1} g\|^2_{L^2(Q)}\Big). \tag{3.8}$$

In particular, the above relation yields $u(T) = 0$.

Proof of Theorem 3.2. Let us begin by defining the sequence

$$\tau_j = T - \frac{T}{q^j} \quad j \geq 0.$$

For each $j \geq 0$, we solve the equation

$$\begin{cases} z_t - \Delta z + k(-\Delta)^\theta z = g & \text{in} \quad \Omega\times(\tau_j,\tau_{j+1}),\\ z = 0 & \text{on} \quad \partial\Omega\times(\tau_j,\tau_{j+1}),\\ z(\tau_j) = 0 & \text{in} \quad \Omega\end{cases} \tag{3.9}$$

and introduce the sequence of data $\{u_0^j\}_{j\geq 0}$, given by

$$u_0^0 = u_0, \quad u_0^{j+1} = z(\tau_{j+1}), \quad \text{for } j \geq 0. \tag{3.10}$$

Then, we consider the controlled system

$$\begin{cases} w_t - \Delta w + k(-\Delta)^\theta w = f_j 1_\omega & \text{in} \quad \Omega\times(\tau_j,\tau_{j+1}),\\ w = 0 & \text{on} \quad \partial\Omega\times(\tau_j,\tau_{j+1}),\\ w(\tau_j) = u_0^j & \text{in} \quad \Omega.\end{cases}$$

From Theorem 1.1, there exists a control function f_j such that

$$w(\tau_{j+1}) = 0 \quad \text{in } \Omega,$$

and satisfying the estimate

$$\|f_j 1_\omega\|^2_{L^2((\tau_j,\tau_{j+1});L^2(\Omega))} \leq C_k^2(\tau_{j+1}-\tau_j)\|u_0^j\|^2_{L^2(\Omega)}.$$

where $C_k(t) = C_1 e^{-C_2 kt} e^{\frac{C_3}{t}}$ is the cost of controllability of the linear system without external force.

By the definition of the function γ, we then have

$$\|f_j 1_\omega\|^2_{L^2((\tau_j,\tau_{j+1});L^2(\Omega))} \ \leq \ \gamma^2(\tau_{j+1}-\tau_j)\|u_0^j\|^2_{L^2(\Omega)}. \tag{3.11}$$

Since Eq. (3.9) satisfies the energy estimate

$$\|z\|^2_{C([\tau_j,\tau_{j+1}];L^2(\Omega))} \le C \int_{\tau_j}^{\tau_{j+1}} \|g(t)\|^2_{L^2(\Omega)} dt,$$

we readily see that

$$\|u_0^{j+1}\|^2_{L^2(\Omega)} \le C \int_{\tau_j}^{\tau_{j+1}} \|g(t)\|^2_{L^2(\Omega)} dt, \tag{3.12}$$

where the constant C does not depend on k.

Now, recalling the definition of ρ_k in (3.6), we have that

$$\rho_k(\tau_{j+2}) = \rho_0(\tau_j)\gamma(\tau_{j+2} - \tau_{j+1}). \tag{3.13}$$

Hence, from (3.11)–(3.13), and the fact that ρ_0 is non-increasing, it is not difficult to see that

$$\|f_{j+1}1_\omega\|^2_{L^2((\tau_{j+1},\tau_{j+2});L^2(\Omega))} \le C\rho_k^2(\tau_{j+2}) \int_{\tau_j}^{\tau_{j+1}} \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt,$$

where the constant C does not depend on k.

In particular, since ρ_k is non-increasing, we conclude that

$$\|\rho_k^{-1}f_{j+1}1_\omega\|^2_{L^2((\tau_{j+1},\tau_{j+2});L^2(\Omega))} \le C \int_{\tau_j}^{\tau_{j+1}} \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt, \tag{3.14}$$

where C does not depend on k.

Let us now define the control

$$f := \sum_{j=0}^{\infty} f_j 1_{(\tau_j,\tau_j+1)}. \tag{3.15}$$

Claim 3.1. $\rho_k^{-1} f \in L^2(Q)$.

Proof of the Claim 3.1. Using (3.11) for $j = 0$, we have that

$$\|f_0 1_\omega\|^2_{L^2((0,\tau_1);L^2(\Omega))} \le \gamma^2(\tau_1)\|u_0\|^2_{L^2(\Omega)}. \tag{3.16}$$

From the fact that ρ_k is constant in $[0, T(1-\frac{1}{q^2})]$, i.e.

$$\rho_k(\tau_1) = \rho_k\big(T(1-\tfrac{1}{q^2})\big),$$

estimate (3.16) implies that

$$\|\rho_k^{-1} f_0 1_\omega\|^2_{L^2((0,\tau_1);L^2(\Omega))} \le e^{-2s_1kT} e^{\frac{2C_3q - 2(C_3q^2 - s_0(q-1))}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)}. \tag{3.17}$$

Thus, from (3.14), (3.17) and the definition of f in (3.15), the following estimate holds

$$\begin{aligned}\|\rho_k^{-1} f 1_\omega\|^2_{L^2((0,T);L^2(\Omega))} \le\ & e^{-2s_1kT} e^{\frac{2C_3q - 2(C_3q^2 - s_0(q-1))}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)} \\ &+ C \int_0^T \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt,\end{aligned} \tag{3.18}$$

where C does not depend on k. This finishes the proof of Claim 3.1. □

Now, we set

$$u = z + w$$

which solves

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u = g + f_j 1_\omega & \text{in} \quad \Omega \times (\tau_j, \tau_{j+1}), \\ u = 0 & \text{on} \quad \partial\Omega \times (\tau_j, \tau_{j+1}), \\ u(\tau_j) = u_0^j & \text{in} \quad \Omega. \end{cases} \tag{3.19}$$

We have following decay in time for u.

Claim 3.2. $\rho_k^{-1} u \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$.

Proof of Claim 3.2. To prove this claim, we use the following result, whose proof is straightforward.

Lemma 3.3. *Let $T > 0$, $u_0 \in L^2(\Omega)$ and let $g \in L^2(Q)$. The solution of*

$$\begin{cases} u_t - \Delta u + k(-\Delta)^\theta u = g & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega \end{cases}$$

satisfies

$$\|u\|^2_{C([0,T];L^2(\Omega))\cap L^2((0,T);H_0^1(\Omega))} \le C\left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|g(t)\|^2_{L^2(\Omega)} dt\right), \tag{3.20}$$

where $C > 0$ does not depend on k.

Furthermore, if $u_0 \in H_0^1(\Omega)$, then

$$\|u\|^2_{C([0,T];H_0^1(\Omega))\cap L^2((0,T);H^2(\Omega))} \le C\left(\|u_0\|^2_{H_0^1(\Omega)} + \int_0^T \|g(t)\|^2_{L^2(\Omega)} dt\right), \tag{3.21}$$

where $C > 0$ does not depend on k.

Applying this lemma to system (3.19), from (3.20) we get

$$\begin{aligned}&\|u\|^2_{C([\tau_j,\tau_{j+1}];L^2(\Omega))\cap L^2((\tau_j,\tau_{j+1});H_0^1(\Omega))} \\ &\quad \le C\left(\|u_0^j\|^2_{L^2(\Omega)} + \int_{\tau_j}^{\tau_{j+1}} \|f_j(t)1_\omega\|^2_{L^2(\Omega)} dt + \int_{\tau_j}^{\tau_{j+1}} \|g(t)\|^2_{L^2(\Omega)} dt\right),\end{aligned} \tag{3.22}$$

with the constant $C > 0$ not depending on k.

For $j \ge 1$, combining (3.11) with (3.12), and using (3.13), estimate (3.22) reads

$$\|u\|^2_{C([\tau_j,\tau_{j+1}];L^2(\Omega))\cap L^2((\tau_j,\tau_{j+1});H_0^1(\Omega))} \le \mathfrak{C}\rho_k^2(\tau_{j+1}) \int_{\tau_{j-1}}^{\tau_{j+1}} \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt,$$

where the constant $\mathfrak{C}$ does not depend on k.

Since ρ_k is non-increasing, this last inequality leads to

$$\|\rho_k^{-1}u\|^2_{C([\tau_j,\tau_{j+1}];L^2(\Omega))\cap L^2((\tau_j,\tau_{j+1});H_0^1(\Omega))} \le \mathfrak{C} \int_{\tau_{j-1}}^{\tau_{j+1}} \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt, \qquad j \ge 1. \tag{3.23}$$

For $j = 0$, we know that

$$\|u\|^2_{C([0,\tau_1];L^2(\Omega))\cap L^2((0,\tau_1);H_0^1(\Omega))} \le C\left(C_1^2 e^{\frac{2C_3q}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)} + \int_0^{\tau_1} \|g(t)\|^2_{L^2(\Omega)} dt\right).$$

Then, using that ρ_k is constant in $[0, \tau_1]$, we see that

$$\rho_k^{-2}(\tau_1) = C_1^{-2} e^{-2s_1kT} e^{-\frac{2(C_3q^2 - s_0(q-1))}{(q-1)T}}.$$

Thus,

$$\begin{aligned}\|\rho_k^{-1}u\|^2_{C([0,\tau_1];L^2(\Omega))\cap L^2((0,\tau_1);H_0^1(\Omega))} =\ & Ce^{-2s_1kT} e^{\frac{2C_3q - 2(C_3q^2 - s_0(q-1))}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)} \\ &+ CC_1^{-2} e^{-\frac{2C_3q^2}{T(q-1)}} \int_0^{\tau_1} \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt.\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we obtain the estimate

$$\begin{aligned}\|\rho_k^{-1}u\|^2_{C([0,T];L^2(\Omega))\cap L^2((0,T);H_0^1(\Omega))} \le\ & Ce^{-2s_1kT} e^{\frac{2C_3q - 2(C_3q^2 - s_0(q-1))}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)} \\ &+ CC_1^{-2} e^{-\frac{2C_3q^2}{T(q-1)}} \int_0^T \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt \\ &+ C \int_0^T \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt.\end{aligned}$$

This last inequality readily implies that

$$\|\rho_k^{-1}u\|^2_{C([0,T];L^2(\Omega))\cap L^2((0,T);H_0^1(\Omega))} \le \hat{\mathfrak{C}}\left(e^{-2s_1kT} e^{\frac{2C_3q - 2(C_3q^2 - s_0(q-1))}{T(q-1)}} \|u_0\|^2_{L^2(\Omega)} + \int_0^T \|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)} dt\right)$$

$$\tag{3.25}$$

where the constant $\hat{\mathfrak{C}}$ does not depend on k. This finishes the proof of the Claim 3.2. □

Finally, combining (3.18) and (3.25), we deduce that

$$\|\rho_k^{-1}u\|^2_{C([0,T];L^2(\Omega))\cap L^2((0,T);H_0^1(\Omega))} + \|\rho_k^{-1}f1_\omega\|^2_{L^2((0,T);L^2(\Omega))} \leq \mathfrak{C}_1\left(e^{-2s_1kT}e^{\frac{2C_3q-2(C_3q^2-s_0(q-1))}{T(q-1)}}\|u_0\|^2_{L^2(\Omega)}+\int_0^T\|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)}dt\right),$$

which proves Theorem 3.2. □

When the initial data u_0 is more regular, we can improve Theorem 3.2. This result will be important in the next section when dealing with the nonlinear problem.

Before stating the result, let us introduce the following Banach space:

$$E=\Big\{(u,f)\ :\ \rho_0^{-1}\left(L_k(u)-f1_\omega\right)\in L^2(Q),\ u=0 \text{ on } \Sigma,\ \rho_k^{-1}u\in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega))\text{and } \rho_k^{-1}f1_\omega\in L^2(Q)\Big\},\tag{3.26}$$

with the norm

$$\|(u,f)\|_E := \|\rho_k^{-1}u\|_{L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\rho_k^{-1}f1_\omega\|_{L^2(Q)}.$$

We have:

Theorem 3.4. *Assume $u_0\in H_0^1(\Omega)$. Under the same assumptions of Theorem 3.2, the state-control pair $(u,f)\in E$ satisfies:*

$$\|\rho_k^{-1}u\|^2_{C([0,T],H_0^1(\Omega))\cap L^2((0,T);H^2(\Omega))} \leq \mathfrak{C}_1\left(e^{-2s_1kT}e^{\frac{\mathfrak{C}_2}{T}}\|u_0\|^2_{H_0^1(\Omega)}+\|\rho_0^{-1}g\|^2_{L^2(Q)}\right),\tag{3.27}$$

for some constants $\mathfrak{C}_1,\mathfrak{C}_2>0$ which does not depend on k.

Proof of Theorem 3.4. Since $u_0\in H_0^1(\Omega)$, the sequence of data in (3.10) satisfies $u_0^j\in H_0^1(\Omega)$ for each $j\geq 0$.

By construction, we know that, for each $j\geq 0$, u solves system (3.19). Then, energy estimate (3.21) gives

$$\|u\|^2_{C([\tau_j,\tau_{j+1}];H_0^1(\Omega))\cap L^2((\tau_j,\tau_{j+1});H^2(\Omega))} \leq C\left(\|u_0^j\|^2_{H_0^1(\Omega)}+\int_{\tau_j}^{\tau_{j+1}}\|f_j(t)1_\omega\|^2_{L^2(\Omega)}dt+\int_{\tau_j}^{\tau_{j+1}}\|g(t)\|^2_{L^2(\Omega)}dt\right),$$

where the constant $C>0$ does not depend on k.

For $j\geq 1$, estimate (3.11) implies

$$\|u\|^2_{C([\tau_j,\tau_{j+1}];H_0^1(\Omega))\cap L^2((\tau_j,\tau_{j+1});H^2(\Omega))} \leq C\left(\gamma^2(\tau_{j+1}-\tau_j)\|u_0^j\|^2_{H_0^1(\Omega)}+\int_{\tau_j}^{\tau_{j+1}}\|g(t)\|^2_{L^2(\Omega)}dt\right).$$

Next, since $u_0^j\in H_0^1(\Omega)$, we have that

$$\|u_0^{j+1}\|^2_{H_0^1(\Omega)}\leq C\int_{\tau_j}^{\tau_{j+1}}\|g(t)\|^2_{L^2(\Omega)}dt.$$

Thus, for each $j\geq 1$, we have

$$\|u\|^2_{C([\tau_j,\tau_{j+1}];H_0^1(\Omega))\cap L^2((\tau_j,\tau_{j+1});H^2(\Omega))} \leq C\gamma^2(\tau_{j+1}-\tau_j)\int_{\tau_{j-1}}^{\tau_{j+1}}\|g(t)\|^2_{L^2(\Omega)}dt.$$

Now we recall that

$$\rho_k(\tau_{j+1})=\rho_0(\tau_{j-1})\gamma(\tau_{j+1}-\tau_j),\quad j\geq 1.$$

Then, it follows that

$$\|u\|^2_{C([\tau_j,\tau_{j+1}];H_0^1(\Omega))\cap L^2((\tau_j,\tau_{j+1});H^2(\Omega))} \leq C\rho_k^2(\tau_{j+1})\int_{\tau_{j-1}}^{\tau_{j+1}}\|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)}dt.$$

Using again that ρ_k is non-increasing, we have that

$$\|\rho_k^{-1}u\|^2_{C([\tau_j,\tau_{j+1}];H_0^1(\Omega))\cap L^2((\tau_j,\tau_{j+1});H^2(\Omega))} \leq C\int_{\tau_{j-1}}^{\tau_{j+1}}\|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)}dt,\quad j\geq 1.\tag{3.28}$$

For $j=0$, we have the estimate

$$\|u\|^2_{C([0,\tau_1];H_0^1(\Omega))\cap L^2((0,\tau_1);H^2(\Omega))} \leq C\left(e^{\frac{2C_3q}{(q-1)T}}\|u_0\|^2_{H_0^1(\Omega)}+\int_0^{\tau_1}\|g(t)\|^2_{L^2(\Omega)}dt\right).$$

Using that ρ_k is constant in $[0,\tau_1]$, i.e.

$$\rho_k^{-2}(\tau_1) = C_1^{-2}e^{-2s_1kT}e^{-\frac{2(C_3q^2-s_0(q-1))}{(q-1)T}},$$

it follows that

$$\|\rho_k^{-1}u\|^2_{C([0,\tau_1];H_0^1(\Omega))\cap L^2((0,\tau_1);H^2(\Omega))} \leq C\left(e^{-2s_1kT}e^{\frac{2C_3q-2(C_3q^2-s_0(q-1))}{(q-1)T}}\|u_0\|^2_{H_0^1(\Omega)}+\int_0^{\tau_1}\|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)}dt\right).\tag{3.29}$$

Finally, combining (3.28) and (3.29), we obtain

$$\|\rho_k^{-1}u\|^2_{C([0,T];H_0^1(\Omega))\cap L^2((0,T);H^2(\Omega))} \leq C\left(e^{-2s_1kT}e^{\frac{2C_3q-2(C_3q^2-s_0(q-1))}{(q-1)T}}\|u_0\|^2_{H_0^1(\Omega)}+\int_0^{T}\|\rho_0^{-1}g(t)\|^2_{L^2(\Omega)}dt\right),$$

which proves Theorem 3.4. □

Remark 3.5. Note that, in Theorem 3.4, we do not have boundedness, with respect to k, for the norm of $\rho_k^{-1}u_t$ in $L^2(Q)$. This has implications in the range of the powers of the nonlinearities we can address in the next section. In particular, it prevents the use of embedding results such as the Aubin–Lions Lemma.

We finish this section with the following result.

Lemma 3.6. *Let $l>1$. Suppose that $1<q<\sqrt{l}$ and that $s_0>\frac{lC_3q^2}{(q-1)(l-q^2)}$. Then, there exists $C>0$ such that*

$$\rho_0^{-2}(t)\rho_k^{2l}(t) \leq Ce^{s_1kT(4q^2-2)},\qquad \forall\ t\in[0,T].$$

Proof. Indeed, we have

$$\rho_0^{-2}(t)\rho_k^{2l}(t)=C_1^{2l}e^{\frac{2lC_3q^2+2s_0(q-1)(q^2-l)}{q^2(q-1)(T-t)}}e^{2s_1k(T-t)(lq^2-1)}.$$

The result then follows from the fact that

$$2lC_3q^2-2s_0(q-1)(l-q^2) < 0.\quad \square$$

Remark 3.7. The choice of the parameter s_0 in the previous lemma readily implies inequality (3.4). Indeed, just notice that, by the choice of q, we have that $\frac{l}{l-q^2}>1$.

4. The nonlinear case

In this section, we prove Theorems 1.5 and 1.6. To do this, we borrow some ideas from [30,32]. We will see that the results obtained in the previous section allow us to locally invert the nonlinear system

(1.1). In fact, the regularity deduced for the solution of the linearized system (1.9) will be sufficient to apply a suitable inverse function theorem (see Theorem 4.1 below).

To address the nonlinear problem, we notice that the norms on the left-hand side of (3.8) and (3.27) depend on k, but we still did not choose the parameter s_1. So, to make things uniform with respect to k, we choose $s_1 = \frac{1}{k}$. Then, for $u_0 \in H_0^1(\Omega)$, we can rewrite estimates (3.8) and (3.27) as

$$\begin{aligned} &\|\rho^{-1}u\|^2_{C([0,T],H_0^1(\Omega))\cap L^2((0,T);H^2(\Omega))} + \|\rho^{-1}f1_\omega\|^2_{L^2(Q)} \\ &\leq \mathfrak{C}\left(\|u_0\|^2_{H_0^1(\Omega)} + \|\rho_0^{-1}g\|^2_{L^2(Q)}\right), \end{aligned} \tag{4.1}$$

where $\mathfrak{C} = \mathfrak{C}(T)$ does not depend on k, and

$$\rho_0(t) = e^{-\frac{s_0}{T-t}}e^{T-t}, \quad \rho(t) = C_1 e^{q^2(T-t)} e^{\frac{C_3 q^2 - s_0(q-1)}{q^2(q-1)(T-t)}}.$$

Moreover, arguing as in Lemma 3.6, one can prove that

$$\rho_0^{-2}(t)\rho^{2l}(t) \leq C,$$

where $l > 1$ and the constant $C > 0$ does not depend on k.

For what follows, we use an inverse map result due to Lawrence M. Graves (see [26]).

Theorem 4.1. *Let E and G be two Banach spaces and let $\mathcal{A} : E \to G$ be a continuous function from E to G defined in $B_\eta(0)$ for some $\eta > 0$ with $\mathcal{A}(0) = 0$. Let Λ be a continuous and linear operator from E onto G and suppose there exists $C_0 > 0$ such that*

$$\|e\|_E \leq C_0\|\Lambda(e)\|_G$$

and that there exists $\delta < C_0^{-1}$ such that

$$\|\mathcal{A}(e_1) - \mathcal{A}(e_2) - \Lambda(e_1 - e_2)\| \leq \delta\|e_1 - e_2\| \tag{4.2}$$

whenever $e_1, e_2 \in B_\eta(0)$. Then, the equation $\mathcal{A}(e) = h$ has a solution $e \in B_\eta(0)$ whenever $\|h\|_G \leq c\eta$, where $c = C_0^{-1} - \delta$.

For a proof of Theorem 4.1 we refer the reader to [35] (see also [26, 36]).

Remark 4.2. In the case where $\mathcal{A} \in C^1(E;G)$, using the mean value theorem, it can be shown, that for any $\delta < C_0^{-1}$, inequality (4.2) is satisfied with $\Lambda = \mathcal{A}'(0)$ and $\eta > 0$ the continuity constant at zero, i.e.,

$$\|\mathcal{A}'(e) - \mathcal{A}'(0)\|_{\mathcal{L}(E;G)} \leq \delta$$

whenever $\|e\|_E \leq \eta$.

The strategy to prove both Theorems 1.5 and 1.6 is as follows. We begin by letting the system evolve freely (i.e., with $f \equiv 0$) over the time interval $\left[0, \frac{T}{2}\right]$, and then apply a control on the interval $\left[\frac{T}{2}, T\right]$. During the first part, we use a precise decay of the system to gain regularity at a precise rate. In the second part, we control the system with a control which is uniformly bounded. By combining these two steps, we are able to obtain a decay estimate for the control.

In what follows, we will use Theorem 4.1 with the space E given by (3.26) and the space

$$G = L^2\left(\rho_0^{-1}; Q\right) \times H_0^1(\Omega),$$

both adapted to the interval $\left[\frac{T}{2}, T\right]$.

The space G is endowed with the norm

$$\|(g, v_0)\|^2_G = \|v_0\|^2_{H_0^1(\Omega)} + \|\rho_0^{-1}g\|^2_{L^2(Q)}.$$

Let us prove Theorem 1.6.

Proof of Theorem 1.6. We divide the proof into two steps. In the first step, we obtain a decay estimate for the uncontrolled system on the time interval $\left[0, \frac{T}{2}\right]$. In the second step, we apply Theorem 4.1 to control the system on the interval $\left[\frac{T}{2}, T\right]$.

We begin with the first step, stated as the following claim:

Claim 4.1. *Let $0 < T \leq 1$ be fixed and $u_0 \in L^2(\Omega)$ with $\|u_0\|^2_{L^2(\Omega)} \leq \gamma\frac{T}{2}$, for some $\gamma \leq 1$ sufficiently small.*

If $f \equiv 0$ in $[0, \frac{T}{2}]$, then the solution u of (1.12) satisfies the following estimates:

For $N = 1$ or $N = 2$, one has

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \leq C_1 e^{-C_2 kT}\gamma, \tag{4.3}$$

for some constants $C_1, C_2 > 0$.

For $N = 3$, one has

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \leq C_1 e^{-C_2 kT}\sqrt{\gamma}, \tag{4.4}$$

for some constants $C_1, C_2 > 0$.

Proof. Multiplying Eq. $(1.12)_1$ by u and integrating by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{\theta}{2}}u(t)\|^2_{L^2(\Omega)} = 0.$$

From this identity, we see that there exists $C > 0$ such that

$$\|u(\tfrac{T}{4})\|_{L^2(\Omega)} \leq e^{-CkT}\|u_0\|_{L^2(\Omega)} \tag{4.5}$$

and that

$$\int_{\frac{T}{4}}^{\frac{T}{2}} \|\nabla u(s)\|^2_{L^2(\Omega)}ds \leq e^{-CkT}\|u_0\|^2_{L^2(\Omega)}, \quad t \in \left[\frac{T}{4}, \frac{T}{2}\right].$$

Hence, by mean value theorem, there exists $\tilde{T} \in \left[\frac{T}{4}, \frac{T}{2}\right]$ such that

$$\left\|\nabla u(\tilde{T})\right\|^2_{L^2(\Omega)} \leq \frac{2}{T}e^{-CkT}\|u_0\|^2_{L^2(\Omega)}.$$

If $\tilde{T} = \frac{T}{2}$, the Claim follows. If $\tilde{T} < \frac{T}{2}$, since $\gamma\frac{T}{2} \leq \gamma$, we see that

$$\left\|\nabla u(\tilde{T})\right\|^2_{L^2(\Omega)} \leq e^{-CkT}\gamma. \tag{4.6}$$

Applying Proposition A.2 with $\delta = e^{-CkT}\gamma$ over the interval $[\tilde{T}, \frac{T}{2}]$, we get:

If $N = 1$ or $N = 2$, there exist $C_1, C_2 > 0$ such that:

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \leq C_1 e^{-C_2 kT}\gamma.$$

If $N = 3$, there exist $C_1, C_2 > 0$ such that:

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \leq C_1 e^{-C_2 kT}\sqrt{\gamma}. \quad \square$$

Let us now proceed with the second step, i.e. control the system from $\frac{T}{2}$ to T. For this, we recall that both spaces E and G are adapted to the interval $\left[\frac{T}{2}, T\right]$.

For each $k > 0$, we steer the solution to zero at time T using the operator

$$\mathcal{A}_k(u, f) = \left(L_k(u) + u\left(\vec{A}\cdot\nabla u\right) - f1_\omega, u\left(\tfrac{T}{2}\right)\right), \quad \forall\, (u, f) \in E.$$

It is not difficult to see that $\mathcal{A}_k(0, 0) = (0, 0)$. Also,

$$[\mathcal{A}_k'(0, 0)](u, f) = \left(L_k(u) - f1_\omega, u\left(\tfrac{T}{2}\right)\right), \quad \forall\, (u, f) \in E.$$

Now, we show that the hypothesis of Theorem 4.1 are fulfilled. This is done in the following claims.

Claim 4.2. $\mathcal{A}_k \in \mathfrak{C}^1(E; G)$.

Proof. First, notice that all terms in $\mathcal{A}_k$ are linear, and consequently are $\mathfrak{C}^1$, except the term $u(\vec{A}\cdot\nabla u)$.

Let E_1 be the projection of E on its first component and consider the operator

$$\mathcal{B} : E_1 \times E_1 \to L^2\left(\rho_0^{-1}; Q\right), \tag{4.7}$$

given by $\mathcal{B}(u,v) = u(\vec{A}\cdot\nabla v)$. Since $\mathcal{B}$ is a bilinear form, and therefore $\mathcal{C}^1$, it suffices to show that $\mathcal{B}$ is continuous.

From

$$\|\mathcal{B}(u,v)\|^2_{L^2\left(\rho_0^{-1};Q\right)} \le C\sum_{i=1}^{N}\int_{\frac{T}{2}}^{T}\int_{\Omega}\rho_0^{-2}\left|u\partial_{x_i}v\right|^2 dxdt,$$

we use Hölder's inequality to see that

$$\int_{\frac{T}{2}}^{T}\int_{\Omega}\rho_0^{-2}\left|u\partial_{x_i}v\right|^2 dxdt \le C\int_{\frac{T}{2}}^{T}\|\rho^{-1}u\|^2_{L^4(\Omega)}\|\rho^{-1}\partial_{x_i}v\|^2_{L^4(\Omega)}dt,$$

for some $C>0$ does not depending on k.

Now, since $1\le N\le 4$, we have that $H^1(\Omega)\hookrightarrow L^4(\Omega)$, and then

$$\|\mathcal{B}(u,v)\|_{L^2\left(\rho_0^{-1};Q\right)} \le C\|\rho^{-1}u\|_{L^\infty\left(\left(\frac{T}{2},T\right),H^1(\Omega)\right)}\|\rho^{-1}v\|_{L^2\left(\left(\frac{T}{2},T\right),H^2(\Omega)\right)}. \tag{4.8}$$

This proves the continuity of the bilinear form $\mathcal{B}$. $\square$

Claim 4.3. *The operator $\mathcal{A}'_k(0,0)$ is surjective and satisfies*

$$\|(u,f)\|_E \le C_0\left\|[\mathcal{A}'_k(0,0)](u,f)\right\|_G \quad \forall (u,f)\in E, \tag{4.9}$$

where the constant $C_0>0$ does not depend on k.

Proof of Claim 4.3. Let $\Lambda_k = \mathcal{A}'_k(0,0)$. To show that Λ_k is surjective we must guarantee that for all $\left(g,u\left(\frac{T}{2}\right)\right)\in G$ there exists $(u,f)\in E$ such that

$$\Lambda_k(u,f) = \left(g,u\left(\tfrac{T}{2}\right)\right). \tag{4.10}$$

From Theorem 3.2, when $u\left(\frac{T}{2}\right)\in H_0^1(\Omega)$ and $g\in L^2(\rho_0^{-1};Q)$, there exists $(u,f)\in E$ satisfying system (3.7), that is to say that expression (4.10) holds and consequently the operator Λ_k is surjective. Furthermore, a direct consequence of Theorem 3.2 is the estimate (4.1) which is equivalent to (4.9) (with $C_0 = \mathfrak{C}^{\frac{1}{2}}$). $\square$

Claim 4.4. *Given $\delta>0$, there exists $\eta>0$, such that*

$$\|\mathcal{A}'_k(v,\tilde f) - \mathcal{A}'_k(0,0)\|_{\mathcal{L}(E;G)} \le \delta$$

whenever

$$\|(v,\tilde f)\|_E \le \eta.$$

Proof of Claim 4.4. Let $(u,f),(v,\tilde f)\in E$. Note that

$$\mathcal{A}'_k(v,\tilde f)(u,f) - \mathcal{A}'_k(0,0)(u,f) = \left(u(\vec{A}\cdot\nabla v)+v(\vec{A}\cdot\nabla u),0\right).$$

Hence, we have that

$$\|\mathcal{A}'_k(v,\tilde f) - \mathcal{A}'_k(0,0)\|_{\mathcal{L}(E;G)} = \sup_{\|(u,f)\|_E\le 1}\|u(\vec{A}\cdot\nabla v)+v(\vec{A}\cdot\nabla u)\|_{L^2(\rho_0^{-1};Q)}. \tag{4.11}$$

Since

$$\|u(\vec{A}\cdot\nabla v)+v(\vec{A}\cdot\nabla u)\|_{L^2(\rho_0^{-1};Q)} \le \|\mathcal{B}(u,v)\|_{L^2(\rho_0^{-1};Q)} + \|\mathcal{B}(v,u)\|_{L^2(\rho_0^{-1};Q)},$$

where $\mathcal{B}$ is the bilinear form defined in (4.7), from (4.8) and (4.11), it follows that

$$\left\|\mathcal{A}'_k(v,\tilde f) - \mathcal{A}'_k(0,0)\right\|_{\mathcal{L}(E;G)} \le 2C\|(v,\tilde f)\|_E,$$

where the constant $C>0$ does not depends on k.

Therefore, the claim follows taking $\eta = \frac{\delta}{2C}$. $\square$

To finish the proof, taking γ sufficiently small, and using estimates (4.3) and (4.4), from Theorem 4.1 it follows that the equation

$$\mathcal{A}_k(u,f) = \left(0,u\left(\tfrac{T}{2}\right)\right)$$

has a solution $(u,f)\in E$, with

$$\|(u,f)\|_E \le C_1 e^{-C_2 kT}.$$

Consequently, the pair $(u,f)\in E$ is such that $u(T)=0$, and in particular

$$\|f\|^2_{L^2(\omega\times(0,T))} \le C_1 e^{-C_2 kT},$$

for some $C_1, C_2>0$.

This finishes the proof of Theorem 1.6. $\square$

Let us now prove Theorem 1.5.

Proof of Theorem 1.5. The proof of Theorem 1.5 is performed following the same steps as in Theorem 1.6.

For the first part, we use the following.

Claim 4.5. *Let $0<T\le 1$ be fixed and $u_0\in L^2(\Omega)$ with $\|u_0\|^2_{L^2(\Omega)}\le\gamma T^2$, for some $\gamma\le 1$ sufficiently small. Then*

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \le C_1 e^{-C_2 kT}\gamma, \tag{4.12}$$

for some constants $C_1, C_2>0$ independent of k.

Proof. Multiplying Eq. (1.10) by u and integrating by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{\theta}{2}}u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^{l+1}_{L^{l+1}(\Omega)} = 0.$$

Then, as before, there exists $\tilde T\in\left[\frac{T}{4},\frac{T}{2}\right]$ such that

$$\left\|\nabla u(\tilde T)\right\|^2_{L^2(\Omega)} \le \frac{2}{T}e^{-CkT}\|u_0\|^2_{L^2(\Omega)}. \tag{4.13}$$

From Proposition A.1, we have the following estimate

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \le C\left(k\left\|\nabla u(\tilde T)\right\|^2_{L^2(\Omega)} + \left\|\nabla u(\tilde T)\right\|^{l+1}_{L^2(\Omega)}\right).$$

Therefore,

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \le C\left(\frac{2k}{T}e^{-CkT}\|u_0\|^2_{L^2(\Omega)} + \frac{2^{\frac{l+1}{2}}}{T^{\frac{l+1}{2}}}e^{-C(l+1)kT}\|u_0\|^{l+1}_{L^2(\Omega)}\right).$$

Now, a straightforward calculation gives

$$\left\|\nabla u\left(\tfrac{T}{2}\right)\right\|^2_{L^2(\Omega)} \le C_1 e^{-C_2 kT}\gamma. \quad \square$$

To finish, we drive the solution to zero from $\frac{T}{2}$ to T using the operator:

$$\mathcal{A}_k(u,f) = \left(L_k(u)+|u|^{l-1}u - f1_\omega, u\left(\tfrac{T}{2}\right)\right), \quad \forall\,(u,f)\in E.$$

Arguing as before, and using that

$$L^\infty\left(\left(\tfrac{T}{2},T\right);H_0^1(\Omega)\right)\cap L^2\left(\left(\tfrac{T}{2},T\right);H^2(\Omega)\right)\hookrightarrow L^{2l}\left(\left(\tfrac{T}{2},T\right);L^{2l}(\Omega)\right)$$

for

$$\begin{cases} l>1, & \text{for } N=1,2;\\ 1<l<5, & \text{for } N=3;\\ 1<l<3, & \text{for } N=4,\end{cases}$$

we conclude that $\mathcal{A}_k\in\mathcal{C}^1(E;G)$ and satisfies the assumptions of Theorem 4.1.

Thus, taking γ sufficiently small, there exists a pair $(u,f)\in E$ such that $u(T)=0$ and that

$$\|f\|^2_{L^2(\omega\times(0,T))} \le C_1 e^{-C_2 kT}.$$

This concludes the proof of Theorem 1.5. $\square$

5. Comments and open problems

In this paper, we have studied the heat equation perturbed by an anomalous diffusion term scaled by a large parameter. We have shown that it is possible to control the linear system using a control whose L^2-norm decays exponentially with respect to the parameter, as conjectured by Lions and Zuazua in [19]. Additionally, when nonlinearity is introduced into the system, it is possible to achieve similar controllability results with respect to the parameter.

The ideas developed in this paper open up the possibility of addressing many other related problems. We mention here some of them that could be of interest:

• *General parabolic operators:* The method developed in this paper can be applied to more general parabolic systems. To be precise, one can consider parabolic systems under the form

$$\left|\begin{array}{lll} u_t + Au + kA^{\theta}u = f 1_{\omega} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega, \end{array}\right. \tag{5.1}$$

with $k > 0$, A a linear positive self-adjoint operator and $0 < \theta < 1$.

In fact, as soon as the operator A has a suitable spectral inequality it is possible to argue as in Section 2 and construct null controls for (5.1) which have exponential decay for the L^2-norm. For example, one might consider A to be the Stokes operator, for which a spectral inequality has been obtained in [8].

• *Local anomalous diffusion term:* To obtain the result for the linear system an important fact we used was the dissipation of the system. An interesting open problem arises when the anomalous diffusion term is localized in a proper nonempty subset $\mathcal{O} \subsetneq \Omega$.

More precisely, it is an open problem to construct null controls that exhibit exponential decay in the L^2-norm for the following system:

$$\left|\begin{array}{lll} u_t - \Delta u + k\left((-\Delta)^{\theta}u\right) 1_{\mathcal{O}} = f 1_{\omega} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega, \end{array}\right.$$

with $\mathcal{O} \subsetneq \Omega$, $k > 0$ and $0 < \theta < 1$.

• *Another limiting problem:* An interesting problem to consider is the case where the Laplace operator have a small diffusion term. Namely, one can ask about uniform controllability results for the following system:

$$\left|\begin{array}{lll} u_t - \delta\Delta u + (-\Delta)^{\theta}u = f 1_{\omega} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega, \end{array}\right. \tag{5.2}$$

with $\delta > 0$ and $0 < \theta < 1$.

For each $\delta > 0$, system (5.2) is null controllable at any time $T > 0$. However, as $\delta \to 0^+$, this formally leads to the fractional parabolic system:

$$\left|\begin{array}{lll} u_t + (-\Delta)^{\theta}u = f 1_{\omega} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega. \end{array}\right. \tag{5.3}$$

It is well-known (see [37,38]) that system (5.3) is null controllable for $\frac{1}{2} < \theta < 1$ at any time $T > 0$ and is not null controllable if $0 < \theta \le \frac{1}{2}$.

Thus, we conjecture that, when $\frac{1}{2} < \theta < 1$, it is possible to obtain the null controllability for (5.3) as a limit of the null controllability for system (5.2).

• *Integral fractional Laplacian:* It would be interesting to analyze whether similar results to those obtained in this paper can be established when the spectral fractional Laplacian is replaced by the integral fractional Laplacian, as considered, for instance, in [39]. That is, one replaces the fractional Laplacian by

$$(-\Delta)^{\theta}u(x) = C_{N,\theta} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\theta}} dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,\theta}$ is an appropriate constant and P.V.denotes the principal value.

We expect that similar results to those presented here may also hold in this setting. However, this remains an open problem.

CRediT authorship contribution statement

F.W. Chaves-Silva: Writing – review & editing, Writing – original draft, Methodology, Investigation. **N. Carreño:** Writing – original draft, Methodology, Investigation. **M.G. Ferreira-Silva:** Writing – review & editing, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Nicolás Carreño reports travel was provided by Fondo Nacional de Desarrollo Científico y Tecnológico. Felipe Wallison Chaves-Silva reports travel was provided by CAPES-MathAmsud. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Some regularity estimates

In this appendix, we give some regularity results we use in the proof of Theorems 1.5 and 1.6.

Proposition A.1. *Let $1 \le N \le 4$ and $0 < T \le 1$. Assume that l satisfy (1.11), and let $k > 0$. Consider the system*

$$\left|\begin{array}{lll} u_t - \Delta u + k(-\Delta)^{\theta}u + |u|^{l-1}u = 0 & \textit{in} & Q, \\ u = 0 & \textit{on} & \Sigma, \\ u(0) = u_0 & \textit{in} & \Omega. \end{array}\right. \tag{A.1}$$

If $u_0 \in H_0^1(\Omega)$, the following estimate holds:

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \le C\left(k\|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla u_0\|^{l+1}_{L^2(\Omega)}\right), \tag{A.2}$$

where $C > 0$ does not depend on k.

Proof of Proposition A.1. Multiplying Eq. $(A.1)_1$ by u_t, and integrating over Ω, yields

$$\|u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{k}{2}\frac{d}{dt}\left\|(-\Delta)^{\frac{\theta}{2}}u(t)\right\|^2_{L^2(\Omega)} + \frac{1}{l+1}\frac{d}{dt}\|u(t)\|^{l+1}_{L^{l+1}(\Omega)} = 0. \tag{A.3}$$

Now, we integrate from 0 to T, and use that $\|(-\Delta)^{\frac{\theta}{2}}u_0\|^2_{L^2(\Omega)} \le C\|\nabla u_0\|^2_{L^2(\Omega)}$, to obtain

$$\begin{aligned} \int_0^T \|u_t(s)\|^2_{L^2(\Omega)} ds + \frac{1}{2}\|\nabla u(T)\|^2_{L^2(\Omega)} + \frac{k}{2}\left\|(-\Delta)^{\frac{\theta}{2}}u(T)\right\|^2_{L^2(\Omega)} & \\ + \frac{1}{l+1}\|u(T)\|^{l+1}_{L^{l+1}(\Omega)} & \\ \le Ck\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{l+1}\|u_0\|^{l+1}_{L^{l+1}(\Omega)}. & \end{aligned} \tag{A.4}$$

The estimate then follows from the embedding $H_0^1(\Omega) \hookrightarrow L^{l+1}(\Omega)$, where l is given by (1.11). □

We now give a regularity result for system (1.12).

Proposition A.2. *Let $1 \leq N \leq 3$, and $0 < T \leq 1$. Assume $A \in \mathbb{R}^N$ and $k > 0$. Consider the following system*

$$\begin{cases} u_t - \Delta u + k(-\Delta)^{\theta} u + u(\vec{A} \cdot \nabla u) = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{A.5}$$

Suppose that $u_0 \in H_0^1(\Omega)$ and that $\|\nabla u_0\|^2_{L^2(\Omega)} \leq \delta$, for some $\delta \leq 1$ sufficiently small.

For $N = 1$ or $N = 2$, there exists $C > 0$, independent of δ, such that:

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \leq C\delta. \tag{A.6}$$

For $N = 3$, there exists $C > 0$, independent of δ, such that:

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \leq C\sqrt{\delta}. \tag{A.7}$$

Proof. Multiplying system (A.5) by u and integrating on Ω, it is not difficult to see that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{\theta}{2}} u(t)\|^2_{L^2(\Omega)} = 0.$$

This implies that

$$\|u(t)\|_{L^2(\Omega)} \leq e^{-Ckt}\|u_0\|_{L^2(\Omega)}, \quad t \in [0, T], \tag{A.8}$$

for some $C > 0$ does not depending on k, and that

$$\int_0^T \|\nabla u(s)\|^2_{L^2(\Omega)} ds \leq \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{A.9}$$

Now, multiplying the system by $-\Delta u$, and integrating over Ω, gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{1+\theta}{2}} u(t)\|^2_{L^2(\Omega)} \leq C\int_\Omega |\nabla u(t)|^3 dx.$$

Since

$$\|\nabla u(t)\|^3_{L^3(\Omega)} \leq \|\nabla u(t)\|_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^4(\Omega)},$$

applying [40, Proposition III.2.35], we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{1+\theta}{2}} u(t)\|^2_{L^2(\Omega)} \leq C\|\nabla u(t)\|^{3-\frac{N}{2}}_{L^2(\Omega)}\|\Delta u(t)\|^{\frac{N}{2}}_{L^2(\Omega)}. \tag{A.10}$$

From now on, we distinguish between the cases $N = 1, 2$ and $N = 3$.

Case 1: We treat the case $N = 2$.

By applying Young's inequality, we see that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Delta u(t)\|^2_{L^2(\Omega)} + k\|(-\Delta)^{\frac{1+\theta}{2}} u(t)\|^2_{L^2(\Omega)} \leq C\|\nabla u(t)\|^4_{L^2(\Omega)}.$$

In particular, for any $\tau \in [0, T]$, it holds that

$$\|\nabla u(\tau)\|^2_{L^2(\Omega)} \leq \|\nabla u_0\|^2_{L^2(\Omega)} + C\int_0^\tau \|\nabla u(s)\|^4_{L^2(\Omega)} ds,$$

and, from [40, Lemma II.4.10], it follows that

$$\|\nabla u(\tau)\|^2_{L^2(\Omega)} \leq \|\nabla u_0\|^2_{L^2(\Omega)} e^{\int_0^\tau \|\nabla u(s)\|^2_{L^2} ds}.$$

Then, using estimate (A.9), and the fact that $\|u_0\|_{L^2(\Omega)} \leq C$, we see that

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \leq C\|\nabla u_0\|^2_{L^2(\Omega)},$$

for some $C > 0$.

Case 2: Let us prove the case $N = 3$.

Again, by Young's inequality, with $p = 4$ and $q = \frac{4}{3}$, it is not difficult to see that

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u(t)\|^2_{L^2(\Omega)} + 2k\|(-\Delta)^{\frac{1+\theta}{2}} u(t)\|^2_{L^2(\Omega)} \leq C_1\|\nabla u(t)\|^6_{L^2(\Omega)}, \tag{A.11}$$

for some $C_1 > 0$.

If we set $y(t) = \|\nabla u(t)\|^2_{L^2(\Omega)}$, we see that

$$y(\tau) \leq \|\nabla u_0\|^2_{L^2(\Omega)} + C_1\int_0^\tau y(s)^3 ds.$$

For any $\delta < \sqrt{\frac{1}{2C_1}}$, if $\|\nabla u_0\|^2_{L^2(\Omega)} \leq \delta$, then

$$T^* = \frac{1}{2C_1\|\nabla u_0\|^4_{L^2(\Omega)}} > 1$$

and, from [40, Lemma II.4.12], we deduce that, for any $T \leq 1 < T^*$, the following estimate holds

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \leq C, \tag{A.12}$$

for some $C > 0$.

From estimate (A.11), if we take $\delta \leq 1$, we get

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2(\Omega)} + C_2\delta\|\nabla u(t)\|^2_{L^2(\Omega)} \leq C_1\|\nabla u(t)\|^6_{L^2(\Omega)},$$

for some positive constants C_1 and C_2.

Finally, using [40, Lemma V.2.9], whenever

$$\|\nabla u_0\|^2_{L^2(\Omega)} \leq \frac{1}{2}\sqrt{\frac{C_2}{2C_1}}\sqrt{\delta},$$

it follows that

$$\|\nabla u(T)\|^2_{L^2(\Omega)} \leq \sqrt{\frac{C_2}{2C_1}}\sqrt{\delta}. \tag{A.13}$$

It is now easy to see that the result follows for any $\delta < \min\left\{1, \frac{C_2}{8C_1}, \sqrt{\frac{1}{2C_1}}\right\}$.

This finishes the proof. □

Data availability

No data was used for the research described in the article.

References

[1] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M.M. Meerschaert, M. Ainsworth, et al., What is the fractional Laplacian? A comparative review with new results, J. Comput. Phys. 404 (2020).

[2] E. Fernández-Cara, E. Zuazua, The cost of approximate controllability for heat equations: The linear case, Adv. Differential Equations 5 (4–6) (2000) 465–514.

[3] I. Boutaayamou, S.-E. Chorfi, L. Maniar, O. Oukdach, The cost of approximate controllability of heat equation with general dynamical boundary conditions, Port. Math. 78 (1) (2021) 65–99.

[4] T.I. Seidman, Two results on exact boundary control of parabolic equations, Appl. Math. Optim. 11 (1984) 145–152.

[5] T.I. Seidman, How violent are fast controls: III, J. Math. Anal. Appl. 339 (1) (2008) 461–468.

[6] L. Miller, A direct lebeau-robbiano strategy for the observability of heat-like semigroups, Discrete Contin. Dyn. Syst. Ser. B 14 (4) (2010) 1465–1485.

[7] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1–2) (1995) 335–356.

[8] F.W. Chaves-Silva, G. Lebeau, Spectral inequality and optimal cost of controllability for the Stokes system, ESAIM Control Optim. Calc. Var. 22 (2016) 1137–1162.

[9] R. Buffe, L. Gagnon, Spectral inequality for an oseen operator in a two dimensional channel, 2021, preprint hal-03200711.

[10] S. Guerrero, G. Lebeau, Singular optimal control for a transport-diffusion equation, Comm. Partial Differential Equations 32 (12) (2007) 1813–1836.

[11] C. Laurent, M. Léautaud, On uniform observability of gradient flows in the vanishing viscosity limit, J. Éc. Polytech.— Math. 8 (2021) 439–506.

[12] J.-M. Coron, S. Guerrero, Singular optimal control: a linear 1-D parabolic–hyperbolic example, Asymptot. Anal. 44 (3–4) (2005) 237–257.

[13] O. Glass, A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit, J. Funct. Anal. 258 (3) (2010) 852–868.

[14] O. Glass, S. Guerrero, Uniform controllability of a transport equation in zero diffusion–dispersion limit, Math. Models Methods Appl. Sci. 19 (9) (2009) 1567–1601.

[15] P. Lissy, Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport–diffusion equation, J. Differential Equations 2591 (10) (2015) 5331–5352.
[16] J.A. Bárcena-Petisco, Uniform controllability of a Stokes problem with a transport term in the zero-diffusion limit, SIAM J. Control Optim. 58 (3) (2020) 1597–1625.
[17] J.A. Bárcena-Petisco, Cost of null controllability for parabolic equations with vanishing diffusivity and a transport term, ESAIM Control Optim. Calc. Var. 27 (2021) 106.
[18] J.A. Bárcena-Petisco, M. Cavalcante, G.M. Coclite, N. De Nitti, E. Zuazua, Control of hyperbolic and parabolic equations on networks and singular limits, Math. Control Relat. Fields 15 (1) (2025) 348–389.
[19] J.-L. Lions, E. Zuazua, The cost of controlling unstable systems: time irreversible systems, Rev. Mat. Univ. Complut. Madr. 10 (2) (1997) 481–523.
[20] Y. Liu, T. Takahashi, M. Tucsnak, Single input controllability of a simplified fluid–structure interaction model, ESAIM Control Optim. Calc. Var. 19 (1) (2013) 20–42.
[21] O.Y. Imanuvilov, T. Takahashi, Exact controllability of a fluid–rigid body system, J. Math. Pures Appl. (9) 87 (4) (2007) 408–437.
[22] V. Hernández-Santamaría, K. Le Balc'h, L. Peralta, Statistical null-controllability of stochastic nonlinear parabolic equations, Stoch. Partial. Differ. Equ. Anal. Comput. 10 (1) (2022) 190–222.
[23] S. Micu, T. Takahashi, Local controllability to stationary trajectories of a Burgers equation with nonlocal viscosity, J. Differential Equations 264 (5) (2018) 3664–3703.
[24] M. González-Burgos, G.R. Souza-Neto, Boundary controllability of a one-dimensional phase-field system with one control force, J. Differential Equations 269 (5) (2020) 4286–4331.
[25] M. Duprez, P. Lissy, Bilinear local controllability to the trajectories of the Fokker–Planck equation with a localized control, Ann. Inst. Fourier (Grenoble) 72 (4) (2022) 1621–1659.
[26] L.M. Graves, Some mapping theorems, Duke Math. J. 17 (1950) 111–114.
[27] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-L. Puel, Local exact controllability of the Navier–Stokes system, J. Math. Pures Appl. 83 (12) (2004) 1501–1542.
[28] E. Fernández-Cara, J. Límaco, Y. Thamsten, D. Menezes, Local null controllability of a quasi-linear system and related numerical experiments, ESAIM Control Optim. Calc. Var. 29 (2023) 34.
[29] A.V. Fursikov, O.Y. Imanuvilov, Controllability of evolution equations, Lecture Notes Ser. 34, Research Institute of Mathematics, Seoul National University, Seoul, 1996.
[30] O.Y. Imanuvilov, Remarks on exact controllability for the Navier–Stokes equations, ESAIM Control Optim. Calc. Var. 6 (2001) 39–72.
[31] C. Montoya, Remarks on local controllability for the Boussinesq system with Navier boundary condition, C. R. Math. Sci. Paris 358 (2) (2020) 169–175.
[32] F.W. Chaves-Silva, S. Guerrero, A uniform controllability result for the Keller–Segel system, Asymptot. Anal. 92 (3–4) (2015) 313–338.
[33] V. Hernández-Santamaría, K. Le Balc'h, Local-controllability of the one-dimensional nonlocal gray-scott model with moving controls, J. Evol. Equ. 21 (4) (2021) 4539–4574.
[34] J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc. 16 (2014) 2433–2475.
[35] V.M. Alekseev, V.M. Tikhomirov, S.V. Fomin, Optimal control, in: Contemporary Soviet Mathematics, Consultants Bureau, 1987.
[36] A.L. Dontchev, The Graves theorem revisited, J. Convex Anal. 3 (1996) 45–53.
[37] S. Micu, E. Zuazua, On the controllability of a fractional order parabolic equation, SIAM J. Control Optim. 44 (6) (2006) 1950–1972.
[38] L. Miller, On the controllability of anomalous diffusions generated by the fractional Laplacian, Math. Control Signals Systems 18 (3) (2006) 260–271.
[39] U. Biccari, V. Hernández-Santamaría, Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects, IMA J. Math. Control Inform. 36 (4) (2019) 1199–1235.
[40] F. Boyer, P. Fabrie, Applied Mathematical Sciences, Springer New York, 2012.