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Simultaneous observability/controllability of uncoupled parabolic systems

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In this paper, we consider two uncoupled parabolic equations with different diffusion coefficients and study the problem of observing both equations from a single observation in a small region. This question is equivalent to the problem of controlling two parabolic equations with the same control. We present positive and negative results which depend on the potentials and some associated elliptic operators. In particular, we see that it may be possible to observe both equations through a single observation even in cases where their sets of eigenvalues are not disjoint, a situation that does not happen for uncoupled equations with the same diffusion coefficient.

Keywords: Simultaneous controllability; simultaneous observability; uncoupled systems; parabolic equations.

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1. Introduction

Controllability of Partial Differential Equations (PDE's) has been intensively studied in the last fifty years. As one can see for instance from [12, 16, 22, 28] and references therein, the literature on this subject is vast and treats many different problems such as exact, approximate and null controllability, distributed and boundary controls, linear and semilinear equations. Regarding the many different types of phenomena, one can encounter in nature, the analysis of controllability

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properties for those modeled by coupled systems of PDE's has become a very important subject. In this context, the application of the methods used to analyze controllability properties for single equations cannot be applied to systems of PDE's in a straightforward manner. In fact, for the later, several new and unexpected controllability results can emerge (see, for instance, [5, 7]).

Among several different problems of controllability for systems of PDE's which are of interest, in this paper, we consider the following one.

Question 1. Assume that a given PDE is controllable (in some appropriate sense) and suppose that you have two of these PDE's. If the two PDE's are uncoupled, is it possible to control both equations with the same control?

In other words, we ask if it possible to control *simultaneously* two uncoupled PDE's with the same input function on both equations. This is the so-called *simultaneous controllability problem*. It is not difficult to see that this question is equivalent to the following simultaneous observability problem:

Question 1'. Assume that a given PDE is observable in a (small) subset and suppose that you have two of these PDE's. If the two PDE's are uncoupled, is it possible to observe both equations from a single observation in the same (small) subset?

To our knowledge, the first result on this topic is due to Russell in [26], where the simultaneous boundary controllability of Maxwell's equations is studied. In there, the simultaneous controllability of two uncoupled wave equations with different boundary conditions is analyzed. Motivated by Russell's work, Lions in [22, Chap. 5], consider the problem of simultaneous controllability for two wave equations with different boundary conditions and with one control acting on both equation not in the same manner (see also [6]). More recently, there has been some interesting works on the simultaneous controllability problem. For instance, we mention the work of Tébou in [27] for semilinear wave equations (see references therein for a panorama on the subject) and the one by Niu in [24] for wave-wave systems with different speeds. Burq and Moyano [10] treated also this problem for two heat equations (without any potential or different diffusions) but with different boundary data: one satisfies Dirichlet and the other Neumann conditions. See also [1] for some related problems. We must also cite [29], in which, inspired by the so-called averaged controllability problem, the author considers the question of observing a given PDE under unknown additive perturbations solving another PDE. This question studied by Zuazua is somewhat close to the simultaneous observability/controllability problem, even though the two concepts are not equivalent.

As one can see from above, most of the works on simultaneous controllability has been carried out in the context of wave or in a slightly different formulation as coupled systems. Thus, in this paper, we are interested in studying the question for the case of uncoupled parabolic systems with different diffusion coefficients. To our knowledge, the only work dealing with simultaneous controllability for uncoupled parabolic system is due to Ouaili in [25]. However, that paper deals with equations with same diffusion coefficients and only treats some very particular case of potentials. In fact, as we will see, having different diffusion coefficients changes the dynamics in such a way that it is even possible to have simultaneous controllability for cases in which it does not hold for systems with the same diffusion coefficient (see Remark 1).

To state precisely the problem we have in mind, let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain whose boundary Γ is regular enough, $T > 0$ be given, $\omega \subset \Omega$ be a (small) nonempty open subset, and consider the following system of two uncoupled parabolic equations:

$$\begin{cases} u_t - \Delta u + au = f1_\omega & \text{in } Q := \Omega \times (0,T), \\ v_t - \theta\Delta v + bv = f1_\omega & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma := \Gamma \times (0,T), \\ u(0) = u_0, \quad v(0) = v_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\theta > 0$ is a diffusion parameter, the potentials $a = a(x)$ and $b = b(x)$ belong to $W^{2,\infty}(\Omega)$, $f \in L^2(\omega \times (0,T))$ is the control to be determined, 1_ω is the characteristic function of the set ω and the pair $(u_0, v_0) \in [L^2(\Omega)]^2$ is prescribed.

We will be concerned with two types of simultaneous controllability problems for (1). The first one is the simultaneous null controllability:

Give conditions on θ, T, a and b in order to guarantee, for any initial data $(u_0, v_0) \in [L^2(\Omega)]^2$, the existence of a control $f \in L^2(\omega \times (0,T))$ such that the solution (u,v) of (1) *satisfies*

$$u(T) = v(T) = 0 \quad \text{in } \Omega. \tag{2}$$

Our second problem is the weaker notion of simultaneous approximate controllability:

Give conditions on θ, T, a and b such that, for any $\epsilon > 0$ and initial and final data $(u_0, v_0), (u_T, v_T) \in [L^2(\Omega)]^2$, there exists a control $f \in L^2(\omega \times (0,T))$ such that the solution (u,v) of (1) *satisfies*

$$\|(u(T), v(T)) - (u_T, v_T)\|_{[L^2(\Omega)]^2} \leq \epsilon.$$

Before continuing, let us notice that, by simply taking $w = u$–v, simultaneous controllability problem for the uncoupled system (1) can be linked to a controllability problem for the coupled system:

$$\begin{cases} v_t - \theta\Delta v + bv = f1_\omega & \text{in } Q, \\ w_t - \Delta w + aw = (b-a)v + (1-\theta)\Delta v & \text{in } Q, \\ v = w = 0 & \text{on } \Sigma, \\ v(0) = v_0, \quad w(0) = w_0 & \text{in } \Omega. \end{cases} \tag{3}$$

When $\theta = 1$, the coupling in the second equation of system (3) reduces to $(b-a)v$ and null controllability for system (3) holds at any $T > 0$ as soon as b–a has a sign in a small set contained in ω (see, for instance, [17]). Also, there may exist a minimal time of null controllability $T_0 > 0$ whenever b–a vanishes in ω, see [5]. Recently, still for $\theta = 1$, Ouaili in [25] studied controllability properties in the one-dimensional setting of (3) in the particular case $a(x) = \mu_1 q(x)$, $b(x) = \mu_2 q(x)$, $\mu_i \in \mathbb{R}$, proving approximate controllability at any time and null controllability for large times under the assumption $\sigma(L_1) \cap \sigma(L_2) = \emptyset$ and $(\operatorname{supp} q) \cap \omega = \emptyset$, where $\sigma(L_i)$ denotes the spectrum of the operator $L_i = -\partial_{xx} + \mu_i q$. Still for approximate controllability, following the techniques in [8], it can be shown that the approximate controllability of (3) holds whenever $(\operatorname{supp}(b–a)) \cap \omega \neq \emptyset$. When this last condition is not satisfied the situation is more delicate as one can see in [19]. For other results in this direction, see [9].

Since the case $\theta = 1$ is fairly well understood, in the rest of this paper, and without loss of generality, we will consider system (1) with $0 < \theta < 1$.

Let $y = [v, w]^T$ and rewrite system (3) in the form

$$y_t = (DR + A)y + Bv1_\omega,$$

where the diagonalizable matrix D, the matrix A, the symmetric second order self-adjoint elliptic operator R and the the control operator B are given by

$$D = \begin{bmatrix} \theta & 0 \\ (1-\theta) & 1 \end{bmatrix}, \quad R = \Delta - \frac{b}{\theta}, \quad A = \begin{bmatrix} 0 & 0 \\ \dfrac{b}{\theta} - a & \dfrac{b}{\theta} - a \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By considering this abstract setting, the works of Ammar-Khodja *et al.* [2, 3] give controllability results for (3) under the global assumption that b–θa is constant or depend only on time.

In this paper, we show that it is still possible to obtain controllability results for (3) (and consequently for (1)) with only local assumptions on b–θa. Moreover, we are also able to treat situations where potentials a and b depend on the space variable x, which is a case that was not analyzed in [2, 3].

For the particular case $a = b = 0$, one can show simultaneous null controllability for system (1) by simply adapting the arguments of Zuazua for the case of averaged controllability problem in [29] (or simply applying the results in [3]). Nevertheless, the arguments in [29] are based on the Fourier expansion of the solution and the fact that two heat operators with different diffusion coefficients commute, cannot be used to treat potentials $a = a(x)$ and $b = b(x)$. Hence, we follow a different strategy to tackle the problem.

As controllability properties for a system are equivalent to appropriate observability inequalities or unique continuation principles for its adjoint system, we introduce the following system:

$$\begin{cases} -\phi_t - \Delta\phi + a\phi = 0 & \text{in } Q, \\ -\xi_t - \theta\Delta\xi + b\xi = 0 & \text{in } Q, \\ \phi = \xi = 0 & \text{on } \Sigma, \\ \phi(T) = \phi_0, \quad \xi(T) = \xi_0 & \text{in } \Omega. \end{cases} \tag{4}$$

The simultaneous null controllability of system (1) is then equivalent to the existence of an universal constant $C > 0$ such that the observability estimate

$$\int_\Omega (|\phi(0)|^2 + |\xi(0)|^2)\,dx \leq C \iint_{\omega\times(0,T)} |\phi + \xi|^2\,dx\,dt \tag{5}$$

holds for any solution to system (4).

On the other hand, the simultaneously approximate controllability property for (1) is equivalent to a unique continuation property for solutions of (4):

$$\phi + \xi = 0 \quad \text{in} \quad \omega \times (0,T) \implies \phi \equiv \xi \equiv 0 \quad \text{in } \Omega \times (0,T). \tag{6}$$

Note that inequality (5) means that we can recover both φ and ξ at time $t = 0$ by knowing the value of the sum $\varphi + \xi$ in $\omega \times (0,T)$. Thus, to expect that the inequality holds true, one needs to be sure that φ and ξ are not evolving with the same frequency in $\omega \times (0,T)$.

Our first main result says that it is possible to have simultaneous null controllability for system (1) with a local condition on b–θa.

Theorem 1. *Let $T > 0$ and let $a, b \in W^{2,\infty}(\Omega)$. Suppose that both operators*

$$-\Delta + a(\cdot) : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$$
$$-\theta\Delta + b(\cdot) : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$$

are coercive and that

$$b - \theta a \equiv 0 \quad \text{in } \omega.$$

Then, for any initial data $(u_0, v_0) \in [L^2(\Omega)]^2$, there exists a control $f \in L^2(\omega \times (0,T))$ such that the corresponding solution to (1) *satisfies* (2).

Remark 1. From Theorem 1, it is possible to construct potentials $q = q(x)$ such that, taking $a(x) = \mu_1 q(x)$ and $b(x) = \mu_2 q(x)$, for appropriate $\mu_1, \mu_2 \in \mathbb{R}$, the spectra of the operators $-\Delta + a(x)$ and $-\theta\Delta + b(x)$ are not disjoint and nevertheless system (1) is simultaneously null controllable. We recall that, as proved by Ouaili in [25], this type of result cannot happen in the case of equations having the same diffusion coefficient, i.e. when $\theta = 1$ the spectra must be disjoint to obtain controllability results. This shows a great difference between the cases $0 < \theta < 1$ and $\theta = 1$.

Note that assumptions in Theorem 1 are only necessary conditions for the simultaneous null controllability problem for (1). In fact, one also has the following result.

Proposition 1. *Let us assume $a, b \in W^{2,\infty}(\Omega)$ with $b - \theta a \equiv c$ in Ω with $c \in \mathbb{R}$, and consider the elliptic system*:

$$\begin{cases} (1-\theta)\Delta u + (b-a)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{7}$$

Then, system (1) *is simultaneously null controllable at time* $T > 0$ *if and only if* (7) *has a unique solution.*

A proof of Proposition 1 can be derived from the results proved in [2, 3]. However, for the sake of completeness, we will give a shorter proof.

From Proposition 1, it is remarkable that the property of simultaneous null controllability can be linked to the uniqueness of the elliptic system (7). In particular, we see that if system (1) is simultaneously null controllable in a time $T_0 > 0$, then it is simultaneously null controllable at any time $T > 0$. Actually, a non uniqueness for system (7) has an even stronger consequence, notably the lack of simultaneous approximate controllability to system (1). This is precisely the content of the next result.

Proposition 2. *Suppose* $a, b \in W^{2,\infty}(\Omega)$ *and* $b - \theta a \equiv c$ *in* Ω *with* $c \in \mathbb{R}$. *If the elliptic system* (7) *has multiple solutions, then system* (1) *is not simultaneously approximately controllable at any time* $T > 0$.

We prove both Propositions 1 and 2 in Sec. 3. Our next result says that the assumption b–θa being constant in Proposition 1 is only a (global) sufficient condition. Indeed, in the one-dimensional case, we have the following result.

Proposition 3. *Let* $\Omega = (0, 1)$ *and assume that* $b(x) - \theta a(x) = cx$, *for all* $x \in \Omega$, *with* $c \in \mathbb{R}$. *If* (7) *has a unique solution, then system* (1) *is simultaneously null controllable at any time* $T > 0$.

Note that, contrary to Theorem 1, in Propositions 1 and 3 we have a global assumption on b–θa. Moreover, it is important to stress that Proposition 3 is a new result that does not follow from the results in [2, 3]. We will prove Proposition 3 in Sec. 5.

It is clear that Theorem 1 and Proposition 1 are complementary and give simultaneous null controllability at any time $T > 0$. Nevertheless, as we will see below, there are situations in which we can only guarantee the simultaneous null controllability for large times.

Let us now give a simultaneous approximate controllability result.

Theorem 2. *Let* $\Omega = (0, 1)$, $\omega = (\alpha, \beta)$, *with* $0 < \alpha < \beta < 1$, *and* $a, b \in L^\infty(\Omega)$. *If*

$$supp(a), supp(b) \subset [0, \alpha] \quad or \quad supp(a), supp(b) \subset [\beta, 1]$$

then, for any $0 < \theta < 1$, *system* (1) *is simultaneous approximately controllable at any time* $T > 0$.

Concerning Theorem 2, it is important to say that for $\theta = 1$, in the one-dimensional case, it was proved by Ouaili in [25] that when $a = \mu_1 q$ and $b = \mu_2 q$, for some potential $q \not\equiv 0$ with $\text{supp}(q) \subset [0, \alpha]$ or $\text{supp}(q) \subset [\beta, 1]$, and real numbers $\mu_1 \neq \mu_2$, the simultaneous approximately controllability for (1) holds if and only if the set of eigenvalues of $L_1 = -\partial_{xx} + \mu_1 q$ and $L_2 = -\partial_{xx} + \mu_2 q$ are disjoint. For the case $0 < \theta < 1$, Theorem 2 says that we can have simultaneous approximately controllability for (1) even if the sets of eigenvalues of L_1 and L_2 are not disjoint. The main reason for this to happen is due to the fact the equations have different diffusions. Indeed, this implies that a solution for the adjoint equation of $(1)_1$ can not be at the same frequency of a solution of the adjoint equation of $(1)_2$ at the same time in ω.

Our last result says that simultaneous null controllability depends on a minimum time of controllability.

Theorem 3. *Let $\Omega = (0,1)$ and let $a, b \in L^\infty(\Omega)$. Consider the following Sturm–Liouville problems:*

$$\begin{cases} -\Delta\varphi_{1,j} + a\varphi_{1,j} = \mu_{1,j}\varphi_{1,j} & \text{in } \Omega, \\ -\theta\Delta\varphi_{2,j} + b\varphi_{2,j} = \mu_{2,j}\varphi_{1,j} & \text{in } \Omega, \\ \varphi_{1,j}(x) = \varphi_{2,j}(x) = 0 & \text{for } x \in \{0,1\}. \end{cases} \tag{8}$$

Assume that $\mu_{1,j} \neq \mu_{2,k}$ for all $j, k \in \mathbb{N}$, and define $\Lambda = \{\mu_{1,j}\}_{j\in\mathbb{N}} \cup \{\mu_{2,k}\}_{k\in\mathbb{N}}$. Let $\{\lambda_n\}_{k\in\mathbb{N}}$ be the increasing enumeration of the elements of Λ, i.e. $\lambda_1 < \lambda_2 < \lambda_3 <, \ldots,$. Define the condensation index of Λ by

$$c(\Lambda) = \limsup_{k\to\infty} \frac{\log \frac{1}{|E'(\lambda_k)|}}{\lambda_k}, \quad \text{with } E(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\lambda_k^2}\right). \tag{9}$$

Then, system (1) is null controllable at time T whenever $T > c(\Lambda)$.

Remark 2. By considering the case where $\mu_{1,j} = \pi^2 j^2$ and $\mu_{2,j} = \theta\pi^2 j^2$, we can argue as in [4] and show that, for each $\tau \in [0, \infty]$, there exists a value of θ such that $c(\Lambda) = \tau$.

Remark 3. In some cases, there is a simpler expression for the condensation index $c(\Lambda)$, given by

$$c(\Lambda) = \limsup_{k\to\infty} -\frac{\log(\lambda_{k+1} - \lambda_k)}{\lambda_k}.$$

For example, this holds when the following conditions are satisfied:

$$\lim_{k\to\infty} |\mu_{1,k+1} - \mu_{2,k}| = \lim_{k\to\infty} |\mu_{2,k+1} - \mu_{1,k}| = \infty \tag{10}$$

and

$$\lim_{k\to\infty} |\mu_{1,k} - \mu_{2,k}| = C > 0. \tag{11}$$

When (10) and (11) hold, one can show that, for sufficiently large n, the ordered eigenvalues λ_n alternate between the two families. This alternating behavior, however, does not always occur. For instance, in the case where $b(x) = \theta a(x)$ with $\int_0^1 a(x)\,dx = 6$ and $\theta = 1/2$, we have

$$\lim_{k\to\infty} |\mu_{1,k} - \mu_{2,k}| = \infty,$$

and it becomes necessary to consider a "grouping" of eigenvalues that depends on θ. In such cases, it is not clear whether a simpler expression than (9) exists. The condensation index (9) can also be described in a different way, using divided differences and groupings of eigenvalues, as done in [7].

Remark 4. The result in Theorem 3 is based on the the moment method and give a minimal controllability time T_0, with $0 \leq T_0 \leq c(\Lambda)$. This minimal time may be either finite or infinite. In the case $T_0 = +\infty$, the system is not simultaneously null controllable at any time. Several examples illustrating this phenomenon can be found in the literature, see [4, 5, 23].

Remark 5. In Theorem 3, it is a fact that operators $-\Delta + a(x)$ and $-\theta\Delta + b(x)$ have disjoint spectra and that $T > c(\Lambda)$ are only sufficient conditions for having simultaneous null controllability of system (1). In fact, from Remark 1 above, we even have cases in which the spectra are not disjoint and the system is simultaneously null controllable for all times. Hence, it would be very interesting to find general conditions to guarantee when the system is simultaneously null controllable. We believe this is a very challenging problem since, as in [4] and [25], it may happen that the system is not controllable for small times.

This paper is organized as follows. In Sec. 2, we introduce some tools that we will use to prove Theorems 1, Propositions 1 and 3. In Sec. 3, we prove Theorem 1 and Proposition 2. We give the proof of Theorem 1 in Sec. 4. We devote Sec. 5 to the one-dimensional results, i.e. Proposition 3 and Theorems 2 and 3. We also discuss why our results are very different from the ones obtained by Ouaili [25] when $\theta = 1$. Finally, Sec. 6 contains some comments and open questions.

2. Some Weighted Estimates and an Extended Adjoint System

In this section, we present some well known estimates for solutions of elliptic and parabolic equations. More precisely, we give some weighted inequalities which, in particular, allow us to recover the solution of the PDE in the whole domain by means local measurements. These type of estimate are called Carleman inequalities and are important tools in Control Theory and Inverse problems (see [11, 14, 16, 21]).

In order to precisely state the inequalities we shall use, we must introduce several weight functions. Thus, given ω a non-empty open set satisfying $\omega \Subset \Omega$, let $\eta_0 = \eta_0(x)$ be a function satisfying

$$\left|\begin{array}{l} \eta_0 \in C^2(\overline{\Omega}), \quad \eta_0 > 0 \quad \text{in} \quad \Omega, \quad \eta_0 = 0 \quad \text{on} \quad \Gamma, \\ |\nabla \eta_0| > 0 \quad \text{in} \quad \overline{\Omega} \setminus \omega. \end{array}\right.$$

The construction of the function η_0 is done via Morse functions and can be found in [16].

Then, for some $\lambda > 0$ we introduce the weights:

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda(\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}, \quad \beta(x,t) = \frac{e^{\lambda(\|\eta^0\|_\infty + \eta^0(x))}}{t(T-t)}.$$

Let us also introduce the following notation:

$$I(\psi) := s^{-1} \iint_Q e^{-2s\alpha}\beta^{-1}(|\psi_t|^2 + |\Delta\psi|^2)\,dx\,dt \\ + s\lambda^2 \iint_Q e^{-2s\alpha}\beta|\nabla\psi|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\psi|^2\,dx\,dt.$$

We now state an estimate for the heat equation that will be very important to our purpose:

Lemma 1. *Let $\omega_0 \Subset \omega$ a nonempty subset and $\sigma \in (0,1]$. There exists a constant $\lambda_0 = \lambda_0(\Omega,\omega,\sigma) \geq 1$ such that for every $\lambda \geq \lambda_0$ there exist positive constants $s_0 = s_0(\Omega,\omega,\lambda,\sigma)$ and $C = C(\Omega,\omega,\lambda,\sigma)$ such that, for every $s \geq s_0(T+T^2)$, the following inequality holds*:

$$I(q) \leq C\left(\iint_Q e^{-2s\alpha}|q_t + \sigma\Delta q|^2 dxdt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|q|^2 dxdt\right), \quad (12)$$

for all $q \in C^2(\overline{Q})$, with $q = 0$ on Σ.

The proof of Lemma 1 is classical and can be deduced from the Carleman inequality for the heat equation with homogeneous Dirichlet boundary conditions and we simply refer the interested reader to [14] and [16].

The next result is an estimate for solutions of elliptic equations.

Lemma 2. *Let $\omega_0 \Subset \omega$ a nonempty subset. There exists a constant $\lambda_0 = \lambda_0(\Omega,\omega) \geq 1$ such that for every $\lambda \geq \lambda_0$ there exists $s_0 = s_0(\Omega,\omega,\lambda)$ and $C = C(\Omega,\omega,\lambda)$ such that, for every $s \geq s_0$, the following inequality holds*:

$$s\lambda^2 \iint_Q e^{-2s\alpha}\beta|\nabla y|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|y|^2\,dx\,dt \\ \leq C\left(\iint_Q e^{-2s\alpha}|\Delta y|^2 dx + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|y|^2\,dx\right), \quad (13)$$

for all $y \in C^2(\overline{Q})$, with $y = 0$ on Σ.

The proof of Lemma 2 readily follows from the proof of the parabolic counterpart given in Lemma 1, see [18] where the technique was introduced. For more details, see [15].

To finish this section, we introduce an *extended adjoint system* to (1). More precisely, we extend the uncoupled system (4) to a coupled system composed of six equations. To do this, we consider the operators

$$L = -\partial_t - \Delta + a, \quad L_\theta = -\partial_t - \theta\Delta + b$$

and define

$$w := \phi + \xi.$$

A straightforward calculation leads to the following augmented system:

$$\begin{cases} L_\theta w = z & \text{in } Q, \\ Lz = g & \text{in } Q, \\ Lw = \widetilde{z} & \text{in } Q, \\ L_\theta \widetilde{z} = \widetilde{g} & \text{in } Q, \\ (1-\theta)\Delta\phi + (b-a)\phi = z & \text{in } Q \\ (\theta-1)\Delta\xi + (a-b)\xi = \widetilde{z} & \text{in } Q \\ w = z = \widetilde{z} = 0 & \text{on } \Sigma, \\ w(T) = w_0, \quad z(T) = z_0, \quad \widetilde{z}(T) = \widetilde{z}_0 & \text{in } \Omega, \end{cases} \tag{14}$$

where

$$g = -\Delta(b-\theta a)\phi - 2\nabla(b-\theta a)\nabla\phi, \quad \widetilde{g} = \Delta(b-\theta a)\xi + 2\nabla(b-\theta a)\nabla\xi$$

and

$$w_0 = \phi_0 + \xi_0, \quad z_0 = (1-\theta)\Delta\phi_0 + (b-a)\phi_0, \quad \widetilde{z}_0 = (\theta-1)\Delta\xi_0 + (a-b)\xi_0.$$

3. The Case b–θa Constant in Ω

This section is devoted to the proofs of Propositions 1 and 2. We begin with the proof of the latter.

Proof of Proposition 2. Suppose that system (7) has multiple solutions and let $\varsigma \neq 0$ be a nontrivial one. In particular, it holds that

$$-\theta\Delta\varsigma + b\varsigma = -\Delta\varsigma + a\varsigma \qquad \text{in } \Omega.$$

Now, consider the adjoint system (4) with initial conditions $(\phi_0, \xi_0) = (\varsigma, -\varsigma)$, i.e.

$$\begin{cases} -\phi_t - \Delta\phi + a\phi = 0 & \text{in } Q, \\ -\xi_t - \theta\Delta\xi + b\xi = 0 & \text{in } Q, \\ \phi = \xi = 0 & \text{on } \Sigma, \\ \phi(T) = \phi_0 = \varsigma, \quad \xi(T) = \xi_0 = -\varsigma & \text{in } \Omega. \end{cases}$$

Then, $w = \phi + \xi$ solves the first equation in (14) with initial datum zero, that is

$$\begin{cases} -w_t - \theta\Delta w + bw = z & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(T) = 0 & \text{in } \Omega. \end{cases} \tag{15}$$

The assumption $b - \theta a \equiv c$, with c being a constant, implies $g = 0$ in (14), and then

$$\begin{cases} -z_t - \Delta z + az = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = (1-\theta)\Delta\varsigma + (b-a)\varsigma = 0 & \text{in } \Omega. \end{cases} \tag{16}$$

Since the parabolic equation (16) has a unique solution, it follows that $z = 0$, and so $w = 0$. Because ϕ and ξ are nonzero, the unique continuation property (6) does not hold. This finishes the proof of Proposition 2. $\square$

Let us now prove Proposition 1.

Proof of Proposition 1. The fact that system (7) having multiple solutions implies that system (1) is not simultaneously null controllable follows directly from Proposition 2.

Reciprocally, let us assume that system (7) has a unique solution. As we mentioned before, to prove that system (1) is simultaneously null controllable at time T, it is sufficient to prove the observability inequality (5).

Now, since b–$\theta a \equiv c$, with c being a constant, it follows that g in (14) is zero and the extended adjoint system (14) gives

$$\begin{cases} L_\theta w = z & \text{in } Q, \\ Lz = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(T) = w_0, \quad z(T) = z_0 & \text{in } \Omega. \end{cases} \tag{17}$$

Applying Lemma 1 to the equations of w and z in (17), with $\sigma = \theta$ and $\sigma = 1$, respectively, after absorbing the lower order terms, we get

$$I(w) + I(z) \leq Cs^3\lambda^4 \iint_{\omega_0 \times (0,T)} e^{-2s\alpha}\beta^3(|w|^2 + |z|^2)\, dx\, dt. \tag{18}$$

Let us estimate the integral on z in the right-hand side of (18). To do this, we introduce cut-off function $\zeta \in C^2(\overline{\Omega})$, such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in ω_0 and $\zeta = 0$ in $\Omega\backslash\omega$ and use a *local energy estimate technique* that will be used all along the paper. Here, and when considered necessary, we will specify the technique.

Multiplying the first equation in (17) by $s^3\lambda^4\beta^3 e^{-2s\alpha}\zeta z$, integrating by parts and using Young's inequality, we obtain, for some $q, p \in \mathbb{N}$ (the precise value is irrelevant in the present problem), that

$$
\begin{aligned}
s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z|^2\,dx\,dt &\le s^3\lambda^4 \iint_{\omega\times(0,T)} \zeta\, e^{-2s\alpha}\beta^3 z\, L_\theta w\,dx\,dt \\
&\le \varepsilon I(z) + C_\varepsilon s^p\lambda^q \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt, \qquad (19)
\end{aligned}
$$

for any $\epsilon > 0$.

Combining (18) and (19), we find

$$
I(w) + I(z) \le C s^p\lambda^q \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt. \qquad (20)
$$

We now add a global term in ϕ on the left-hand side of (20). This is done using the fact that this function solves the elliptic problem

$$
\begin{cases}
(1-\theta)\Delta\phi + (b-a)\phi = z & \text{in } \Omega,\\
\phi = 0 & \text{on } \Gamma.
\end{cases} \qquad (21)
$$

In fact, defining $e^{-2s\widehat{\alpha}}\widehat{\beta}^3 = \min_{x\in\Omega} e^{-2s\alpha}\beta^3$ and using energy estimates for the solution of (21) and Poincaré inequality for ϕ, we show that

$$
\iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt \le C\iint_Q e^{-2s\alpha}\beta^3|z|^2\,dx\,dt.
$$

This last inequality and (20) imply

$$
\iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt \le C s^p\lambda^q \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt. \qquad (22)
$$

Arguing in the same way for ξ, we deduce that

$$
\begin{aligned}
&\iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt + \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\xi|^2\,dx\,dt \\
&\qquad \le C s^p\lambda^q \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt. \qquad (23)
\end{aligned}
$$

Finally, from (23), we conclude the proof by using similar arguments as in [17, Proposition 3.1], which guarantees us that there exists a constant $C > 0$, such that

$$
\int_\Omega (|\phi(0)|^2 + |\xi(0)|^2)\,dx \le C\iint_{\omega\times(0,T)} |w|^2\,dx\,dt. \qquad (24)
$$

This inequality is in fact (5) and, thus, we conclude the proof of Proposition 1.

□

4. The Case b–$\theta a = 0$ in ω

In this section, we prove Theorem 1.

Proof of Theorem 1. Here again, our aim is to prove the observability inequality given by (5). By applying Lemma 1 to the first two equations in (14), we get

$$\begin{aligned} &I(w) + I(z) \\ &\quad \le C\left(\iint_Q e^{-2s\alpha}|g|^2\,dx\,dt + s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3(|w|^2+|z|^2)\,dx\,dt\right). \end{aligned} \tag{25}$$

As in the proof of Proposition 1, we have the estimate

$$\begin{aligned} s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z|^2\,dx\,dt &\le s^3\lambda^4\iint_{\omega\times(0,T)} \zeta\, e^{-2s\alpha}\beta^3 z\, L_\theta w\,dx\,dt \\ &\le \varepsilon I(z) + C_\varepsilon s^p\lambda^q\iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt, \end{aligned} \tag{26}$$

for any $\epsilon > 0$.

Combining (25) and (26), we find

$$I(w) + I(z) \le C\left(\iint_Q e^{-2s\alpha}|g|^2\,dx\,dt + s^p\lambda^q\iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^p|w|^2\,dx\,dt\right). \tag{27}$$

In what follows, we estimate the integral on g in the right-hand side of (27). This is done using that $g = -\Delta(b-\theta a)\phi - 2\nabla(b-\theta a)\nabla\phi$ and the system

$$\begin{cases} Lz = g & \text{in } Q, \\ (1-\theta)\Delta\phi + (b-a)\phi = z & \text{in } Q. \end{cases} \tag{28}$$

Taking the time derivative, we obtain

$$\begin{cases} L(z_t) = -\Delta(b-\theta a)\phi_t - 2\nabla(b-\theta a)\nabla\phi_t & \text{in } Q, \\ (\theta-1)\Delta\phi_t = -z_t + (b-a)\phi_t & \text{in } Q. \end{cases} \tag{29}$$

By applying Lemma 2 to $(29)_2$, we get

$$\begin{aligned} &s^3\lambda^4\iint_Q e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s\lambda^2\iint_Q e^{-2s\alpha}\beta|\nabla\phi_t|^2\,dx\,dt \\ &\quad \le C\left(\iint_Q e^{-2s\alpha}|z_t|^2\,dx\,dt + s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt\right). \end{aligned}$$

Now, applying Lemma 1 to the parabolic equation $(29)_1$, we obtain

$$I(z_t) \leq C\Big(\iint_Q e^{-2s\alpha}(|\phi_t|^2 + |\nabla\phi_t|^2)\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z_t|^2\,dx\,dt.$$

For $s >> 1$, these last two inequalities give us

$$\begin{aligned}
&I(z_t) + s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s\lambda^2 \iint_Q e^{-2s\alpha}\beta|\nabla\phi_t|^2\,dx\,dt \\
&\quad \leq C\left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z_t|^2\,dx\,dt\right).
\end{aligned} \tag{30}$$

Claim 1. *There exists a positive constant $C = C(\|a\|_\infty, \Omega, T)$ such that*

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|z|^2\,dx\,dt \leq Cs^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt. \tag{31}$$

Proof of Claim 1. Since $\phi_t = -\Delta\phi + a\phi$, we have

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|-\Delta\phi + a\phi|^2\,dx\,dt = s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt.$$

Let $e^{-2s\widehat{\alpha}}\widehat{\beta}^3 = \min_{x\in\Omega} e^{-2s\alpha}\beta^3$. From the coercivity of the operator $-\Delta + a$,

$$\begin{aligned}
&s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3(|\Delta\phi|^2 + |\nabla\phi|^2 + |\phi|^2)\,dx\,dt \\
&\quad \leq s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|-\Delta\phi + a\phi|^2\,dx\,dt
\end{aligned}$$

and then

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3(|\Delta\phi|^2 + |\nabla\phi|^2 + |\phi|^2)\,dx\,dt \leq Cs^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt. \tag{32}$$

Using the second equation in (28) and estimate (32), we readily have (31). □

Combining (30)–(32), we obtain

$$\begin{aligned}
&s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|z|^2\,dx\,dt + I(z_t) + s^3\lambda^4 \iint_Q e^{-2s\alpha}\beta^3|\phi|^2\,dx\,dt \\
&\quad \leq C\left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z_t|^2\,dx\,dt\right).
\end{aligned} \tag{33}$$

Now, using that for some $k, l \in \mathbb{N}$ and every $\epsilon > 0$, one has

$$s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|z_t|^2\,dx\,dt$$
$$\leq Cs^k\lambda^l \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^k|z|^2\,dx\,dt + \epsilon s^{-1}\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^{-1}|z_{tt}|^2\,dx\,dt,$$

we can deduce by (33) that

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|z|^2\,dx\,dt + I(z_t) + s^3\lambda^4\iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt$$
$$\leq C\left(s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s^k\lambda^l\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^k|z|^2\,dx\,dt\right), \tag{34}$$

for some $k, l \in \mathbb{N}$

Using that

$$-\Delta z + az = z_t - \Delta(b-\theta a)\phi - 2\nabla(b-\theta a)\nabla\phi,$$

and the fact that the operator $-\Delta + a$ is coercive together with (32) and (34), we show that

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\Delta z|^2\,dx\,dt$$
$$\leq C\left(s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + s^k\lambda^l\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\beta^k|z|^2\,dx\,dt\right). \tag{35}$$

Let us now estimate the second term on the right-hand side of (35). For this, we take $e^{-2s\alpha^*}(\beta^*)^k = \max_{x\in\Omega} e^{-2s\alpha}\beta^k$ the maximum of the weight, and then

$$\iint_{\omega_0\times(0,T)} e^{-2s\alpha^*}(\beta^*)^k|z|^2\,dx\,dt$$
$$= \iint_{\omega_0\times(0,T)} e^{-2s\alpha^*}(\beta^*)^k z(-w_t - \theta\Delta w + bw)\,dx\,dt.$$

Then, performing local energy estimates in ω, it is not difficult to prove that

$$\iint_{\omega_0\times(0,T)} e^{-2s\alpha^*}(\beta^*)^k|z|^2\,dx\,dt$$
$$\leq C\iint_{\omega\times(0,T)} |w|^2\,dx\,dt + \epsilon I(z_t) + \epsilon s^3\lambda^4\iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\Delta z|^2\,dx\,dt, \tag{36}$$

for every $\epsilon > 0$.

In this way, putting (36) into (34), we conclude that

$$\begin{aligned} &s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|z|^2\,dx\,dt + I(z_t) + s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt \\ &\quad \le C\left(s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt + \iint_{\omega\times(0,T)} |w|^2\,dx\,dt\right). \end{aligned} \tag{37}$$

Now, we are going to estimate the first term on the right-hand side of (37). We notice that by combining the first equation in (4) and the second one in (28), we find

$$z = -(1-\theta)\phi_t + (b-\theta a)\phi.$$

Because $b–\theta a = 0 \ \text{ in } \ \omega$, we see that

$$z = -(1-\theta)\phi_t \quad \text{in } \omega$$

and, therefore

$$s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^3|\phi_t|^2\,dx\,dt \le Cs^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\beta^3|z|^2\,dx\,dt.$$

This last inequality together with (36) and (37) imply

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|z|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt \le C\iint_{\omega\times(0,T)} |w|^2\,dx\,dt. \tag{38}$$

Arguing in the same way for $(\xi, \widetilde{z})$ instead of (ϕ, z), we have the following similar estimate:

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\widetilde{z}|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\xi|^2\,dx\,dt \le C\iint_{\omega\times(0,T)} |w|^2\,dx\,dt. \tag{39}$$

These last two inequalities give us

$$s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\phi|^2\,dx\,dt + s^3\lambda^4 \iint_Q e^{-2s\widehat{\alpha}}\widehat{\beta}^3|\xi|^2\,dx\,dt \le C\iint_{\omega\times(0,T)} |w|^2\,dx\,dt. \tag{40}$$

Proceeding as in the proof of Proposition 1, we can prove inequality (24) and, consequently, the observability one (5). This completes the proof.

5. One Dimensional Setting

This section is devoted to prove the one dimensional results, namely Proposition 3 and Theorems 2 and 3.

5.1. *Proof of Proposition 3*

Without loss of generality, we give the proof in the case $a = 0$ and $c = 1$. Note that, with this condition, the auxiliary adjoint system (14) reads

$$\begin{cases} -w_t - \theta w_{xx} + bw = z & \text{in } Q, \\ -z_t - z_{xx} = -2\phi_x & \text{in } Q, \\ -w_t - w_{xx} = \widetilde{z} & \text{in } Q, \\ -\widetilde{z}_t - \theta\widetilde{z}_{xx} + b\widetilde{z} = 2\xi_x & \text{in } Q, \\ w = z = \widetilde{z} = 0 & \text{on } \Sigma, \\ w(T) = w_0, \quad z(T) = z_0, \quad \widetilde{z}(T) = \widetilde{z}_0 & \text{in } \Omega \end{cases} \tag{41}$$

with

$$z = (1-\theta)\phi_{xx} + x\phi \quad \text{and} \quad \widetilde{z} = (\theta - 1)\xi_{xx} + x\xi. \tag{42}$$

Let us consider a subset $\omega_0 \Subset \omega$, with $0 \notin \overline{\omega_0}$ and apply Lemma 1 to the parabolic equation verified by ξ in (4). This gives

$$I(\xi) \le Cs^3\lambda^4 \iint_{\omega_0\times(0,T)} \beta^3 e^{-2s\alpha}|\xi|^2\, dx\, dt. \tag{43}$$

Now, by applying the same inequality to the equation verified by $\widetilde{z}$ in (41), we get

$$I(\widetilde{z}) \le C\left(\iint_Q e^{-2s\alpha}|\xi_x|^2\, dx\, dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} \beta^3 e^{-2s\alpha}|\widetilde{z}|^2\, dx\, dt\right). \tag{44}$$

Combining (43) and (44), we obtain, for s large enough, that

$$I(\xi) + I(\widetilde{z}) \le Cs^3\lambda^4\left(\iint_{\omega_0\times(0,T)} \beta^3 e^{-2s\alpha}|\xi|^2\, dx\, dt + \iint_{\omega_0\times(0,T)} \beta^3 e^{-2s\alpha}|\widetilde{z}|^2\, dx\, dt\right). \tag{45}$$

Considering now two intermediate subsets $\omega_0 \Subset \omega_1 \Subset \omega_2 \Subset \omega$. Using local energy estimates to the elliptic equation satisfied by ξ in (42), we find, for some $l, p \in \mathbb{N}$, that

$$\iint_{\omega_0\times(0,T)} \beta^3 e^{-2s\alpha}|\xi|^2\, dx\, dt \le Cs^l\lambda^p \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\beta^l\left(|\xi_x|^2 + |\widetilde{z}|^2\right)\, dx\, dt. \tag{46}$$

This last estimate is the reason why we must take $0 \notin \overline{\omega_0}$.

Estimates (45) and (46) yield

$$I(\xi) + I(\widetilde{z}) \le Cs^l\lambda^p \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\beta^l\left(|\xi_x|^2 + |\widetilde{z}|^2\right)\, dx\, dt. \tag{47}$$

Let us estimate the local integral on ξ_x. For this, we take $\zeta \in C^2([0,1])$ such that $0 \le \zeta \le 1$, $\zeta = 1$ in ω_1 and $\zeta = 0$ in $\Omega\backslash\omega_2$ and multiply the fourth equation in

(41) by $s^l \lambda^p \beta^l e^{-2s\alpha} \zeta \xi_x$. Integration by parts, together with estimate (47), gives

$$\begin{aligned} s^l \lambda^p \iint_{\omega_1 \times (0,T)} & e^{-2s\alpha} \beta^l |\xi_x|^2 \, dx\, dt \\ & \leq C s^m \lambda^q \iint_{\omega_2 \times (0,T)} e^{-2s\alpha} \beta^m \left(|\widetilde{z}|^2 + |\widetilde{z}_x|^2 \right) \, dx\, dt, \end{aligned} \tag{48}$$

for some $m, q \in \mathbb{N}$.

By the same reasoning, one can show that

$$s^m \lambda^q \iint_{\omega_2 \times (0,T)} e^{-2s\alpha} \beta^m |\widetilde{z}_x|^2 dxdt \leq C s^\ell \lambda^r \iint_{\omega \times (0,T)} e^{-2s\alpha} \beta^\ell |\widetilde{z}|^2 dxdt, \tag{49}$$

for appropriate $\ell, r \in \mathbb{N}$.

Thus, the following estimate holds:

$$I(\xi) + I(\widetilde{z}) \leq C s^\ell \lambda^r \iint_{\omega \times (0,T)} e^{-2s\alpha} \beta^\ell |\widetilde{z}|^2 dxdt, \tag{50}$$

for appropriate $\ell, r \in \mathbb{N}$.

Finally, we apply estimate in Lemma 1 to the equation verified by w in (41) and combine it with (50) to get

$$I(w) + I(\xi) + I(\widetilde{z}) \leq C \iint_{\omega \times (0,T)} |w|^2 dxdt. \tag{51}$$

Since this last inequality readily implies the observability inequality we need, the proof of Proposition 3 is finished.

5.2. *Proof of Theorems 2 and 3*

In this section, following the ideas in [25], we prove Theorems 2 and 3. Before this, we recall the following results (see e.g. [20, Th 5.4, p. 173, Th 5.10, p. 183 and Th 5.11, p. 184]):

Theorem 4. *Let $q \in L^2(0,1)$, $\lambda \in \mathbb{C}$. For $0 < \theta \leq 1$, consider the Cauchy problem:*

$$\begin{cases} -\theta p'' + qp = \lambda p & \text{in } (0,1), \\ p(0) = 0, \quad p'(0) = 1. \end{cases} \tag{52}$$

Then (52) has a unique solution $p \in H^2(0,1)$ satisfying

$$p(x, \lambda, q, \theta) = \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}} x\right)}{\sqrt{\lambda}} \sqrt{\theta} + \int_0^x \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}} (x-y)\right)}{\sqrt{\lambda}\sqrt{\theta}} q(y) p(y) dy, \quad x \in (0,1).$$

Taking into account Theorem 4 and introducing the operators

$$\mathcal{L}(\varphi_{1,j}) := -\partial_{xx}\varphi_{1,j} + a\varphi_{1,j} \quad \text{and} \quad \mathcal{L}_\theta(\varphi_{2,j}) := -\theta \partial_{xx}\varphi_{2,j} + b\varphi_{2,j},$$

with domain $H^2(0,1)\cap H_0^1(0,1)$, the following results associated to Sturm–Liouville problems (8) hold.

Proposition 4. *Let $a \in L^\infty(0,1)$. The operator $(\mathcal{L}, D(\mathcal{L}))$ is self-adjoint and has an increasing of eigenvalues $\{\mu_{1,k}\}_{k\in\mathbb{N}}$ satisfying*

$$\mu_{1,1} < \mu_{1,2} < \mu_{1,3} <, \dots, \quad \textit{with} \quad \lim_{k\to\infty} \mu_{1,k} = +\infty.$$

Moreover, we have the following asymptotic behavior:

$$\mu_{1,k} = \pi^2 k^2 + \int_0^1 a(x)dx - \int_0^1 \cos(2k\pi x)a(x)dx + \mathcal{O}(1/k), \quad k \to +\infty.$$

Also, denoting by $\varphi_{1,k}$ the normalized eigenfunction associated to $\mu_{1,k}$, then $\{\varphi_{1,k}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1)$

$$\varphi_{1,k}(x) = \frac{p(x,\mu_{1,k},a,1)}{\|p(\cdot,\mu_{1,k},a,1)\|_{L^2(0,1)}}$$

and

$$p(x,\mu_{1,k},a,1) = \frac{\sin(\sqrt{\mu_{1,k}}x)}{\sqrt{\mu_{1,k}}} + \mathcal{O}(1/k^2) \quad \textit{with}$$

$$\|p(\cdot,\mu_{1,k},a,1)\|_{L^2(0,1)} = \frac{1}{\sqrt{2\mu_{1,k}}}\sqrt{1+\mathcal{O}(1/k)}.$$

Also, for $x \in [0,1]$, we have, as $k \to +\infty$, that

$$\varphi_{1,k}(x) = \sqrt{2}\sin(k\pi x) + \mathcal{O}(1/k), \quad \varphi_{1,k}'(x) = \sqrt{2}k\pi\cos(k\pi x) + \mathcal{O}(1).$$

Proposition 5. *Let $b \in L^\infty(0,1)$. The operator $(\mathcal{L}_\theta, D(\mathcal{L}_\theta))$ is self-adjoint and has an increasing sequence of eigenvalues $\{\mu_{2,k}\}_{k\in\mathbb{N}}$ satisfying*

$$\mu_{2,1} < \mu_{2,2} < \mu_{2,3} <, \dots, \quad \textit{with} \quad \lim_{k\to\infty} \mu_{2,k} = +\infty.$$

Moreover, we have the following asymptotic behavior:

$$\mu_{2,k} = \theta\pi^2 k^2 + \int_0^1 b(x)dx - \int_0^1 \cos(2k\pi x)b(x)dx + \mathcal{O}(1/k), \quad k \to +\infty.$$

Also, denoting by $\varphi_{2,k}$ the normalized eigenfunction associated to $\mu_{2,k}$, then $\{\varphi_{2,k}\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1)$,

$$\varphi_{2,k}(x) = \frac{p(x,\mu_{2,k},b,\theta)}{\|p(\cdot,\mu_{2,k},b,\theta)\|_{L^2(0,1)}}$$

and

$$p(x,\mu_{2,k},b,\theta) = \sqrt{\theta}\frac{\sin\left(\frac{\sqrt{\mu_{2,k}}}{\sqrt{\theta}}x\right)}{\sqrt{\mu_{2,k}}} + \mathcal{O}(1/k^2) \quad \textit{with}$$

$$\|p(\cdot,\mu_{2,k},b,\theta)\|_{L^2(0,1)} = \frac{\sqrt{\theta}}{\sqrt{2\mu_{2,k}}}\sqrt{1+\mathcal{O}(1/k)}.$$

Also, for $x \in [0,1]$, we have, as $k \to +\infty$, that

$$\varphi_{2,k}(x) = \sqrt{2}\sin(k\pi x) + \mathcal{O}(1/k), \quad \varphi_{2,k}'(x) = \sqrt{2}k\pi\cos(k\pi x) + \mathcal{O}(1).$$

Now, let (ϕ, ξ) be the solution to (4) corresponding to the initial datum $(\phi_0, \xi_0) \in L^2(0,1) \times L^2(0,1)$. Then $(\phi, \xi) \in [L^2(0,T; H_0^1(0,1)) \cap C^0([0,T], L^2(0,1))]^2$ and can be written as

$$(\phi(\cdot,t), \xi(\cdot,t)) = \sum_{k=1}^{\infty} \left(e^{-\mu_{1,k}(T-t)} \langle \phi_0, \varphi_{1,k}\rangle \varphi_{1,k},\, e^{-\mu_{2,k}(T-t)} \langle \xi_0, \varphi_{2,k}\rangle \varphi_{2,k}\right).$$

5.2.1. *Proof of Theorem* 2

Let us prove that (6) holds for all solutions of (4). For this, let us consider the set $\Lambda = \{\mu_{1,k}\}_{k\in\mathbb{N}} \cup \{\mu_{2,j}\}_{j\in\mathbb{N}} = \{\lambda_m\}_{m\in\mathbb{N}}$, which we order Λ in an increasing manner:

$$\lambda_1 < \lambda_2 <, \ldots, .$$

For each $m \in \mathbb{N}$, we have only the following possibilities:

- There exists a unique $k_0 \in \mathbb{N}$ and a unique $j_0 \in \mathbb{N}$ such that $\lambda_m = \mu_{1,k_0} = \mu_{2,j_0}$.
- There exists a unique $k_0 \in \mathbb{N}$ such that $\lambda_m = \mu_{1,k_0}$ but $\lambda_m \neq \mu_{2,j}, \quad \forall j \in \mathbb{N}$.
- There exists a unique $j_0 \in \mathbb{N}$ such that $\lambda_m = \mu_{2,j_0}$ but $\lambda_m \neq \mu_{1,k}, \quad \forall k \in \mathbb{N}$.

Thus, we may write $\phi + \xi$ as

$$\phi + \xi = \sum_{m=1}^{+\infty} a_0^m e^{-\lambda_m(T-t)} e_m, \tag{53}$$

where, for each $m = 1, 2, \ldots$, we have one of the following:

- There exists a unique $k_0 \in \mathbb{N}$ and a unique $j_0 \in \mathbb{N}$ such that
$$a_0^m e_m = \langle \phi_0, \varphi_{1,k_0}\rangle \varphi_{1,k_0} + \langle \xi_0, \varphi_{2,j_0}\rangle \varphi_{2,j_0}.$$
- There exists a unique $k_0 \in \mathbb{N}$ such that $a_0^m e_m = \langle \phi_0, \varphi_{1,k_0}\rangle \varphi_{1,k_0}$.
- There exists a unique $j_0 \in \mathbb{N}$ such that $a_0^m e_m = \langle \xi_0, \varphi_{2,j_0}\rangle \varphi_{2,j_0}$.

The proof of Theorem 2 is a consequence from the following claim.

Claim. *If*

$$\phi + \xi = 0 \quad in\ \omega \times (0,T),$$

then

$$\langle \phi_0, \varphi_{1,k}\rangle = 0 = \langle \xi_0, \varphi_{2,j}\rangle, \quad \forall j,k \in \mathbb{N}.$$

In particular,

$$\phi + \xi = 0 \quad in \quad \omega \times (0,T) \implies \varphi \equiv 0 \equiv \xi.$$

Proof of the Claim. By the time analyticity, from (53) it follows that

$$(\phi+\xi)1_\omega = \sum_{m=1}^{+\infty} a_0^m e^{-\lambda_m (T-t)} 1_\omega e_m = 0 \quad \text{in } (-\infty, T). \tag{54}$$

For each $j = 2, 3, \ldots$, let $\sigma_j > 0$ be defined as $\sigma_j = \lambda_j - \lambda_1$. Then

$$e^{-\lambda_1 (T-t)} a_0^1 1_\omega e_1 + e^{-\lambda_1 (T-t)} \sum_{m=2}^{+\infty} a_0^m e^{-\sigma_m (T-t)} 1_\omega e_m = 0 \quad \text{in } (-\infty, T).$$

Multiplying this last identity by $e^{\lambda_1 (T-t)}$, we get

$$a_0^1 1_\omega e_1 + \sum_{m=2}^{+\infty} a_0^m e^{-\sigma_m (T-t)} 1_\omega e_m = 0 \quad \text{in } (-\infty, T),$$

and the limit as $t \to -\infty$ gives

$$a_0^1 1_\omega e_1 = 0.$$

Arguing the same way, it is not difficult to see that

$$a_0^m 1_\omega e_m = 0, \quad \forall m \in \mathbb{N}.$$

To prove the Claim, for each $m \in \mathbb{N}$, we have three situations to consider:

(1) $a_0^m e_m = \langle \phi_0, \varphi_{1,k_0} \rangle \varphi_{1,k_0} + \langle \xi_0, \varphi_{2,j_0} \rangle \varphi_{2,j_0} = 0 \quad \text{in } \omega.$
(2) $a_0^m e_m = \langle \phi_0, \varphi_{1,k_0} \rangle \varphi_{1,k_0} = 0 \quad \text{in } \omega.$
(3) $a_0^m e_m = \langle \xi_0, \varphi_{2,j_0} \rangle \varphi_{2,j_0} = 0 \quad \text{in } \omega.$

Since the last two cases are straightforward, we only have to prove that

$$\langle \phi_0, \varphi_{1,k_0} \rangle \varphi_{1,k_0} + \langle \xi_0, \varphi_{2,j_0} \rangle \varphi_{2,j_0} = 0 \quad \text{in } \omega \implies \langle \phi_0, \varphi_{1,k_0} \rangle = \langle \xi_0, \varphi_{2,j_0} \rangle = 0. \tag{55}$$

In what follows, we must use that we are in the situation where $\mu_{1,k_0} = \mu_{2,j_0}$ and the following auxiliary lemma.

Lemma 3. *Let $q \in L^\infty(0,1)$, $\omega = (\alpha, \beta)$, with $0 < \alpha < \beta < 1$. In ω, all solutions of problem* (52) *that verify $p(1) = 0$ can be written as*

$$p(x) = \frac{\sqrt{\theta}}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}} (x-1) \right) p'(1), \quad \forall x \in \omega, \quad \textit{if } supp\,(q) \subset [0, \alpha]$$

and

$$p(x) = \frac{\sqrt{\theta}}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}} x \right), \quad \forall x \in \omega, \quad \textit{if } supp\,(q) \subset [\beta, 1].$$

For the rest of the proof, let us assume that $\operatorname{supp}(a), \operatorname{supp}(b) \subset [\beta, 1]$. Noticing that $\varphi'(0) \neq 0$ for any eigenfunction of either operators $\mathcal{L}$ or $\mathcal{L}_\theta$, Lemma 3 guarantees the existence of $C_1, C_2 > 0$ such that

$$\varphi_{1,k_0} = C_1 \frac{1}{\sqrt{\mu_{1,k_0}}} \sin(\sqrt{\mu_{1,k_0}}x) \quad \text{in } \omega$$

and

$$\varphi_{2,j_0} = C_2 \frac{\sqrt{\theta}}{\sqrt{\mu_{2,j_0}}} \sin\left(\frac{\sqrt{\mu_{2,j_0}}}{\sqrt{\theta}}x\right) \quad \text{in } \omega.$$

Hence, the proof of (55) boils down to

$$A\frac{1}{\sqrt{\mu_{1,k_0}}} \sin(\sqrt{\mu_{1,k_0}}x) + B\frac{\sqrt{\theta}}{\sqrt{\mu_{2,j_0}}} \sin\left(\frac{\sqrt{\mu_{2,j_0}}}{\sqrt{\theta}}x\right) = 0 \quad \text{in } \omega \implies A = B = 0. \tag{56}$$

Since $\mu_{1,k_0} = \mu_{2,j_0}$, sentence (56) follows directly from fact that if $\zeta > 0$ and $0 < \theta < 1$ then

$$A\sin(\zeta x) + B\sin\left(\frac{\zeta}{\theta}x\right) = 0 \quad \text{in } \omega \implies A = B = 0.$$

The case $\operatorname{supp}(a), \operatorname{supp}(b) \subset [0, \alpha]$ is analogous and the proof of Theorem 2 is finished. □

Let us now prove Lemma 3.

Proof of Lemma 3. By Theorem 4, the solution of (52) is given by

$$p(x) = \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}x\right)}{\sqrt{\lambda}}\sqrt{\theta} + \int_0^x \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-t)\right)}{\sqrt{\lambda}\sqrt{\theta}} q(t)p(t)dt, \quad x \in (0,1). \tag{57}$$

Since

$$\int_0^1 \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-t)\right)}{\sqrt{\lambda}\sqrt{\theta}} \left[-\theta p''(t) + qp(t)\right]dt = \lambda \int_0^1 \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-t)\right)}{\sqrt{\lambda}\sqrt{\theta}} p(t)dt,$$

we use the fact that $p'(0) = 1$ and integrate by parts to deduce that

$$\int_0^1 \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-t)\right)}{\sqrt{\lambda}\sqrt{\theta}} [q(t)p(t)]dt = \frac{\sqrt{\theta}}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-1)\right) p'(1) - \frac{\sqrt{\theta}}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}x\right).$$

Hence, we rewrite the solution as

$$p(x) = \frac{\sqrt{\theta}}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-1)\right) p'(1) + \int_1^x \frac{\sin\left(\frac{\sqrt{\lambda}}{\sqrt{\theta}}(x-t)\right)}{\sqrt{\lambda}\sqrt{\theta}} q(t)p(t)dt, \quad x \in (0,1). \tag{58}$$

From (57) and (58), the result follows. □

5.2.2. *Proof of Theorem* 3

We prove Theorem 3 restricting the control to the following form:

$$f(x, T-t) = \widetilde{f}(x,t) = \sum_{k=1}^{\infty} q_{1,k}(t)\varphi_{1,k}(x) + q_{2,k}(t)\varphi_{2,k}(x). \tag{59}$$

Let $\Lambda = \{\mu_{1,j}\}_{j\in\mathbb{N}} \cup \{\mu_{2,k}\}_{k\in\mathbb{N}}$. Since $\mu_{1,j} \neq \mu_{2,k}$ for all $j,k \in \mathbb{N}$, we can order the set as $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_1 < \lambda_2 < \lambda_3 <, \ldots,$.

Defining

$$q_n(t)\phi_n(x) = \begin{cases} q_{1,k}(t)\varphi_{1,k}(x) & \text{if } \lambda_n = \mu_{1,k}, \\ q_{2,k}(t)\varphi_{2,k}(x) & \text{if } \lambda_n = \mu_{2,k}, \end{cases}$$

then we can write

$$\widetilde{f}(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x).$$

From Propositions 4 and 5, we have

$$\lambda_n > 0, \quad \forall n \in \mathbb{N} \quad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|} < \infty.$$

As in [4], the condensation index (see (9)) is given by

$$c(\Lambda) = \limsup_{k\to\infty} \frac{\log \frac{1}{|E'(\lambda_k)|}}{\lambda_k}$$

with $E(z) = \prod_{k=1}^{\infty}(1 - \frac{z^2}{\lambda_k^2})$.

Following the arguments in [4], it can be shown that there exists a family $q_n(t)$ biorthogonal to $e^{-\lambda_n t}$ such that

$$\|q_n\|_{L^2(0,T)} \leq C_\varepsilon e^{(c(\Lambda)+\varepsilon)\lambda_n}, \quad \forall n \in \mathbb{N}.$$

This last inequality will imply, in particular, the simultaneous null controllability of Eq. (1) for $T > c(\Lambda)$, see [4] and [25]. Observe that the particular structures of $\mu_{1,k}$ and $\mu_{2,k}$ imply that for different values of θ, $c(\Lambda) = 0$ or $0 < c(\Lambda) < +\infty$ or $c(\Lambda) = +\infty$, see [4, 5, 23].

6. Some Remarks

In the preceding sections, we mainly obtained sufficient conditions to the simultaneous null controllability problem for two uncoupled parabolic equations. Nevertheless, since the assumptions on our results are quite broad, it is clear that the problem is complex and we believe that it is not possible to obtain necessary and sufficient conditions to encompass all cases.

Let us finish with some remarks.

Remark 6. The coercivity of both operators

$$-\Delta + a(x) \quad \text{and} \quad -\theta\Delta + b(x)$$

is a necessary condition for the simultaneous null controllability of (1) when $b-\theta a \equiv 0$ in ω. In fact, just take $\theta = 1/2$, $a = -\lambda_1$ and $b = -\lambda_1/2$, with λ_1 being the first eigenvalue of the Dirichlet Laplacian. In this situation, the assumptions of Proposition 1 are satisfied and hence the approximate controllability property does not hold.

It remains an open question what happens if one of the operators is not coercive but the other one is, and $b - \theta a \equiv 0$ in ω.

Remark 7. The coercivity of both operators

$$-\Delta + a(x) \quad \text{and} \quad -\theta\Delta + b(x)$$

is not sufficient to guarantee the simultaneous null controllability of (1). Indeed, take $M > \max\{1, \lambda_1\}$, λ_1 being again the first eigenvalue of the Dirichlet Laplacian. Take also $a = M - \lambda_1$, $b = M - \frac{\lambda_1}{M}$ and $\theta = 1/M$. Thus, $b - \theta a = -1$ in ω and the system is not simultaneous null controllable because the system

$$\begin{cases} -\Delta u - \lambda_1 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{60}$$

has multiple solutions.

Note that it is also possible to take $b - \theta a \neq 0$ in ω and still have simultaneous null controllability. Indeed, the simultaneous null controllability of (1) holds in the case $a = b$, with $a \in \mathbb{R}_+$.

Remark 8. It is also possible to study the simultaneous null controllability problem for (1) when $a = a(x,t)$ and $b = b(x,t)$. For instance, since Proposition 1 and Theorem 1 are based on Carleman estimates, their proofs can easily generalized to the case of potentials which depend also on time.

Remark 9. It would be interesting to study the boundary simultaneous controllability of two heat equations. Notice that, in the one-dimensional case, when $\theta = 1$ and $b-a \equiv c$ with $c \in \mathbb{R}\backslash\{0\}$, the results in [13] give a positive result. For $0 < \theta < 1$, the problems seems to be very difficult to tackle. In fact, even in the case of coupled parabolic system with less controls than equations, most of the methods are restricted to the one-dimensional case.

Remark 10. It is also possible to consider the simultaneous controllability problem for uncoupled nonlinear parabolic equations. It would be interesting to know if our ideas could be adapted to treat such problems. Notice that the application of our results are not straightforward even to obtain local results. For instance, in Theorem 1 we assume some regularity for the potentials, nevertheless, to treat nonlinear problems one usually must be able to deal with the case of L^∞ potentials. On the other hand, Theorem 3 is based on the moment method and there is not a direct application to nonlinear problems.

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