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An Insensitizing Control Result for the Keller–Segel System

F. W. Chaves-Silva[1] · **J. Prada**[2]

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Abstract

The Keller–Segel system is a classical model in chemotaxis, widely used in biological and physical contexts, but also a challenging prototype for nonlinear PDE analysis. Our focus studies an insensitizing control problem for the nonlinear parabolic-parabolic Keller–Segel system, which models chemotactic behavior in biological systems. Our goal is to find a control that makes a certain functional of the solution insensitive to small perturbations in the initial data. We show that this problem is equivalent to achieving partial null controllability for a related cascade system that reflects the main structure of the original dynamics. Thanks to this equivalence, we focus on analyzing the controllability of the cascade system. We begin by studying the linearized version of the problem. Using a duality approach, along with carefully selected weighted estimates and energy techniques, we establish a suitable observability inequality. This key result enables us to move on to the nonlinear case. We address the nonlinear system through a local inverse mapping argument, relying on the continuity and differentiability of the control-to-state map in an appropriate functional framework, along with other key assumptions.

Keywords Keller–Segel system · Insensitizing controls · Controllability · Weighted estimates

Mathematics Subject Classification 93B05 · 93B07 · 35K65 · 93C20

✉ J. Prada
jefferson.prada@uc.cl

F. W. Chaves-Silva
fchaves@mat.ufpb.br

[1] Departamento de Matemática, Universidade Federal da Paraíba, Campus I, João Pessoa, Paraíba 58051-900, Brazil

[2] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Macul, 7820436 Santiago, Chile

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and $\tilde{\omega}$ and ω be two (small) nonempty subsets of Ω with $\tilde{\omega} \Subset \omega$ where ω is called as the *control domain*. We will use the notation $Q = \Omega \times (0, T)$, and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $\nu(x)$ the outward normal to Ω at the point $x \in \partial\Omega$.

We will be concerned with the Keller–Segel system:

$$\begin{cases} u_t - \Delta u = -\nabla \cdot (u\nabla v) + F, & \text{in } Q, \\ v_t - \Delta v = au - bv + g\chi_1 + G, & \text{in } Q, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \Sigma, \\ u(\cdot, 0) = M_1 + \tau\hat{u}_0,\ v(\cdot, 0) = M_2 + \tau\hat{v}_0, & \text{in } \Omega. \end{cases}, \tag{1.1}$$

where $u = u(x, t)$ and $v = v(x, t)$ represent, respectively, the concentrations of species (i.e, the population density) and that of the chemical (i.e., concentration of the chemical substance). For more details about the Keller–Segel system see, for instance, [1–7].

Throughout the article, we assume that a, b, M_1 and M_2 are positive real constants such that $aM_1 - bM_2 = 0$ and that the initial data $(u(\cdot, 0), v(\cdot, 0))$ is partially unknown in the following sense:

- M_1 and M_2 are known;
- $\tau \in \mathbb{R}$ a unknown small number;
- $\hat{u}_0 \in L^2(\Omega)$ and $\hat{v}_0 \in L^2(\Omega)$ are unknown with $\|\hat{u}_0\|_{L^2(\Omega)} = \|\hat{v}_0\|_{L^2(\Omega)} = 1$, and have mean equal to zero, i.e.,

$$(\hat{u}_0)_\Omega = \frac{1}{|\Omega|}\int_\Omega \hat{u}_0(x)\mathrm{d}x = 0 \ \text{ and } \ (\hat{v}_0)_\Omega = \frac{1}{|\Omega|}\int_\Omega \hat{v}_0(x)\mathrm{d}x = 0.$$

We consider the external forces $F = F(x, t)$ and $G = G(x, t)$ belonging to an appropriate Banach space and having exponential decay at $t = 0$, and that $(F)_\Omega(t) = 0$ for a.e. $t \in (0, T)$. The external force g is an internal control acting on ω and $\chi_1 \in C_c^\infty(\omega)$ such that $\mathrm{supp}(\chi_1) \Subset \omega$, $0 \le \chi_1 \le 1$ and $\chi_1 \equiv 1$ in $\tilde{\omega}$.

For a given nonempty open set $\mathcal{O} \subset \Omega$ (called the *observation set*), we introduce the observation functional:

$$J_\tau(u, v) := \frac{1}{2}\iint\limits_{\mathcal{O}\times(0,T)} |u|^2\chi_2 \mathrm{d}x\, \mathrm{d}t, \tag{1.2}$$

where $\chi_2 \in C_c^\infty(\mathcal{O})$ such that $0 \le \chi_2 \le 1$, $\mathrm{supp}(\chi_2) \Subset \mathcal{O}$ and $\chi_2 \equiv 1$ in a nonempty open set $\mathcal{O}_0 \Subset \mathcal{O}$, and the pair (u, v) solves the Keller–Segel system (1.1).

The goal of this work is to prove the existence of controls that insensitize the functional J_τ. In other words, we aim to find a control g such that the influence of the unknown data $(\hat{u}_0, \hat{v}_0)$ is not perceptible for the functional J_τ, at least at the first order, that is,

$$\left.\frac{\partial J_\tau(u,v)}{\partial \tau}\right|_{\tau=0} = 0, \tag{1.3}$$

for any $(\hat{u}_0, \hat{v}_0) \in [L^2(\Omega)]^2$ satisfying $\|\hat{u}_0\|_{L^2(\Omega)} = \|\hat{v}_0\|_{L^2(\Omega)} = 1$ and $(\hat{u}_0)_\Omega = (\hat{v}_0)_\Omega = 0$. It is not difficult to see that condition (1.3) is equivalent to the following (partial) null controllability problem: *Find a control function g such that*

$$z(\cdot,0) = q(\cdot,0) = 0 \text{ in } \Omega \tag{1.4}$$

where (w, z, r, q) solves the following cascade system (for more details, see Appendix A):

$$\begin{cases} w_t - \Delta w = -\nabla\cdot(w\nabla r) + F, & \text{in } Q,\\ -z_t - \Delta z = aq + [\nabla z\cdot\nabla r - (\nabla z\cdot\nabla r)_\Omega] + w\chi_2 - (w\chi_2)_\Omega, & \text{in } Q,\\ r_t - \Delta r = aw - br + g\chi_1 + G, & \text{in } Q,\\ -q_t - \Delta q = -bq - \nabla\cdot(w\nabla z), & \text{in } Q,\\ \partial_\nu w = 0,\ \partial_\nu z = 0,\ \partial_\nu r = 0,\ \partial_\nu q = 0, & \text{on } \Sigma,\\ w(\cdot,0) = M_1,\ z(\cdot,T) = 0,\ r(\cdot,0) = M_2,\ q(\cdot,T) = 0, & \text{in } \Omega. \end{cases} \tag{1.5}$$

To study this null controllability problem, a key observation is that, given any solution (w, z, r, q) of (1.5), one has

$$(q)_\Omega(t) = (z)_\Omega(t) = 0 \quad \text{and} \quad (w)_\Omega(t) = M_1, \quad \text{for } t \in [0,T]. \tag{1.6}$$

To show this, let us consider $V(t) := \int_\Omega q(x,t)\mathrm{d}x$ and use equation $(1.5)_4$, then V solves

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}[V(t)] = -bV(t),\\ V(T) = 0, \end{cases}$$

which directly implies $(q)_\Omega(t) = 0$ for $t \in [0,T]$. The other two identities in (1.6) follow from $(1.5)_1$ and $(1.5)_2$, and the fact that $(F)_\Omega(t) = 0$ for a.e. $t \in (0,T)$.

These observations regarding the mean of solutions (w, z, r, q) of cascade system (1.5) will be taken into account latter when proving the (partial) null controllability of (1.5).

As far as we know, the first result regarding the controllability of the Keller–Segel system was provided in [8], where local controllability around constant trajectories was achieved through a control acting on the equation of the chemical. We also reference [9], in which the controllability of the Keller–Segel system (1.1) around a given regular trajectory is attained by applying control to the equation of the species, which is not intuitive, as there is no practical way to act directly on the equation of the species. Another significant contribution can be found in [10], where the authors established that, in the two-dimensional case, any solution globally bounded in time for system (1.1) converges to a single equilibrium as time approaches infinity. Finally, we mention [11], where the authors proved the local exact controllability for the Keller–Segel system coupled with the Navier–Stokes system around certain specific trajectories.

Interest in insensitizing control problems began with the works of J.-L. Lions, specifically in [12] and [13], where he explored this type of problem and posed numerous pertinent questions. These works sparked our interest in investigating the existence of insensitizing controls for the Keller–Segel system. While there exists an extensive body of literature on similar problems for parabolic system—such as the heat equation in [14–17], the Stokes system in [18], the Navier–Stokes system in [19–21], and the Boussinesq system in [22, 23]—there are, to our knowledge, no available results specifically addressing the insensitizing problem for the Keller–Segel system (1.1).

It is also important to note that most insensitizing results found in the literature involve Dirichlet-type boundary condition. There has been comparatively less research on other types of boundary conditions, such as Neumann boundary condition, which is the case for the Keller–Segel system. For example, [24] examines the phase field system with Neumann boundary conditions, and [25] addresses the heat equation with dynamic boundary conditions.

From a physical perspective, the system aims to control the amount of a certain chemical substance under small perturbations, in response to a virus triggered by a population of bacteria. For instance, it may represent the intravenous administration of a medication under small doses (perturbations), with the goal of controlling the spread of the virus. For this reason, we focus on the L^2-norm of the state variable $u = u(x, t)$ which physically represents the concentration of the chemical substance with a specific biological purpose.

As mentioned before, the main goal of this article is to establish the existence of controls for the Keller–Segel system (1.1) that act on the species equation and effectively insensitize functional (1.2). To achieve this, we will prove the existence of controls such that (1.4) holds for system (1.5). More precisely, we prove the following result.

Theorem 1.1 *Assume that $\tilde{\omega} \cap \mathcal{O} \neq \emptyset$ and let $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$ and let $m \geq 4$. Then, there exist $\delta > 0$ and $C > 0$ depending on Ω, ω, $\tilde{\omega}$, $\mathcal{O}$ and T such that, for any*[1] *$(F, G) \in L^2_0(Q) \times L^2(0, T; H^1(\Omega))$, satisfying $\|e^{C/t^m}(F, G)\|_{L^2_0(Q)\times L^2(0,T;H^1(\Omega))} < \delta$, there exists a localized control function $g\chi_1 \in L^2(0, T; H^1(\Omega))$ and a corresponding solution (w, z, r, q) of (1.5) satisfying*

$$z(\cdot, 0) = q(\cdot, 0) \text{ in } \Omega.$$

As a consequence, we readily infer that functional (1.2) can be insensitized.

Corollary 1.1 *There are insensitizing controls g for functional J_τ given by (1.2).*

To prove Theorem 1.1 we follow an approach introduced in [26]. This method consists of two main steps:

[1] $L^2_0(Q) := \{u \in L^2(Q) \mid (u)_\Omega(t) = 0 \text{ for a.e. } t \in (0, T)\}$.

Step 1. We prove (partial) null controllability result for the linear system:

$$\begin{cases} w_t - \Delta w = -M_1 \Delta r + F + h_1, & \text{in } Q, \\ -z_t - \Delta z = aq + w\chi_2 - (w\chi_2)_\Omega + h_2, & \text{in } Q, \\ r_t - \Delta r = aw - br + g\chi_1 + G + h_3, & \text{in } Q, \\ -q_t - \Delta q = -bq - M_1 \Delta z + h_4, & \text{in } Q, \\ \partial_\nu w = 0,\ \partial_\nu z = 0,\ \partial_\nu r = 0,\ \partial_\nu q = 0, & \text{on } \Sigma, \\ w(\cdot,0) = 0,\ z(\cdot,T) = 0,\ r(\cdot,0) = 0,\ q(\cdot,T) = 0, & \text{in } \Omega. \end{cases} \tag{1.7}$$

with F, G, h_1, h_2, h_3 and h_4 belonging to certain Banach spaces, exhibiting exponential decay to zero at $t = 0$, and such that $(F)_\Omega(t) = (h_1)_\Omega(t) = (h_2)_\Omega(t) = (h_4)_\Omega(t) = 0$ for a.e. $t \in (0, T)$. That is to say, we prove that there exists a control g such that solution of (1.7) satisfies

$$z(\cdot, 0) = q(\cdot, 0) = 0 \ \text{ in } \Omega.$$

Moreover, we show that the pair state-control constructed in this step belong to an appropriate Banach space, which will be sufficient to proceed to Step 2.

Step 2. We apply a suitable inverse mapping theorem to recover the local (partial) null controllability of (1.5) from the (partial) null controllability of (1.7).

Remark 1.1 Note that system (1.7) is the linearization of system (1.5) around the trajectory $(M_1, 0, M_2, 0)$.

To obtain the (partial) null controllability for system (1.7) in Step 1, we show an estimate for the solutions of the adjoint system, namely,

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi\chi_2 + a\xi + g_1, & \text{in } Q, \\ \psi_t - \Delta\psi = -M_1 \Delta\eta + g_2, & \text{in } Q, \\ -\xi_t - \Delta\xi = -M_1 \Delta\varphi - b\xi + g_3, & \text{in } Q, \\ \eta_t - \Delta\eta = a\psi - b\eta + g_4, & \text{in } Q, \\ \partial_\nu\varphi = 0,\ \partial_\nu\psi = 0,\ \partial_\nu\xi = 0,\ \partial_\nu\eta = 0, & \text{on } \Sigma, \\ \varphi(\cdot,T) = 0,\ \psi(\cdot,0) = \psi_0,\ \xi(\cdot,T) = 0,\ \eta(\cdot,0) = \eta_0, & \text{in } \Omega. \end{cases} \tag{1.8}$$

with $\psi_0, \eta_0 \in L^2_0(\Omega)$[2] and the functions g_1, g_2, g_3 and g_4 having suitable regularity properties and such that $(g_2)_\Omega(t) = (g_4)_\Omega(t) = 0$ for a.e. $t \in (0, T)$. In particular, notice that, from $(1.8)_2$ and $(1.8)_4$, we have that $(\psi)_\Omega(t) = (\eta)_\Omega(t) = 0$, for $t \in [0, T]$.

Thus, the second main result of this paper will be a weighted estimate for (1.8) which has the following form:

$$\begin{aligned} &\iint_Q \rho_1^2(t)\Big(|\varphi - (\varphi)_\Omega|^2 + |\psi|^2 + |\xi|^2 + |\eta|^2\Big)\mathrm{d}x\,\mathrm{d}t \\ &\qquad\qquad \leq C\bigg(\iint_{\omega\times(0,T)} \rho_2^2(t)|\chi_1\xi|^2\mathrm{d}x\,\mathrm{d}t + \Big\|\rho_3(t)(g_1, g_2, g_3, g_4)\Big\|_Y^2\bigg), \end{aligned} \tag{1.9}$$

[2] $L^2_0(\Omega) := \{u \in L^2(\Omega) \mid (u)_\Omega = 0\}$.

where $\rho_k,\ k \in \{1, 2, 3\}$, are appropriate positive weight functions having exponential decay at $t = 0$ and $t = T$. The constant $C > 0$ only depends on Ω, ω, $\mathcal{O}$ and T, and Y is a suitable Banach space (see Proposition 3.1 for more details).

Remark 1.2 We can assume that $(g_2)_\Omega(t) = (g_4)_\Omega(t) = 0$ for a.e. $t \in (0, T)$ as consequence of the fact system (1.8) is the dual of (1.7), and that $(h_2)(t) = (h_4)(t) = 0$ for a.e. $t \in (0, T)$. We do not need to assume this for the mean of g_1 because, either way, from (1.8) we can not conclude that φ has average zero for a.e. $t \in (0, T)$.

This paper is divided as follows: In Sect. 2, we present some well-known estimates we need to obtain the (partial) null controllability for cascade system (1.5). Then, in Sect. 3, we prove a weighted estimate. This estimate is important for the rest of the article. In Sect. 4 we will prove the null controllability of linear system (1.7). Finally, in Sect. 5, we will prove the main result of this work, Theorem 2.1.

2 Some well-known estimates

In this section, we present some estimates that we will need to prove (1.9). These inequalities have been established in previous studies and we provided precise references where each one of them can be found. To state the estimates, we introduce several weight functions that will be useful in the sequel. The basic weight function, denoted by κ_0 belongs to $C^2(\overline{\Omega})$ and satisfies the following conditions:

$$\kappa_0(x) > 0 \text{ for } x \in \Omega, \quad \kappa_0 \equiv 0 \text{ on } \partial\Omega, \quad |\nabla\kappa_0(x)| > 0, \ \forall x \in \overline{\Omega \setminus \omega_0},$$

where $\omega_0 \Subset \tilde{\omega} \cap \mathcal{O}$ is a nonempty open set. The existence of such function κ_0 is proved in [26]. Then, for some $\lambda \in \mathbb{R}_+$ and $m \geq 4$, we introduce:

$$\begin{aligned}
\phi(x,t) &= \frac{e^{\lambda\kappa_0(x)}}{t^m(T-t)^m}, & \alpha(x,t) &= \frac{e^{\lambda\kappa_0(x)} - e^{2\lambda\|\kappa_0\|_\infty}}{t^m(T-t)^m},\\
\hat{\phi}(t) &= \min_{x\in\overline{\Omega}} \phi(x,t), & \phi^*(t) &= \max_{x\in\overline{\Omega}} \phi(x,t),\\
\hat{\alpha}(t) &= \min_{x\in\overline{\Omega}} \alpha(x,t), & \alpha^*(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t).
\end{aligned}$$

From the definition, it is not difficult to see that for any $n \in \mathbb{N}$ and every N-multi-index ι, there exists $C > 0$, depending only on $T,\ \Omega,\ \lambda$ and κ_0, such that

$$|\partial_t^n \partial_x^\iota \alpha| + |\partial_t^n \partial_x^\iota \phi| \leq C\phi^{n+|\iota|+n/m}. \tag{2.1}$$

Also, note that for every $a > 0$, and every $b, c \in \mathbb{R}$ the function $s^b e^{as\alpha}\xi^c$ is bounded in Q. Then, for any given $C > 0$, one has

$$s^b e^{as\alpha}\xi^c \leq C, \quad \text{for } s >> 1. \tag{2.2}$$

Throughout this article, we will use the following notation:

$$I_{M,\beta}(s,u) := s^{\beta+3}\iint\limits_{Q} e^{Ms\alpha}\phi^{\beta+3}|u|^2\mathrm{d}x\,\mathrm{d}t + s^{\beta+1}\iint\limits_{Q} e^{Ms\alpha}\phi^{\beta+1}|\nabla u|^2\mathrm{d}x\,\mathrm{d}t$$
$$+ s^{\beta-1}\iint\limits_{Q} e^{Ms\alpha}\phi^{\beta-1}\left(|u_t|^2 + \sum_{i,j=1}^{N}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2\right)\mathrm{d}x\,\mathrm{d}t,$$

where M, s and β are real numbers and $u = u(x,t)$.

The following result is a Carleman estimate for the heat equation with homogeneous Neumann boundary conditions. Its proof can be deduced by following the steps given in [26] for the case $M = 2$ and $\beta = 0$.

Lemma 2.1 *There exist positive constants $C = C(\Omega, \omega_0)$ and $\lambda_0 = \lambda_0(\Omega, \omega_0)$ such that, for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega, \omega_0, \lambda)$ for which, for any $s \geq s_0(T^m + T^{2m})$, any $u_0 \in L^2(\Omega)$ and any $f \in L^2(Q)$, the solution to*

$$\begin{cases} u_t - \Delta u = f, & \text{in } Q, \\ \partial_\nu u = 0, & \text{on } \Sigma, \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases}$$

satisfies

$$I_{8,\beta}(s,u) \leq C\left(s^{\beta+3}\iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^{\beta+3}|u|^2\mathrm{d}x\,\mathrm{d}t + s^{\beta}\iint\limits_{Q} e^{8s\alpha}\phi^{\beta}|f|^2\mathrm{d}x\,\mathrm{d}t\right),$$

for all $\beta \in \mathbb{R}$.

Next, we present a Carleman estimate for a coupled parabolic system with homogeneous Neumann boundary conditions. This result was obtained in Theorem 2.2 in [27].

Theorem 2.1 *There exist $C = C(\Omega, \omega_0)$ and $\lambda_0 = \lambda_0(\Omega, \omega_0)$ such that, for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega, \omega_0, \lambda)$ such that, for any $s \geq s_0(T^m + T^{2m})$, any $(u_0, v_0) \in L^2_0(\Omega) \times L^2(\Omega)$ and any $f_1, f_2 \in L^2(Q)$, the solution (u, v) of system*

$$\begin{cases} u_t - \Delta u = av + f_1, & \text{in } Q, \\ v_t - \Delta v = -bv - M_1\Delta u + f_2, & \text{in } Q, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \Sigma, \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0, & \text{in } \Omega, \end{cases}$$

satisfies

$$s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta u|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,v) \le C\left(s^{18}\iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|v|^2\mathrm{d}x\,\mathrm{d}t\right.$$
$$\left.+s^{10}\iint\limits_Q e^{9s\alpha}\phi^{10}|f_1|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_Q e^{9s\alpha}\phi^3|f_2|^2\mathrm{d}x\,\mathrm{d}t\right).$$

3 An estimate for the adjoint system

In this section, we are going to prove a weighted estimate for adjoint system (1.8), when g_1, g_2, g_3, g_4 and ψ_0, η_0 are in a suitable Banach spaces. More precisely, we prove the following result.

Proposition 3.1 *Assume that $\tilde{\omega}\cap\mathcal{O}\neq\emptyset$. Then, there exist positive constants $C = C(\lambda,\Omega,M_1,M_2,a,b,\omega,\omega_0)$ and $\lambda_0=\lambda_0(\Omega,\omega_0)$ such that, for every $\lambda\ge\lambda_0$, there exists $s_0=s_0(\Omega,\omega_0,\lambda)$ for which, for any $s\ge s_0$, for any $\psi_0,\eta_0\in L^2_0(\Omega)$, and any*

$$\begin{aligned}&g_1\in L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega)),\ g_2\in L^2(0,T;H^2(\Omega)),\\&g_3\in H^1(0,T;L^2(\Omega)),\ g_4\in L^2(0,T;H^4(\Omega)),\end{aligned}$$

with

$$\partial_\nu g_2=\partial_\nu g_4=\partial_\nu\Delta g_4=0,\ \textit{on}\ \Sigma,$$

and

$$(g_2)_\Omega(t)=(g_4)_\Omega(t)=0,\ \textit{for a.e. } t\in(0,T),$$

the solution (φ,ψ,ξ,η) of (1.8) satisfies

$$\begin{aligned}&I_{9,1}(s,\xi)+I_{9,1}(s,\xi_t)+I_{8,8}(s,\Delta\psi)+I_{8,6}(s,\Delta^2\eta)\\&\quad+s^3\iint\limits_Q e^{9s\hat\alpha}(\hat\phi)^3|\varphi-(\varphi)_\Omega|^2\mathrm{d}x\,\mathrm{d}t+s^{11}\iint\limits_Q e^{8s\hat\alpha}(\hat\phi)^{11}|\psi|^2\mathrm{d}x\,\mathrm{d}t\\&\quad+s^9\iint\limits_Q e^{8s\hat\alpha}(\hat\phi)^9|\eta|^2\mathrm{d}x\,\mathrm{d}t+s^3\iint\limits_Q e^{9s\alpha}\phi^3(|\Delta\varphi|^2+|\Delta\varphi_t|^2)\mathrm{d}x\,\mathrm{d}t\\&\le C\Bigg(s^{40}\iint\limits_{\tilde\omega\times(0,T)} e^{7s\alpha}\phi^{40}|\xi|^2\mathrm{d}x\,\mathrm{d}t+\left\|e^{7/2s\alpha^*}g_1\right\|^2_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))}\\&\quad+\left\|e^{7/2s\alpha^*}g_2\right\|^2_{L^2(0,T;H^2(\Omega))}+\left\|e^{4s\alpha^*}g_3\right\|^2_{H^1(0,T;L^2(\Omega))}\\&\quad+\left\|e^{7/2s\alpha^*}g_4\right\|^2_{L^2(0,T;H^4(\Omega))}\Bigg),\end{aligned}\tag{3.1}$$

for every $s \geq s_0$.

Proof We begin applying Theorem 2.1 to equations $(1.8)_1$ and $(1.8)_3$, as follows:

$$
\begin{aligned}
& s^3 \iint\limits_{Q} e^{9s\alpha}\phi^3|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) \\
& \quad \leq C\left(s^{18} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|\xi|^2 \mathrm{d}x\,\mathrm{d}t \right. \\
& \qquad + s^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}\phi^{10}|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\
& \qquad \left. + s^{10} \iint\limits_{Q} e^{9s\alpha}\phi^{10}|g_1|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_{Q} e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\,\mathrm{d}t \right).
\end{aligned}
$$

Next, we use equations $(1.8)_2$ and $(1.8)_4$ to obtain an estimate for ψ and η. Without loss of generality, we assume that $\psi_0,\ \eta_0$ are sufficiently regular. It is then straightforward to see that $\Delta\psi$ and $\Delta^2\eta$ satisfy the following two equations:

$$
\begin{cases}
(\Delta\psi)_t - \Delta^2\psi = -M_1\Delta^2\eta + \Delta g_2, & \text{in } Q, \\
\partial_\nu \Delta\psi = 0, & \text{on } \Sigma, \\
\Delta\psi(\cdot,0) = \Delta\psi_0, & \text{in } \Omega,
\end{cases}
\tag{3.2}
$$

and

$$
\begin{cases}
(\Delta^2\eta)_t - \Delta^3\eta = a\Delta^2\psi - b\Delta^2\eta + \Delta^2 g_4, & \text{in } Q, \\
\partial_\nu \Delta^2\eta = 0, & \text{on } \Sigma, \\
\Delta^2\eta(\cdot,0) = \Delta^2\eta_0, & \text{in } \Omega.
\end{cases}
\tag{3.3}
$$

Applying Lemma 2.1 to systems (3.2) and (3.3), with $\beta = 8$ and $\beta = 6$, respectively, we have

$$
\begin{aligned}
I_{8,8}(s,\Delta\psi) \leq C\Bigg(& s^{11} \iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^{11}|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^8 \iint\limits_{Q} e^{8s\alpha}\phi^8|\Delta^2\eta|^2 \mathrm{d}x\,\mathrm{d}t \\
& + s^8 \iint\limits_{Q} e^{8s\alpha}\phi^8|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t \Bigg),
\end{aligned}
\tag{3.4}
$$

and

$$I_{8,6}(s,\Delta^2\eta) \le C\Big(s^9 \iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^9|\Delta^2\eta|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint\limits_{Q} e^{8s\alpha}\phi^6|\Delta^2\psi|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint\limits_{Q} e^{8s\alpha}\phi^6|\Delta^2 g_4|^2\mathrm{d}x\,\mathrm{d}t\Big). \tag{3.5}$$

Combining estimates (3.4) and (3.5), we can absorb the second term on the right-hand side of (3.5) by the term on the left-hand side of (3.4), and absorb the second term on the right-hand side of (3.4) by the term on left-hand side of (3.5). This gives

$$\begin{aligned} &I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta)\\ &\quad\le C\left(s^{11}\iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^{11}|\Delta\psi|^2\mathrm{d}x\,\mathrm{d}t\right.\\ &\qquad + s^9\iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^9|\Delta^2\eta|^2\mathrm{d}x\,\mathrm{d}t\\ &\qquad \left. + s^8\iint\limits_{Q} e^{8s\alpha}\phi^8|\Delta g_2|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint\limits_{Q} e^{8s\alpha}\phi^6|\Delta^2 g_4|^2\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{3.6}$$

Next, we are going to absorb the local term of $\Delta^2\eta$ in (3.6) using the coupling between η and ψ in equation $(1.8)_2$. We set:

$$J^1 := s^9\iint\limits_{\omega_0\times(0,T)} e^{8s\alpha}\phi^9|\Delta^2\eta|^2\mathrm{d}x\,\mathrm{d}t.$$

Applying the Laplacian operator to equation $(1.8)_2$ yields

$$\Delta^2\eta = \frac{1}{M_1}(\Delta^2\psi - (\Delta\psi)_t + \Delta g_2), \text{ in } Q.$$

For an open subset ω_1 satisfying $\omega_0 \Subset \omega_1 \Subset \tilde{\omega} \cap \mathcal{O}_0$ and $\theta_1 \in C_c^\infty(\omega_1)$ such that $\theta_1 \equiv 1$ in ω_0 and $0 \le \theta_1 \le 1$, we then have

$$\begin{aligned}
J^1 &\le s^9 \iint\limits_{\omega_1 \times (0,T)} \theta_1 e^{8s\alpha} \phi^9 |\Delta^2 \eta|^2 \mathrm{d}x\, \mathrm{d}t \\
&= \frac{s^9}{M_1} \iint\limits_{\omega_1 \times (0,T)} \theta_1 e^{8s\alpha} \phi^9 \left(\Delta^2 \eta \Delta^2 \psi - \Delta^2 \eta (\Delta \psi)_t + \Delta^2 \eta \Delta g_2 \right) \mathrm{d}x\, \mathrm{d}t \\
&:= \frac{1}{M_1} \sum_{k=1}^{3} J_k^1.
\end{aligned}$$

Using integration by parts, and the properties of the weight functions given in (2.1), we have that

$$J_1^1 + J_2^1 \le \epsilon I_{8,6}(s, \Delta^2 \eta) + C(\epsilon) s^{13} \iint\limits_{\omega_1 \times (0,T)} e^{8s\alpha} \phi^{13} |\Delta \psi|^2 \mathrm{d}x\, \mathrm{d}t,$$

for every $s \ge s_0$ and any $\epsilon > 0$.

For J_3^1, we have that

$$J_3^1 \le \epsilon I_{8,6}(s, \Delta^2 \eta) + C(\epsilon) s^9 \iint\limits_{Q} e^{8s\alpha} \phi^9 |\Delta g_2|^2 \mathrm{d}x\, \mathrm{d}t,$$

for every $s \ge s_0$ and any $\epsilon > 0$. for a.e. $t \in (0, T)$. Since $(g_2)_\Omega(t) = 0$, from $(1.8)_2$ and $\partial_\nu \psi(t)|_{\partial\Omega} = 0$, one can deduce that $(\psi)_\Omega(t) = 0$ for a.e. $t \in (0, T)$, then, after integration by parts and applying Poincaré inequality yield

$$\int_\Omega |\psi(t)|^2 \mathrm{d}x \le C \int_\Omega |\Delta \psi(t)|^2 \mathrm{d}x, \quad \text{a.e. } t \in (0, T)$$

and, in particular,

$$s^{11} \iint\limits_{Q} e^{8s\hat{\alpha}} (\hat{\phi})^{11} |\psi|^2 \mathrm{d}x\, \mathrm{d}t \le C s^{11} \iint\limits_{Q} e^{8s\hat{\alpha}} (\hat{\phi})^{11} |\Delta \psi(t)|^2 \mathrm{d}x\, \mathrm{d}t. \tag{3.7}$$

Combining the estimates to J_k^1, for $j = 1, 2, 3,$, together with (3.6) and (3.7), and taking ϵ sufficiently small, we get

$$\begin{aligned} & s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta) \\ & + s^{11} \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2 \mathrm{d}x\,\mathrm{d}t \le C\Bigg(s^{18} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|\xi|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}\phi^{10}|\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^{13} \iint\limits_{\omega_1\times(0,T)} e^{8s\alpha}\phi^{13}|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|g_1|^2 \mathrm{d}x\,\mathrm{d}t + s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\,\mathrm{d}t + s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\,\mathrm{d}t \Bigg), \end{aligned} \tag{3.8}$$

for every $s \ge s_0$.

Using an integration by parts, Poincaré inequality, and the properties of the weight functions, the following estimate holds

$$\begin{aligned} s^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}\phi^{10}|\psi|^2 \mathrm{d}x\,\mathrm{d}t & \le Cs^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha^*}(\phi^*)^{10}|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ & \le Cs^{10} \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^{10}|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.9}$$

for every $s \ge s_0$. Therefore, using (3.9) in (3.8), we estimate the local term associated with ψ on the right-hand side by the third term of the left-hand side. Thus, for $s \ge s_0$, we have:

$$\begin{aligned} & s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta) \\ & + s^{11} \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2 \mathrm{d}x\,\mathrm{d}t \le C\Bigg(s^{18} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|\xi|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^{13} \iint\limits_{\omega_1\times(0,T)} e^{8s\alpha}\phi^{13}|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|g_1|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$+s^3 \iint\limits_Q e^{9s\alpha}\phi^3|g_3|^2\mathrm{d}x\,\mathrm{d}t + s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2\mathrm{d}x\,\mathrm{d}t\Bigg). \tag{3.10}$$

Now, we are going to absorb the local term in $\Delta\psi$ that appears on the right-hand side of (3.10). To do this, we apply the Laplace operator to equation $(1.8)_1$ within $\mathcal{O}$, in particular $\mathcal{O}_0$, to see that

$$\Delta\psi = -(\Delta\varphi)_t - \Delta^2\varphi - a\Delta\xi - \Delta g_1, \quad \text{in } \mathcal{O}_0 \times (0,T).$$

We also consider an open subset ω_2 satisfying $\omega_0 \Subset \omega_1 \Subset \omega_2 \Subset \tilde{\omega} \cap \mathcal{O}_0$ and a non-negative function $\theta_2 \in C_c^\infty(\omega_2)$ such that $\theta_2 \equiv 1$ in ω_1. It is straightforward that

$$\begin{aligned}
&s^{13} \iint\limits_{\omega_1\times(0,T)} e^{8s\alpha}\phi^{13}|\Delta\psi|^2\mathrm{d}x\,\mathrm{d}t \le\; s^{13} \iint\limits_{\omega_2\times(0,T)} \theta_2 e^{8s\alpha}\phi^{13}|\Delta\psi|^2\mathrm{d}x\,\mathrm{d}t\\
&= s^{13} \iint\limits_{\omega_2\times(0,T)} \theta_2 e^{8s\alpha}\phi^{13}\Big(-\Delta\psi(\Delta\varphi)_t - \Delta\psi\Delta^2\varphi - a\Delta\psi\Delta\xi - \Delta\psi\Delta g_1\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&:= \sum_{k=1}^{4} J_k^2.
\end{aligned}$$

Estimating each one of the terms J_k^2, for $k = 1, 2, 3, 4$, yield

$$\begin{aligned}
J_1^2 + J_2^2 &\le \epsilon I_{8,8}(s,\Delta\psi) + C(\epsilon)s^{19} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t,\\
J_3^2 &\le \epsilon I_{8,8}(s,\Delta\psi) + C(\epsilon)s^{19} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\xi|^2\mathrm{d}x\,\mathrm{d}t,\\
J_4^2 &\le \epsilon I_{8,8}(s,\Delta\psi) + C(\epsilon)s^{19} \iint\limits_{Q} e^{8s\alpha}\phi^{19}|g_1|^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

for every $s \ge s_0$ and any $\epsilon > 0$.

Taking ϵ small enough, from (3.10) and the estimates for J_k^2 for $k = 1, 2, 3, 4$, we obtain

$$\begin{aligned}
&s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta)\\
&+s^{11} \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2\mathrm{d}x\,\mathrm{d}t \le C\Bigg(s^{19} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\xi|^2\mathrm{d}x\,\mathrm{d}t\\
&+s^{19} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t + s^{19} \iint\limits_Q e^{8s\alpha}\phi^{19}|g_1|^2\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$+s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\,\mathrm{d}t$$

$$+s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\,\mathrm{d}t\Bigg), \tag{3.11}$$

for every $s \geq s_0$.

The rest of the proof consists in estimating the local term in $\Delta\varphi$ on the right-hand side of (3.11). Before, we apply the operator $\partial_t(\cdot)$ to equations $(1.8)_1$ and $(1.8)_3$, to get the following system:

$$\begin{cases} -(\varphi_t)_t - \Delta\varphi_t = \psi_t\chi_2 + a\xi_t + (g_1)_t, & \text{in } Q, \\ -(\xi_t)_t - \Delta\xi_t = -M_1\Delta\varphi_t - b\xi_t + (g_3)_t, & \text{in } Q, \\ \partial_\nu\varphi_t = 0,\ \partial_\nu\xi_t = 0, & \text{on } \Sigma, \\ \varphi_t(\cdot, T) = 0,\ \xi_t(\cdot, T) = 0, & \text{in } \Omega. \end{cases} \tag{3.12}$$

Applying Theorem 2.1 to system (3.12), we get:

$$s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi_t|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi_t) \leq C\left(s^{18} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|\xi_t|^2 \mathrm{d}x\,\mathrm{d}t\right.$$

$$+s^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}\phi^{10}|\psi_t|^2 \mathrm{d}x\,\mathrm{d}t + s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|(g_1)_t|^2 \mathrm{d}x\,\mathrm{d}t$$

$$\left.+s^3 \iint\limits_Q e^{9s\alpha}\phi^3|(g_3)_t|^2 \mathrm{d}x\,\mathrm{d}t\right). \tag{3.13}$$

Using integration by parts, we estimate the local term in ξ_t appearing on the right-hand side of (3.13) as follows:

$$\begin{aligned} &s^{18} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{18}|\xi_t|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leq \epsilon\left(I_{9,1}(s,\xi) + I_{9,1}(s,\xi_t)\right) + C(\epsilon)s^{38+2/m} \iint\limits_{\omega_0\times(0,T)} e^{9s\alpha}\phi^{38+2/m}|\xi|^2 \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.14}$$

for every $s \geq s_0$ and any $\epsilon > 0$.

Using again Poincaré inequality and integrating by parts, we estimate the term in ψ_t on the right-hand side of (3.13):

$$s^{10} \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}\phi^{10}|\psi_t|^2 \mathrm{d}x\,\mathrm{d}t \leq Cs^{10} \iint\limits_Q e^{9s\alpha^*}(\phi^*)^{10}|\Delta\psi_t|^2 \mathrm{d}x\,\mathrm{d}t. \tag{3.15}$$

From (3.11), (3.13), (3.14) and (3.15), we see that:

$$
\begin{aligned}
& s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta) + I_{9,1}(s,\xi_t) \\
& \quad + s^{11} \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi_t|^2 \mathrm{d}x\,\mathrm{d}t \\
& \le C\Bigg(s^{38+2/m} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{38+2/m}|\xi|^2 \mathrm{d}x\,\mathrm{d}t + s^{19} \iint\limits_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t \\
& \quad + s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t + s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\,\mathrm{d}t \\
& \quad + s^{19} \iint\limits_Q e^{8s\alpha}\phi^{19}|g_1|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\,\mathrm{d}t \\
& \quad + s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|(g_1)_t|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|(g_3)_t|^2 \mathrm{d}x\,\mathrm{d}t \Bigg),
\end{aligned} \tag{3.16}
$$

for every $s \ge s_0$.

Considering equation $(1.8)_1$ in $\mathcal{O}_0$, and applying the operator $\Delta(\cdot)$ to it, yields:

$$
\Delta^2\varphi = -\Delta\varphi_t - \Delta\psi - a\Delta\xi - \Delta g_1, \quad \text{in } \mathcal{O}_0 \times (0,T).
$$

Therefore, it follows that, for $s \ge s_0$, the following inequality holds:

$$
\begin{aligned}
& \iint\limits_{\mathcal{O}_0\times(0,T)} e^{9s\alpha}|\Delta^2\varphi|^2 \mathrm{d}x \\
& \quad \le C\Bigg(s^3 \iint\limits_Q e^{9s\alpha}\phi^3|\Delta\varphi_t|^2 \mathrm{d}x\,\mathrm{d}t + I_{8,8}(s,\Delta\psi) + I_{9,1}(s,\xi) \Bigg) \\
& \quad\quad + \iint\limits_{\mathcal{O}_0\times(0,T)} e^{9s\alpha}|\Delta g_1|^2 \mathrm{d}x.
\end{aligned} \tag{3.17}
$$

Hence, we can add a term of $\Delta^2\varphi$ to the left-hand side of (3.16), namely:

$$\begin{aligned}
&\iint_{\mathcal{O}_0\times(0,T)} e^{9s\alpha}|\Delta^2\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta) + I_{9,1}(s,\xi_t)\\
&+ s^{11}\iint_Q e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^3\iint_Q e^{9s\alpha}\phi^3(|\Delta\varphi_t|^2+|\Delta\varphi|^2)\mathrm{d}x\,\mathrm{d}t\\
&\le C\left(s^{38+2/m}\iint_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{38+2/m}|\xi|^2 \mathrm{d}x\,\mathrm{d}t + s^{19}\iint_{\omega_2\times(0,T)} e^{8s\alpha}\phi^{19}|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t\right.\\
&+ s^{19}\iint_Q e^{8s\alpha}\phi^{19}|g_1|^2 \mathrm{d}x\,\mathrm{d}t + s^{10}\iint_Q e^{9s\alpha}\phi^{10}|(g_1)_t|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\alpha}|\Delta g_1|^2 \mathrm{d}x\,\mathrm{d}t\\
&+s^9\iint_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\,\mathrm{d}t + s^3\iint_Q e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\,\mathrm{d}t\\
&\left.+s^3\iint_Q e^{9s\alpha}\phi^3|(g_3)_t|^2 \mathrm{d}x\,\mathrm{d}t + s^6\iint_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\,\mathrm{d}t\right),
\end{aligned} \tag{3.18}$$

for every $s \ge s_0$.

Now, we are able to address the local term associated with $\Delta\varphi$ on the right-hand side of (3.18). For this, we use $(1.8)_3$, to see that:

$$\Delta\varphi = \frac{1}{M_1}(\xi_t - b\xi + g_3 + \Delta\xi), \quad \text{in } Q.$$

Let ω_3 be an open subset satisfying $\omega_0 \Subset \omega_1 \Subset \omega_2 \Subset \omega_3 \Subset \tilde{\omega}\cap\mathcal{O}_0 \Subset \Omega$ and let $\theta_3 \in C_c^\infty(\omega_3)$ be a non-negative function such that $\theta_3 \equiv 1$ in ω_2 and $0 \le \theta_3 \le 1$. As we have done before, we write:

$$\begin{aligned}
s^{19}\iint_{\omega_3\times(0,T)} \theta_3^2 e^{8s\alpha}\phi^{19}|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t &= \frac{s^{19}}{M_1}\iint_{\omega_3\times(0,T)} \theta_3^2 e^{8s\alpha}\phi^{19}\Delta\varphi\,(\xi_t - b\xi + g_3 + \Delta\xi)\,\mathrm{d}x\,\mathrm{d}t\\
&:= \sum_{k=1}^4 J_k^3.
\end{aligned}$$

For the first three terms, one has the estimate:

$$J_1^3 + J_2^3 + J_3^3 \le \epsilon\left(s^3\iint_{\omega_3\times(0,T)} e^{9s\alpha}\phi^3|\Delta\varphi|^2 \mathrm{d}x\,\mathrm{d}t s^3\iint_{\omega_3\times(0,T)} e^{9s\alpha}\phi^3|\Delta\varphi_t|^2 \mathrm{d}x\,\mathrm{d}t\right)$$

$$+ C(\epsilon)\left(s^{35}\iint\limits_{Q} e^{7s\alpha}\phi^{35}|g_3|^2\mathrm{d}x\,\mathrm{d}t + s^{37+2/m}\iint\limits_{\omega_3\times(0,T)} e^{7s\alpha}\phi^{37+2/m}|\xi|^2\mathrm{d}x\,\mathrm{d}t\right),$$

for every $s \geq s_0$ and any $\epsilon > 0$. We treat J_4^3 as follows:

$$\begin{aligned}
J_4^3 =&\frac{s^{19}}{M_1}\iint\limits_{\omega_3\times(0,T)} \theta_3^2 e^{8s\alpha}\phi^{19}\Delta\varphi\Delta\xi\mathrm{d}x\,\mathrm{d}t\\
=&\frac{s^{19}}{M_1}\iint\limits_{\omega_3\times(0,T)} [\Delta(\theta_3^2 e^{8s\alpha}\phi^{19})\Delta\varphi + 2\nabla(\theta_3^2 e^{8s\alpha}\phi^{19})\nabla\Delta\varphi + \theta_3^2 e^{8s\alpha}\phi^{19}\Delta^2\varphi]\xi\mathrm{d}x\,\mathrm{d}t\\
\leq&\, \epsilon\left(\iint\limits_{\omega_3\times(0,T)} \theta_3^2 e^{9s\alpha}|\nabla\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t + \iint\limits_{\mathcal{O}\times(0,T)} e^{9s\alpha}|\Delta^2\varphi|^2\mathrm{d}x\,\mathrm{d}t\right.\\
&\left.+s^3\iint\limits_{Q} e^{9s\alpha}\phi^3|\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t\right) + C(\epsilon)s^{40}\iint\limits_{\omega_3\times(0,T)} e^{7s\alpha}\phi^{40}|\xi|^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{3.19}$$

for every $s \geq s_0$ and any $\epsilon > 0$.

Finally, we observe that after a integration by parts and applying Young's inequality, we will estimate the first term on the right-hand side of (3.19) namely, $\nabla\Delta\varphi$, as follows

$$\begin{aligned}
\iint\limits_{\omega_3\times(0,T)} \theta_3^2 e^{9s\alpha}|\nabla\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t \leq&\ \frac{1}{2}\iint\limits_{\omega_3\times(0,T)} \theta_3^2 e^{9s\alpha}|\nabla\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t\\
&+ C\left(s^{-1}\iint\limits_{\omega_3\times(0,T)} e^{9s\alpha}\phi^{-1}|\Delta^2\varphi|^2\mathrm{d}x\,\mathrm{d}t + s^2\iint\limits_{\omega_3\times(0,T)} e^{9s\alpha}\phi^2|\Delta\varphi|^2\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}\tag{3.20}$$

for every $s \geq s_0$.

Replacing (3.20) in (3.19) and combining with (3.18), to then take ϵ small enough in the estimates of J_k^3, $k \in \{1, 2, 3, 4\}$, we can absorb the desired local term on $\Delta\varphi$. Then, from (3.18), one has:

$$\begin{aligned}
&\iint\limits_{\mathcal{O}_0\times(0,T)} e^{9s\alpha}|\Delta^2\varphi|^2\mathrm{d}x\,\mathrm{d}t + I_{9,1}(s,\xi) + I_{8,8}(s,\Delta\psi) + I_{8,6}(s,\Delta^2\eta) + I_{9,1}(s,\xi_t)\\
&+s^{11}\iint\limits_{Q} e^{8s\hat{\alpha}}(\hat{\phi})^{11}|\psi|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_{Q} e^{9s\alpha}\phi^3(|\Delta\varphi|^2 + |\Delta\varphi_t|^2)\mathrm{d}x\,\mathrm{d}t\\
&\leq\ C\left(s^{40}\iint\limits_{\omega_3\times(0,T)} e^{7s\alpha}\phi^{40}|\xi|^2\mathrm{d}x\,\mathrm{d}t + s^{19}\iint\limits_{Q} e^{8s\alpha}\phi^{19}|g_1|^2\mathrm{d}x\,\mathrm{d}t\right.
\end{aligned}$$

$$+ s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|(g_1)_t|^2 \mathrm{d}x\, \mathrm{d}t + \iint\limits_Q e^{9s\alpha}|\Delta g_1|^2 \mathrm{d}x\, \mathrm{d}t + s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\, \mathrm{d}t$$

$$+ s^3 \iint\limits_Q e^{9s\alpha}\phi^3|g_3|^2 \mathrm{d}x\, \mathrm{d}t + s^3 \iint\limits_Q e^{9s\alpha}\phi^3|(g_3)_t|^2 \mathrm{d}x\, \mathrm{d}t + s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\, \mathrm{d}t \Bigg), \tag{3.21}$$

for every $s \geq s_0$.

Now, using (2.2), we see that

$$s^{19} \iint\limits_Q e^{8s\alpha}\phi^{19}|g_1|^2 \mathrm{d}x\, \mathrm{d}t + s^{10} \iint\limits_Q e^{9s\alpha}\phi^{10}|(g_1)_t|^2 \mathrm{d}x\, \mathrm{d}t + \iint\limits_Q e^{9s\alpha}|\Delta g_1|^2 \mathrm{d}x\, \mathrm{d}t$$
$$\leq C \left\| e^{7/2s\alpha^*} g_1 \right\|^2_{L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))},$$

$$s^9 \iint\limits_Q e^{8s\alpha}\phi^9|\Delta g_2|^2 \mathrm{d}x\, \mathrm{d}t \leq C \left\| e^{7/2s\alpha^*} g_2 \right\|^2_{L^2(0,T;H^2(\Omega))},$$

$$s^3 \iint\limits_Q e^{9s\alpha}\phi^3 \left(|g_3|^2 + |(g_3)_t|^2 \right) \mathrm{d}x\, \mathrm{d}t \leq C \left\| e^{4s\alpha^*} g_3 \right\|^2_{H^1(0,T;L^2(\Omega))},$$

$$s^6 \iint\limits_Q e^{8s\alpha}\phi^6|\Delta^2 g_4|^2 \mathrm{d}x\, \mathrm{d}t \leq C \left\| e^{7/2s\alpha^*} g_4 \right\|^2_{L^2(0,T;H^4(\Omega))},$$

for every $s \geq s_0$.

Finally, we add a global term of η in the left-hand side of (3.21). Indeed, from equation $(1.8)_4$, we have that $(\eta)_\Omega(t) = (\Delta\eta)_\Omega(t) = 0$ for a.e. $t \in (0, T)$ and $\partial_\nu \Delta\eta|_\Sigma = \partial_\nu \eta|_\Sigma = 0$. Thus, by Poincaré's inequality and integration by parts, there exists $C > 0$ such that

$$\int_\Omega |\eta(t)|^2 \mathrm{d}x \leq C \int_\Omega |\Delta\eta(t)|^2 \mathrm{d}x \leq C \int_\Omega |\Delta^2\eta(t)|^2 \mathrm{d}x, \qquad \forall t \in (0, T).$$

Now, using the fact that the weight functions $\hat{\alpha}$ and $\hat{\phi}$ only depend on t, we obtain that

$$s^9 \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^9|\eta|^2 \mathrm{d}x\, \mathrm{d}t \leq C s^9 \iint\limits_Q e^{8s\hat{\alpha}}(\hat{\phi})^9|\Delta^2\eta|^2 \mathrm{d}x\, \mathrm{d}t.$$

Similarly, using the term in $\Delta\varphi$, we can add the global term of $[\varphi - (\varphi)_\Omega]$ in (3.21).

This concludes the proof of Proposition 3.1. □

In the following result, we assume more regularity on the external forces and uniformize the weight functions in Proposition 3.1. This will allow us to work more effectively in the next section.

Let us fix the notation:

$$
\begin{aligned}
I(s;\varphi,\psi,\xi,\eta) := & \iint_Q e^{9s\hat{\alpha}}|\varphi-(\varphi)_\Omega|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\hat{\alpha}}|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\
& + \iint_Q e^{9s\hat{\alpha}}|\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\hat{\alpha}}|\xi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\hat{\alpha}}|\nabla\xi|^2 \mathrm{d}x\,\mathrm{d}t \\
& + \iint_Q e^{9s\hat{\alpha}}|\eta|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Corollary 3.1 *Let s and λ be as in Proposition 3.1. Assume that*

$$
\begin{aligned}
& g_1 \in L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega)), \quad g_2 \in L^2(0,T;H^3(\Omega)) \cap H^{3/2}(0,T;L^2(\Omega)), \\
& g_3 \in L^2(0,T;H^3(\Omega)) \cap H^{3/2}(0,T;L^2(\Omega)), \quad g_4 \in L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega)), \\
& \partial_\nu g_2 = \partial_\nu g_4 = \partial_\nu \Delta g_4 = 0, \quad \text{on } \Sigma, \\
& (g_2)_\Omega(t) = (g_4)_\Omega(t) = 0, \quad \text{for all } t \in (0,T).
\end{aligned}
$$

Then, there exists a positive constant C depending on T, s and λ, such that every solution of (1.8) verifies:

$$
\begin{aligned}
I(s;\varphi,\psi,\xi,\eta) \le\ & C\Bigg(\iint_{\omega\times(0,T)} e^{13/2s\hat{\alpha}}|\chi_1|^2|\xi|^2 \mathrm{d}x\,\mathrm{d}t \\
& + \left\| e^{13/4s\hat{\alpha}} g_1 \right\|^2_{L^2(0,T;H^4(\Omega))\cap H^2(0,T;L^2(\Omega))} + \left\| e^{13/4s\hat{\alpha}} g_2 \right\|^2_{L^2(0,T;H^3(\Omega))\cap H^{3/2}(0,T;L^2(\Omega))} \\
& + \left\| e^{13/4s\hat{\alpha}} g_3 \right\|^2_{L^2(0,T;H^3(\Omega))\cap H^{3/2}(0,T;L^2(\Omega))} + \left\| e^{13/4s\hat{\alpha}} g_4 \right\|^2_{L^2(0,T;H^4(\Omega)\cap H^2(0,T;L^2(\Omega))} \Bigg).
\end{aligned}
$$

The proof of Corollary 3.1 follows directly from estimate (3.1), and the properties of the weight functions.

4 Null controllability of the linear system

In this section we solve the null controllability problem for system (1.7) with a right-hand side which decays exponentially as $t \to 0^+$. This result will be used to prove the local controllability of (1.7) in the next section. Indeed, we would like to find a control g such that the solution to

$$\begin{cases} \mathcal{L}(w,r) = \big(-M_1\Delta r + F + h_1,\ aw + g\chi_1 + G + h_3\big), & \text{in } Q, \\ \mathcal{T}(z,q) = \big(-M_1\Delta z + h_4,\ aq + w\chi_2 - (w\chi_2)_\Omega + h_2\big), & \text{in } Q, \\ \partial_\nu w = \partial_\nu z = \partial_\nu r = \partial_\nu q = 0, & \text{on } \Sigma, \\ w(\cdot,0) = 0,\ z(\cdot,T) = 0,\ r(\cdot,0) = 0,\ q(\cdot,T) = 0, & \text{in } \Omega, \end{cases} \tag{4.1}$$

satisfies

$$z(\cdot,0) = q(\cdot,0) = 0, \text{ in } \Omega. \tag{4.2}$$

Here,

$$\mathcal{L}(w,r) := \big(\mathcal{L}_1 w,\ \mathcal{L}_2 r\big) \text{ and } \mathcal{T}(z,q) := \big(\mathcal{L}_2^* q,\ \mathcal{L}_1^* z\big),$$

where

$$\mathcal{L}_1 := \partial_t - \Delta, \quad \mathcal{L}_2 := \partial_t - \Delta + b,$$

and $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ are the adjoint operators of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, namely:

$$\mathcal{L}_1^* := -\partial_t - \Delta, \quad \mathcal{L}_2^* := -\partial_t - \Delta + b.$$

We prove the existence of a solution of the previous problem in an appropriate weighted space. Before introducing the spaces where we solve problem (4.1)–(4.2), we improve the estimate obtained in the previous section. This new estimate will contain only weight functions that do not vanish at $t = T$. In order to introduce these weights, let us consider a function $l \in C^\infty([0,T])$ such that

$$l(t) = \begin{cases} t(T-t), & 0 \le t \le T/4, \\ T^2/4, & 3T/4 < t \le T, \end{cases}$$

and define our new weight functions to be

$$\begin{aligned} \beta(x,t) &= \frac{e^{\lambda\eta^0(x)} - e^{2\lambda\|\eta_0\|_\infty}}{l^m(t)}, & \gamma(x,t) &= \frac{e^{\lambda\eta_0(x)}}{l^m(t)}, \\ \hat{\gamma}(t) &= \min_{x\in\overline{\Omega}} \gamma(x,t), & \gamma^*(t) &= \max_{x\in\overline{\Omega}} \gamma(x,t), \\ \beta^*(t) &= \max_{x\in\overline{\Omega}} \beta(x,t), & \hat{\beta}(t) &= \min_{x\in\overline{\Omega}} \beta(x,t). \end{aligned}$$

We also introduce some notation. Denoting by $Y_0 := L^2(Q)$, $X_0 := L^2(0,T;H^1(\Omega))$ and, for $n \in \mathbb{N}$, we define the spaces Y_n and X_n as follows:

$$Y_n := L^2(0,T;H^{2n}(\Omega)) \cap H^n(0,T;L^2(\Omega)),$$

and

$$X_n := L^2(0,T;H^{2n+1}(\Omega)) \cap H^{\frac{2n+1}{2}}(0,T;L^2(\Omega)),$$

given by the norms

$$\|u\|_{Y_n}^2 := \|u\|_{L^2(0,T;H^{2n}(\Omega))}^2 + \|u\|_{H^n(0,T;L^2(\Omega))}^2,$$

and

$$\|u\|^2_{X_n} := \|u\|^2_{L^2(0,T;H^{2n+1}(\Omega))} + \|u\|^2_{H^{\frac{2n+1}{2}}(0,T;L^2(\Omega))}.$$

We also set

$$Y^+_{n,0} := \{u \in Y_n : \partial_\nu[\mathcal{L}^k_i u]|_\Sigma = 0, \ [\mathcal{L}^k_i u](\cdot, 0) = 0, \ k = 0, \dots, n-1, \ i = 1, 2\},$$

$$Y^-_{n,0} := \{u \in Y_n : \partial_\nu[\mathcal{L}^k_i u]|_\Sigma = 0, \ [\mathcal{L}^k_i u](\cdot, T) = 0, \ k = 0, \dots, n-1, \ i = 1, 2\},$$

and

$$X^+_{n,0} := \{u \in X_n : \partial_\nu[\mathcal{L}^k_i u]|_\Sigma = 0, \ [\mathcal{L}^k_i u](\cdot, 0) = 0, \ k = 0, \dots, n-1, \ i = 1, 2\},$$

$$X^-_{n,0} := \{u \in X_n : \partial_\nu[\mathcal{L}^k_i u]|_\Sigma = 0, \ [\mathcal{L}^k_i u](\cdot, T) = 0, \ k = 0, \dots, n-1, \ i = 1, 2\},$$

endowed with the equivalent norms

$$\|u\|^2_{Y^+_{n,0}} = \|u\|^2_{Y^-_{n,0}} = \|u\|^2_{X^+_{n,0}} = \|u\|^2_{X^-_{n,0}} := \|\mathcal{L}^n_1 u\|^2_{L^2(Q)}.$$

We now state a refined estimate.

Lemma 4.1 *Let s and λ be like in Proposition 3.1 and assume that*

$$\begin{aligned}
&g_1 \in Y_2 : e^{13/4s\hat\beta} g_1 \in Y^-_{2,0};\\
&g_2 \in X_2 : e^{13/4s\hat\beta} g_2 \in X^+_{2,0}, \ (g_2)_\Omega(t) = 0, \quad \forall t \in [0, T];\\
&g_3 \in X_2 : e^{13/4s\hat\beta} g_3 \in X^-_{2,0};\\
&g_4 \in Y_2 : e^{13/4s\hat\beta} g_4 \in Y^+_{2,0}, \ (g_4)_\Omega(t) = 0, \ \forall t \in [0, T].
\end{aligned}$$

Then, there exists a positive constant C depending on T, s and λ, such that every solution of (1.8) verifies:

$$\begin{aligned}
&\iint_Q e^{9s\hat\beta}|\varphi - (\varphi)_\Omega|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\hat\beta}(|\psi|^2 + |\nabla\psi|^2)\mathrm{d}x\,\mathrm{d}t\\
&\quad + \iint_Q e^{9s\hat\beta}(|\xi|^2 + |\nabla\xi|^2)\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{9s\hat\beta}|\eta|^2 \mathrm{d}x\,\mathrm{d}t\\
&\leq C\Bigg(\iint_{\omega\times(0,T)} e^{13/2s\hat\beta}|\chi_1|^2|\xi|^2 \mathrm{d}x\,\mathrm{d}t + \left\| e^{13/4s\hat\beta} g_1\right\|^2_{Y^-_{2,0}} + \left\| e^{13/4s\hat\beta} g_2\right\|^2_{X^+_{2,0}}\\
&\quad + \left\| e^{13/4s\hat\beta} g_3\right\|^2_{X^-_{2,0}} + \left\| e^{13/4s\hat\beta} g_4\right\|^2_{Y^+_{2,0}}\Bigg), \qquad (4.3)
\end{aligned}$$

where

$$(\varphi)_\Omega(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)\mathrm{d}x.$$

Proof The proof of this lemma is standard. It combines energy estimates, using a cut-off function, with Corollary 3.1 together with the fact that $\alpha \leq \beta$ in Q. A similar proof is given by Lemma 1 in [28]. □

Now, we proceed to the definition of the spaces where (4.1)–(4.2) will be solved. The main space will be:

$$
\begin{aligned}
E = \Bigg\{ & (w, z, r, q, v) \in E_0 : \ \partial_\nu w = \partial_\nu z = \partial_\nu r = \partial_\nu q = 0 \text{ on } \Sigma, \\
& e^{-9/2s\hat{\beta}} \left((\mathcal{L}(w, r))_1 + M_1 \Delta r \right) \in L^2(Q), \\
& e^{-9/2s\hat{\beta}} \left((\mathcal{L}(w, r))_2 - aw - g\chi_1 \right) \in L^2(0, T; H^1(\Omega)), \\
& e^{-9/2s\hat{\beta}} \left((\mathcal{T}(z, q))_2 - aq - w\chi_2 + (w\chi_2)_\Omega \right) \in L^2(0, T; H^1(\Omega)), \\
& e^{-9/2s\hat{\beta}} \left((\mathcal{T}(z, q))_1 + M_1 \Delta z \right) \in L^2(Q), \\
& \int_\Omega \left((\mathcal{L}(w, r))_1 + M_1 \Delta r \right) \mathrm{d}x = 0, \ \int_\Omega \Big(\big(\mathcal{T}(z, q)\big)_1 + M_1 \Delta z \Big) \mathrm{d}x = 0, \\
& \int_\Omega \left((\mathcal{T}(w, r))_2 - aq - w\chi_2 + (w\chi_2)_\Omega \right) \mathrm{d}x = 0 \Bigg\},
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
E_0 = \big\{ & (w, z, r, q, g) : \ e^{-13/4s\hat{\beta}} (w, r) \in [L^2 \left(0, T; H^2(\Omega)\right) \cap L^\infty \left(0, T; H^1(\Omega)\right)]^2, \\
& e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-5-5/m} (z, q) \in [L^2(Q)]^2, \ e^{-13/4s\hat{\beta}} g\chi_1 \in L^2(Q), \\
& e^{-2s\hat{\beta}} r \in L^2 \left(0, T; H^3(\Omega)\right) \cap L^\infty \left(0, T; H^2(\Omega)\right), \\
& e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-6-6/m} q \in L^2 \left(0, T; H^2(\Omega)\right) \cap L^\infty \left(0, T; H^1(\Omega)\right), \\
& e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-7-7/m} z \in L^2 \left(0, T; H^3(\Omega)\right) \cap L^\infty \left(0, T; H^2(\Omega)\right), \\
& e^{-2s\hat{\beta}} g\chi_1 \in L^2 \left(0, T; H^1(\Omega)\right) \big\}.
\end{aligned}
$$

Notice that E is a Banach space for the norm:

$$
\begin{aligned}
\|(w, z, r, q, g)\|_E^2 := & \left\| e^{-13/4s\hat{\beta}} (w, r) \right\|^2_{[L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))]^2} \\
& + \left\| e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-5-5/m} (z, q) \right\|^2_{[L^2(Q)]^2} \\
& + \left\| e^{-13/4s\hat{\beta}} g\chi_1 \right\|^2_{L^2(Q)} + \left\| e^{-2s\hat{\beta}} r \right\|^2_{L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega))} \\
& + \left\| e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-6-6/m} q \right\|^2_{L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))} \\
& + \left\| e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-7-7/m} z \right\|^2_{L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega))}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\| e^{-9/2s\hat{\beta}} \left((\mathcal{L}(w,r))_1 + M_1 \Delta r \right) \right\|^2_{L^2(Q)} \\
&+ \left\| e^{-9/2s\hat{\beta}} \left((\mathcal{L}(w,r))_2 - aw - g\chi_1 \right) \right\|^2_{L^2(0,T;H^1(\Omega))} \\
&+ \left\| e^{-9/2s\hat{\beta}} \left((\mathcal{T}(z,q))_2 - aq - w\chi_2 + (w\chi_2)_\Omega \right) \right\|^2_{L^2(0,T;H^1(\Omega))} \\
&+ \left\| e^{-9/2s\hat{\beta}} \left((\mathcal{T}(z,q))_1 + M_1 \Delta z \right) \right\|^2_{L^2(Q)} \\
&+ \left\| e^{-2s\hat{\beta}} g\chi_1 \right\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
$$

Proposition 4.1 *Let $M_1, M_2 \in \mathbb{R}_+$ be such that $aM_1 - bM_2 = 0$. Assume that $e^{-9/2s\hat{\beta}}(F, G) \in L^2_0(Q) \times L^2(0, T; H^1(\Omega))$, and consider that*

$$
\begin{aligned}
&e^{-9/2s\hat{\beta}}\,(h_1, h_3) \in L^2(0, T; L^2_0(\Omega)) \times L^2(0, T; H^1(\Omega)),\\
&e^{-9/2s\hat{\beta}}\,(h_2, h_4) \in L^2(0, T; L^2_0(\Omega) \cap H^1(\Omega)) \times L^2(0, T; L^2_0(\Omega)).
\end{aligned}
\tag{4.5}
$$

Then, there exists a control $\hat{g} \in L^2(0, T; H^1(\Omega))$, such that, if $(\hat{w}, \hat{z}, \hat{r}, \hat{q})$ is the associated solution to (4.1), one has $(\hat{w}, \hat{z}, \hat{r}, \hat{q}, \hat{g}) \in E$. In particular, (4.2) holds.

Proof of Proposition 4.1 The proof follows the same strategy used in [21–23], although the arguments were introduced in [26] and [29]. We introduce the space P_0 the functions $(\varphi, \psi, \xi, \eta) \in C^\infty(\overline{Q})^4$ such that

- $\varphi(\cdot, T) = \xi(\cdot, T) = 0, \text{ in } \Omega,$
- $\partial_\nu \varphi = \partial_\nu \psi = \partial_\nu \xi = \partial_\nu \eta = 0, \text{ on } \Sigma,$
- $(\psi_0)_\Omega = (\eta_0)_\Omega = 0,$
- $\int_\Omega (\mathcal{L}_1 \psi + M_1 \Delta \eta) \mathrm{d}x = \int_\Omega (\mathcal{L}_2 \eta - a\psi) \mathrm{d}x = 0, \quad \forall t \in [0, T],$
- $\partial_\nu \big((\mathcal{L}_1^*)^l [e^{13/4s\hat{\beta}} \big(\mathcal{L}_1^* \varphi - \psi\chi_2 - a\xi \big)] \big)|_\Sigma = 0, \;\; l = 0, 1,$
- $\big((\mathcal{L}_1^*)^l [e^{13/4s\hat{\beta}} \big(\mathcal{L}_1^* \varphi - \psi\chi_2 - a\xi \big)] \big)(\cdot, T) = 0, \;\; l = 0, 1,$
- $\partial_\nu \big(\mathcal{L}_1^k [e^{13/4s\hat{\beta}} (\mathcal{L}_1 \psi + M_1 \Delta \eta)] \big)|_\Sigma = 0, \;\; k = 0, 1,$
- $(\mathcal{L}_1^k [e^{13/4s\hat{\beta}} (\mathcal{L}_1 \psi + M_1 \Delta \eta)])(\cdot, 0) = 0, \;\; k = 0, 1,$
- $\partial_\nu \big((\mathcal{L}_2^*)^l [e^{13/4s\hat{\beta}} (\mathcal{L}_2^* \xi + M_1 \Delta \varphi)] \big)|_\Sigma = 0, \;\; l = 0, 1,$
- $\big((\mathcal{L}_2^*)^l [e^{13/4s\hat{\beta}} (\mathcal{L}_2^* \xi + M_1 \Delta \varphi)] \big)(\cdot, T) = 0, \;\; l = 0, 1,$
- $\partial_\nu \big(\mathcal{L}_2^k [e^{13/4s\hat{\beta}} (\mathcal{L}_2 \eta - a\psi)] \big)|_\Sigma = 0, \;\; k = 0, 1,$
- $(\mathcal{L}_2^k [e^{13/4s\hat{\beta}} (\mathcal{L}_2 \eta - a\psi)])(\cdot, 0) = 0, \;\; k = 0, 1.$

We define the bilinear form

$$
\begin{aligned}
& a\big((\tilde{\varphi}, \tilde{\psi}, \tilde{\xi}, \tilde{\eta}), (\varphi, \psi, \xi, \eta)\big) \\
& \quad := \iint_Q (\mathcal{L}_1^*)^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_1^* \tilde{\varphi} - \tilde{\psi} \chi_2 - a\tilde{\xi} \big) \big] \cdot (\mathcal{L}_1^*)^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_1^* \varphi - \psi \chi_2 - a\xi \big) \big] \mathrm{d}x \, \mathrm{d}t \\
& \qquad + \iint_Q \mathcal{L}_1^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_1 \tilde{\psi} + M_1 \Delta \tilde{\eta} \big) \big] \cdot \mathcal{L}_1^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_1 \psi + M_1 \Delta \eta \big) \big] \mathrm{d}x \, \mathrm{d}t \\
& \qquad + \iint_Q (\mathcal{L}_2^*)^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_2^* \tilde{\xi} + M_1 \Delta \tilde{\varphi} \big) \big] \cdot (\mathcal{L}_2^*)^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_2^* \xi + M_1 \Delta \varphi \big) \big] \mathrm{d}x \, \mathrm{d}t \\
& \qquad + \iint_Q \mathcal{L}_2^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_2 \tilde{\eta} - a\psi \big) \big] \cdot \mathcal{L}_2^2 \big[e^{13/4s\hat{\beta}} \big(\mathcal{L}_2 \eta - a\psi \big) \big] \mathrm{d}x \, \mathrm{d}t \\
& \qquad + \iint_{\omega \times (0,T)} e^{13/2s\hat{\beta}} |\chi_1|^2 \tilde{\xi} \xi \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
$$

and a linear form

$$
\langle l, (\varphi, \psi, \xi, \eta) \rangle := \iint_Q \left[(F + h_1)\varphi + h_2 \psi + (G + h_3)\xi + h_4 \eta \right] \mathrm{d}x \, \mathrm{d}t.
$$

Thanks to (4.3), we have that $a(\cdot, \cdot) : P_0 \times P_0 \to \mathbb{R}$ is a symmetric, definite positive, bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot, \cdot)$. Then, $a(\cdot, \cdot)$ is well-defined, continuous and positive definite on P. Furthermore, in view of the estimate (4.3) and the assumptions (4.5), the linear form $(\varphi, \psi, \xi, \eta) \mapsto \langle l, (\varphi, \psi, \xi, \eta) \rangle$ is defined and continuous on P. Hence, from Lax-Milgram lemma, we deduce that the variational problem:

$$
\begin{cases}
\text{Find } (\tilde{\varphi}, \tilde{\psi}, \tilde{\xi}, \tilde{\eta}) \in P \text{ such that} \\
a\big((\tilde{\varphi}, \tilde{\psi}, \tilde{\xi}, \tilde{\eta}), (\varphi, \psi, \xi, \eta)\big) = \langle l, (\varphi, \psi, \xi, \eta) \rangle, \qquad \forall (\varphi, \psi, \xi, \eta) \in P,
\end{cases}
\tag{4.6}
$$

possesses exactly one solution $(\hat{\varphi}, \hat{\psi}, \hat{\xi}, \hat{\eta}) \in P$.

Let $\hat{g}$ be given by

$$
\hat{g} := -e^{13/2s\hat{\beta}} \hat{\xi} \chi_1, \quad \text{in } Q. \tag{4.7}
$$

From (4.6) and (4.7) we get that

$$
\iint_Q \big(|\tilde{w}|^2 + |\tilde{z}|^2 + |\tilde{r}|^2 + |\tilde{q}|^2 \big) \mathrm{d}x \, \mathrm{d}t + \iint_Q e^{-13/2s\hat{\beta}} |\chi_1|^2 |\hat{g}|^2 \mathrm{d}x \, \mathrm{d}t < +\infty,
$$

where $\tilde{w}$, $\tilde{z}$, $\tilde{r}$ and $\tilde{q}$ are given by

$$
\begin{cases}
\tilde{w} := (\mathcal{L}_1^*)^2\big[e^{13/4s\hat{\beta}}\big(\mathcal{L}_1^*\hat{\varphi} - \hat{\psi}\chi_2 - a\hat{\xi}\big)\big],\\
\tilde{z} := \mathcal{L}_1^2\big[e^{13/4s\hat{\beta}}\big(\mathcal{L}_1\hat{\psi} + \Delta\hat{\eta}\big)\big],\\
\tilde{r} := (\mathcal{L}_2^*)^2\big[e^{13/4s\hat{\beta}}\big(\mathcal{L}_2^*\hat{\xi} + M_1\Delta\hat{\varphi}\big)\big],\\
\tilde{q} := \mathcal{L}_2^2\big[e^{13/4s\hat{\beta}}\big(\mathcal{L}_2\hat{\eta} - a\hat{\psi}\big)\big].
\end{cases}
\tag{4.8}
$$

In particular, notice that $e^{-13/4s\hat{\beta}}\hat{g}\chi_1 \in L^2(Q)$.

Let us call $(\hat{w}, \hat{z}, \hat{r}, \hat{q})$ the weak solution of (4.1) with $g = \hat{g}$, that is to say

$$
\begin{cases}
\mathcal{L}_1\hat{w} + M_1\Delta\hat{r} = F + h_1, & \text{in } Q,\\
\mathcal{L}_1^*\hat{z} - a\hat{q} - \hat{w}\chi_2 + (\hat{w}\chi_2)_\Omega = h_2, & \text{in } Q,\\
\mathcal{L}_2\hat{r} - a\hat{w} = \hat{g}\chi_1 + G + h_3, & \text{in } Q,\\
\mathcal{L}_2^*\hat{q} + M_1\Delta\hat{z} = h_4, & \text{in } Q,\\
\partial_\nu\hat{w} = \partial_\nu\hat{z} = \partial_\nu\hat{r} = \partial_\nu\hat{q} = 0, & \text{on } \Sigma,\\
\hat{w}(\cdot,0) = 0,\ \hat{z}(\cdot,T) = 0,\ \hat{r}(\cdot,0) = 0,\ \hat{q}(\cdot,T) = 0, & \text{in } \Omega.
\end{cases}
\tag{4.9}
$$

Using the decay of the right-hand sides and the properties of the weight functions, it is not difficult to see that:

Claim 1 It holds that $(e^{-13/4s\hat{\beta}}\hat{w}, e^{-13/4s\hat{\beta}}\hat{r}, \hat{z}, \hat{q}) \in [L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))]^4$.

Proof of the Claim 1 Since system (4.9) is in cascade form, we first work with the forward equations, i.e., equations for $\hat{w}$ and $\hat{r}$.

Indeed, we multiply the first and third equation in (4.9) by $e^{-13/2s\hat{\beta}}\hat{w}$ and $e^{-13/2s\hat{\beta}}\hat{r}$, respectively, and then the third one again by $e^{-13/2s\hat{\beta}}\Delta\hat{r}$, to see that there exists $C > 0$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|e^{-13/4s\hat{\beta}}(\hat{w},\hat{r})\|^2_{L^2(\Omega)^2} + \|e^{-13/4s\hat{\beta}}\nabla\hat{r}\|^2_{L^2(\Omega)}\right)\\
&+s\hat{\beta}_t e^{-13/4s\hat{\beta}}\big(\|(\hat{w},\hat{r})\|^2_{L^2(\Omega)^2} + \|\nabla\hat{r}\|^2_{L^2(\Omega)}\big) + \|e^{-13/4s\hat{\beta}}\hat{r}\|^2_{L^2(\Omega)}\\
&+\|e^{-13/4s\hat{\beta}}(\nabla\hat{w},\nabla\hat{r})\|^2_{L^2(\Omega)^2} + \|\Delta(e^{-13/4s\hat{\beta}}\hat{r})\|^2_{L^2(\Omega)} \le C\big(\|e^{-13/4s\hat{\beta}}(\hat{w},\hat{r})\|^2_{L^2(\Omega)^2}\\
&+\|\nabla(e^{-13/4s\hat{\beta}}\hat{r})\|^2_{L^2(\Omega)} + \|e^{-13/4s\hat{\beta}}(F+h_1, \hat{g}\chi_1 + G + h_3)\|^2_{L^2(\Omega)}\big).
\end{aligned}
$$

Integrating this identity in $(0, T/4)$, and using the fact that $\hat{\beta}_t \ge 0$ in $[0, T/4]$, we obtain the desired regularity for $(\hat{w}, \hat{r})$, combined with the fact that $\mathcal{L}_1\hat{w} = -M_1\Delta\hat{r} + F + h_1$.

Notice that the other two equations in (4.9) are backwards, for this reason a similar estimate with the exponential weight as above does not work for them. Neither, we cannot simply make the change of variable $t \to T - t$ in the backward system because then we would need also to change variable in the weight functions. However, it is enough to see the backward system as a coupled system with right-hand side in $L^2(Q)$ to obtain regularity for $\hat{z}$ and $\hat{q}$. □

Let us now show how to obtain the exponential decay for $(\hat{z}, \hat{q})$. Indeed, we prove:

Claim 2 It holds that $e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}\left(\hat{z}, \hat{q}\right) \in [L^2(Q)]^2$.

Proof of Claim 2 First, we will prove that $(\tilde{w}, \tilde{z}, \tilde{r}, \tilde{q})$ given by (4.8) is actually the solution (in the sense of transposition) of

$$\begin{cases} e^{13/4s\hat{\beta}}(\mathcal{L}_1)^2\tilde{w} = \hat{w}, \text{ in } Q, \\ e^{13/4s\hat{\beta}}(\mathcal{L}_1^*)^2\tilde{z} = \hat{z}, \quad \text{in } Q, \\ e^{13/4s\hat{\beta}}(\mathcal{L}_2)^2\tilde{r} = \hat{r}, \quad \text{in } Q, \\ e^{13/4s\hat{\beta}}(\mathcal{L}_2^*)^2\tilde{q} = \hat{q}, \quad \text{in } Q, \end{cases} \tag{4.10}$$

with

$$\begin{cases} \partial_\nu(\mathcal{L}_1)^l\tilde{w} = 0, & \text{on } \Sigma, \ l = 0, 1, \\ (\mathcal{L}_1)^l\tilde{w}(\cdot, 0) = 0, & \text{in } \Omega, \ l = 0, 1, \\ \partial_\nu(\mathcal{L}_1^*)^k\tilde{z} = 0, & \text{on } \Sigma \ \ k = 0, 1, \\ (\mathcal{L}_1^*)^k\tilde{z}(\cdot, T) = 0, & \text{in } \Omega \ \ k = 0, 1, \\ \partial_\nu(\mathcal{L}_2)^l\tilde{r} = 0, & \text{on } \Sigma, \ l = 0, 1, \\ (\mathcal{L}_2)^l\tilde{r}(\cdot, 0) = 0, & \text{in } \Omega, \ l = 0, 1, \\ \partial_\nu(\mathcal{L}_2^*)^k\tilde{q} = 0, & \text{on } \Sigma, \ k = 0, 1, \\ (\mathcal{L}_2^*)^k\tilde{q}(\cdot, T) = 0, & \text{in } \Omega, \ k = 0, 1. \end{cases} \tag{4.11}$$

Indeed, $(\tilde{w}, \tilde{z}, \tilde{r}, \tilde{q})$ is the transposition solution of (4.10)–(4.11) if for any $e^{-9/2s\hat{\beta}}(F + h_1, h_2, G + h_3, h_4) \in L^2(Q)^4$, with $(F)_\Omega(t) = (h_1)_\Omega(t) = (h_2)_\Omega(t) = (h_4)_\Omega(t) = 0$ for a.e. $t \in (0, T)$, it holds that

$$\begin{aligned} &\iint_Q \tilde{w}(F + h_1)\mathrm{d}x\,\mathrm{d}t + \iint_Q \tilde{z}h_2\mathrm{d}x\,\mathrm{d}t + \iint_Q \tilde{r}(G + h_3)\mathrm{d}x\,\mathrm{d}t + \iint_Q \tilde{q}h_4\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q \hat{w}\Phi^w\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{z}\Phi^z\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{r}\Phi^r\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{q}\Phi^q\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.12}$$

where $(\Phi^w, \Phi^z, \Phi^r, \Phi^q)$ is the solution of

$$\begin{cases} (\mathcal{L}_1^*)^2[e^{13/4s\hat{\beta}}\Phi^w] = F + h_1, & \text{in } Q, \\ \mathcal{L}_2^2[e^{13/4s\hat{\beta}}\Phi^z] = h_2, & \text{in } Q, \\ (\mathcal{L}_3^*)^2[e^{13/4s\hat{\beta}}\Phi^r] = G + h_3, & \text{in } Q, \\ \mathcal{L}_4^2[e^{13/4s\hat{\beta}}\Phi^q] = h_4, & \text{in } Q, \end{cases}$$

such that

$$\begin{cases} \partial_\nu (\mathcal{L}_1^*)^\ell \big(e^{13/4s\hat\beta}\Phi^w\big) = 0, & \text{on } \Sigma,\ \ell = 0, 1, \\ (\mathcal{L}_1^*)^\ell \big(e^{13/4s\hat\beta}\Phi^w\big)(\cdot, T) = 0, & \text{in } \Omega,\ \ell = 0, 1, \\ \partial_\nu \mathcal{L}_1^k \big(e^{13/4s\hat\beta}\Phi^z\big) = 0, & \text{on } \Sigma \ \ k = 0, 1, \\ \mathcal{L}_1^k \big(e^{13/4s\hat\beta}\Phi^z\big)(\cdot, 0) = 0, & \text{in } \Omega \ \ k = 0, 1, \\ \partial_\nu (\mathcal{L}_2^*)^\ell \big(e^{13/4s\hat\beta}\Phi^r\big) = 0, & \text{on } \Sigma,\ \ell = 0, 1, \\ (\mathcal{L}_2^*)^\ell \big(e^{13/4s\hat\beta}\Phi^r\big)(\cdot, T) = 0, & \text{in } \Omega,\ \ell = 0, 1, \\ \partial_\nu \mathcal{L}_2^k \big(e^{13/4s\hat\beta}\Phi^q\big) = 0, & \text{on } \Sigma,\ k = 0, 1, \\ \mathcal{L}_2^k \big(e^{13/4s\hat\beta}\Phi^q\big)(\cdot, 0) = 0, & \text{in } \Omega,\ k = 0, 1. \end{cases}$$

Using (4.6), (4.7), (4.8) and (4.9), a simple calculation shows that for every $(\varphi, \psi, \xi, \eta) \in P_0$ one has

$$\begin{aligned}
&\iint_Q \tilde w (\mathcal{L}_1^*)^2 \big[e^{13/4s\hat\beta}\big(\mathcal{L}_1^*\varphi - \psi\chi_2 - a\xi\big)\big] \mathrm{d}x\, \mathrm{d}t + \iint_Q \tilde z \mathcal{L}_1^2 \big[e^{13/4s\hat\beta}\big(\mathcal{L}_1\psi + M_1\Delta\eta\big)\big] \mathrm{d}x\, \mathrm{d}t \\
&\quad + \iint_Q \tilde r (\mathcal{L}_2^*)^2 \big[e^{13/4s\hat\beta}\big(\mathcal{L}_2^*\xi + M_1\Delta\varphi\big)\big] \mathrm{d}x\, \mathrm{d}t + \iint_Q \tilde q \mathcal{L}_2^2 \big[e^{13/4s\hat\beta}\big(\mathcal{L}_2\eta - a\psi\big)\big] \mathrm{d}x\, \mathrm{d}t \\
&= \iint_Q \varphi\big(\mathcal{L}_1\hat w + M_1\Delta\hat r\big) \mathrm{d}x\, \mathrm{d}t + \iint_Q \psi\big(\mathcal{L}_1^*\hat z - a\hat q - \hat w\chi_2 + (\hat w\chi_2)_\Omega\big) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \iint_Q \xi\big(\mathcal{L}_2\hat r - a\hat w\big) \mathrm{d}x\, \mathrm{d}t + \iint_Q \eta\big(\mathcal{L}_2^*\hat q + M_1\Delta\hat z\big) \mathrm{d}x\, \mathrm{d}t \\
&= \iint_Q \hat w\big(\mathcal{L}_1^*\varphi - \psi\chi_2 - a\xi\big) \mathrm{d}x\, \mathrm{d}t + \iint_Q \hat z\big(\mathcal{L}_1\psi + M_1\Delta\eta\big) \mathrm{d}x\, \mathrm{d}t \\
&\quad + \iint_Q \hat r\big(\mathcal{L}_2^*\xi + M_1\Delta\varphi\big) \mathrm{d}x\, \mathrm{d}t + \iint_Q \hat q\big(\mathcal{L}_2\eta - a\psi\big) \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

This last identity readily implies that (4.12) holds and that $(\tilde w, \tilde z, \tilde r, \tilde q)$ is indeed the unique transposition solution of (4.10)–(4.11). Next, we define the following functions:

$$\begin{aligned} \big(z_{*,0}, q_{*,0}\big) &:= e^{-13/4s\hat\beta}(\hat\gamma)^{-3-3/m}\left(\hat z, \hat q\right), \\ \left(f_{*,0}^z, f_{*,0}^q\right) &:= e^{-13/4s\hat\beta}(\hat\gamma)^{-3-3/m}\left(h_2, h_4\right). \end{aligned}$$

From (4.5) and Claim 1, we have that $\left(f^z_{*,0}, f^q_{*,0}\right) \in [L^2(Q)]^2$ and that $e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega] \in L^2(Q)$. By (4.9), $\left(z_{*,0}, q_{*,0}\right)$ satisfies

$$
\begin{cases}
\mathcal{L}_1^* z_{*,0} - a q_{*,0} = f^z_{*,0} + e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega] \\
\qquad\qquad - (e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m})_t \hat{z}, & \text{in } Q, \\
\mathcal{L}_2^* q_{*,0} + M_1 \Delta z_{*,0} = f^q_{*,0} - (e^{-13/s\hat{\beta}}(\hat{\gamma})^{-3-3/m})_t \hat{q}, & \text{in } Q, \\
\partial_\nu z_{*,0} = \partial_\nu q_{*,0} = 0, & \text{on } \Sigma, \\
z_{*,0}(\cdot, T) = 0, \ q_{*,0}(\cdot, T) = 0, & \text{in } \Omega.
\end{cases}
$$

where the last term in the right-hand side can be written as

$$
\left(e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m}\right)_t \left(\hat{z}, \hat{q}\right) = c_1(t)\left((\mathcal{L}_1^*)^2\tilde{z}, (\mathcal{L}_2^*)^2\tilde{q}\right), \quad \text{with } c_1 \in C^2[0, T].
$$

Hence, for any $(\varsigma_1, \varsigma_2) \in [Y^+_{1,0}]^2$, we have

$$
\begin{aligned}
\iint_Q \left(z_{*,0}, q_{*,0}\right) \cdot (\varsigma_1, \varsigma_2)\,\mathrm{d}x\,\mathrm{d}t &= \iint_Q \left(f^z_{*,0}, f^q_{*,0}\right) \cdot (\Phi_1, \Phi_2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_Q e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega]\Phi_1\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint_Q c_1(t)\left((\mathcal{L}_1^*)^2\tilde{z}, (\mathcal{L}_2^*)^2\tilde{q}\right) \cdot (\Phi_1, \Phi_2)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where (Φ_1, Φ_2) solves

$$
\begin{cases}
\mathcal{L}_1 \Phi_1 + M_1 \Delta \Phi_2 = \varsigma_1, & \text{in } Q, \\
\mathcal{L}_2 \Phi_2 - a \Phi_1 = \varsigma_2, & \text{in } Q, \\
\partial_\nu \Phi_1 = \partial_\nu \Phi_2 = 0, & \text{on } \Sigma, \\
\Phi_1(\cdot, 0) = \Phi_2(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases}
\tag{4.13}
$$

Using (4.11), we can integrate by parts to obtain

$$
\begin{aligned}
\iint_Q \left(z_{*,0}, q_{*,0}\right) \cdot (\varsigma_1, \varsigma_2)\,\mathrm{d}x\,\mathrm{d}t &= \iint_Q \left(f^z_{*,0}, f^q_{*,0}\right) \cdot (\Phi_1, \Phi_2)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \iint_Q e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-3-3/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega]\Phi_1\mathrm{d}x\,\mathrm{d}t \\
&\quad - \iint_Q (\tilde{z}, \tilde{q}) \cdot \left((\mathcal{L}_1)^2[c_1(t)\Phi_1], (\mathcal{L}_2)^2[c_1(t)\Phi_2]\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\Big((\mathcal{L}_1^*)^2[c_1(t)\Phi_1], (\mathcal{L}_2^*)^2[c_1(t)\Phi_2]\Big)\\
&\quad = \Big(c_1^{''}(t)\Phi_1 + 2c_1^{'}(t)\mathcal{L}_1^*\Phi_1 + c_1(t)(\mathcal{L}_1^*)^2\Phi_1\ ,\ c_1^{''}(t)\Phi_2\\
&\qquad +2c_1^{'}(t)\mathcal{L}_2^*\Phi_2 + c_1(t)(\mathcal{L}_2^*)^2\Phi_2\Big)
\end{aligned}
$$

and, by Proposition B.1, we see that

$$
\|(\Phi_1,\Phi_2)\|_{[Y_2]^2} \leq\ C\|(\varsigma_1,\varsigma_2)\|_{[Y_{1,0}^+]^2},
$$

for some $C > 0$. Thus, for $(\varsigma_1,\varsigma_2) \in [Y_{1,0}^+]^2$, we obtain

$$
\begin{aligned}
&\iint\limits_Q \big(z_{*,0}, q_{*,0}\big)\cdot(\varsigma_1,\varsigma_2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq\ C\left(\left\|\left(f_{*,0}^z, f_{*,0}^q\right)\right\|_{[L^2(Q)]^2}\right.\\
&\qquad\left. + \|(\tilde{z},\tilde{q})\|_{[L^2(Q)]^2} + \|e^{-13/4s\hat{\beta}}\hat{w}\|_{L^2(Q)}\right)\|(\varsigma_1,\varsigma_2)\|_{[Y_{1,0}^+]^2}\,.
\end{aligned}
\tag{4.14}
$$

Now let,

$$
\begin{aligned}
\big(z_{*,1}, q_{*,1}\big) &:= e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}\left(\hat{z},\hat{q}\right),\\
\left(f_{*,1}^z, f_{*,1}^q\right) &:= e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}\left(h_2, h_4\right).
\end{aligned}
$$

Similarly as before, $(z_{*,1}, q_{*,1})$ satisfies

$$
\begin{cases}
\mathcal{L}_1^* z_{*,1} - a q_{*,1} = f_{*,1}^z + e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega] & \\
\qquad -(e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m})_t\hat{z}, & \text{in } Q,\\
\mathcal{L}_2^* q_{*,1} + M_1\Delta z_{*,1} = f_{*,0}^q - (e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m})_t\hat{q}, & \text{in } Q,\\
\partial_\nu z_{*,1} = \partial_\nu q_{*,1} = 0, & \text{on } \Sigma,\\
z_{*,1}(\cdot,T) = 0,\ q_{*,1}(\cdot,T) = 0, & \text{in } \Omega.
\end{cases}
$$

By taking $(\varsigma_1,\varsigma_2) \in [Y_0]^2$ in (4.13) and arguing as before, we get

$$
\begin{aligned}
&\iint\limits_Q \big(z_{*,1}, q_{*,1}\big)\cdot(\varsigma_1,\varsigma_2)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \iint\limits_Q \left(f_{*,1}^z, f_{*,1}^q\right)\cdot(\Phi_1,\Phi_2)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \iint\limits_Q e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}[\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega]\Phi_1\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$-\iint_Q (e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m})_t \left(\hat{z}, \hat{q}\right) \cdot (\Phi_1, \Phi_2)\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, by writing

$$\iint_Q \left(e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}\right)_t \left(\hat{z}, \hat{q}\right) \cdot (\Phi_1, \Phi_2)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \iint_Q c_2(t)\,(\Phi_1, \Phi_2) \cdot \left(z_{*,0}, q_{*,0}\right)\mathrm{d}x\,\mathrm{d}t,$$

with $c_2 \in C^2[0, T]$, we see that

$$\iint_Q c_1(t)\,(\Phi_1, \Phi_2) \cdot \left(z_{*,0}, q_{*,0}\right)\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\left(\left\|\left(f^z_{*,0}, f^q_{*,0}\right)\right\|_{[L^2(Q)]^2} + \|(\tilde{z}, \tilde{q})\|_{[L^2(Q)]^2} + \|e^{-13/4s\hat{\beta}}\hat{w}\|_{L^2(Q)}\right)$$
$$\|c_2(t)\,(\Phi_1, \Phi_2)\|_{[Y^+_{1,0}]^2}\,.$$

Here, we have applied (4.14) with $c_1(t)\,(\Phi_1, \Phi_2)$ instead of $(\varsigma_1, \varsigma_2)$ (notice that $c_1(t)\,(\Phi_1, \Phi_2) \in [Y^+_{1,0}]^2$). Then, it follows that

$$\iint_Q \left(z_{*,1}, q_{*,1}\right) \cdot (\varsigma_1, \varsigma_2)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\left(\left\|\left(f^z_{*,0}, f^q_{*,0}\right)\right\|_{[L^2(Q)]^2} + \|(\tilde{z}, \tilde{q})\|_{[L^2(Q)]^2} + \|e^{-13/4s\hat{\beta}}\hat{w}\|_{L^2(Q)}\right)$$
$$\|(\Phi_1, \Phi_2)\|_{[Y^+_{1,0}]^2}\,.$$

Now, since

$$\|(\Phi_1, \Phi_2)\|_{[Y^+_{1,0}]^2} \leq C\,\|(\varsigma_1, \varsigma_2)\|_{[Y_0]^2}\,,$$

we obtain

$$\iint_Q \left(z_{*,1}, q_{*,1}\right) \cdot (\varsigma_1, \varsigma_2)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\left(\left\|\left(f^z_{*,0}, f^q_{*,0}\right)\right\|_{[L^2(Q)]^2} + \|(\tilde{z}, \tilde{q})\|_{[L^2(Q)]^2} + \|e^{-13/4s\hat{\beta}}\hat{w}\|_{L^2(Q)}\right)\|(\varsigma_1, \varsigma_2)\|_{[Y_0]^2}\,,$$

for all $(\varsigma_1, \varsigma_2) \in [Y_0]^2$. Thus, we deduce that

$$e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-5-5/m}\left(\hat{z}, \hat{q}\right) \in L^2(Q), \tag{4.15}$$

and the proof of Claim 2 is finished. □

To finish Proposition 4.1, we show the following:

Claim 3 We have the following regularity:

$$\begin{cases} e^{-2s\hat{\beta}}\hat{w} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \\ e^{-2s\hat{\beta}}\hat{r} \in L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega)). \end{cases}$$

Proof of the Claim 3 We begin noticing that, from the definition of $\hat{g}$ and (4.3), one has

$$\iint_Q |\nabla(e^{-2s\hat{\beta}}\hat{g})|^2 \mathrm{d}x\,\mathrm{d}t < +\infty.$$

Hence, it follows that $e^{-2s\hat{\beta}}\hat{g} \in L^2(0,T;H^1(\Omega))$. Also, from Claim 1, we already have that

$$e^{-13/4s\hat{\beta}}\left(\hat{w},\hat{r}\right) \in [L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))]^2. \tag{4.16}$$

Let us then establish the remaining regularity. To do this, let

$$(w_*, r_*) := e^{-2s\hat{\beta}}\left(\hat{w},\hat{r}\right), \qquad \left(f_*^w, f_*^r\right) := e^{-2s\hat{\beta}}\left(F+h_1, \hat{g}\chi_1 + G + h_3\right),$$

which solves

$$\begin{cases} \mathcal{L}_1 w_* + M_1\Delta r_* = f_*^w + (e^{-2s\hat{\beta}})_t\hat{w}, & \text{in } Q, \\ \mathcal{L}_2 r_* - a w_* = f_*^r + (e^{-2s\hat{\beta}})_t\hat{r}, & \text{in } Q, \\ \partial_\nu w_* = \partial_\nu r_* = 0, & \text{on } \Sigma, \\ w_*(\cdot,0) = 0,\ r_*(\cdot,0) = 0, & \text{in } \Omega. \end{cases}$$

From (4.16), the terms $(e^{-2s\hat{\beta}})_t\hat{w}$ and $(e^{-2s\hat{\beta}})_t\hat{r}$ both belong to $L^2(0,T;H^1(\Omega))$ and, by Proposition B.1, the Claim follows. □

We also have the following result:

Claim 4 It holds that

$$\begin{cases} e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-6-6/m}\hat{q} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \\ e^{-13/4s\hat{\beta}}(\hat{\gamma})^{-7-7/m}\hat{z} \in L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega)). \end{cases}$$

Proof of the Claim 4 Let

$$(z_*, q_*) := \rho(t)\left(\hat{z},\hat{q}\right), \qquad \left(f_*^z, f_*^q\right) := \rho(t)\left(\hat{w}\chi_2 - (\hat{w}\chi_2)_\Omega + h_2, h_4\right),$$

which solves

$$\begin{cases} \mathcal{L}_1^* z_* - a q_* = f_*^z - \rho_t \hat{z}, & \text{in } Q, \\ \mathcal{L}_2^* q_* + M_1 \Delta z_* = f_*^q - \rho_t \hat{q}, & \text{in } Q, \\ \partial_\nu z_* = \partial_\nu q_* = 0, & \text{on } \Sigma, \\ z_*(\cdot, T) = 0, \ q_*(\cdot, T) = 0, & \text{in } \Omega. \end{cases}$$

We will consider two cases:

Case 1: $\rho(t) = e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-6-6/m}$.

From (4.15), the terms $\rho_t \hat{z}$ and $\rho_t \hat{q}$ both belong to $L^2(Q)$ and, by Proposition B.1, we deduce that

$$e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-6-6/m} (\hat{z}, \hat{q}) \in [L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))]^2.$$

Case 2: $\rho(t) = e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-7-7/m}$.

Since there exists $C > 0$ such that

$$|\rho_t| \le C e^{-13/4s\hat{\beta}} (\hat{\gamma})^{-6-6/m},$$

we have that $\rho_t \hat{z}$ and $\rho_t \hat{q}$ belong to $L^2(0, T; H^1(\Omega))$. Moreover, given that $\left(f_*^z, f_*^q\right) \in L^2(0, T; H^1(\Omega)) \times L^2(Q)$, the result follows. □

Remark 4.1 From Claim 4, we also have that $e^{-5/2\hat{\beta}} \hat{z} \in L^2(0, T; H^3(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$. This will be used in the next section.

Finally, from Claim 1, Claim 2, Claim 3, and Claim 4, we see that $(\hat{w}, \hat{z}, \hat{r}, \hat{q}, \hat{g}) \in E$.

This finishes the proof of Proposition 4.1. □

5 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. The results obtained in the previous section allow us to locally invert the nonlinear system (1.5). In particular, the regularity deduced for the solution of the linearized system (4.1) is sufficient to apply a suitable inverse function theorem (see Theorem 5.1 below). Our approach borrow some ideas from [28, 29].

Let us set $w = M_1 + W$ and $r = M_2 + R$, and rewrite the equations in (1.5) accordingly. For simplicity, we will abuse notation by identifying W and R with w and r, respectively. This leads to the system:

$$\begin{cases} \mathcal{L}(w, r) = \big(- M_1 \Delta r - \nabla \cdot (w \nabla r) + F, a w + g \chi_1 + G \big), & \text{in } Q, \\ \mathcal{T}(z, q) = \big(- M_1 \Delta z - \nabla \cdot (w \nabla z), & \\ \qquad a q + w \chi_2 - (w \chi_2)_\Omega + \nabla z \cdot \nabla r - (\nabla z \cdot \nabla r)_\Omega \big), & \text{in } Q, \\ \partial_\nu w = \partial_\nu z = \partial_\nu r = \partial_\nu q = 0, & \text{on } \Sigma, \\ w(\cdot, 0) = 0, \ z(\cdot, T) = 0, \ r(\cdot, 0) = 0, \ q(\cdot, T) = 0, & \text{in } \Omega, \end{cases} \tag{5.1}$$

In this way, we have reduced our problem to proving a local null controllability result for the solution (w, z, r, q) of the nonlinear problem (5.1). To proceed, we need the following inverse mapping theorem from [30]:

Theorem 5.1 *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Banach spaces and let $\mathcal{F} : \mathcal{G}_1 \to \mathcal{G}_2$ satisfy $\mathcal{F} \in C^1(\mathcal{G}_1, \mathcal{G}_2)$. Assume that $g_1 \in \mathcal{G}_1$, $\mathcal{F}(g_1) = g_2$ and that $\mathcal{F}^{'}(g_1) : \mathcal{G}_1 \to \mathcal{G}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $\bar{g} \in \mathcal{G}_2$ satisfying $\|\bar{g} - g_2\| < \delta$, there exists a solution of the equation*

$$\mathcal{F}(g_1) = \bar{g}, \quad g \in \mathcal{G}_1.$$

We start by defining the spaces:

$$L^2\left(e^{-9/2s\hat{\beta}}(0,T); L_0^2(\Omega)\right) := \{u \in L^2(Q) : e^{-9/2s\hat{\beta}}u \in L^2(Q) \text{ and } (u)_\Omega = 0\},$$

$$L^2\left(e^{-9/2\hat{\beta}}(0,T); H^1(\Omega)\right) := \{u \in L^2(0,T; H^1(\Omega)) : e^{-9/2s\hat{\beta}}u \in L^2(0,T; H^1(\Omega))\}.$$

Let E the space given in (4.4). We apply Theorem 5.1 with the spaces

$$\mathcal{G}_1 := E,$$

$$\begin{aligned}\mathcal{G}_2 :=& L^2\left(e^{-9/2s\hat{\beta}}(0,T); L_0^2(\Omega)\right) \times L^2\left(e^{-9/2s\hat{\beta}}(0,T); H^1(\Omega)\right)\\ &\times L^2\left(e^{-9/2s\hat{\beta}}(0,T); L_0^2(\Omega)\right) \times L^2\left(e^{-9/2s\hat{\beta}}(0,T); H^1(\Omega)\right).\end{aligned}$$

and the operator

$$\begin{aligned}&\mathcal{F}(w,z,r,q,g)\\ &:= \big(\mathcal{L}(w,r) + \big(M_1\Delta r + \nabla\cdot(w\nabla r), -aw - g\chi_1\big),\\ &\quad \mathcal{T}(z,q) + \big(M_1\Delta z + \nabla\cdot(w\nabla z), -aq - w\chi_2\\ &\quad +(w\chi_2)_\Omega - \nabla z\cdot\nabla r + (\nabla z\cdot\nabla r)_\Omega\big)\big).\end{aligned}$$

for $(w, z, r, q, g) \in \mathcal{G}_1$.

With this setting, we have the following lemma:

Lemma 5.1 *The operator $\mathcal{F}$ is the class $C^1(\mathcal{G}_1; \mathcal{G}_2)$.*

Proof Since all the terms in $\mathcal{F}$ are linear, except for $\nabla\cdot(w\nabla r)$, $\nabla\cdot(w\nabla z)$, and $\nabla z\cdot\nabla r - (\nabla z\cdot\nabla r)_\Omega$, it is sufficient to prove the continuity, from from $\mathcal{G}_1 \times \mathcal{G}_1$ to $\mathcal{G}_2$, of the operators

$$((w_1,z_1,r_1,q_1,g_1),(w_2,z_2,r_2,q_2,g_2)) \mapsto (\nabla\cdot(w_1\nabla r_2),0,0,0),$$

$$((w_1,z_1,r_1,q_1,g_1),(w_2,z_2,r_2,q_2,g_2)) \mapsto (0,0,\nabla\cdot(w_1\nabla z_2),0),$$

$$((w_1,z_1,r_1,q_1,g_1),(w_2,z_2,r_2,q_2,g_2)) \mapsto (0,0,0,\nabla z_1\cdot\nabla r_2 - (\nabla z_1\cdot\nabla r_2)_\Omega).$$

For the first operator, the continuity follows from the estimate:

$$\begin{aligned}
&\left\| e^{-9/2s\hat{\beta}} \nabla \cdot (w_1 \nabla r_2) \right\|_{L^2(Q)} \\
&\leq C \left(\| e^{-13/4s\hat{\beta}} w_1 \|^2_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} \| e^{-2s\hat{\beta}} r_2 \|^2_{L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))} \right),
\end{aligned}$$

for some $C > 0$. Similarly, one can show that the second operator is continuous.

Finally, to estimate the third operator, a simple calculation gives:

$$\begin{aligned}
&\left\| e^{-9/2s\hat{\beta}} \left[\nabla z_1 \cdot \nabla r_2 - (\nabla z \cdot \nabla r)_\Omega \right] \right\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C \left(\| e^{-5/2s\hat{\beta}} z_1 \|^2_{L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))} \| e^{-2s\hat{\beta}} r_2 \|^2_{L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))} \right).
\end{aligned}$$

for some $C > 0$.

Thus, the proof of Lemma 5.1 is finished. □

Let us now prove Theorem 1.1.

Proof of Theorem 1.1 Since $\mathcal{F}'(0) : \mathcal{G}_1 \to \mathcal{G}_2$ is given by

$$\begin{aligned}
\mathcal{F}'(0)(w, z, r, q, g) = \big(&\mathcal{L}(w, r) + \big(M_1 \Delta r, -aw - g\chi_1 \big), \\
&\mathcal{T}(z, q) + \big(M_1 \Delta z, -aq - w\chi_2 + (w\chi_2)_\Omega \big) \big),
\end{aligned}$$

from Proposition 4.1, we deduce that $\mathcal{F}'(0)$ is surjective.

Applying Theorem 5.1, with $g_1 = 0$ and $g_2 = 0$, there exist $\delta > 0$ and $C > 0$ such that, whenever $\left\| e^{C/t^m} (F, G) \right\|_{L^2_0(Q) \times L^2(0,T;H^1(\Omega))} < \delta$, there exists $(w, z, r, q, g) \in \mathcal{G}_1$ solution of system (5.1). In particular, $z(\cdot, 0) = z(\cdot, 0) = 0$. Consequently, the proof of Theorem 1.1 is complete. □

6 Comments and open problems

In this work, we have shown that it is possible to insensitize the component of the species in the Keller–Segel model by means of controls acting only on the chemical equation. This work, leads to some natural questions. Let us mention two of them.

- **Positivity of the constructed solution.**
  We believe that, if $(F, G) \in \left(L^2_0(Q) \cap L^2(0, T; H^1(\Omega)) \right) \times L^2(0, T; H^2(\Omega))$ with $\| e^{C/t^m} (F, G) \|_{L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^2(\Omega))} < \delta$ for some $C > 0$ and $\delta > 0$ small enough, and the unknown data $(\hat{u}_0, \hat{v}_0) \in H^1(\Omega) \times H^2(\Omega)$, the solution (u, v) we construct for system (1.1) remains positive for all time.
  Indeed, by a Taylor expansion with respect to τ, one has

$$u \sim u|_{\tau=0} + \tau (\partial_\tau u)|_{\tau=0} \quad \text{and} \quad v \sim v|_{\tau=0} + \tau (\partial_\tau v)|_{\tau=0}.$$

Let

$$w := u|_{\tau=0}, \quad u^\tau := \left.\frac{\partial u}{\partial \tau}\right|_{\tau=0}, \quad r := v|_{\tau=0}, \quad v^\tau := \left.\frac{\partial v}{\partial \tau}\right|_{\tau=0}.$$

In Appendix A, we show that (w, r) solves (A.3) and (u^τ, v^τ) solves (A.2).
By construction (see Sect. 5), we have $w = M_1 + W$, $r = M_2 + R$, where (W, Z, R, Q) is a controlled solution of the nonlinear system (5.1), associated to some control g.
If $\|e^{C/t^m}(F, G)\|_{L^2(0,T;H^1(\Omega))\times L^2(0,T;H^2(\Omega))} < \delta$, the local inverse mapping argument yields a constant $C_1 > 0$ such that $\|(W, Z, R, Q, g)\|_E \leq C_1\delta$.
On the other hand, since (u^τ, v^τ) solves system (A.2), it follows that it is bounded whenever $(\hat{u}_0, \hat{v}_0)$ belongs to $H^1(\Omega) \times H^2(\Omega)$.
Therefore, assuming sufficient regularity of F, G, $\hat{u}_0$ and $\hat{v}_0$, the solution (u, v) for system (1.1) must be continuous in space and time, and satisfies

$$u \geq M_1 - C_1\delta - \tau K \quad \text{and} \quad v \geq M_2 - C_1\delta - \tau K,$$

for some $K > 0$. This suggests that (u, v) remains positive provided that δ and τ are sufficiently small.

- **Bilinear control.**
Instead of using a linear control term in the v-equation, one may consider a bilinear control of the form $gv\mathbb{1}_\omega$. In this setting, g acts as a proliferation/degradation coefficient, and the positivity of the solution is preserved regardless of the sign of g, but it significantly complicates the controllability analysis. Indeed, the most widely used techniques for bilinear controllability of parabolic equations rely on spectral methods and the Lebeau-Robbiano strategy (see, e.g., [31, 32]), which are not well suited to systems of parabolic equations. Consequently, this issue represents a challenging problem and, to the best of our knowledge, remains open.

Appendix A: Equivalence Between Insensitizing and Controllability Problems

In this appendix, we show the equivalence between condition (1.3) for system (1.1) and the (partial) null controllability problem for system (1.5).

Proof From condition (1.3), we obtain

$$0 = \left.\frac{\partial J_\tau(u, v)}{\partial \tau}\right|_{\tau=0} = \iint_{\mathcal{O}\times(0,T)} (u|_{\tau=0}) \left(\left.\frac{\partial u}{\partial \tau}\right|_{\tau=0}\right) \chi_2 \mathrm{d}x\, \mathrm{d}t = \iint_Q u^\tau w \chi_2 \mathrm{d}x\, \mathrm{d}t. \tag{A.1}$$

Here, we have used the notation

$$w := u|_{\tau=0}, \quad u^\tau := \left.\frac{\partial u}{\partial \tau}\right|_{\tau=0}, \quad r := v|_{\tau=0}, \quad v^\tau := \left.\frac{\partial v}{\partial \tau}\right|_{\tau=0},$$

where (u^τ, v^τ) satisfies

$$\begin{cases} u_t^\tau - \Delta u^\tau = -\nabla \cdot (u^\tau \nabla r) - \nabla \cdot (w \nabla v^\tau), & \text{in } Q, \\ v_t^\tau - \Delta v^\tau = a u^\tau - b v^\tau, & \text{in } Q, \\ \partial_\nu u^\tau = 0, \ \partial_\nu v^\tau = 0, & \text{on } \Sigma, \\ u^\tau(\cdot, 0) = \hat{u}_0, \quad v^\tau(\cdot, 0) = \hat{v}_0, & \text{in } \Omega. \end{cases} \tag{A.2}$$

and (w, r) solves

$$\begin{cases} w_t - \Delta w = -\nabla \cdot (w \nabla r) + F, & \text{in } Q, \\ r_t - \Delta r = a w - b r + g\chi_1 + G, & \text{in } Q, \\ \partial_\nu w = 0, \ \partial_\nu r = 0, & \text{on } \Sigma, \\ w(\cdot, 0) = M_1, \ r(\cdot, 0) = M_2, & \text{in } \Omega. \end{cases} \tag{A.3}$$

Integrating $(\text{A.2})_1$ in Ω, and using that $(\hat{u}_0)_\Omega = 0$, it follows that $(u^\tau)_\Omega = 0$. Now, integrating $(\text{A.2})_2$ in Ω, since $(\hat{v}_0)_\Omega = 0$, we deduce that $(v^\tau)_\Omega = 0$.

Then, we rewrite (A.1) as

$$0 = \left.\frac{\partial J_\tau(u, v)}{\partial \tau}\right|_{\tau=0} = \langle u^\tau, w\chi_2 - (w\chi_2)_\Omega \rangle_{L_0^2(Q)}. \tag{A.4}$$

Let (z, q) be the solution of

$$\begin{cases} -z_t - \Delta z = aq + [\nabla z \cdot \nabla r - (\nabla z \cdot \nabla r)_\Omega] + w\chi_2 - (w\chi_2)_\Omega, & \text{in } Q, \\ -q_t - \Delta q = -bq - \nabla \cdot (w \nabla z), & \text{in } Q, \\ \partial_\nu z = 0, \ \partial_\nu q = 0, & \text{on } \Sigma, \\ z(\cdot, T) = 0, \ q(\cdot, T) = 0, & \text{in } \Omega. \end{cases} \tag{A.5}$$

Multiplying the first equation of (A.2) by z and the second one by q, to see, after an integration by parts, that

$$\begin{aligned} &\iint_Q u^\tau(-z_t - \Delta z) \mathrm{d}x\, \mathrm{d}t - \int_\Omega z(\cdot, 0)\hat{u}_0 \mathrm{d}x \\ &= \iint_Q u^\tau \, [\nabla z \cdot \nabla r - (\nabla z \cdot \nabla r)_\Omega] \, \mathrm{d}x\, \mathrm{d}t \\ &\quad - \iint_Q v^\tau \nabla \cdot (w \nabla z) \mathrm{d}x\, \mathrm{d}t, \end{aligned} \tag{A.6}$$

and

$$\iint_Q v^\tau(-q_t - \Delta q)\mathrm{d}x\,\mathrm{d}t - \int_\Omega q(\cdot,0)\hat{v}_0\mathrm{d}x = a\iint_Q u^\tau q\mathrm{d}x\,\mathrm{d}t - b\iint_Q v^\tau q\mathrm{d}x\,\mathrm{d}t. \tag{A.7}$$

Notice that second term of right-hand side of (A.6) is zero, indeed

$$\iint_Q u^\tau(\nabla z\nabla r)_\Omega \mathrm{d}x\,\mathrm{d}t = \int_0^T (\nabla z\nabla r)_\Omega\left(\int_\Omega u^\tau\mathrm{d}x\right)\mathrm{d}t = 0.$$

Therefore, combining (A.6) with (A.7) and using (A.5),

$$\int_\Omega z(\cdot,0)\hat{u}_0\mathrm{d}x + \int_\Omega w(\cdot,0)\hat{v}_0\mathrm{d}x = \iint_Q u^\tau(w\chi_2 - (w\chi_2)_\Omega)\mathrm{d}x\,\mathrm{d}t. \tag{A.8}$$

But, by (A.4) and the last identity, we get

$$\begin{gathered}\int_\Omega z(\cdot,0)\hat{u}_0\mathrm{d}x + \int_\Omega w(\cdot,0)\hat{v}_0\mathrm{d}x = 0,\\ \big\langle \big(z(\cdot,0), w(\cdot,0)\big), \big(\hat{u}_0,\hat{v}_0\big)\big\rangle_{[L^2_0(\Omega)]^2} = 0,\end{gathered}$$

for any $(\hat{u}_0,\hat{v}_0) \in [L^2_0(\Omega)]^2$, with $\|\hat{u}_0\|_{L^2(\Omega)} = \|\hat{v}_0\|_{L^2(\Omega)} = 1$.

Hence, we deduce that

$$z(\cdot,0) = q(\cdot,0) = 0, \text{ in } \Omega.$$

□

Remark A.1 Notice that (A.3) and (A.5) form the cascade system given in (1.5).

Appendix B: Regularity Results

In this section, we are going to present some regularity results for the heat equation with homogeneous Neumann boundary conditions which are used throughout the paper.

Lemma B.1 ([33, Chap. 4, p. 32]) *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be bounded connected open set with regular boundary.*

Given $f \in L^2(Q)$ and $u_0 \in H^1(\Omega)$, there exists a unique solution

$$u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$$

to the heat equation

$$\begin{cases} u_t - \Delta u = f, & \textit{in } Q,\\ \partial_\nu u = 0, & \textit{on } \Sigma,\\ u(\cdot,0) = u_0, & \textit{in } \Omega.\end{cases} \tag{B.1}$$

Moreover, there exists a constant $C > 0$ *such that*

$$\|u\|^2_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \leq C\left(\|f\|^2_{L^2(Q)} + \|u_0\|^2_{H^1(\Omega)}\right).$$

Also, given $f \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ *and* $u_0 \in H^3(\Omega)$ *with* $\partial_\nu u_0 = 0$ *on* $\partial\Omega$*, there exists a unique solution to* (B.1) *such that*

$$u \in L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^3(\Omega))$$

and there exists a constant $C > 0$ *such that*

$$\begin{aligned}&\|u\|^2_{L^2(0,T;H^4(\Omega))\cap H^2(0,T;L^2(\Omega))\cap L^\infty(0,T;H^3(\Omega))}\\ &\quad\leq C\left(\|f\|^2_{L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} + \|u_0\|^2_{H^3(\Omega)}\right).\end{aligned}$$

Lemma B.2 ([33, Chap. 4, p. 32]) *Let* $T > 0$, $\Omega \subset \mathbb{R}^N$ *be bounded connected open set with regular boundary. Given* $f \in L^2(0,T;H^1(\Omega))$ *and* $u_0 \in H^2(\Omega)$ *with* $\partial_\nu u_0 = 0$ *on* $\partial\Omega$*, there exists a unique solution to* (B.1) *such that*

$$u \in L^2(0,T;H^3(\Omega)) \cap H^{3/2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ *such that*

$$\begin{aligned}&\|u\|^2_{L^2(0,T;H^3(\Omega))\cap H^{3/2}(0,T;L^2(\Omega))\cap L^\infty(0,T;H^2(\Omega))}\\ &\quad\leq C\left(\|f\|^2_{L^2(0,T;H^1(\Omega))} + \|u_0\|^2_{H^2(\Omega)}\right).\end{aligned}$$

We conclude this section with a regularity result for a coupled system that has been used throughout the article.

Proposition B.1 *Let* $T > 0$, $\Omega \subset \mathbb{R}^N$ *be bounded connected open set with regular boundary. Given,* $(f_1, f_2) \in [L^2(Q)]^2$*, there exists a unique solution* $(u, v) \in [L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))]^2$ *to the system*

$$\begin{cases} u_t - \Delta u + M_1 \Delta v = f_1, & \text{in } Q, \\ v_t - \Delta v + bv - au = f_2, & \text{in } Q, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \Sigma, \\ u(\cdot,0) = v(\cdot,0) = 0, & \text{in } \Omega. \end{cases} \tag{B.2}$$

Moreover, there exists $C > 0$ *such that*

$$\|(u,v)\|_{[L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))]^2} \leq C\|(f_1,f_2)\|_{[L^2(Q)]^2}.$$

Given $(f_1, f_2) \in L^2(Q) \times L^2(0,T;H^1(\Omega))$*, there exists a unique solution* (u, v) *to system* (B.2) *with*

$$(u, v) \in \left[L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))\right] \\ \times \left[L^2(0,T;H^3(\Omega)) \cap H^{3/2}(0,T;L^2(\Omega))\right].$$

Also, there exists $C > 0$ *such that*

$$\|(u, v)\|^2_{\left[L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))\right]\times\left[L^2(0,T;H^3(\Omega))\cap L^\infty(0,T;H^2(\Omega))\right]} \\ \leq C\|(f_1, f_2)\|_{L^2(Q)\times L^2(0,T;H^1(\Omega))}.$$

For any $(f_1, f_2) \in [L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))]^2$ *there exists a unique solution* (u, v) *to system* (B.2) *with*

$$(u, v) \in \left[L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega))\right]^2.$$

and such that

$$\|(u, v)\|^2_{[L^2(0,T;H^4(\Omega))\cap H^2(0,T;L^2(\Omega))]^2} \leq C\|(f_1, f_2)\|_{[L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))]^2}.$$

Proof Multiplying $(B.2)_1$ by u, and $(B.2)_2$, by v, and then by $-\Delta v$, after several integration by parts, we deduce that there exists $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(u, v)\|^2_{L^2(\Omega)^2} + \|\nabla v\|^2_{L^2(\Omega)}\right) + \|v\|^2_{L^2(\Omega)} + \|(\nabla u, \nabla v)\|^2_{L^2(\Omega)^2} + \|\Delta v\|^2_{L^2(\Omega)} \\ \leq C\left(\|(u, v)\|^2_{L^2(\Omega)^2} + \|\nabla v\|^2_{L^2(\Omega)} + \|(f_1, f_2)\|^2_{L^2(\Omega)}\right). \tag{B.3}$$

Now, multiplying $(B.2)_1$ by $-\Delta u$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2_{L^2(\Omega)}\right) + \|\Delta u\|^2_{L^2(\Omega)} \leq C\left(\|\Delta v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right). \tag{B.4}$$

From (B.3) and (B.4), the first estimate follows when $(f_1, f_2) \in [L^2(Q)]^2$.

When $f_2 \in L^2(0,T;H^1(\Omega))$, from $(B.2)_2$ and Lemma B.2, we have the second estimate.

If $(f_1, f_2) \in [L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))]^2$, we already have that $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and then, using equation $(B.2)_2$ and Lemma (B.1), it follows that

$$v \in L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^3(\Omega)).$$

Now, using $(B.2)_1$ it yields the regularity for u. □

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Data and Code Availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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