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A Direct Approach to the Boundary Controllability of the Hyperbolic Stokes System

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Abstract

The boundary null controllability for the parabolic Stokes system and the hyperbolic Stokes system is, up to now, obtained via an artificial argument consisting in enlarging the domain, using the results of distributed controllability in a small domain contained in the enlarged part, and then obtaining the boundary controllability by restriction. In the present paper we give in a first part some direct results of boundary controllability for the hyperbolic Stokes system in a star-shaped domain. We then apply the transmutation method to obtain a result of null controllability for the parabolic Stokes system. One interest is that we can characterize the control that we use and this gives a rather surprising result.

Keywords Stokes system · Hyperbolic Stokes · Boundary controllability · Transmutation

Mathematics Subject Classification Primary: 93B05 · 93B07; Secondary: 35K40 · 35L51

1 Introduction

The controllability of fluid equations has drawn significant attention from the mathematical community since the late 1990s (see, for instance, Coron 1996; Fabre 1995; Fursikov and Imanuvilov 1999 and Imanuvilov 2001). Control problems involving

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fluid flow arise in a wide range of practical situations, such as climate modeling, optimal design of aircraft shapes, flow regulation in water channels, blood flow regulation, among others (see Fernández-Cara 2020 and Mallea-Zepeda et al. 2019 and references therein). These problems are challenging due to the complex behavior of fluids, particularly turbulence effects and the intrinsic existence of the pressure, which make the mathematical analysis and control of such systems notably difficult.

The distributed null controllability of the Stokes system, which is a parabolic system, has been extensively studied in the last period (see Fernández-Cara et al. 2004; Fursikov and Imanuvilov 1996; Imanuvilov et al. 2009 for example) but, up to now, the boundary controllability is not obtained directly. By an artificial argument, one can enlarge the physical domain and extend all data, then consider a problem of distributed controllability with a control acting in a small open subset of the enlarged part of the domain, and then restrict the whole system to the actual domain which gives a result of boundary null controllability. One drawback of this method is that we have no characterization of the control and also the method is really artificial. In the present paper, we intend to study directly the boundary controllability problem. First, we study the hyperbolic Stokes system (see below for the exact description), for which we can obtain a result of exact controllability and then, using the transmutation method, we obtain a result of boundary null controllability for the Stokes system with a geometric condition which, of course, is not necessary but is a restriction of the method we use.

The strategy applied for the hyperbolic Stokes system is quite classical and is based on Lions' Hilbert Uniqueness Method (HUM), see Lions (1988). It turns out that the arguments used for the wave equation can be adapted to the hyperbolic Stokes system. Nevertheless, the result that we obtain for the observability inequality is rather surprising at first sight, as it seems to be in contradiction with a result of unique continuation by C. Fabre and G. Lebeau, see Fabre and Lebeau (2002). We will explain later on why there is no contradiction (see Remark 3.2).

In what follows, the set Ω is open, bounded and of class C^2 of $\mathbb{R}^N$, $N = 2$ or 3. Throughout the article, even if some of the results do not require this assumption, we assume that Ω is star-shaped and without loss of generality we can assume that Ω is star-shaped around $0 \in \Omega$.

We denote by Γ the boundary of Ω, by Γ_0 an open subset of Γ and by ν the outward pointing unit normal vector. We introduce the classical functional spaces

$$V = \{\psi \in H_0^1(\Omega)^N,\ \mathrm{div}\,\psi = 0\} \text{ and } H = \{\psi \in L^2(\Omega)^N,\ \mathrm{div}\,\psi = 0,\ \psi.\nu = 0\}. \tag{1}$$

It is well known that the space

$$\mathcal{V} = \{\psi \in C_0^\infty(\Omega)^N,\ \mathrm{div}\,\psi = 0\},$$

is dense in V and H (cf Temam 1977).

Let us now describe precisely the problem of exact controllability for the hyperbolic Stokes system. We consider the following system that we call hyperbolic Stokes system

$$(HSS)\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y + \nabla p = 0 & \text{in } \Omega \times (0,T), \\ \operatorname{div} y = 0 & \text{in } \Omega \times (0,T), \\ y = v & \text{on } \Gamma_0 \times (0,T), \\ y = 0 & \text{on } \Gamma \setminus \Gamma_0 \times (0,T), \\ y(0) = y_0, \ \frac{\partial y}{\partial t}(0) = y_1 & \text{in } \Omega. \end{cases} \tag{2}$$

First of all, we would like to have existence and uniqueness of a solution for this system with $y_0 \in H,\ y_1 \in V'$ and the control $v \in L^2(\Gamma_0 \times (0,T))$. The divergence free condition implies of course that for every $t \in (0,T)$

$$\int_{\Gamma_0} v(t) \cdot \nu \, d\gamma = 0.$$

Of course, with the given regularity we can only expect solutions in weak sense, which will be achieved with a stronger condition on v (see Theorem 2.6).

Now, assuming that we have a solution of the direct problem (2), the exact boundary controllability problem is the following: *given* $y^0 \in H$ *and* $y^1 \in V'$, *find* v *in the correct class such that, at time* T, *we have*

$$y(T) = y^0 \text{ and } \frac{\partial y}{\partial t}(T) = y^1. \tag{3}$$

Since system (2) is reversible in time, this problem is equivalent to find v such that

$$y(T) = 0 \text{ and } \frac{\partial y}{\partial t}(T) = 0. \tag{4}$$

The strategy that we will use follows the lines of HUM and is based on the obtainment of a suitable observability inequality for the adjoint system.

This paper is organized as follows: in Section 2, we will give preliminary results on the adjoint system and prove an existence and uniqueness result for a weak solution of HSS with a surprising condition on the control. In Section 3, we will obtain an observability inequality for the adjoint system and then the main result of exact boundary controllability for HSS. In Section 4, we will use the transmutation method to obtain a direct result of boundary null controllability for the (parabolic) Stokes system (SS) with a geometric control condition.

2 The Adjoint system and Well-Posedness for HSS

To study the controllability problem for (2), let us introduce the generalized adjoint system. Indeed, for $h \in L^1(0,T,H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$, we look for (φ, q) solution of the following system

$$\begin{cases} \frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + \nabla q = h & \text{in } \ \Omega \times (0,T), \\ \operatorname{div}\varphi = 0 & \text{in } \ \Omega \times (0,T), \\ \varphi = 0 & \text{on } \Gamma \times (0,T), \\ \varphi(0) = \varphi_0, \frac{\partial\varphi}{\partial t}(0) = \varphi_1 & \text{in } \ \Omega. \end{cases} \tag{5}$$

We have the following existence and regularity result.

Proposition 2.1 *For every $h \in L^1(0,T;H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$, there exists a unique solution φ to (5) with*

$$\varphi \in C([0,T];V) \cap C^1([0,T];H).$$

If we write the energy

$$E_\varphi(t) = \frac{1}{2}\left(\left|\frac{\partial\varphi}{\partial t}(t)\right|_H^2 + |\nabla\varphi(t)|_H^2\right),$$

then

$$\sqrt{E_\varphi(t)} \le \sqrt{E_\varphi(0)} + \frac{1}{\sqrt{2}}\int_0^t |h|_H ds. \tag{6}$$

When $h = 0$ we have conservation of energy

$$E_\varphi(t) = E_\varphi(0) \ \ \textit{for every} \ \ t \in (0,T).$$

Moreover, writing $W := V \cap H^2(\Omega)^N$, if $h \in L^1(0,T;V)$, $\varphi_0 \in W$ and $\varphi_1 \in V$, then

$$\varphi \in C([0,T];W) \cap C^1([0,T];V) \ \textit{and} \ q \in L^1(0,T;H^1(\Omega)).$$

Proof The proof is classical and based on the Fourier method. For this reason, we just give a sketch of the proof.

Let λ_n be the ordered eigenvalues (with $\lambda_n \to \infty$ as $n \to \infty$) and w_n the corresponding normalized eigenfunctions in H (with π_n being the corresponding pressure) for the stationary Stokes operator

$$\begin{cases} -\Delta w_n + \nabla\pi_n = \lambda_n w_n & \text{in } \ \Omega, \\ \operatorname{div} w_n = 0 & \text{in } \ \Omega, \\ w_n = 0 & \text{on } \Gamma. \end{cases} \tag{7}$$

We have

$$(w_n, w_m)_H = \delta_{n,m}.$$

From Temam (1977), we know that $w_n \in W$ and that $\pi_n \in H^1(\Omega)$. Also, $\{w_n\}_n$ is a Hilbert basis in H, $\{\frac{w_n}{\sqrt{\lambda_n}}\}_n$ is a Hilbert basis in V and $\{\frac{w_n}{\lambda_n}\}_n$ is a Hilbert basis in W.

We then know that

$$h(t) = \sum_{k=1}^{+\infty} h_k(t) w_k, \quad \int_0^T \left(\sum_{k=1}^{+\infty} |h_k(t)|^2\right)^{\frac{1}{2}} dt = ||h||_{L^1(0,T;H)},$$

$$\varphi_0 = \sum_{k=1}^{+\infty} a_{k,0} w_k, \quad \sum_{k=1}^{+\infty} \lambda_k |a_{k,0}|^2 = ||\varphi_0||_V^2 = ||\varphi_0||^2,$$

and

$$\varphi_1 = \sum_{k=1}^{+\infty} b_{k,1} w_k, \quad \sum_{k=1}^{+\infty} |b_{k,1}|^2 = |\varphi_1|_H^2 = |\varphi_1|^2.$$

We consider the truncated series

$$h^K(t) = \sum_{k=1}^{K} h_k(t) w_k,$$
$$\varphi_0^K = \sum_{k=1}^{K} a_{k,0} w_k,$$

and

$$\varphi_1^K = \sum_{k=1}^{K} b_{k,1} w_k.$$

We now look for an associated solution

$$\varphi^K(t) = \sum_{k=1}^{K} a_k(t) w_k \quad \text{and} \quad q^K = \sum_{k=1}^{K} c_k \pi_k$$

for the truncated problem

$$\begin{cases} \frac{\partial^2 \varphi^K}{\partial t^2} - \Delta \varphi^K + \nabla q^K = h^K & \text{in } \Omega \times (0, T), \\ \operatorname{div} \varphi^K = 0 & \text{in } \Omega \times (0, T), \\ \varphi^K = 0 & \text{on } \Gamma \times (0, T), \\ \varphi^K(0) = \varphi_0^K, \ \frac{\partial \varphi^K}{\partial t}(0) = \varphi_1^K & \text{in } \Omega. \end{cases} \tag{8}$$

For $k = 1, ..., K$, we have the diagonal system

$$\begin{cases} \frac{d^2 a_k}{dt^2}(t) + \lambda_k a_k(t) = h_k(t), & t \in (0, T), \\ a_k(0) = a_{k,0}, \ \frac{da_k}{dt}(0) = b_{k,1}. \end{cases} \tag{9}$$

It is easy to see that, for each k, there exists a unique solution $a_k \in C^1([0, T])$. With the first hypotheses on h, φ_0 and φ_1, after multiplication by $\dfrac{da_k}{dt}$, we see that φ_K is a Cauchy sequence in $C([0, T]; V) \cap C^1([0, T]; H)$. Therefore, φ_K converges to a function φ in these spaces and φ is a solution of (5) associated with a pressure q.

Uniqueness is standard as, if $\bar{\varphi}$ is another solution and if its decomposition on the basis $\{w_n\}$ is

$$\bar{\varphi} = \sum_{k=1}^{+\infty} \bar{a}_k w_k,$$

multiplying the equation by w_k shows that $\bar{a_k}$ satisfies the same equation as a_k.

The energy inequality (6) is straightforward.

Concerning the regularity result, with the second set of hypotheses in Proposition 2.1 we have to multiply the equation for a_k by $\lambda \dfrac{da_k}{dt}$ to obtain the desired result for φ. Using the equation for a_k we see that $\dfrac{\partial^2 \varphi}{\partial t^2} \in L^1(0, T; H)$ which shows that $\nabla q \in L^1(0, T; L^2(\Omega)^N)$ and this implies $q \in L^1(0, T; H^1(\Omega))$. □

Remark 2.2 When $h \in L^1(0, T; V)$, $\varphi_0 \in W$, and $\varphi_1 \in V$, we have seen that $\varphi \in C([0, T]; W)$, which implies $\operatorname{div} \varphi \in C([0, T]; H^1(\Omega))$ and, in particular, $\operatorname{div} \varphi|_\Gamma \in C([0, T]; H^{1/2}(\Gamma))$ and therefore $\operatorname{div} \varphi|_\Gamma = 0$. Since $\varphi|_\Gamma = 0$, writing $\varphi = (\varphi^1, ..., \varphi^N)$, then

$$\frac{\partial \varphi^k}{\partial x_k} = \frac{\partial \varphi^k}{\partial \nu} \nu_k \quad \text{on} \quad \Gamma.$$

Hence

$$\sum_{k=1}^{N} \frac{\partial \varphi^k}{\partial \nu} \nu_k = 0 \quad \text{on} \quad \Gamma. \tag{10}$$

We will now show a hidden regularity result on the solution of system (5). First of all we derive a multiplier identity. Unlike the case of the wave equation, we will only consider a multiplier of the form $m(x) = x - x_0$, because for general multipliers we would need information on the pressure which is not available.

Lemma 2.3 *Let $x_0 \in \mathbb{R}^N$ and denote by Γ_{x_0} the set*

$$\Gamma_{x_0} = \{x \in \Gamma,\ (x - x_0) \cdot \nu > 0\}.$$

Let us denote by $m(x)$ the multiplier

$$m(x) = x - x_0.$$

Then, for every solution of (5) with $h \in L^1(0, T; V)$, $\varphi_0 \in W$ and $\varphi_1 \in V$ we have, using the summation convention for repeated indices, the following multiplier identity

$$\begin{aligned}
&\left(\frac{\partial \varphi^k}{\partial t}(T), m.\nabla \varphi^k(T)\right)_{L^2(\Omega)} - (\varphi_1^k, m.\nabla \varphi_0^k)_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\int_0^T \int_\Omega (|\frac{\partial \varphi^k}{\partial t}|^2 + |\nabla \varphi^k|^2)\, dx dt + \frac{(N-1)}{2} \int_0^T \int_\Omega (|\frac{\partial \varphi^k}{\partial t}|^2 - |\nabla \varphi^k|^2)\, dx dt \\
&\quad - \int_0^T \int_\Omega h^k m.\nabla \varphi^k\, dx dt - \frac{1}{2}\int_0^T \int_{\Gamma \setminus \Gamma_{x_0}} (m.\nu) |\frac{\partial \varphi^k}{\partial \nu}|^2\, d\gamma dt \\
&= \frac{1}{2}\int_0^T \int_{\Gamma_{x_0}} (m.\nu) |\frac{\partial \varphi^k}{\partial \nu}|^2\, d\gamma dt.
\end{aligned}$$

Proof We multiply the equation of φ^k by $m(x).\nabla\varphi^k$ and we integrate on $\Omega \times (0.T)$. We obtain by integration by parts

$$\left(\frac{\partial\varphi^k}{\partial t}(T), m_i \frac{\partial\varphi^k}{\partial x_i}(T)\right)_{L^2(\Omega)} - (\varphi_1^k, m_i \frac{\partial\varphi_0^k}{\partial x_i})_{L^2(\Omega)}$$
$$-\frac{1}{2}\int_0^T\int_\Omega m_i \frac{\partial}{\partial x_i}|\frac{\partial\varphi^k}{\partial t}|^2\,dxdt - \int_0^T\int_\Gamma \frac{\partial\varphi^k}{\partial x_j}\nu_j m_i \frac{\partial\varphi^k}{\partial x_i}\,d\gamma dt$$
$$+\int_0^T\int_\Omega \frac{\partial\varphi^k}{\partial x_j}\frac{\partial m_i}{\partial x_j}\frac{\partial\varphi^k}{\partial x_i}\,dxdt + \frac{1}{2}\int_0^T\int_\Omega m_i \frac{\partial}{\partial x_i}|\frac{\partial\varphi^k}{\partial x_j}|^2\,dxdt$$
$$+\int_0^T\int_\Gamma q m_i \nu_k \frac{\partial\varphi^k}{\partial x_i}\,d\gamma dt - \int_0^T\int_\Omega q\frac{\partial m_i}{\partial x_k}\frac{\partial\varphi^k}{\partial x_i}\,dxdt = \int_0^T\int_\Omega h^k m.\nabla\varphi^k\,dxdt.$$

Now, we know that $\dfrac{\partial m_i}{\partial x_j} = \delta_{i,j}$, also that on the boundary $\dfrac{\partial\varphi^k}{\partial x_i} = \dfrac{\partial\varphi^k}{\partial\nu}.\nu_i$ and from Remark 2.2 that we have (10). This gives the multiplier identity for regular data. Notice that due to our choice of multiplier and Remark 2.2, all terms including the pressure have disappeared. □

Theorem 2.4 *For every $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$, the solution φ of (5) satisfies*

$$\frac{\partial\varphi}{\partial\nu} \in L^2(0, T; L^2(\Gamma)^N). \tag{11}$$

Moreover, the mapping

$$(h, \varphi_0, \varphi_1) \in L^1(0, T; H) \times V \times H \to \frac{\partial\varphi}{\partial\nu} \in L^2(0, T; L^2(\Gamma)^N)$$

is linear continuous.

Proof Let us start with a dense set of regular data so that the previous Lemma applies. We here take $x_0 = 0$ and we recall the assumptiom that Ω is star-shaped around 0 so that $\Gamma_{x_0} = \Gamma$. The multiplier identity implies

$$\frac{1}{2}\int_0^T\int_\Gamma (m.\nu)|\frac{\partial\varphi^k}{\partial\nu}|^2\,d\gamma dt = \left(\frac{\partial\varphi^k}{\partial t}(T), m.\nabla\varphi^k(T)\right)_{L^2(\Omega)} - (\varphi_1^k, m.\nabla\varphi_0)_{L^2(\Omega)}$$
$$+\frac{1}{2}\int_0^T\int_\Omega (|\frac{\partial\varphi^k}{\partial t}|^2 + |\nabla\varphi^k|^2)\,dxdt + \frac{(N-1)}{2}\int_0^T\int_\Omega (|\frac{\partial\varphi^k}{\partial t}|^2 - |\nabla\varphi^k|^2)\,dxdt$$
$$-\int_0^T\int_\Omega h^k m.\nabla\varphi^k\,dxdt.$$

Because of the bound for the energy of φ, all terms in the right hand side are bounded by

$$C(||\varphi_0||_V^2 + |\varphi_1|_H^2 + |h|^2_{L^1(0,T;H)}),$$

where C is a constant independent of the data. By continuity, we obtain that

$$\frac{\partial \varphi}{\partial \nu} \in L^2(0, T; L^2(\Gamma)^N),$$

and that the mapping

$$(h, \varphi_0, \varphi_1) \in L^1(0, T; H) \times V \times H \to \frac{\partial \varphi}{\partial \nu} \in L^2(0, T; L^2(\Gamma)^N)$$

is linear continuous. □

Corollary 2.5 *For every $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$, the solution φ of (5) satisfies*

$$\sum_{k=1}^{N} \frac{\partial \varphi^k}{\partial \nu}.\nu_k = 0 \text{ a.e. on } \Gamma \times (0, T). \tag{12}$$

Proof From Remark 2.2, we know that the result is true for the dense set of regular data $h \in L^1(0, T; V)$, $\varphi_0 \in W$, and $\varphi_1 \in V$ and, from Theorem 2.4, we can extend this result to the case $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$. □

We can now obtain the existence result for a weak solution of (2). For this, we need to introduce the following space

$$L^2_0(\Gamma_0)^N = \{v \in L^2(\Gamma_0)^N, \quad v \cdot \nu = \sum_{k=1}^{N} v^k \nu_k = 0\}. \tag{13}$$

Theorem 2.6 *For every $(y_0, y_1) \in H \times V'$ and any $v \in L^2(0, T; L^2_0(\Gamma_0)^N)$, there exists a unique solution y of (2), in the sense that*

$$y \in C([0, T]; H) \cap C^1([0, T]; V')$$

and that

$$\begin{aligned} &\int_0^T \int_\Omega h.y \, dxdt - (\varphi_1, y(T))_H + < \varphi_0, \frac{\partial y}{\partial t}(T) >_{V,V'} \\ &=< \varphi(0), y_1 >_{V,V'} - \left(\frac{\partial \varphi}{\partial t}(0), y_0\right)_H - \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu}.v \, d\gamma dt, \end{aligned} \tag{14}$$

for every $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$

Proof We will argue using the transposition method.

In a first step, let us take a dense set of regular data so that the existence of a (regular) solution for (2) is known, for example, by extension to the whole domain of the (regular) boundary control.

Let us now take $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$ and consider the solution (φ, q) of the reversed generalized adjoint system (with the change of time $t \to (T-t)$).

$$\begin{cases} \frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi + \nabla q = h & \text{in } \Omega \times (0, T), \\ \operatorname{div} \varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \Gamma \times (0, T), \\ \varphi(T) = \varphi_0, \frac{\partial \varphi}{\partial t}(T) = \varphi_1 & \text{in } \Omega. \end{cases} \tag{15}$$

Let us multiply the Eq. (15) by the (regular) solution y of (2) and integrate by parts. This gives

$$\begin{aligned} &(\varphi_1, y(T))_H - \left(\frac{\partial \varphi}{\partial t}(0), y_0\right)_H - < \varphi_0, \frac{\partial y}{\partial t}(T) >_{V,V'} + < \varphi(0), y_1 >_{V,V'} \\ &\quad + \int_0^T \int_\Omega \varphi . \left(\frac{\partial^2 y}{\partial t^2} - \Delta y\right) dxdt - \int_0^T \int_{\Gamma_0} \left(\frac{\partial \varphi}{\partial \nu} - q\nu\right) \cdot v \, d\gamma dt \\ &= \int_0^T \int_\Omega h.y \, dxdt. \end{aligned}$$

We know that $v \cdot \nu = 0$ and that $\int_0^T \int_\Omega \nabla p.\varphi \, dxdt = 0$, which yields

$$\begin{aligned} &\int_0^T \int_\Omega h.y \, dxdt - (\varphi_1, y(T))_H + < \varphi_0, \frac{\partial y}{\partial t}(T) >_{V,V'} \\ &\quad =< \varphi(0), y_1 >_{V,V'} - \left(\frac{\partial \varphi}{\partial t}(0), y_0\right)_H - \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} \cdot v \, d\gamma dt \end{aligned}$$

For $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$, we define a functional $\mathcal{L}$ by

$$\mathcal{L}(h, \varphi_0, \varphi_1) =< \varphi(0), y_1 >_{V,V'} - \left(\frac{\partial \varphi}{\partial t}(0), y_0\right)_H - \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial \nu} \cdot v \, d\gamma dt. \tag{16}$$

From Proposition 2.1 and Theorem 2.4, we have

$$|\mathcal{L}(h, \varphi_0, \varphi_1)| \leq C(|h|_{L^1(0,T;H)} + ||\varphi_0||_V + |\varphi_1|_H)(|y_0|_H + ||y_1||_{V'} + |v|_{L^2(0,T;L^2(\Gamma_0))}). \tag{17}$$

Therefore, $\mathcal{L}$ is a linear continuous form on $L^1(0, T; H) \times V \times H$. Then, there exists a unique triplet $(y, \bar{y}_T, \partial \bar{y}_T) \in L^\infty(0, T; H) \times H \times V'$ such that for every $h \in L^1(0, T; H)$, $\varphi_0 \in V$ and $\varphi_1 \in H$,

$$\mathcal{L}(h, \varphi_0, \varphi_1) = \int_0^T \int_\Omega y.h \, dxdt - (\bar{y}_T, \varphi_1)_H + < \partial \bar{y}_T, \varphi_0 >_{V',V} . \tag{18}$$

Moreover, we have

$$|y|_{L^\infty(0,T;H)} + |\bar{y}_T|_H + ||\partial \bar{y}_T||_{V'} \leq C(|y_0|_H + ||y_1||_{V'} + |v|_{L^2(0,T;L^2(\Gamma_0))}).$$

For a dense set of regular data, we have $y \in C([0,T];H)$, $\bar{y}_T = y(T)$ and $\partial \bar{y}_T = \frac{\partial y}{\partial t}(T)$. Therefore, using a Cauchy sequence of regular data, we conclude that the solution $(y, \bar{y}_T)$ of (18) satisfy

$$y \in C([0.T];H) \text{ and } \bar{y}(T) = y(T).$$

Now we prove that $y \in C^1([0,T];V')$, which will ensure that $\partial \bar{y}_T = \frac{\partial y}{\partial t}(T)$.

In order to do that, we take $h \in \mathcal{D}(]0,T[;\mathcal{V}) = C_0^\infty(]0,T[;\mathcal{V})$ (which is a dense set in $L^1(0,T;V)$) and $\varphi_0 = 0$, $\varphi_1 = 0$. Then the corresponding solution (φ, q) of (5) is regular.

We only have to prove that there exists $C > 0$ such that

$$|\mathcal{L}\left(\frac{\partial h}{\partial t}, 0, 0\right)| \leq C||h||_{L^1(0,T;V)}.$$

Let us write

$$w = \frac{\partial \varphi}{\partial t}.$$

We have

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} - \Delta w + \nabla r = \frac{\partial h}{\partial t} & \text{in } \Omega \times (0,T), \\ \operatorname{div} w = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \Gamma \times (0,T), \\ w(T) = 0, \ \frac{\partial w}{\partial t}(T) = -\nabla q(T) & \text{in } \Omega, \end{cases} \tag{19}$$

and

$$\mathcal{L}\left(\frac{\partial h}{\partial t}, 0, 0\right) = < w(0), y_1 >_{V,V'} - \left(\frac{\partial w}{\partial t}(0), y_0\right)_H - \int_0^T \int_{\Gamma_0} \frac{\partial w}{\partial \nu} \cdot v \, d\gamma dt. \tag{20}$$

From the regularity of φ, we know that

$$||w(0)||_V = ||\frac{\partial \varphi}{\partial t}(0)||_V \leq C||h||_{L^1(0,T;V)},$$

and

$$|\frac{\partial w}{\partial t}(0)|_H = |\Delta\varphi(0) - \nabla q(0)|_H = |\Delta\varphi(0)|_{L^2(\Omega)^N} \leq C||h||_{L^1(0,T;V)}.$$

For the third term on the right-hand side of (20), we have:

$$\left|\frac{\partial w}{\partial \nu}\right|_{L^2(0,T:L^2(\Gamma)} \leq C||h||_{L^1(0,T;V)}.$$

To prove this, we use for w the same multiplier identity as in Lemma 2.3. This gives

$$
\begin{aligned}
I_1 = &-\left(\frac{\partial w}{\partial t}(0), m.\nabla w(0)\right)_{L^2(\Omega)^N} + \frac{N}{2}\int_0^T\int_\Omega (|\frac{\partial w}{\partial t}|^2 - |\nabla w|^2)\, dxdt \\
&+ \int_0^T\int_\Omega |\nabla w|^2\, dxdt \\
&- \frac{1}{2}\int_0^T\int_\Gamma (m.\nu)|\frac{\partial w}{\partial \nu}|^2\, d\gamma dt = \int_0^T\int_\Omega \frac{\partial h}{\partial t} m.\nabla w\, dxdt \\
= &\int_0^T\int_\Omega (m.\nabla h + Nh).\frac{\partial w}{\partial t}\, dxdt.
\end{aligned}
$$

Using the equation for φ, we have

$$
\begin{aligned}
I_1 = &\int_0^T\int_\Omega (m.\nabla h.h - m.\nabla h.\nabla q + m.\nabla h.\Delta\varphi)\, dxdt \\
&+ N\int_0^T\int_\Omega (|h|^2 - h.\nabla q + h.\Delta\varphi)\, dxdt \\
= &\frac{N}{2}\int_0^T\int_\Omega |h|^2\, dxdt + N\int_0^T\int_\Omega h.\Delta\varphi\, dxdt + \int_0^T\int_\Omega m.\nabla h.\Delta\varphi\, dxdt
\end{aligned}
$$

We also have

$$
\begin{aligned}
I_2 = &\frac{N}{2}\int_0^T\int_\Omega |\frac{\partial w}{\partial t}|^2\, dxdt = \frac{N}{2}\int_0^T |\frac{\partial^2\varphi}{\partial t^2}|_H^2\, dt = \frac{N}{2}\int_0^T |h - \nabla q + \Delta\varphi|_H^2\, dt \\
= &\frac{N}{2}\int_0^T\int_\Omega (|h|^2 + 2h.\Delta\varphi)\, dxdt + \frac{N}{2}\int_0^T |\Delta\varphi - \nabla q|_H^2\, dt).
\end{aligned}
$$

Using once again that $|\Delta\varphi(t) - \nabla q(t)|_H^2 = \int_\Omega |\Delta\varphi(t)|^2\, dx$, and the regularity of φ, we see that

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_{\Gamma_{x_0}} (m.\nu)|\frac{\partial w}{\partial \nu}|^2\, d\gamma dt \le I_2 - I_1 + |\left(\frac{\partial w}{\partial t}(0), m.\nabla w(0)\right)_{L^2(\Omega)^N}| \\
&+ \frac{(N-2)}{2}\int_0^T\int_\Omega |\nabla\frac{\partial\varphi}{\partial > t}|^2\, dxdt \le C||h||^2_{L^1(0,T;V)}.
\end{aligned}
$$

Using the extension by continuity, we obtain

$$
\int_0^T\int_\Gamma |\frac{\partial w}{\partial \nu}|^2\, d\gamma dt \le C||h||^2_{L^1(0,T;V)}, \quad \forall h \in L^1(0,T;V). \tag{21}
$$

This finishes the proof of Theorem 2.6. □

Remark 2.7 In order to obtain a solution for the direct problem (2), we have strongly used the assumption $v.\nu = 0$ on Γ_0 for the external force. The reason was to get rid of all terms including the pressure and, in particular, the trace of the pressure. If we do not have this assumption we would need a more regular solution for the adjoint system and therefore, a weaker notion of solution for (2). We will see below that the restriction on v does not affect the process of exact controllability.

3 Exact Boundary Controllability

We will use the Hilbert Uniqueness Method (HUM) of J.-L. Lions (see Lions 1988), which relies on an observability inequality for the adjoint system.

3.1 Observability inequality

For $\varphi_0 \in V$ and $\varphi_1 \in H$, consider the adjoint system

$$\begin{cases} \frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi + \nabla q = 0 & \text{in } \ \Omega \times (0, T), \\ \operatorname{div} \varphi = 0 & \text{in } \ \Omega \times (0, T), \\ \varphi = 0 & \text{on } \Gamma \times (0, T), \\ \varphi(0) = \varphi_0, \frac{\partial\varphi}{\partial t}(0) = \varphi_1 & \text{in } \ \Omega. \end{cases} \tag{22}$$

We know that $\varphi \in C([0, T]; V) \cap C^1([0, T]; H)$ and that $\dfrac{\partial\varphi}{\partial\nu} \in L^2(0, T; L^2(\Gamma))$. We also know that the energy of φ is conserved and, from Corollary 2.5, that $\dfrac{\partial\varphi}{\partial\nu}.\nu = 0$.

The following Theorem gives us the observability inequality for the solution φ of (22).

Theorem 3.1 *For $x_0 \in \mathbb{R}^N$ let $R(x_0) = \max_{x\in\Omega} |x - x_0|$. If*

$$\exists\, x_0 \in \mathbb{R}^N \text{ such that } \Gamma_{x_0} \subset \Gamma_0, \text{ and } T > 2R(x_0), \tag{23}$$

we have

$$2\frac{(T - 2R(x_0))}{R(x_0)} E_\varphi(0) \le \int_0^T \int_{\Gamma_0} |\frac{\partial\varphi}{\partial\nu}|^2 \, d\gamma dt. \tag{24}$$

Proof Let $x_0 \in \mathbb{R}^N$ such that $\Gamma_{x_0} \subset \Gamma_0$. We use the multiplier identity obtained in Lemma 2.3 with the multiplier $m(x) = (x - x_0)$ and we proceed as for the case of the wave equation. Indeed, we have

$$\left(\frac{\partial\varphi}{\partial t}(T), m.\nabla\varphi(T)\right)_{L^2(\Omega)^N} - (\varphi_1, m.\nabla\varphi_0)_{L^2(\Omega)^N}$$
$$+\frac{1}{2}\int_0^T \int_\Omega (|\frac{\partial\varphi}{\partial t}|^2 + |\nabla\varphi|^2)\, dxdt + \frac{(N-1)}{2}\int_0^T \int_\Omega (|\frac{\partial\varphi}{\partial t}|^2 - |\nabla\varphi|^2)\, dxdt$$

$$-\frac{1}{2}\int_0^T\int_{\Gamma\backslash\Gamma_{x_0}}(m.\nu)|\frac{\partial\varphi}{\partial\nu}|^2\,d\gamma dt = \frac{1}{2}\int_0^T\int_{\Gamma_{x_0}}(m.\nu)|\frac{\partial\varphi}{\partial\nu}|^2\,d\gamma dt$$
$$\leq \frac{R(x_0)}{2}\int_0^T\int_{\Gamma_0}|\frac{\partial\varphi}{\partial\nu}|^2\,d\gamma dt.$$

The energy of φ being conserved, give us

$$\frac{1}{2}\int_0^T\int_\Omega(|\frac{\partial\varphi}{\partial t}|^2+|\nabla\varphi|^2)\,dxdt = TE_\varphi(0).$$

On the other hand, multiplying the adjoint equation (22) by φ, we obtain

$$\int_0^T\int_\Omega(|\frac{\partial\varphi}{\partial t}|^2-|\nabla\varphi|^2)\,dxdt = \left(\frac{\partial\varphi}{\partial t}(T),\varphi(T)\right)_H-(\varphi_0,\varphi_1)_H,$$

so that

$$\left(\frac{\partial\varphi}{\partial t}(T), m.\nabla\varphi(T)+\frac{(N-1)}{2}\varphi(T)\right)_{L^2(\Omega)^N}-(\varphi_1, m.\nabla\varphi_0+\frac{(N-1)}{2}\varphi_0)_{L^2(\Omega)^N}$$
$$+TE_\varphi(0)\leq\frac{R(x_0)}{2}\int_0^T\int_{\Gamma_0}|\frac{\partial\varphi}{\partial\nu}|^2\,d\gamma dt.$$

Now

$$\left(\frac{\partial\varphi}{\partial t}(T), m.\nabla\varphi(T)+\frac{(N-1)}{2}\varphi(T)\right)_{L^2(\Omega)^N}-(\varphi_1, m.\nabla\varphi_0+\frac{(N-1)}{2}\varphi_0)_{L^2(\Omega)^N}$$
$$\leq 2\max_{t\in[0,T]}|\left(\frac{\partial\varphi}{\partial t}(t), m.\nabla\varphi(t)+\frac{(N-1)}{2}\varphi(t)\right)_{L^2(\Omega)^N}|$$
$$\leq 2\max_{t\in(0,T)}\left(\frac{R(x_0)}{2}|\frac{\partial\varphi}{\partial t}(t)|^2+\frac{1}{2R(x_0)}|m.\nabla\varphi(t)+\frac{(N-1)}{2}\varphi(t)\right)_{L^2(\Omega)^N}|^2,$$

and

$$|m.\nabla\varphi(t)+\frac{(N-1)}{2}\varphi(t))|^2_{L^2(\Omega)^N} = |m.\nabla\varphi(t)|^2_{L^2(\Omega)^N}$$
$$-\left(\frac{N(N-1)}{2}-\frac{(N-1)^2}{4}\right)|\varphi(t)|^2_H$$
$$\leq|m.\nabla\varphi(t)|^2_{L^2(\Omega)^N}\leq R(x_0)^2|\nabla\varphi(t)|^2_{L^2(\Omega)^N}.$$

Therefore, we obtain

$$(T-2R(x_0))E_\varphi(0)\leq\frac{R(x_0)}{2}\int_0^T\int_{\Gamma_0}|\frac{\partial\varphi}{\partial\nu}|^2\,d\gamma dt,$$

which gives the result of Theorem 3.1. □

Remark 3.2 The previous result implies, in particular, a unique continuation result. When the normal derivative of φ vanishes on $\Gamma_0 \times (0, T)$, with T large enough, then the solution φ vanishes everywhere in $\Omega \times (0, T)$. We can notice that the pressure q does not enter in this condition. At first sight, this seems in contradiction with the classical result of C. Fabre and G. Lebeau (cf Fabre and Lebeau 2002), which says that the pressure must intervene to obtain a local result of unique continuation. But, the result in Fabre and Lebeau (2002) is, as already said, a local one and the Stokes operator (classical Stokes or hyperbolic Stokes) is not local because of the divergence free condition. Also, notice that the counterexample given in Fabre and Lebeau (2002) does not work in a bounded domain.

Remark 3.3 The observability inequality (24) was previouly obtained by the first author in Chaves-Silva (2015). Nevertheless, he only used it as a mean to achieve a distributed null controllability result. In fact, as seen in this paper, the use of the observability inequality (24) in the obtainment of a boundary control for the hyperbolic Stokes system is not as straightforward as one might think.

3.2 Exact boundary controllability result

We now proceed along the lines of HUM. Given $\varphi_0 \in V$ and $\varphi_1 \in H$ we solve (22) to obtain a solution $\varphi \in C([0, T]; V) \cap C^1([0, T]; H)$. Then, we consider the solution ψ of the system:

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + \nabla r = 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} \psi = 0 & \text{in } \Omega \times (0, T), \\ \psi = \frac{\partial \varphi}{\partial \nu} & \text{on } \Gamma_0 \times (0, T), \\ \psi = 0 & \text{on } \Gamma \setminus \Gamma_0 \times (0, T), \\ \psi(T) = 0, \ \frac{\partial \psi}{\partial t}(T) = 0 \text{ in } \Omega. \end{cases} \tag{25}$$

As we have $\dfrac{\partial \varphi}{\partial \nu}.\nu = 0$, we know from Theorem 2.6 that we have a solution to this system with

$$\psi \in C([0, T]; H) \cap C^1([0, T]; V').$$

For each $(\varphi_0, \varphi_1) \in V \times H$, we define

$$\Lambda(\varphi_0, \varphi_1) = \left(\frac{\partial \psi}{\partial t}(0), -\psi(0) \right) \in V' \times H. \tag{26}$$

From the exact definition of ψ (see Theorem 2.6), we have

$$< \Lambda(\varphi_0, \varphi_1), (\varphi_0, \varphi_1) >_{V' \times H, V \times H} = \int_0^T \int_{\Gamma_0} |\frac{\partial \varphi}{\partial \nu}|^2 \, d\gamma dt. \tag{27}$$

When Γ_0 satisfies the geometric condition (23), the observability inequality (24) holds and we can apply Lax-Milgram's Theorem to obtain

$$\forall (y_0, y_1) \in H \times V', \ \exists (\varphi_0, \varphi_1) \in V \times H \text{ such that } \Lambda(\varphi_0, \varphi_1) = (y_1, -y_0). \quad (28)$$

In particular,

$$\psi(0) = y_0 \text{ and } \frac{\partial \psi}{\partial t}(0) = y_1,$$

and thus ψ is the desired solution of (2).

Hence, we can state our main result:

Theorem 3.4 *Assume that the geometric condition (23) is satisfied for an $x_0 \in \mathbb{R}^N$. Then, for every $(y_0, y_1) \in H \times V'$, there exists a boundary control $v \in L^2(0, T; L^2(\Gamma_0))$ with $v.\nu = 0$ on Γ_0 such that the solution y of (2) satisfies*

$$y(T) = 0 \textit{ and } \frac{\partial y}{\partial t}(T) = 0.$$

Moreover, we can choose the control as being the normal derivative of the solution of a suitable adjoint system.

Remark 3.5 One main interest of the previous result is the fact that we can characterize a boundary control and choose it as being the normal derivative of the solution of an adjoint system without addition of a pressure term. This was not expected at all when we started to work on the subject.

4 Application to the Stokes System

We are now interested in the boundary controllability of the (parabolic) Stokes system.

For $z_0 \in H$, we look for a boundary control $v \in L^2(0, T; L^2(\Gamma_0))$, with $\int_{\Gamma_0} v(t).\nu = 0$ for every $t \in (0, T)$, and such that the solution z of the Stokes system

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z + \nabla r = 0 & \text{in } \ \Omega \times (0, T), \\ \operatorname{div} z = 0 & \text{in } \ \Omega \times (0, T), \\ z = v & \text{on } \Gamma_0 \times (0, T), \\ z = 0 & \text{on } \Gamma \setminus \Gamma_0 \times (0, T), \\ z(0) = z_0, & \text{in } \ \Omega \end{cases} \quad (29)$$

satisfies

$$z(T) = 0.$$

As before, we consider the adjoint system

$$\begin{cases} -\frac{\partial \psi}{\partial t} - \Delta \psi + \nabla q = 0 & \text{in } \ \Omega \times (0, T), \\ \operatorname{div} \psi = 0 & \text{in } \ \Omega \times (0, T), \\ \psi = 0 & \text{on } \Gamma \times (0, T), \\ \psi(T) = \psi_0, & \text{in } \ \Omega, \end{cases} \quad (30)$$

and prove a suitable observability inequality.

Notice that, when $\psi_0 \in H$, there exists a unique solution $\psi \in C([0,T];H) \cap L^2(0,T;V)$ of the adjoint system (30), but we have no information, a priori, on $\dfrac{\partial \psi}{\partial \nu}$. Nevertheless, if $\rho \in C^1([0,T])$ with $\rho(T) = 0$, then we have $\rho \frac{\partial \psi}{\partial \nu} \in L^2(0,T;H^{\frac{1}{2}}(\Gamma))$.

In what follows, we will use the transmutation method developed (in different ways) by several authors (see Ervedoza and Zuazua 2011a and the references therein) and we will follow the strategy adopted in Ervedoza and Zuazua (2011a, b) which ensures that, for any $S > 0$ and any $T > 0$, there exists a smooth function $k = k(t,s)$ such that

$$\begin{cases} \frac{\partial k}{\partial t} + \frac{\partial^2 k}{\partial s^2} = 0 & \text{in } (0,T) \times (-S,S), \\ k(0,s) = k(T,s) = 0 & \text{on } (-S,S), \\ k(t,0) = 0 & \text{on } (0,T), \\ \frac{\partial k}{\partial s}(t,0) = e^{-\alpha \frac{T}{t(T-t)}} & \text{on } (0,T), \end{cases} \tag{31}$$

with $\alpha > 2S^2$, satisfying

$$|k(s,t)| \leq |s|\rho(t), \quad \forall t \in (0,T),$$

where ρ is an appropriate smooth function satisfying $\rho(0) = \rho(T) = 0$.

In order to perform all calculations below, we first take ψ_0 in a dense set of regular functions and we consider the functions

$$\theta(s,x) = \int_0^T k(t,s)\psi(t,x)dt, \ \ \pi(s,x) = \int_0^T k(t,s)q(t,x)dt.$$

It is standard to see that (θ, p) satisfies the following hyperbolic Stokes system

$$\begin{cases} \frac{\partial^2 \theta}{\partial s^2} - \Delta\theta + \nabla\pi = 0 & \text{in } \Omega \times (-S,S), \\ \operatorname{div}\theta = 0 & \text{in } \Omega \times (-S,S), \\ \theta = 0 & \text{on } \Gamma \times (-S,S), \\ \theta(0,x) = 0 & \text{in } \Omega, \\ \frac{\partial \theta}{\partial s}(0,x) = \theta_1 = \int_0^T \frac{\partial k}{\partial s}(t,0)\psi(t,x)dx = \int_0^T e^{-\alpha\frac{T}{t(T-t)}}\psi(t,x)dt & \text{in } \Omega. \end{cases} \tag{32}$$

The energy of θ is conserved and is equal to $E_\theta(0) = \dfrac{1}{2}|\theta_1|_H^2$. Assuming that Γ_0 satisfies the geometric control condition (23), we apply Theorem 3.1 on $\Omega \times (-S,S)$, with $S > R(x_0)$, to obtain

$$\frac{2(S - R(x_0))}{R(x_0)}|\theta_1|_H^2 \leq \int_{-S}^{S}\int_{\Gamma_0}\left|\frac{\partial\theta}{\partial\nu}\right|^2 d\gamma ds \leq \int_{-S}^{S}\int_{\Gamma_0}\left|\int_0^T k(t,s)\frac{\partial\psi}{\partial\nu}dt\right|^2 d\gamma ds$$
$$\leq CS^3\int_0^T\int_{\Gamma_0}\left|\rho\frac{\partial\psi}{\partial\nu}\right|^2 d\gamma dt.$$

Now, using the eigenvalues and eigenfunctions, namely (λ_n, w_n), of the stationary Stokes operator with Dirichlet boundary condition (see system (7)), we have

$$\psi_0 = \sum_{n=1}^{+\infty} a_n w_n,$$

$$\psi(t) = \sum_{n=1}^{+\infty} a_n e^{-\lambda_n (T-t)} w_n,$$

and therefore

$$\psi(0) = \sum_{n=1}^{+\infty} a_n e^{-\lambda_n T} w_n.$$

On the other hand we have

$$|\theta_1|_H^2 = \int_\Omega \left| \int_0^T e^{-\frac{\alpha T}{t(T-t)}} \psi(t,x)\, dt \right|^2 dx = \int_\Omega \left| \int_0^T \sum_{n=1}^{+\infty} e^{-\frac{\alpha T}{t(T-t)}} e^{-\lambda_n (T-t)} a_n w_n \, dt \right|^2 dx.$$

Using that $e^{-\lambda_n (T-t)} \geq e^{-\lambda_n T}$ and since $\{w_n\}_n$ is a Hilbert basis in H, we have

$$|\theta_1|_H^2 \geq \left(\int_0^T e^{-\frac{\alpha T}{t(T-t)}} \, dt \right)^2 \sum_{n=1}^{+\infty} e^{-2\lambda_n T} |a_n|^2 = C(T) |\psi(0)|_H^2,$$

where

$$C(T) = \left(\int_0^T e^{-\frac{\alpha T}{t(T-t)}} \, dt \right)^2 > 0.$$

By density, we then obtain the observability inequality

$$|\psi(0)|_H^2 \leq \frac{S^3 R(x_)}{2C(T)(S - R(x_0))} \int_0^T \int_{\Gamma_0} \rho^2 |\frac{\partial \psi}{\partial \nu}|^2 \, d\gamma dt, \quad \forall \psi_0 \in H. \tag{33}$$

Now we give the result of boundary null controllability for the Stokes system.

Theorem 4.1 *Let $T > 0$ and assume that Γ_0 satisfies the geometric control condition (23). For every $z_0 \in H$, there exists a control $v \in L^2(0, T; L^2(\Gamma_0))$, with $v \cdot \nu = 0$ in Γ_0 a.e., and such that the solution z of the Stokes system (29) satisfies*

$$z(T) = 0.$$

Proof Let $\epsilon > 0$. Using the observability inequality (33), the functional $J : H \longrightarrow \mathbb{R}$ given by

$$J(\psi_0) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \rho^2 |\frac{\partial \psi}{\partial \nu}|^2 \, d\gamma dt - < z_0, \psi(0) >_H + \frac{\epsilon}{2} |\psi_0|_H^2$$

is continuous, strictly convex and coercive.

In fact, the continuity follows from the well-posedness of the system. The convexity follows from the fact that the functional is quadratic. Using (33) and that $z_0 \in H$ is fixed, the coercivity follows from the fact that there exists $C > 0$ such that

$$J(\psi_0) \geq \frac{\epsilon}{2}|\psi_0|_H^2 - C|z_0|_H^2.$$

For each ϵ, let $\widehat{\psi}_0^\epsilon \in H$ be the minimun of J and $\widehat{\psi}_\epsilon$ the corresponding solution to (30). Take $v_\epsilon = \rho^2 \frac{\partial \widehat{\psi}_\epsilon}{\partial \nu}$ as a control in (29). Then, using the fact that $\rho^2 \frac{\partial \widehat{\psi}_\epsilon}{\partial \nu} \cdot \nu = 0$ on Γ for a.e. t, the duality between (29) and (30) gives

$$(z_\epsilon(T), \psi_0)_H - (z_0, \psi(T))_H + \int_0^T \int_{\Gamma_0} \rho^2 \frac{\partial \widehat{\psi}_\epsilon}{\partial \nu} \frac{\partial \psi}{\partial \nu} = 0, \quad \forall \psi_0 \in H, \tag{34}$$

where z_ϵ is the solution of (29) associated to v_ϵ.

Since $J(\widehat{\psi}_0^\epsilon) \leq J(0)$, the obsevability inequality (33) implies that

$$\frac{1}{4}\int_0^T \int_{\Gamma_0} \rho^2 \Big|\frac{\partial \widehat{\psi}^\epsilon}{\partial \nu}\Big|^2 d\gamma dt + \frac{\epsilon}{2}|\widehat{\psi}_0^\epsilon|_H^2 \leq C|z_0|_H^2, \tag{35}$$

with $C > 0$ does not depending on ϵ.

Thus, from (34) and (35), and the fact that ρ is bounded, there exists $C > 0$, does not depending on ϵ, such that

$$\int_0^T \int_{\Gamma_0} |v_\epsilon|^2 d\gamma dt \leq C$$

and

$$|z_\epsilon(T)|_H^2 \leq C\epsilon.$$

Let v be the weak limit of v_ϵ in $L^2(0, T; L^2(\Gamma_0))$ and take v as a boundary control in (29). Passing to the limit in (34), we get

$$-(z_0, \psi(T))_H + \int_0^T \int_{\Gamma_0} v \frac{\partial \psi}{\partial \nu} = 0, \quad \forall \psi_0 \in H, \tag{36}$$

which implies that the solution z of (29) associated to v satisfies $z(T) = 0$. □

Remark 4.2 The result of Theorem 4.1 is, at least for us, surprising. In fact, it shows that under the strong assumption of the Geometric Control Condition, one can essentially choose the boundary control for the Stokes system as limit of a sequence of sole normal derivatives $\frac{\partial \psi}{\partial \nu}$ of an adjoint system. Recall that, by duality, the natural boundary control one expects is a limit of a sequence of the form $\frac{\partial \psi}{\partial \nu} - q\nu$, with q being the associated pressure to ψ.

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Declarations

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