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On the stabilization of a hyperbolic Stokes system under geometric control condition

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Abstract. In this article, we study the stabilization problem for a hyperbolic-type Stokes system posed on a bounded domain. We show that when the damping effects are restricted to a subdomain satisfying the geometric control condition, the energy of the system decays exponentially. The result is a consequence of a new quasi-mode estimate for the Stokes system.

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough, ω be a small subset of Ω and let $T > 0$. In this article, we are interested in the stabilization problem for the following hyperbolic Stokes system:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0 & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V \times H, \end{cases} \tag{1.1}$$

where V and H are the usual spaces in the context of fluid mechanics:

$$V = \left\{ u \in H_0^1(\Omega)^d : \text{div } u = 0 \right\}$$

and

$$H = \left\{ u \in L^2(\Omega)^d : \text{div } u = 0, u \cdot \nu|_{\partial\Omega} = 0 \right\},$$

and $\nu(x)$ is the outward normal to Ω at the point $x \in \partial\Omega$. In (1.1), the damping term $a \in L^\infty(\Omega)$ and satisfies $a(x) \geq 0$, for all $x \in \Omega$.

If $u = u(x,t)$ is a (sufficiently smooth) solution of the system, we define its energy as

$$E[u](t) = \frac{1}{2}\int_\Omega \left(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \right) \mathrm{d}x, \quad \forall t \in \mathbb{R},$$

and when there is no damping, namely $a \equiv 0$, the energy is conserved, while in general we only have that $E[u](t)$ is non-increasing:

$$\frac{\mathrm{d}E[u]}{\mathrm{d}t} = -\int_\Omega a(x)|\partial_t u(t,x)|^2 \mathrm{d}x \leq 0.$$

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As for other hyperbolic systems, the stabilization problem for (1.1) concerns the decay rate in time of the energy $E[u](t)$ under appropriate assumptions on the damping term.

It is well known that stabilization problems are closely related to observability and exact controllability problems in abstract settings. In fact, if we consider the undamped system

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V\times H, \end{cases} \tag{1.2}$$

we say that (1.2) is observable at time T with observation in ω if there exists $C > 0$ such that

$$||u_0||_V^2 + ||v_0||_H^2 \le C \int_0^T \int_\omega |\partial_t u(t,x)|^2 \mathrm{d}x\mathrm{d}t, \tag{1.3}$$

for every $(u_0, v_0) \in V\times H$.

When (1.3) holds, one can show that for any $(u_0, v_0) \in V\times H$ there exists $f \in L^2((0,T)\times\omega)^d$ such that the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = f\mathbf{1}_\omega & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V\times H, \end{cases} \tag{1.4}$$

satisfies

$$u(T,x) = 0, \quad \partial_t u(T,x) = 0,$$

that is to say, system (1.4) is exact controllable at T with control localized in ω. Nevertheless, it is important to mention that a complete characterization of the sets ω for which (1.3) is true remains open. A partial answer to this question was given by the first author in [6].

The motivation for studying the stabilization of system (1.1) is twofold. First, system (1.2) is the hyperbolic counterpart of Stokes system, which is the linearized version of the well-known Navier–Stokes equation in fluid mechanics. In fact, if we know that system (1.2) is exact controllable at some time $T > 0$, with control applied to some control region ω, then the so-called control transmutation method can be applied to obtain the null controllability at any time and the optimal cost of controllability (in time) for the Stokes system (for more details, see [6]). On the other hand, system (1.2) comes from simple models of dynamical elasticity for incompressible materials. More precisely, it can be derived as a limit model of the Lamé system in the linear elastic theory when one parameter tends to infinity [14]. For the sake of completeness, in "Appendix" we give a derivation of the system (1.2) from the Lamé system. It is important to remark that the stabilization problem for the Lamé system has been already studied in [5].

To state our main result, let us introduce several concepts. Some terminologies and notations will be clear in the next section.

Definition 1.1. We say that the support of a nonnegative function $a \in C(\overline{\Omega})$ satisfies the geometric control condition (GCC in short) if there exists $T > 0$, such that each generalized bicharacteristic ray $\gamma(t)$ with speed 1 issued from a point $\rho \in^b T^*\overline{\Omega}$ enters the set $\{x \in \overline{\Omega} : a(x) > 0\}$ in a time $t < T$.

We recall that an open set Ω has no infinite order of contact, if there is no geodesic tangent to $\partial\Omega$ of infinite order. More precisely, in the decomposition

$$T^*\partial\Omega = \mathcal{E}\cup\mathcal{H}\cup\mathcal{G},$$

we have

$$\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j.$$

Here, the sets $\mathcal{E}, \mathcal{H}, \mathcal{G}$ are called the elliptic region, the hyperbolic region and the glancing surface, respectively, and $\mathcal{G}^j$ is the set of points in $\mathcal{G}$ with jth order of contact. The precise definition will be given in the next section.

The first main result in this article is as follows:

Theorem 1.2. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set with no infinite order of contact and $a \in C(\overline{\Omega})$ is a nonnegative function whose support satisfies the geometric control condition. Then, there exist positive constants C_0 and α such that for any $(u_0, v_0) \in V \times H$, the energy of the corresponding solution $u(t)$ to* (1.1) *decays exponentially:*

$$E[u](t) \leq C_0 E[u](0) \mathrm{e}^{-\alpha t} \tag{1.5}$$

for all $t \geq 0$.

In what follows, we say that the stabilization of (1.1) holds if (1.5) holds true.

Remark 1.3. As a by-product of the proof of Theorem 1.2, we obtain the null (exact) controllability at some time T of system (1.4). Namely, there exist $T > 0$ and a control $f \in L^2([0,T] \times \omega)$ such that the corresponding solution u to (1.4) satisfies $(u(T), \partial_t u(T)) = (0,0)$. However, we do not describe the control time T explicitly, since we prove the observability inequality (1.3) by reducing it to a quasi-mode estimate.

Let us mention that if a is supported in a neighborhood of the boundary $\partial\Omega$, the same result is true by adapting the strategy in [6], where the author has proved the exact controllability of the system (1.4) with ω be a neighborhood of $\partial\Omega$. Our result is a generalization of the result in [6].

The pioneering work of Rauch and Taylor [19] related the exponential decay of damped wave equations to the geometric control condition (GCC) of damped region on compact Riemannian manifold without boundary. Until the celebrated work of Bardos–Lebeau–Rauch [2], the presence of the boundary has been understood and the exactly controllability for wave equations as well as the exponential stabilization are obtained under (GCC). The proof mainly relies on the propagation of singularities along the Melrose–Sjöstrand flow. Later on, the tool of micro-local defect measure, introduced by Gérard and Tartar independently, has been used to simplify the proof of these results and to adapt to many other problems, see, for example, [5] for the Lamé systems and [8] for a coupled wave system. The key ingredient of the measure-based proof is the propagation formula, which can be viewed as a transport equation for the defect measure. This is a more quantitative version of the classical propagation of singularities, which usually describes the invariance of the wave front set along the Melrose–Sjöstrand flow.

For the present system (1.1), the presence of the pressure term ∇p introduces non-trivial difficulties if we want to adapt the strategy in [5] directly, due to the rough regularity of time-dependent harmonic function $p(t,x)$. However, following [11], the exponentially stabilization of (1.1) can be reduced to the following semi-classical version of the observation inequality:

Proposition 1.4. *Assume that $a \in L^\infty(\Omega) \cap C^0(\overline{\Omega})$ and $\int_\Omega a\,dx > 0$. Then, the stabilization of* (1.1) *holds if the following statement is true:*

$$\exists h_0 > 0, C > 0 \text{ such that for all } 0 < h < h_0, \text{ if } (u,q,f) \in H^2(\Omega) \cap V \times H^1(\Omega) \times H$$

solves the equation

$$-h^2 \Delta u - u + h \nabla q = f, \tag{1.6}$$

then the inequality

$$\|u\|_{L^2(\Omega)} \leq C\Big(\|a^{1/2} u\|_{L^2(\Omega)} + \frac{1}{h}\|f\|_{L^2(\Omega)}\Big). \tag{1.7}$$

holds.

Note that the system (1.7) is just a quasi-mode equation of the stationary Stokes system. In particular, if $f = 0$, the solution $u(h)$ is a eigenfunction of the Stokes operator corresponding to the eigenvalues h^{-2}.

The proof of (1.7) is based on the propagation of semi-classical measure μ in the recent work [20] of the second author. We give a brief recall here. The sequence of pressures $q(h)$ is harmonic, and their impact on the solution only occurs at the boundary. It has been shown that the measure is propagated along bicharacteristic rays which is invariant under the Melrose–Sjöstrand flow. When a ray reaches the boundary, careful analysis between the wave-like propagation phenomenon and the impact of the pressure allows us to deduce the propagation of the support of the measure μ along generalized bicharacteristic rays defined in [16].

We organize this article as follows. In Sect. 2, we give some notations, definitions and classical results. In Sect. 3, we follow the strategy in [4] to reduce the stabilization to the semi-classical observability (1.7). In Sect. 4, we prove the semi-classical observability by adapting the propagation result. Finally in Appendix, we give the derivation from the Lamé system to the hyperbolic Stokes system (1.1).

2. Preliminary

2.1. Notations

For a manifold M, we let TM be its tangent bundle and T^*M be the cotangent bundle with canonical projection

$$\pi : TM(\text{ or } T^*M) \to M.$$

In the tubular neighborhood of boundary, we can identify Ω locally as $[0, \epsilon_0) \times X$, $X = \{x' \in \mathbb{R}^{d-1} : |x'| < 1\}$. For $x \in \overline{\Omega}$, we note $x = (y, x')$, where $y \in [0, \epsilon_0), x' \in X$, and $x \in \partial\Omega$ if and only if $x = (0, x')$. In this coordinate system, the Euclidean metric $\mathrm{d}x^2$ can be written as matrices

$$g = \begin{pmatrix} 1 & 0 \\ 0 & M(y, x') \end{pmatrix}, g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(y, x') \end{pmatrix},$$

with $|\xi'|^2_{\alpha(y,x'))} = \langle \xi', \alpha(y, x')\xi' \rangle_{\mathbb{C}^{d-1}}$ be the induced metric on $T^*\partial\Omega$, parametrized by y. Note that $|\xi'|^2_{\alpha(0,x')} = \langle \xi', \alpha(0, x')\xi' \rangle_{\mathbb{C}^{d-1}}$ is the natural norm on $T^*\partial\Omega$, dual of the norm on $T\partial\Omega$, induced by the canonical metric on $\overline{\Omega}$. Write $(x, \xi) = (y, x', \eta, \xi')$ and denote by $|\xi|$ the Euclidean norm on $T^*\mathbb{R}^d$.

We define the L^2 norms and inner product on $[0, \epsilon_0) \times X$ via

$$\|u\|^2_{L^2_{y,x'}} := \int_0^{\epsilon_0} \int_X |u|^2 d_{g(y,\cdot)}x' \mathrm{d}y,$$

$$(u|v)_{L^2_{y,x'}} := (u|v)_\Omega := \int_0^{\epsilon_0} \int_X u \cdot \overline{v} d_{g(y,\cdot)}x' \mathrm{d}y,$$

$$\|u(y,\cdot)\|^2_{L^2_{x'}} := \int_X |u(y,\cdot)|^2 d_{g(y,\cdot)}x',$$

$$(u|v)_{L^2_{x'}}(y) := \int_X u(y,\cdot) \cdot \overline{v(y,\cdot)} d_{g(y,\cdot)}x',$$

where the measure $d_{g(y,\cdot)}x'$ is the induced measure on X, parametrized by $y \in [0, \epsilon_0)$ such that $d_{g(y,\cdot)}x' \mathrm{d}y = \mathcal{L}(\mathrm{d}x)$, the Lebesgue measure on $\mathbb{R}^d$. Note that the measure $d_{g(0,\cdot)}x'$ is nothing but the surface measure on $\partial\Omega$. In certain situations, we perform using global notation for inner product:

$$(u|v)_\Omega := \int_\Omega u \cdot \overline{v} \mathrm{d}x,$$

$$(f|g)_{\partial\Omega} := \int_{\partial\Omega} f \cdot \overline{g} \mathrm{d}\sigma(x)$$

In the tubular neighborhood, we can write a vector field $X = (X_\parallel, X_\perp)$, where $X_\parallel$ stands for the components parallel to the boundary, while $X_\perp$ stands for the normal component with the following convention: $(0, a) = -a\nu$.

As in [7], we will write down system (1.1) in the tubular neighborhood. For $u = (u_\parallel, u_\perp)$, equation (1.1) can be rewritten:

$$\begin{cases} (-h^2\Delta_\parallel - 1)u_\parallel + h\nabla_{x'} q = f_\parallel, \\ (-h^2\Delta_g - 1)u_\perp + h\partial_y q = f_\perp, \\ h \text{ div }_\parallel u_\parallel + \dfrac{h}{\sqrt{\det g}}\partial_y(\sqrt{\det g} u_\perp) = 0 \end{cases} \tag{2.1}$$

where

$$\begin{aligned} h^2\Delta_\parallel &= h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\parallel(y, x', hD_x') + hM_1(y, x')h\partial_y, \\ h^2\Delta_g &= h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\perp(y, x', hD_x') + hN_1(y, x')h\partial_y, \\ h \text{ div }_\parallel u_\parallel &= \frac{h}{\sqrt{\det g}} \sum_{j=1}^{N-1} \partial_{x'_j}(\sqrt{\det g} u_{\parallel,j}). \end{aligned}$$

Note that $h^2\Lambda^2(y, x', hD_{x'})$ has the symbol $\lambda^2 = |\xi'|^2_{g(y,\cdot)}$, $M_{\parallel,\perp}$ are both first-order matrix-valued semiclassical differential operators, and M_1, N_1 are zero-order matrix-valued functions.

2.2. Geometric preliminaries

Denote by ${}^bT\overline{\Omega}$ the vector bundle whose sections are the vector fields $X(p)$ on $\overline{\Omega}$ with $X(p) \in T_p\partial\Omega$ if $p \in \partial\Omega$. Moreover, denote by ${}^bT^*\overline{\Omega}$ the Melrose's compressed cotangent bundle which is the dual bundle of ${}^bT\overline{\Omega}$. Let

$$j : T^*\overline{\Omega} \to^b T^*\overline{\Omega}$$

be the canonical map. In our geodesic coordinate system near $\partial\Omega,^b T\overline{\Omega}$ is generated by the vector fields $\frac{\partial}{\partial x'_1}, \cdots, \frac{\partial}{\partial x'_{d-1}}, y\frac{\partial}{\partial y}$ and thus j is defined by

$$j(y, x'; \eta, \xi') = (y, x'; v = y\eta, \xi').$$

The principal symbol of operator $P_h = -(h^2\Delta + 1)$ is

$$p(y, x', \eta, \xi') = \eta^2 + |\xi'|^2_{\alpha(y,x')} - 1.$$

By $\mathrm{Car}(P)$ we denote the characteristic variety of p:

$$\mathrm{Car}(P) := \{(x, \xi) \in T^*\mathbb{R}^d|_{\overline{\Omega}} : p(x, \xi) = 0\}, Z := j(\mathrm{Car}(P)).$$

By writing in another way

$$p = \eta^2 - r(y, x', \xi'), r(y, x', \xi') = 1 - |\xi'|^2_\alpha,$$

we have the decomposition

$$T^*\partial\Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G},$$

according to the value of $r_0 := r|_{y=0}$ where

$$\mathcal{E} = \{r_0 < 0\}, \mathcal{H} = \{r_0 > 0\}, \mathcal{G} = \{r_0 = 0\}.$$

We call $\mathcal{E}, \mathcal{H}, \mathcal{G}$ elliptic, hyperbolic and glancing sets, with respectively.

For a symplectic manifold S with local coordinate (z, ζ), a Hamiltonian vector field associated with a real function f is given by

$$H_f = \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \zeta}.$$

Now for $(x, \xi) \in \Omega$ far away from the boundary, the Hamiltonian vector field associated with the characteristic function p is given by

$$H_p = 2\xi\frac{\partial}{\partial x}.$$

We call the trajectory of the flow

$$\phi_s : (x, \xi) \mapsto (x + s\xi, \xi)$$

bicharacteristic or simply ray, provided that the point $x + s\xi$ is still in the interior.

To classify different scenarios as a ray approaching the boundary, we need more accurate decomposition of the glancing set $\mathcal{G}$. Let $r_1 = \partial_y r|_{y=0}$ and define

$$\mathcal{G}^{k+3} = \{(x', \xi') : r_0(x', \xi') = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k; H_{r_0}^{k+1}(r_1) \neq 0\}, k \geq 0$$
$$\mathcal{G}^{2,\pm} := \{(x', \xi') : r_0(x', \xi') = 0, \pm r_1(x', \xi') > 0\}, \mathcal{G}^2 := \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}.$$

No infinite order of contact means that we can decompose $\mathcal{G}$ into

$$\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j.$$

Given a ray $\gamma(s)$ with $\pi(\gamma(0)) \in \Omega$ and $\pi(\gamma(s_0)) \in \partial\Omega$ be the first point who attaches the boundary. If $\gamma(s_0) \in \mathcal{H}$, then $\eta_\pm(\gamma(s_0)) = \pm\sqrt{r_0(\gamma(s_0))}$ be the two different roots of $\eta^2 = r_0$ at this point. Notice that the ray starting with direction η_- will leave Ω, while the ray with direction η_+ will enter the interior of Ω. This motivates the following definition of broken bicharacteristic:

Definition 2.1. [12] A broken bicharacteristic arc of p is a map:

$$s \in I \setminus B \mapsto \gamma(s) \in T^*\Omega \setminus \{0\},$$

where I is an interval on $\mathbb{R}$ and B is a discrete subset, such that

(1) If J is an interval contained in $I \setminus B$, then $s \in J \mapsto \gamma(s)$ is a bicharacteristic of P_h over Ω.
(2) If $s \in B$, then the limits $\gamma(s^+)$ and $\gamma(s^-)$ exist and belongs to $T_x^*\overline{\Omega} \setminus \{0\}$ for some $x \in \partial\Omega$, and the projections in $T_x^*\partial\Omega \setminus \{0\}$ are the same hyperbolic point.

When a ray $\gamma(s)$ arrives at some point $\rho_0 \in \mathcal{G}$, there are several situations. If $\rho_0 \in \mathcal{G}^{2,+}$, then the ray passes transversally over ρ_0 and enters $T^*\Omega$ immediately. If $\rho_0 \in \mathcal{G}^{2,-}$ or $\rho_0 \in \mathcal{G}^k$ for some $k \geq 3$, then we can continue it inside $T^*\partial\Omega$ as long as it cannot leave the boundary along the trajectory of the Hamiltonian flow of H_{-r_0}. We now give the precise definition.

Definition 2.2. [12] A generalized bicharacteristic ray of p is a map:

$$s \in I \setminus B \mapsto \gamma(s) \in (T^*\overline{\Omega} \setminus T^*\partial\Omega) \cup \mathcal{G}$$

where I is an interval on $\mathbb{R}$ and B is a discrete set of I such that $p \circ \gamma = 0$ and the following:

(1) $\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\overline{\Omega} \setminus T^*\partial\Omega$ or $\gamma(s) \in \mathcal{G}^{2,+}$.
(2) Every $s \in B$ is isolated, $\gamma(s) \in T^*\overline{\Omega} \setminus T^*\partial\Omega$ if $s \neq t$ and $|s - t|$ is small enough, the limits $\gamma(s^\pm)$ exist and are different points in the same fiber of $T^*\partial\Omega$.

(3) $\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{ds} = H_p(\gamma(s)) + \frac{H_p^2 y}{H_y^2 p} H_y$ if $\gamma(s) \in \mathcal{G} \setminus \mathcal{G}^{2,+}$, where y is the boundary defining function.

Remark 2.3. The definition above does not depend on the choice local coordinate, and in the geodesic coordinate system, the map

$$s \mapsto (y(s), \eta^2(s), x'(s), \xi'(s))$$

is always continuous and

$$s \mapsto (x'(s), \xi'(s))$$

is always differentiable and satisfies the ordinary differential equations

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = -\frac{\partial r}{\partial \xi'}, \frac{\mathrm{d}\xi'}{\mathrm{d}t} = \frac{\partial r}{\partial x'},$$

the map $s \mapsto y(s)$ is left and right differentiable with derivative $2\eta(s^{\pm})$ for any $s \in B$ (hyperbolic point).

Moreover, there is also the continuous dependence with the initial data, namely the map

$$(s, \rho) \mapsto (y(s, \rho), \eta^2(s, \rho), x'(s, \rho), \xi'(s, \rho))$$

is continuous. We denote the flow map by $\gamma(s, \rho)$.

Remark 2.4. Under the map $j : T^*\overline{\Omega} \to^b T^*\overline{\Omega}$, one could regard $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\overline{\Omega}$, and it is called the Melrose–Sjöstrand flow. We will also call each trajectory generalized bicharacteristic or simply ray in the sequel.

It is well known that if there is no infinite contact in $\mathcal{G}$, a generalized bicharacteristic is uniquely determined by any one of its points. In other words, the Melrose–Sjöstrand flow is globally well defined. See [12] for more discussion.

3. Review of semi-classical propagation of singularity

3.1. Definition of defect measure

We follow closely [3] here, see also [9] for a little different but more comprehensive introduction.

Define the partial symbol class $S^m_{\xi'}$ and the class of boundary h-pseudo-differential operators $\mathcal{A}^m_h$ as follows:

$$S^m_{\xi'} := \{a(y, x', \xi') : \sup_{\alpha,\beta,y\in[0,\epsilon_0]} |\partial^\alpha_{x'} \partial^\beta_{\xi'} a(y, x', \xi')| \leq C_{m,\alpha,\beta}(1 + |\xi'|)^{m-\beta}\}.$$

$$\mathcal{A}^m_h =: \mathrm{Op}^{comp}_h(S^m) + \mathrm{Op}_h(S^m_{\xi'}) := \mathcal{A}^m_{h,i} + \mathcal{A}^m_{h,,\partial}.$$

Denote by U a tubular neighborhood of $\partial\Omega$. Consider functions of the form $a = a_i + a_\partial$ with $a_i \in C^\infty_c(\Omega \times \mathbb{R}^d)$ which can be viewed as a symbol in S^0, and $a_\partial \in C^\infty_c(U \times \mathbb{R}^{d-1})$ can be viewed as a symbol in $S^0_{\xi'}$. We quantize a via the formula (in local coordinate)

$$\begin{aligned}\mathrm{Op}_h(a)f(y, x') =& \frac{1}{(2\pi h)^d} \int\limits_{\mathbb{R}^{2d}} \mathrm{e}^{\frac{i(x-z)\xi}{h}} a_i(x, \xi) f(z) \mathrm{d}z\mathrm{d}\xi \\ &+ \frac{1}{(2\pi h)^{d-1}} \int\limits_{\mathbb{R}^{2(d-1)}} \mathrm{e}^{\frac{i(x'-z')\xi'}{h}} a_\partial(y, x', \xi') f(y, z') \mathrm{d}z'\mathrm{d}\xi'.\end{aligned}$$

Notice that the action of the tangential operator $\mathrm{Op}_h(a_\partial)$ can be viewed as pseudo-differential operator on the manifold $\partial\Omega$, parametrized by the parameter $y \in [0, \epsilon_0)$. No doubt that the definition of the operator $\mathrm{Op}_h(a_\partial)$ depends on the choice of local coordinate of $\partial\Omega$. However, the bounded family of operators $\mathcal{A}^m_{h,\partial}$ is defined uniquely up to a family of operators with norms uniformly dominated by Ch, as $h \to 0$. See [9] for more details. Moreover, for any family (A_h), such that

$$\|A_h - \mathrm{Op}_h(a_\partial)\|_{L^2\to L^2} = O(h),$$

the principal symbol $\sigma(A)$ is determined uniquely as a function on $T^*\partial\Omega$, smoothly depending on y, i.e., $\sigma(A) \in C^\infty([0,\epsilon_0) \times T^*\partial\Omega)$.

When we deal with vector-valued functions, we could require the symbol a to be matrix-valued. Now for any sequence of vector-valued function w_k, uniformly bounded in $L^2(\Omega)$, there exists a subsequence (still denoted w_k for simplicity), and a nonnegative definite Hermitian matrix-valued measure μ_i on $T^*\Omega$ such that

$$\lim_{k\to 0}(\mathrm{Op}_{h_k}(a_i)w_k|w_k)_{L^2} = \langle \mu_i, a\rangle := \int\limits_{T^*\Omega} \mathrm{tr}\ (a\mathrm{d}\mu_i).$$

For a proof, see, for example, [3], and the micro-local version appeared in [10].

From now on, we will only deal with scalar-valued operator, even though we will encounter vector-valued functions in the analysis. Suppose u_k be a sequence of solutions to (5.1), under the assumptions below:

$$\begin{aligned} &\|u_k\|_{L^2(\Omega)} = O(1), f_k \in H \text{ and } \|f_k\|_{L^2(\Omega)} = o(h_k),\\ &\|h\nabla q_k\|_{L^2(\Omega)} = O(1), \int\limits_\Omega q_k \mathrm{d}x = 0, \end{aligned} \tag{3.1}$$

The following result shows that the interior measure μ_i is supported on the $\mathrm{Car}(P)$.

Proposition 3.1. *Let $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$ be equal to 0 near $\mathrm{Car}(P)$, then we have*

$$\lim_{k\to\infty}(\mathrm{Op}_{h_k}(a_i)u_k|u_k)_{L^2} = 0.$$

Proof. Note that the symbol $b(x,\xi) = \frac{a_i(x,\xi)}{|\xi|^2-1} \in S^0$ is well defined from the assumption on a_i. From symbolic calculus, we have

$$\mathrm{Op}_{h_k}(a_i) = B_{h_k}(-h_k^2\Delta - 1) + O_{L^2\to L^2}(h_k).$$

Therefore,

$$\begin{aligned} (B_{h_k}(-h_k^2\Delta - 1)u_k|u_k)_{L^2} &= (B_{h_k}f_k|u_k)_{L^2} - (B_{h_k}h_k\nabla q_k|u_k)_{L^2}\\ &= o(1) + ([h_k\nabla, B_{h_k}]q_k|u_k)_{L^2} - (h_k\nabla B_{h_k}q_k|u_k)_{L^2}\\ &= o(1), \text{ as } k\to\infty, \end{aligned}$$

where in the last line we have used the symbolic calculus, integrating by part, and Lemma 5.3. □

Now we denote by $Z = j(\mathrm{Car}(P))$. Proposition 3.1 indicates that the interior defect measure μ_i is supported on Z. To define the defect measure up to the boundary, we have to check that if $a_\partial \in C_c^\infty(U \times \mathbb{R}^{d-1})$ vanishing near Z (i.e., a_∂ is supported in the elliptic region for all y small) then

$$\lim_{k\to\infty}(\mathrm{Op}_{h_k}(a_\partial)u_k|u_k)_{L^2} = 0.$$

Indeed, this can be ensured by the analysis of boundary value problem in the elliptic region, and the reader can consult section 6. Now for any family of operator $A_h \in \mathcal{A}_h^0$, let $a = \sigma(A_h)$ be the principal symbol of A_h and we define $\kappa(a) \in C^0(Z)$ via $\kappa(a)(\rho) := a(j^{-1}(\rho))$. Note that Z is a locally compact metric space and the set

$$\{\kappa(a) : a = \sigma(A_h), A_h \in \mathcal{A}_h^0\}$$

is a locally dense subset of $C^0(Z)$. We then have the following proposition, which guarantees the existence of a Radon measure on Z:

Proposition 3.2. *There exists a subsequence of* u_k, h_k *and a nonnegative definite Hermitian matrix-valued Radon measure* μ*, such that*

$$\lim_{k\to\infty}(A_{h_k}u_k|u_k)_{L^2} = \langle\mu,\kappa(a)\rangle, a=\sigma(A_h), \forall A_h\in\mathcal{A}_h^0.$$

The proof of this result can be found in [3], see also [5] and [10] for its micro-local counterpart. Notice that if we write $a=a_i+a_\partial$, then

$$(A_k u_k|u_k)\to\int\limits_{T^*\Omega}\text{ tr }(a_i(\rho)\mathrm{d}\mu_i(\rho))+\int\limits_{Z}\text{ tr }(a_\partial(\rho)\mathrm{d}\mu(\rho)).$$

The following result shows that information about frequencies higher than the scale h_k^{-1} is not lost, and the measure μ contains the relevant information of the sequence (u_k).

Proposition 3.3. [20] *The sequence of solution* (u_k) *is* h_k-*oscillating in the following sense:*

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int\limits_{|\xi|\geq Rh_k^{-1}}|\widehat{\psi u_k}(\xi)|^2\,d\xi=0, \forall\psi\in C_c^\infty(\Omega),$$

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int\limits_0^{\epsilon_0}dy\int\limits_{|\xi'|\geq Rh_k^{-1}}|\widehat{\psi u_k}(y,\xi')|^2\,d\xi'=0, \forall\psi\in C_c^\infty(\overline{\Omega}),$$

where in the second formula, the Fourier transform involved is only the x' *direction.*

A direct consequence is the following:

Corollary 3.4. *Suppose* $a^{1/2}u_k\to 0$ *in* $L^2(\Omega)$*, and* μ *is the defect measure associated with* (u_k,h_k)*, then*

$$\langle\mu,a\rangle=0.$$

3.2. Recall of propagation theorem

Now let us recall the several results proved in [20].

In the interior, the full transport property of defect measure is proved.

Proposition 3.5. [20] *For any real-valued scalar function* $a\in C_c^\infty(\Omega\times\mathbb{R}^d)$ *vanishing near* $\xi=0$*, we have*

$$\frac{d}{ds}\langle\mu,a\circ\gamma(s,\cdot)\rangle=0.$$

The following proposition illustrates that near a elliptic point on the boundary, there is no accumulation of singularity.

Proposition 3.6. [20] $\mu\mathbf{1}_{\mathcal{E}}=0$*. If we let* ν *be the semi-classical defect measure of the sequence* $(h_k\partial_\nu u_k|_{\partial\Omega}, h_k)$*, then* $\nu\mathbf{1}_{\mathcal{E}}=0$*.*

When a ray travels near a hyperbolic point or a point on the glancing surface, the knowledge of the singularity is much less known. Nevertheless, we have

Theorem 3.7. [20] *Assume that* Ω *is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose* (u_k) *is a sequence of solutions to the quasi-mode problem* (1.1) *with semi-classical parameters* $h=h_k$*. Assume that* $f_k\in H$*,* $\|f_k\|_{L^2(\Omega)}=o(h_k)$ *and* u_k *converges weakly to* 0 *in* $L^2(\Omega)$*. Assume that* μ *is any semi-classical measure associated with some subsequence of* (u_k,h_k)*, then* $\text{supp}\mu$ *is invariant under Melrose–Sjöstrand flow.*

4. Reduction to semi-classical observability

This section is devoted to the proof of Proposition 1.4. In fact, it is classical from [11] that stabilization or observability of a self-adjoint evolution system is equivalent to resolvent estimates. See also [4,23].

Recall that the damped system is given by

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u + \nabla p = 0, (t,x) \in \mathbb{R} \times \Omega \\ \mathrm{div} u = 0, \text{in } \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{4.1}$$

We always assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain (open, connected set). We use ν to denote the outward normal vector on $\partial\Omega$ and the damping term $a \in L^\infty(\Omega)$ with $a(x) \geq 0$.

We also consider the undamped system

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = 0, (t,x) \in \mathbb{R} \times \Omega \\ \mathrm{div} u = 0, \text{in } \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{4.2}$$

4.1. Some functional analysis preliminaries

We work with a Hilbert space $\mathcal{H} := V \times H$, equipped with the norm

$$\|(f,g)^t\|_{\mathcal{H}}^2 := \|\nabla f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2.$$

and denote $\Pi : L^2(\Omega)^N \to H$ be the orthogonal projector (Leray projector) and $A = \Pi\Delta$ be the Stokes operator. We consider the operator:

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & -\Pi a \end{pmatrix} \tag{4.3}$$

with domain

$$D(\mathcal{A}) = (V \cap H^2(\Omega)) \times V.$$

In order to use semi-group theory, we first show that for some $\lambda > 0$, the operator $(\mathcal{A} - \lambda)$ is invertible: Take $(f,g) \in V \times H$, and consider the system

$$\begin{cases} v - \lambda u = f \\ Au - (\Pi a + \lambda)v = g \end{cases} \tag{4.4}$$

We consider the bilinear form

$$B(u_1, u_2) = \int_\Omega \nabla u_1 \cdot \nabla u_2 \mathrm{d}x + \int_\Omega (\lambda^2 + \lambda a(x)) u_1 \cdot u_2 \mathrm{d}x,$$

defined on $V \times V$. We then conclude from Lax–Milgram that for $\lambda > 0$, there exists $u \in V$ such that for any $w \in V$, we have

$$B(u,w) = -\int_\Omega (g \cdot w + (a(x) + \lambda) f \cdot w) \mathrm{d}x.$$

Set $v = \lambda u + f$, we have solved the system (4.4) in weak sense. Standard regularity argument gives that for $\lambda > 0$.

$$(\mathcal{A} - \lambda)^{-1} : \mathcal{H} \to D(\mathcal{A}),$$

is a bounded, and $(\mathcal{A} - \lambda)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. Moreover, if $\lambda \in \mathrm{Spec}(\mathcal{A})$, we must have $\Re\lambda < 0$. This will be clear in the proof of Proposition 4.6.

However, since the operator $\mathcal{A}$ is not maximal dissapative, the Hille–Yoshida theorem is not applicable. A slightly general modification ensures the existence of semi-group $\mathrm{e}^{t\mathcal{A}}$ which evolves the initial data in $D(\mathcal{A})$ and solves Eq. (4.1) with more regular data.

For solution $u, \partial_t u$ to (4.1), we consider the energy functional

$$E[u](t) := \frac{1}{2} \int\limits_{\Omega} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2) \mathrm{d}x,$$

and we calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E[u](t) &= \int\limits_{\Omega} \partial_t u \cdot (\partial_t^2 u - \Delta u) \mathrm{d}x \\ &= - \int\limits_{\Omega} \partial_t u \cdot \nabla p \mathrm{d}x - \int\limits_{\Omega} a(x) |\partial_t u|^2 \mathrm{d}x \\ &\leq 0, \end{aligned}$$

thus

$$E[u](t) \leq E[u](s), \forall s \leq t. \tag{4.5}$$

By density argument, we can solve (4.1) with initial data in $\mathcal{H}$ such that the energy dissipation (4.5) still holds.

4.2. Observability and Stabilization

In this section, we will prove the stabilization for damped system is equivalent to observability for undamped system. For this part, we follow closely in the appendix of [4] in which the authors have sketched the standard argument for damped wave equation.

We first introduce the quantity

$$D[u](T) = \int\limits_0^T \int\limits_{\Omega} a(x) |\partial_t u(t,x)|^2 \mathrm{d}x \mathrm{d}t,$$

and it quantifies the dissipation of the energy:

$$E[u](T) = E[u](0) - D[u](T).$$

Proposition 4.1. *The following assertions are equivalent:*

(1) *Stabilization: There exists $C_0, \alpha > 0$, such that for every solution $u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the damped system* (4.1), *we have*

$$E[u](t) \leq C_0 E[u](0) \mathrm{e}^{-\alpha t}, \forall t \geq 0.$$

(2) *Observability: There exists $C > 0$ and $T > 0$, such that, for every solution $v \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the undamped system* (4.2), *the observability inequality holds:*

$$E[v](0) \leq C D[v](T).$$

Proof. We first claim that the stabilization of damped system is equivalent to the observability of damped system.

It is clear that

$$E[u](0) = E[u](t) - D[u](t).$$

Let us first assume the stabilization of damped system. Argue by contradiction, suppose the observability of damped system does not hold. We first choose $T_0 > 0$ large enough such that $C_0 \mathrm{e}^{-\alpha T_0} < \frac{1}{2}$. We can select a sequence of solutions (u_k) and such that

$$E[u_k](0) = 1, D[u_k](T_0) \to 0, \text{ as } k \to \infty.$$

We thus have

$$\frac{1}{2} > C_0 \mathrm{e}^{-\alpha T_0} \geq E[u_k](T_0) = E[u_k](0) - D[u_k](T_0) = 1 + o(1), \text{ as } k \to \infty,$$

which leads to a contradiction.

Let us now assume the observability for damped system, i.e.,

$$E[u](0) \leq C D[u](T),$$

We may assume that $C > 1$, from the energy dissipation and observability, we have

$$E[u](2T) = E[u](0) - D[u](2T) \leq \left(1 - \frac{1}{C}\right) E[u](0).$$

For any $t > 0$, we write $m = \left\lfloor \frac{t}{2T} \right\rfloor$; therefore, we have

$$E[u](t) \leq E[u](m) \leq \left(1 - \frac{1}{C}\right)^m E[u](0),$$

after choosing C_0, α appropriately, we have the stabilization of damped system.

Our second step is to justify the equivalence between observability of damped system (4.1) and undamped system (4.2). To do this, we denote u and v be solutions of the damped and of the undamped system, respectively, with the same initial data at $t = 0$. Let $w = u - v$, and simple calculations yield

$$\partial_t^2 w - \Delta w = -a \partial_t u - \nabla q,$$
$$\partial_t^2 w - \Delta w + a \partial_t w = -a \partial_t v - \nabla q,$$

with some pressure function q.

We calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} E[w](t) = -\int_\Omega a(x) |\partial_t u|^2 \mathrm{d}x + \int_\Omega a(x) \partial_t u \cdot \partial_t v \mathrm{d}x - \int_\Omega \partial_t w \cdot \nabla q \mathrm{d}x,$$

and the last term of left-hand side vanishes, thanks to $\partial_t w \in C(\mathbb{R}; V)$. Thus, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} E[w](t) = -\int_\Omega a(x) |\partial_t u|^2 \mathrm{d}x + \int_\Omega a(x) \partial_t u \cdot \partial_t v \mathrm{d}x$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} E[w](t) = -\int_\Omega a(x) \partial_t u \cdot \partial_t w \mathrm{d}x.$$

Integrating the two expressions above and using the inequality of the type

$$|ab| \leq \epsilon |a|^2 + C(\epsilon) |b|^2, \forall \epsilon > 0,$$

one easily gets

$$E[w](T) \leq B \min\,(D[u](T), D[v](T)), \forall T > 0, \tag{4.6}$$

where B is another absolute constant.

Now suppose we have observability for the damped system (4.1), if $D[u](T) \leq D[v](T)$, the observability of undamped system (4.2) is trivial. Now assume that $D[u](T) > D[v](T)$, we deduce from (4.6) that

$$\begin{aligned} E[v](0) = E[u](0) &\leq CD[u](T) \leq CD[W](T) + CD[v](T) \\ &\leq C(E[w](T) + D[v](T)) \leq CD[v](T). \end{aligned}$$

The derivation of observability from undamped system to the damped follows in the same way, and we omit the details. □

Remark 4.2. Since the domain $D(\mathcal{A})$ is dense in $\mathcal{H}$ and the observability and energy decay only involves the L^2 norm of ∇u and $\partial_t u$, the same result of Proposition 4.1 holds if we replace $u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to $u \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; H)$.

4.3. Resolvent estimates and stabilization

Recall that from the previous sections, the study of damped system (4.1) is equivalent to the project system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & \mathring{Id} \\ A & -\Pi a \end{pmatrix}\begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \\ &\begin{pmatrix} u \\ \partial_t u \end{pmatrix} \in C(\mathbb{R}; V \times H). \end{aligned} \tag{4.7}$$

We will use the notation $U = (u, \partial_t u)^t$ in the sequel.

In this part, we follow almost the same way as in the appendix of [4], only to pay attention to the changing of working spaces (appearance of the pressure term and divergence free structure). Moreover, we add some technical details which may seems standard to experts in analysis but not disposable for many applied people.

From last section, we know that the observability of undamped system (4.2) is equivalent to the stabilization of damped system (4.1); therefore, we will concentrate ourselves to the study of stabilization of (4.1). The following result is standard in semi-group theory:

Proposition 4.3. *Consider a semi-group e^{tL} on a Hilbert space $\mathcal{X}$, with infinitesimal generator L defined on $D(L)$. Then, if there exists $C > 0, \delta > 0$ such that the resolvent of L, $(L - \lambda)^{-1}$ exists for $\Re\lambda \geq -\delta$ and satisfies*

$$\forall \lambda \in \mathbb{C}^\delta := \{z \in \mathbb{C} : \Re z > -\delta\}, \|(L - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C.$$

Then, there exists $M > 0$ such that for any $t > 0$,

$$\|\mathrm{e}^{tL}\|_{\mathcal{L}(\mathcal{X})} \leq M\mathrm{e}^{-\frac{\delta t}{2}}.$$

We need a lemma from complex analysis. We temporarily use the convention of Fourier transform

$$\widehat{u}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-it\tau} u(t)\mathrm{d}t.$$

Lemma 4.4. *Let u, v be two continuous functions with support in $\mathbb{R}_+ = (0, \infty)$. Assume moreover that $u, v \in L^2(\mathbb{R}_+)$ and v has compact support. From Winer–Paley theory, we know that the Fourier transform $\widehat{v}$ admits a holomorphic extension to $\mathbb{C}$ and of exponential type. Given $a_0 > 0$, suppose that the Fourier transform $\widehat{u}$ is also holomorphic in $S_{a_0} = \{z \in \mathbb{C} : \Im z < a_0\}$ and satisfies*

$$|\widehat{u}(z)| \leq C|\widehat{v}(z)|, \forall z \in S_{a_0}.$$

Then, for any $a < a_0$, $\mathrm{e}^{at}u(t) \in L^2(\mathbb{R}_+)$ *and*

$$\int_0^\infty \mathrm{e}^{2at}|u(t)|^2\,dt = \int_{-\infty}^\infty |\widehat{u}(\tau+ia)|^2\,d\tau.$$

Proof. We first claim that

$$\int_0^\infty \mathrm{e}^{2at}|v(t)|^2\mathrm{d}t = \int_{-\infty}^\infty |\widehat{v}(\tau+ia)|^2\mathrm{d}\tau, \forall a \in \mathbb{R}. \tag{4.8}$$

Indeed, since v is compactly supported,

$$\widehat{v}(\tau+ia) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{e}^{at}\mathrm{e}^{-it\tau}v(t)\mathrm{d}t$$

which is analytic in a and rapidly decreasing in τ for each fixed $a \in \mathbb{R}$. Thus, one easily deduces from the Plancherel (or calculate the integral directly) that (4.8) is true.

As a consequence, $\widehat{u}(.+ia) \in L^2(\mathbb{R})$ for each $a < a_0$. Notice also that $u \in L^2(\mathbb{R}_+)$, thus for each a with $\Re a < 0$, the formula

$$\int_0^\infty \mathrm{e}^{2at}|u(t)|^2\mathrm{d}t = \int_{-\infty}^\infty |\widehat{u}(\tau+ia)|^2\mathrm{d}\tau \tag{4.9}$$

holds true and analytic with respect to a. In particular, $|\widehat{u}(z)| \leq C|\widehat{v}(z)|, z \in S_{a_0}$ implies that $\widehat{u}(\tau+ia)$ is rapidly decreasing in τ for each fixed $a < a_0$. For $z = \tau + ia$ with $a < a_0$, consider the integral

$$F(a,t) = \frac{\mathrm{e}^{at}}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{itz}\widehat{u}(z)\mathrm{d}\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{it\tau}\widehat{u}(z)\mathrm{d}\tau \in L^2(\mathbb{R}).$$

From Cauchy integral theorem, we have that

$$F(a,t) = \frac{\mathrm{e}^{at}}{\sqrt{2\pi}}\int_{-\infty}^\infty \widehat{u}(\tau)\mathrm{e}^{it\tau}\mathrm{d}\tau = \mathrm{e}^{at}u(t) \in L^2(\mathbb{R}).$$

From this, we conclude that (4.9) follows for all $a < a_0$. □

Remark 4.5. In the previous lemma, the same results hold true if we replace u, v to be Hilbert-space valued functions.

Proof of proposition 4.3. The basic tool to prove this proposition is the Fourier–Laplace transform in time variable. From the property of strongly continuous semi-group, we know that there exists $\omega_0 > 0$ such that (see [21])

$$\|\mathrm{e}^{tL}\|_{\mathcal{L}(\mathcal{X})} \leq \mathrm{e}^{\omega_0 t}, \forall t \geq 1.$$

Take $u_0 \in D(L)$, and pick a nonnegative cut-off $\chi \in C^\infty(\mathbb{R})$ such that $\chi \equiv 0, \forall t \leq 1$ and $\chi \equiv 1, \forall t > 2$. We define $u(t) := \chi(t)\mathrm{e}^{tL-\omega t}u_0$ for some $\omega > \omega_0$ and thus $u \in L^\infty(\mathbb{R};\mathcal{X})$. Moreover, we have the equation

$$(\partial_t + \omega - L)u = \chi'(t)\mathrm{e}^{tL-\omega t}u_0 =: v(t).$$

By taking Fourier transform, we get

$$(i\tau + \omega - L)\widehat{u} = \widehat{v}(\tau).$$

Since v is compactly supported in positive axis in time variable, the $\widehat{v}(\tau)$ has a holomorphic and bounded extension in any domain of the form

$$S_\alpha = \{\tau \in \mathbb{C} : \Im\tau < \alpha\}, \alpha > 0.$$

From the assumption on the resolvent, we deduce that $(i\tau + \omega - L)$ is invertible if $\tau \in S_{\delta+\omega}$ and thus $\widehat{u}(\tau)$ admits a bounded holomorphic extension to $S_{\delta+\omega}$ which satisfies

$$\|\widehat{u}(\tau)\|_{\mathcal{X}} \leq C\|\widehat{v}(\tau)\|_{\mathcal{X}}.$$

Apply Lemma 4.4, we deduce that

$$\begin{aligned}\int_{-\infty}^{\infty} \|\mathrm{e}^{(\omega_0+\delta)t}u\|_{\mathcal{X}}^2 \mathrm{d}t &= \int_{-\infty}^{\infty} \|\widehat{u}(\xi + i(\omega_0+\delta))\|_{\mathcal{X}}^2 \mathrm{d}\xi \\ &\leq C \int_{-\infty}^{\infty} \|\widehat{v}(\xi + i(\omega_0+\delta))\|_{\mathcal{X}}^2 \mathrm{d}\xi \\ &\leq C \int_{-\infty}^{\infty} \|\mathrm{e}^{(\omega_0+\delta)t}v\|_{\mathcal{X}}^2 \mathrm{d}t \leq C\|u_0\|_{\mathcal{X}}^2.\end{aligned}$$

We remark that one need use various types of Winer–Paley theorems to justify the above calculations, thanks to the fact that $u(t), v(t)$ is supported on $[1,\infty)$ and furthermore $v(t)$ has compact support. Take $\omega < \omega_0 + \frac{\delta}{2}$ in the definition of u, we have that

$$\|\mathrm{e}^{\frac{\delta t}{2}}\mathrm{e}^{tL}u_0\|_{L^2(\mathbb{R}_+;\mathcal{X})} \leq C_1\|u_0\|_{\mathcal{X}}.$$

Thanks to the semi-group structure and uniform bound principal, we have that there exists $M_0 > 0$, such that for any interval $I \subset (0,+\infty)$ of length 1,

$$\sup_{t\in I, s>0, t+s\in I} \frac{|f(t+s)|}{|f(t)|} \leq M_0.$$

with $f(t) = \|\mathrm{e}^{tL}u_0\|_{\mathcal{X}}$. Therefore, for any $T > 0$,

$$\int_T^{T+1} \mathrm{e}^{\delta t}|f(t)|^2 \mathrm{d}t \geq \mathrm{e}^{\delta T} \min_{t\in[T,T+1]} |f(t)|^2.$$

Therefore,

$$|f(T+1)|^2 \leq M_0^2 \min_{t\in[T,T+1]} |f(t)|^2 \leq \mathrm{e}^{-\delta T} \int_T^{T+1} \mathrm{e}^{\delta u}|f(t)|^2 \mathrm{d}t.$$

This implies the exponential decay

$$\|\mathrm{e}^{tL}u_0\|_{\mathcal{X}} \leq M\mathrm{e}^{-\frac{\delta t}{2}}\|u_0\|_{\mathcal{X}}.$$

□

Now we can introduce the semi-classical observability

Proposition 4.6. *Assume that $a \in L^\infty(\Omega) \cap C^0(\overline{\Omega})$ and $\int_\Omega a dx > 0$. Then, the stabilization of system (4.1) is implied by the following statement:*

$$\exists h_0 > 0, C > 0 \text{ such that } \forall 0 < h < h_0, \forall (u,q,f) \in H^2(\Omega)\cap V \times H^1(\Omega) \times H$$

solves the equation

$$-h^2\Delta u - u + h\nabla q = f,$$

we have

$$\|u\|_{L^2(\Omega)} \leq C\left(\|a^{1/2}u\|_{L^2(\Omega)} + \frac{1}{h}\|f\|_{L^2(\Omega)}\right). \tag{4.10}$$

For the proof, we need two lemmas.

Lemma 4.7. *Let L be a linear operator on Hilbert space $\mathcal{X}$ with a compact resolvent $(L-\mathrm{Id})^{-1}$. Suppose the spectrum Spec$(L) \subset \{z : \Re z < 0\}$ and satisfies that for any $\sigma \in \mathbb{R}, L - i\sigma$ is invertible and satisfies the uniform bound*

$$\sup_{\sigma\in\mathbb{R}} \|(L-i\sigma)^{-1}\| < \infty.$$

Then, there exists $\delta > 0$, such that

$$\sup_{\lambda\in\mathbb{C}^\delta} \|(L-\lambda)^{-1}\| < \infty,$$

where $\mathbb{C}^\sigma := \{z \in \mathbb{C} : \Re z > -\sigma\}$ for any $\sigma \in \mathbb{R}$.

Proof. Write

$$\sup_{\sigma\in\mathbb{R}} \|(L-i\sigma)^{-1}\| = C$$

We denote $R(z) = (L-z)^{-1}$ for $z \in \rho(L) := \{z : z \in \mathbb{C} \setminus \text{ Spec}(L)\}$. Take $z_0 \in \rho(L)$, we write

$$L - z = (L - z_0)(\mathrm{Id} + (L-z_0)^{-1}(z_0 - z)),$$

and for $|z - z_0| < \frac{1}{\|(L-z_0)^{-1}\|}$, we have

$$\|R(z)\| \leq \|R(z_0)\| \sum_{n=0}^{\infty} |z-z_0|^n \|(L-z_0)^{-1}\|^n \leq R(z_0).$$

Therefore, for λ with $|\Re\lambda| \leq \delta$, where $0 < \delta < \frac{1}{2C}$, we have $\|R(\lambda)\| \leq C$. To conclude, we only need show that there exists $C_1 > 0$, such that

$$\sup_{\Re z > \delta} \|(L-z)^{-1}\| \leq C_1.$$

Consider the holomorphic equivalence $\varphi : \mathbb{C}^0 \to \mathbb{D}, \psi = \varphi^{-1}$.

$$\varphi(z) = \frac{z-1}{z+1}, \psi(\zeta) = \frac{1+\zeta}{1-\zeta},$$

where $\mathbb{D} := \{\zeta : |\zeta| < 1\}$ be the unit disk. One easily verifies that the operator-valued function

$$\Phi(\zeta) = R(\psi(\zeta)) : \mathbb{D} \to \mathcal{L}(\mathcal{X})$$

is analytic and satisfies the Cauchy integral formula

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\Phi(\zeta)}{\zeta - \zeta_0} \mathrm{d}\zeta, \forall \zeta_0 \in \mathbb{D}.$$

Since dist $(\partial\mathbb{D}, \varphi(\mathbb{C}^{-\delta})) \geq \epsilon_0 > 0$ for some ϵ_0 depends only on δ, we deduce that for any $z \in \mathbb{C}^{-\delta}$,

$$\|R(z)\| \leq \left\| \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\Phi(\zeta)}{\zeta - \varphi(z)} \mathrm{d}\zeta \right\| \leq \frac{C}{\epsilon_0}.$$

$\square$

Lemma 4.8. (Unique continuation of Stoke operator) *Let $\sigma > 0$ and $u \in V$ satisfies that*

$$Au = \sigma^2 u.$$

Then, if $u|_\omega \equiv 0$, we must have $u \equiv 0$.

Proof. It is equivalent to write

$$-\Delta u + \nabla p = \sigma^2 u, \text{ div } u = 0, u \in V, \int\limits_\Omega p \mathrm{d}x = 0.$$

Take divergence of the equation, we have $\Delta p = 0$. The vanishing of u in ω implies that p equals a constant in a component of ω. Now since Ω is connected, the maximum principal implies that $p \equiv 0$ in Ω. Therefore, we have reduced to unique continuation of eigenfunction of Laplace operator, and this implies that $u \equiv 0$ in Ω. □

Proof of proposition 1.4. We need show that the semi-classical observability implies the stabilization.

Note that the operator $(\mathcal{A} - \lambda)$ is invertible for any $\lambda > 0$. One write

$$\mathcal{A} - z = (\mathrm{Id} + (1 - z)(\mathcal{A} - \mathrm{Id})^{-1})(\mathcal{A} - 1), \forall z \in \mathbb{C}.$$

Since $\mathrm{Id} + (1 - z)(\mathcal{A} - \mathrm{Id})^{-1}$ is Fredholm with index 0, we infer that $\mathcal{A} - z$ is invertible iff it is injective. In light of the previous lemmas and Proposition 4.3, we only have to prove the fact that

$$\exists C > 0, \text{ such that } \forall \sigma \in \mathbb{R}, U \in D(\mathcal{A}), F \in V \times H, (\mathcal{A} - i\sigma)U = F$$
$$\text{implies } \|U\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}.$$

We argue by contradiction. If it is not true, then we can find sequences (σ_n), (U_n), and (F_n) such that

$$(\mathcal{A} - i\sigma_n)U_n = F_n, \|U_n\|_{\mathcal{H}} = 1, \|F_n\|_{\mathcal{H}} < \frac{1}{n}.$$

After extracting subsequences we may assume that $\sigma_n \to \sigma$, and we write

$$U_n = (u_n, v_n)^t, F_n = (f_n, g_n)^t.$$

We have several cases to analyse, according to the limit value σ.

(1) $\sigma = 0$: In this case, we have $\mathcal{A}U_n = o(1)_{\mathcal{H}}$, which is equivalent to

$$v_n = o(1)_{H_0^1}, Au_n - \Pi a v_n = o(1)_{L^2},$$

thus $Au_n = o(1)_{L^2}$. Taking inner product with u_n and integrating by part we have

$$\int\limits_\Omega |\nabla u_n|^2 \mathrm{d}x = o(1).$$

This contradicts to $\|U_n\|_{\mathcal{H}} = 1$.

(2) $0 < |\sigma| < \infty$: In this case, we have $\mathcal{A}U_n - i\sigma U_n = o(1)_{\mathcal{H}}$, or equivalently,

$$v_n - i\sigma u_n = o(1)_{H_0^1}, Au_n - (i\sigma + \Pi a)v_n = o(1)_{L^2}.$$

Thanks to Poincaré inequality, we deduce that

$$Au_n - i\sigma(i\sigma + \Pi a)u_n = o(1)_{L^2}.$$

Applying Rellich compact embedding theorem followed by extracting to suitable subsequences, we may assume that

$$u_n \to u, \text{ in } L^2(\Omega), u_n \rightharpoonup u, \text{ in } V.$$

Taking inner product with u_n, we have

$$-\int\limits_\Omega |\nabla u_n|^2 \mathrm{d}x = -\sigma \int\limits_\Omega |u_n^2| \mathrm{d}x + i\sigma \int\limits_\Omega a(x)|u_n|^2 \mathrm{d}x + o(1),$$

which implies that $au \equiv 0$ in Ω. Thus, we can conclude that u is an eigenfunction of Stokes operator A and vanishes in a non-trivial open subset of Ω. The unique continuation property for the system

$$-\Delta u + \nabla p = \sigma^2 u, \ \operatorname{div}\, u = 0$$

implies that $u \equiv 0$. As a consequence, we have that $u_n = o(1)_{H_0^1}, v_n = o(1)_{L^2}$. This contradicts to the original assumptions.

(3) $|\sigma| = \infty$: We only study the case $\sigma_n \to +\infty$ (the other one is obtained by considering $\overline{U_n}$). Let $h_n = \sigma_n^{-1}$, and we deduce from the system $\mathcal{A}U_n - i\sigma_n U_n = o(1)_{\mathcal{H}}$:

$$\begin{aligned} &h_n^2 A u_n + u_n - i h_n \Pi a u_n = h_n^2 \Pi a f_n + i h_n f_n + h_n^2 g_n = o_{L^2}(h_n)\\ &h_n v_n - i u_n = h_n f_n = o(h_n)_{H_0^1},\\ &h_n^2 A v_n + v_n - i h_n \Pi a v_n = i h_n g_n - h_n^2 A f_n = o_{L^2}(h_n) + o_{H^{-1}}(h_n). \end{aligned}$$

Define the operator $P_h = h^2 A + \mathrm{Id} - ih\Pi a$ on H with domain $H^2(\Omega) \cap V$, we have (dropping the subindex n for the moment)

$$(P_h u|u)_{L^2} = \|u\|_{L^2(\Omega)}^2 - \|h\nabla u\|_{L^2(\Omega)}^2 - ih\|a^{1/2}u\|_{L^2(\Omega)}^2.$$

Taking imaginary part, we have

$$\|a^{1/2}u\|_{L^2(\Omega)}^2 \le C\frac{\|P_h u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}}{h}.$$

Applying the semi-classical observability to the equation

$$h^2 A u + u = ih\Pi a u + \tilde{f}$$

with $\tilde{f} = o_{L^2}(h)$, we have

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 &\le \left(\|a^{1/2}u\|_{L^2(\Omega)}^2 + \frac{1}{h^2}(\|f\|_{L^2(\Omega)}^2 + h^2\|a^{1/2}u\|_{L^2(\Omega)}^2) \right)\\ &\le \frac{C}{h}\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \frac{C}{h^2}\|f\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.11}$$

This implies that

$$\|u_n\|_{L^2(\Omega)} \le \frac{C}{h_n}\|f_n\|_{L^2(\Omega)} = o(1).$$

To conclude, observe that v_n satisfies

$$h_n^2 A v_n + v_n = o_H(h_n) + o_{H^{-1}(\Omega)}(h_n^2),$$

and we claim that if $(h^2 A + 1)v = f_1 + f_2$, then

$$\begin{aligned} &\|v\|_{L^2(\Omega)} + \|h\nabla v\|_{L^2(\Omega)}\\ &\le C\left(\|a^{1/2}v\|_{L^2} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2} \right). \end{aligned} \tag{4.12}$$

Assume the claim for the moment, we thus have $\|h_n\nabla v_n\|_{L^2(\Omega)} = o(1)$, and $\|\nabla u_n\|_{L^2(\Omega)} = o(1)$, thanks to $u_n + ih_n v_n = ih_n f_n$. This contradicts to the original assumption.

Now we turn to the proof of the claim. By density, (4.10) still valid when $v \in V$. Taking inner product of v with $P_h v$, we have

$$(P_h v|v)_{L^2} = \|v\|_{L^2(\Omega)}^2 - \|h\nabla v\|_{L^2(\Omega)}^2 - ih\|a^{1/2}v\|_{L^2(\Omega)}^2.$$

Therefore, by taking real part and injecting (4.10), we have

$$\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \le C\left(\|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|P_h v\|_{L^2(\Omega)}}{h} \right). \tag{4.13}$$

By taking real and imaginary part of $(P_h v|v)_{L^2}$, we have

$$\|a^{1/2}v\|^2_{L^2(\Omega)} \leq \frac{\|P_h v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}}{h}, \tag{4.14}$$

Substituting (4.14) into (4.13), we obtain that

$$\|h\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq C\left(\frac{\|P_h v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}}{h} + \frac{\|P_h v\|^2_{L^2(\Omega)}}{h^2}\right),$$

and this implies that

$$\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C\frac{\|P_h v\|_{L^2(\Omega)}}{h}.$$

Thus, P_h is bijective from $H^2(\Omega)\cap V$ to H and hence invertible. From the fact that

$$P_h = (1 + (2 - ih\Pi a)(h^2A - 1)^{-1})(h^2A - 1),$$

P_h can be written as composition of a positive operator and a Fredholh operator of index 0. From the estimate above, we conclude that

$$\|P_h^{-1}\|_{L^2\to L^2} \leq \frac{C}{h}, \|P_h^{-1}\|_{L^2\to H^1} \leq \frac{C}{h^2}.$$

Now come back to the equation $(h^2A + 1)v = f_1 + f_2$. Taking $g \in H$, and letting $w = P_h^{-1}g$, we have

$$\begin{aligned}(v|g)_{L^2} &= ((h^2A + 1)v|w)_{L^2} + ih(v|\Pi a w)_{L^2}\\ &= (f_1 + f_2|w)_{L^2} + ih(av|w)_{L^2}\\ &\leq \|f_1\|_{L^2(\Omega)}\|P_h^{-1}g\|_{L^2()\Omega)}\\ &\quad + \|f_2\|_{H^{-1}(\Omega)}\|P_h^{-1}g\|_V + h\|av\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\ &\leq C\left(\|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2}\right)\|g\|_{L^2(\Omega)}.\end{aligned} \tag{4.15}$$

This completes the proof.

□

5. A priori estimates for the quasi-mode system

Now we consider the quasi-modes of Stokes system

$$\begin{cases} -h_k^2\Delta u_k - u_k + h_k\nabla q_k = f_k, (u_k, f_k) \in (H^2(\Omega)\cap V)\times H,\\ h_k \mathrm{div} u_k = 0, \text{in } \Omega\end{cases} \tag{5.1}$$

To simplify the notation, we drop the sub index k and just keep the semi-classical parameter h everywhere. Note that the functions u, v, etc., should be understood as $u(h), v(h)$, etc. We fix the geometric assumption on the domain $\Omega \subset \mathbb{R}^d$ is smooth and connected and $\partial\Omega = \cup_{j=1}^N \Gamma_j$ with $\Gamma_j \cap \Gamma_k = \varnothing, i \neq k$ and each Γ_j is smooth and connected.

Now assume that

$$\|u\|_{L^2(\Omega)} = O(1), \|f\|_{L^2(\Omega)} = o(h).$$

Taking inner product with u and integrate by part, we have

$$\|h\nabla u\|_{L^2(\Omega)} = O(1).$$

One can always assume that $\int_\Omega q\mathrm{d}x = 0$, since $q \in L^2(\Omega)/\mathbb{R}$. From the regularity theory of steady Stokes system, (see [22, p. 33]), and Poincaré inequality, we have

$$\|h^2\nabla^2 u\|_{L^2(\Omega)} = O(1), \|q\|_{L^2(\Omega)} = O(h^{-1}), \|h\nabla q\|_{L^2(\Omega)} = O(1).$$

We now give some estimates on the trace. Write $q_0 = q|_{\partial\Omega}$,

Lemma 5.1. $\|q\|_{L^2(\Omega)} = O(h^{-1}), \|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1}), \|q_0\|_{L^2(\partial\Omega)} = O(h^{-1})$.

Proof. Since q is harmonic function, one can apply trace theorem $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ for any $s \in \mathbb{R}$. Hence, the conclusions follows from these and interpolations. □

Lemma 5.2. *$h\partial_\nu u|_{\partial\Omega} = (h\partial_\nu u_\parallel, 0)$, and $\|h\partial_{\mathbf{n}} u|_{\partial\Omega}\|_{L^2(\partial\Omega)} = O(1)$.*

Proof. The first assertion follows from $h\mathrm{div}u = 0$ and Dirichlet boundary condition, while we apply a multiplier method to prove the second. From the geometric assumption on Ω, we can find a vector field $L \in C^1(\overline{\Omega})$ such that $L|_{\partial\Omega} = \nu$ (see [15, p. 36]). In global coordinate system, we write $L = L_j(x)\partial_{x_j}$. By using the equation, we have

$$\int_\Omega Lu \cdot f\mathrm{d}x = \int_\Omega Lu \cdot (-h^2\Delta u - u + h\nabla q)\mathrm{d}x$$

$$\begin{aligned} -\int_\Omega Lu \cdot u\mathrm{d}x &= -\int_\Omega L_j(x)\partial_{x_j} u^i u^i \mathrm{d}x \\ &= -\int_\Omega \partial_{x_j}\left(L_j(x)u^i\right)u^i\mathrm{d}x + \int_\Omega \mathrm{div}\ L(x)|u|^2\mathrm{d}x \\ &= \int_\Omega L_j(x)u^i(x)\partial_{x_j}u^i\mathrm{d}x + \int_\Omega \mathrm{div}\ L(x)|u|^2\mathrm{d}x \\ &= \int_\Omega Lu\cdot u\mathrm{d}x + \int_\Omega \mathrm{div}\ L(x)|u|^2\mathrm{d}x, \end{aligned}$$

thus

$$\begin{aligned} h\int_\Omega Lu\cdot\nabla q\mathrm{d}x &= -h\int_\Omega u^i\partial_{x_j}\left(L_j\partial_{x_i}q\right)\mathrm{d}x \\ &= -h\int_\Omega u\cdot L(\nabla q)\mathrm{d}x - h\int_\Omega (\mathrm{div}\ L(x))u\cdot\nabla q\mathrm{d}x \\ &= -h\int_\Omega u\cdot[L,\nabla]q\mathrm{d}x - h\int_\Omega \mathrm{div}\ L(x)u\cdot\nabla q\mathrm{d}x \\ &= O(1), \end{aligned}$$

and $\displaystyle\int_\Omega Lu\cdot u\mathrm{d}x = -\frac{1}{2}\int_\Omega \mathrm{div}\ L(x)|u|^2\mathrm{d}x = O(1)$,

$$\begin{aligned} -h^2\int_\Omega Lu^i\Delta u^i\mathrm{d}x = &-h^2\int_{\partial\Omega}\left|\partial_\nu u^i\right|^2\mathrm{d}\sigma + h^2\int_\Omega \nabla L(\nabla u^i,\nabla u^i)\mathrm{d}x \\ &+h^2\int_\Omega L_j(x)\partial^2_{x_jx_k}u^i\partial_{x_k}u^i \end{aligned}$$

$$
\begin{aligned}
&= -h^2 \int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 \mathrm{d}\sigma + h^2 \int_\Omega \nabla L(x)(\nabla u^i, \nabla u^i)\mathrm{d}x \\
&\quad + h^2 \int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i \mathrm{d}x - h^2 \int_\Omega \operatorname{div} L(x) \nabla u^i \cdot \nabla u^i(x)\mathrm{d}x,
\end{aligned}
$$

$$
h^2 \int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i \mathrm{d}x = h^2 \int_{\partial\Omega} L\cdot\nu \left|\partial_\nu u^i\right|^2 \mathrm{d}\sigma - h^2 \int_\Omega L_j(x)\partial_{x_k} u^i \partial^2_{x_j x_k} u^i \mathrm{d}x,
$$

$$
-h^2 \int_\Omega L u^i \Delta u^i \mathrm{d}x = -\frac{h^2}{2}\int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 \mathrm{d}\sigma + \int_\Omega \nabla L(x)(h\nabla u^i, h\nabla u^i)\mathrm{d}x - \frac{h^2}{2}\int_\Omega \operatorname{div} L(x)|\nabla u^i|^2 \mathrm{d}x.
$$

Observing that $\int_\Omega Lu\cdot f\mathrm{d}x = o(1)$, we have

$$
\int_{\partial\Omega} |h\partial_\nu u|^2 \mathrm{d}\sigma = O(1).
$$

□

Lemma 5.3. *Under additional assumption that*

$$
\|a^{1/2}u_k\|_{L^2(\Omega)} = o(1),
$$

after extracting to subsequences, we have $h_k\nabla q_k \rightharpoonup 0$ $L^2(\Omega)$ and $u_k \rightharpoonup 0$ weakly in $L^2(\Omega)$. Therefore, from Rellich theorem, we have $hq \to 0$, strongly in $L^2(\Omega)$.

Proof. We may assume that $h\nabla q \rightharpoonup r$, weakly in $L^2(\Omega)$, and Rellich theorem implies that $hq \to P$, strongly in $L^2(\Omega)$, and thus $\nabla P = r$, with the property $\int_\Omega P = 0$. Now we claim that $\Delta P = 0$ in Ω.

Indeed, take any $\varphi \in C_0^\infty(\Omega)$,

$$
\int_\Omega \nabla P\cdot\nabla\varphi = \lim_{h\to 0}\int_\Omega h\nabla q\cdot\nabla\varphi = 0.
$$

Now suppose $u_k \to U$, weakly in $L^2(\Omega)$, $w_k = h_k^2 u_k \to W$, weakly in $H^2(\Omega)$, by taking the weak limit in the equation, we have

$$
-\Delta W - U + \nabla P = 0, \text{ in } L^2(\Omega).
$$

Notice that $a^{1/2}u_k \to 0, a^{1/2}w_k \to 0$, strongly in $L^2(\Omega)$, and this implies that $U|_\omega = W|_\omega = 0$. Therefore, in a connect component ω' of ω, we have $\nabla P \equiv 0$. However, P is a harmonic function, then $P \equiv$const., thanks to the fact that Ω is connected. Note that $\int_\Omega P = 0$, hence $P\equiv 0$. Moreover, from Rellich theorem that $w_k \to W$ strongly in $L^2(\Omega)$, and on the other hand $\|h_k^2 u_k\|_{L^2(\Omega)} = o(1)$ we must have $W = 0$. Therefore, $U = 0$. □

6. Proof of the observability estimates

In this part, we will prove Proposition 1.4 under the assumption in Theorem 1.2 on Ω and ω.

We argue by contradiction, suppose (1.7) is not true, we can then choose a sequence $(u_n, h_n, q_n, f_n) \in H^2(\Omega)\cap V \times \mathbb{R}_+ \times H^1(\Omega)\times H$ satisfies equation

$$
- h_n^2 \Delta u_n - u_n + h_n \nabla q_n = f_n \tag{6.1}
$$

with the following properties:

$$
\|u_n\|_{L^2(\Omega)} = 1, \|f_n\|_{L^2(\Omega)} = o(h_n), \|a^{1/2}u_n\|_{L^2(\Omega)} = o(1), n\to\infty.
$$

Up to extracting to subsequence, we can associate (u_n, h_n) with a semi-classical defect measure μ. Therefore, we have $\omega \cap \pi(\mathrm{supp}(\mu)) = \emptyset$ from Corollary 3.4, where we denote $\pi : T^*\overline{\Omega} \to \overline{\Omega}$ be the canonical projection.

Denote $\phi(s, \rho)$ be the globally defined generalized bicharacteristic flow, thanks to the geometric assumption that Ω has no infinite contact. Pick any point ρ_0 with $\pi(\rho_0) \in \omega$. For any time segment $[0, s_0]$, there are several situations:

Either $\phi([0, s_0], \rho_0) \subset \Omega$, or there exist $\pi(\phi([0, s_0], \rho_0)) \cap \partial\Omega \neq \emptyset$, then from the assumption on Ω, all points $\phi(s, \rho_0)$ with $s \in [0, s_0]$ and $\pi(\phi(s, \rho_0)) \in \partial\Omega$ must lie in $\mathcal{H} \cup \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-} \cup \bigcup_{k\geq 3} \mathcal{G}^k$. Now Theorem 3.7 implies that

$$\mathrm{supp}\ (\phi(s, \cdot)_* \mu) \subset \ \mathrm{supp}\ (\mu).$$

Therefore, we have

$$\phi([0, s_0], \rho_0) \cap \ \mathrm{supp}\ (\mu) = \emptyset.$$

We now invoke the geometric control condition to deduce that

$$\overline{\Omega} \subset \pi\left(\bigcup_{\rho_0 \in \omega} \phi([0, T_0], \rho_0)\right)$$

for some $T_0 > 0$ and thus $\mu = 0$. This contradicts to the assumption that

$$\int\limits_{\Omega} |u_n(x)|^2 \mathrm{d}x = 1.$$

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7. Appendix

We will derive the hyperbolic Stokes system (1.2) from certain limit procedure of Lamé system from elastic theory:

$$\begin{cases} \partial_t^2 w - \mu\Delta w - (\lambda + \mu)\nabla \mathrm{div} w = 0, (t, x) \in [0, T] \times \Omega \\ w(t, .)|_{\partial\Omega} = 0 \\ (w(0), \partial_t w(0)) = (w_0, z_0) \in (H_0^1(\Omega) \times L^2(\Omega))^d \end{cases} \tag{7.1}$$

where the solution $w(t, x)$ is vector-valued.

Define $u(t, x) := w(t/\sqrt{\mu}, x)$, then we find that

$$\partial_t^2 u - \Delta u - \frac{\lambda + \mu}{\mu} \nabla \mathrm{div} u = 0.$$

We let $\epsilon = \frac{\mu}{\mu + \lambda} \ll 1$, in the case that $\lambda \gg \mu > 0$. Thus, we obtain a family of equations

$$\begin{cases} \partial_t^2 u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = 0, (t, x) \in [0, T] \times \Omega \\ u_\epsilon(t, .)|_{\partial\Omega} = 0 \\ (u_\epsilon(0), \partial_t u_\epsilon(0)) = (u_{0,\epsilon}, v_{0,\epsilon}) \in (H_0^1(\Omega) \times L^2(\Omega))^d \end{cases} \tag{7.2}$$

where $p_\epsilon = -\frac{1}{\epsilon}\mathrm{div} u_\epsilon$ and satisfies $\int_\Omega p_\epsilon \mathrm{d}x = 0$.

We make further assumption on the family of initial data $(u_{0,\epsilon}, v_{0,\epsilon})$ so that

$$\|(u_{0,\epsilon}, v_{0,\epsilon}) - (u_0, v_0)\|_{H^1\times L^2} \leq C\epsilon$$

for some divergence free data $(u_0, v_0) \in V \times H$. In particular, we have

$$\|\mathrm{div} u_{0,\epsilon}\|_{L^2(\Omega)} \leq C\epsilon.$$

From the well-posedness of Lame system, we have that $u_\epsilon \in C([0,T]; H_0^1(\Omega)), \partial_t u_\epsilon \in C([0,T]; L^2(\Omega))$, and $p_\epsilon \in C([0,T]; L^2(\Omega))$. Moreover, we have the conservation of energy

$$E[u_\epsilon] = \frac{1}{2}\int\limits_\Omega \left(|\partial_t u_\epsilon|^2 + |\nabla u_\epsilon|^2 + \epsilon |p_\epsilon|^2\right) \mathrm{d}x$$

and therefore

$$E[u_\epsilon] = \frac{1}{2}\int\limits_\Omega \left(|u_{0,\epsilon}|^2 + |v_{0,\epsilon}|^2 + \frac{1}{\epsilon}|\mathrm{div} u_{0,\epsilon}|^2\right) \mathrm{d}x.$$

From this, we have, up to some subsequence of $(u_\epsilon, \partial_t u_\epsilon)$

$$\begin{aligned}
&\mathrm{div} u_\epsilon \to 0, \text{ in } L^\infty([0,T]; L^2(\Omega)),\\
&u_\epsilon * \rightharpoonup u, *\text{weakly in } L^\infty([0,T]; H_0^1(\Omega)),\\
&\partial_t u_\epsilon * \rightharpoonup \partial_t u, *\text{weakly in } L^\infty([0,T]; L^2(\Omega)).
\end{aligned}$$

From the uniform bound of $\|\partial_t u_\epsilon\|_{L^\infty([0,T];L^2(\Omega))}$, apply Ascoli theorem, we have that (up to some subsequence)

$$u_\epsilon \to u, \text{ in } C([0,T]; L^2(\Omega)).$$

Using the equation, we conclude that $\|\nabla p_\epsilon\|_{L^\infty([0,T];H^{-1}(\Omega))}$ is uniformly bounded. Combine with the fact $\int_\Omega p_\epsilon = 0$, we have that $\|p_\epsilon\|_{L^\infty([0,T];L^2(\Omega))}$ is uniformly bounded, thus up to some subsequence, we may assume that

$$p_\epsilon * \rightharpoonup p, *\text{weakly in } L^\infty([0,T]; L^2(\Omega)).$$

Now it is not difficult to verify that (u, p) is a weak solution to (1.1). Moreover, p satisfies the zero mean condition

$$\int\limits_\Omega p \mathrm{d}x = 0.$$

References

[1] Alinhac, S., Gérard, P.: Opérateurs pseudo-différentiels et théorème de Nash-Moser. EDP Sciences (1991)

[2] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**(5), 1024–1065 (1992)

[3] Burq, N.: Semi-classical estimates for the resolvent in nontrapping geometries. Int. Math. Res. Not. **5**, 221–241 (2002)

[4] Burq, N., Gérard, P.: Stabilisation of wave equations on the torus with rough dampings. arXiv:1801.00983

[5] Burq, N., Lebeau, G.: Mesures de défaut de compacité, application au système de Lamé, Ann. Scient. Éc. Norm. Sup., 4^e série, t.34, 817–870 (2001)

[6] Chaves-Silva, F.W.: A hyperbolic system and the cost of the null controllability for the Stokes system. Comput. Appl. Math. **34**(3), 1057–1074 (2015)

[7] Chaves-Silva, F.W., Lebeau, G.: Spectral inequality and optimal cost of controllability for the Stokes system. Control Optim. Calcul. Var. **22**(4), 1137–1162 (2016)

[8] Dehman, B., Le Rousseau, J., Léautaud, M.: Controllability of two coupled wave equations on a compact manifold. Arch. Rat. Mech. Anal. **211**(1), 113–187 (2014)

[9] Gérard, P., Leichtnam, E.: Ergodic properties of eigenfunctions for the Dirichlet problem. Duke. Math. J. **71**(2), 559–607 (1993)

[10] Gérard, P.: Microlocal defect measures. Commun. Partial Differ. Equ. **10**, 1347–1382 (1985)

[11] Gearhart, L.: Spectral theory for contraction semigroups on Hilbert space. Trans. Am. Math. Soc. **236**, 385–394 (1978)

[12] Hörmander, L.: Analysis of Linear Differential Operators, vol. 3. Springer, Berlin

[13] Lebeau, G.: Équation des ondes amorties. Boutet de Monvel, A. (Eds.), Algebraic and Geometric Methods in Mathematical Physics. Kluwer Academic, Dordrecht, pp. 73–109 (1996)

[14] Lions, J.L.: On some hyperbolic equations with a pressure term. In: Proceedings of the Conference Dedicated to Louis Nirenberg, Trento. Pitman Research Notes in Mathematics Series, vol. 269. Longman Scientific and Technical, Harlow, pp. 196–208

[15] Medeiros, L.A., Miranda, M.M., Lourêdo, A.T.: Introduction to Exact Control Theory, Method HUM, eduepb (2013)

[16] Melrose, R.B., Sjöstrand, J.: Singularities of boundary value problems I, II. Comm. Pure Appl. Math. **31**, 593–617 (1978)

[17] Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)

[18] Tartar, L.: H-measures: a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. Proc. R. Soc. Edinb. Sect. A **115**, 193–230 (1990)

[19] Rauch, J., Taylor, M.: Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J. **24**, 79–86 (1974)

[20] Sun, C.: Semi-classical propagation of singularities for the Stokes system. Commun. Partial Differ. Equ. **45**(8), 970–1030 (2020)

[21] Taylor, M.E.: Partial Differential Equations, I, II, Second Edition, Applied Mathematical Sciences. Springer (2011)

[22] Teman, R.: Navier–Stokes Equations, Theory and Numerical Analysis. North Holland, Amsterdam (1977)

[23] Burq, N., Zworski, M.: Geometric control in the presence of a black box. J. Am. Math. Soc. **17**(2), 443–471 (2004)

[24] Zworski, M.: Semi-Classical Analysis, Graduate Studies in Mathematics, vol. 138 (2012)

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