

UNIFORM NULL CONTROLLABILITY FOR A DEGENERATING REACTION-DIFFUSION SYSTEM APPROXIMATING A SIMPLIFIED CARDIAC MODEL*

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Abstract. This paper is devoted to the analysis of the uniform null controllability for a family of nonlinear reaction-diffusion systems approximating a parabolic-elliptic system which models the electrical activity of the heart. The uniform—with respect to the degenerating parameter—null controllability of the approximating system by means of a single control is shown. The proof is based on the combination of Carleman estimates and weighted energy inequalities.

Key words. reaction-diffusion system, monodomain model, Carleman estimates, uniform null controllability, observability

AMS subject classifications. 35K57, 93B05, 93B07, 93C10

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1. Introduction. Let $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) be a bounded connected open set whose boundary, $\partial\Omega$, is sufficiently regular. Let $T > 0$, and let ω and $\mathcal{O}$ be two (small) nonempty subsets of Ω , which we will refer to as *control domains*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

The main objective of this paper is to study the properties of controllability and observability for a family of nonlinear reaction-diffusion systems which degenerates into a nonlinear parabolic-elliptic system which models the electrical activity in the cardiac tissue.

To state the model, we let $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ represent the *intracellular* and *extracellular* electric potentials, respectively. Their difference, $v = u_i - u_e$, is called the *transmembrane* potential. The anisotropic properties of the media are modeled by intracellular and extracellular conductivity tensors $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$.

The widely accepted model (see [11, 21, 31]) describing the electrical activity in the cardiac tissue reads as follows:

$$(1.1) \quad \begin{cases} c_m \partial_t v - \operatorname{div}(\mathbf{M}_i(x) \nabla u_i) + h(v) = f 1_\omega & \text{in } Q, \\ c_m \partial_t v + \operatorname{div}(\mathbf{M}_e(x) \nabla u_e) + h(v) = g 1_{\mathcal{O}} & \text{in } Q, \end{cases}$$

where $c_m > 0$ is the surface capacitance of the membrane, the nonlinear function $h : \mathbb{R} \rightarrow \mathbb{R}$ is the transmembrane ionic current (the most interesting case being when h is a cubic polynomial), and f and g are stimulation currents applied, respectively, to ω and $\mathcal{O}$.

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System (1.1) is known as the *bidomain model* and is completed with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$(1.2) \quad u_i = u_e = 0 \text{ on } \Sigma$$

and initial data for the transmembrane potential

$$(1.3) \quad v(0, x) = v_0(x), \quad x \in \Omega.$$

We point out that realistic models describing electrical activities in the heart also include a system of ODEs for computing the ionic current as a function of the transmembrane potential and a series of additional “gating variables” which aim to model the ionic transfer across the cell membrane (see [5, 7, 8, 22, 23, 29, 30]).

In the case where $f1_\omega = g1_\phi$ and $\mathbf{M}_i = \mu\mathbf{M}_e$, for some constant $\mu \in \mathbb{R}$, the bidomain model is simplified into the following parabolic-elliptic system:

$$(1.4) \quad \begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \operatorname{div}(\mathbf{M}_e(x) \nabla v) + h(v) = f1_\omega & \text{in } Q, \\ -\operatorname{div}(\mathbf{M}(x) \nabla u_e) = \operatorname{div}(\mathbf{M}_i(x) \nabla v) & \text{in } Q, \\ v = u_e = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

where $M = M_i + M_e$.

System (1.4) is known as the *monodomain model* and is a very interesting model from the implementation point of view, since it conserves some of the essential features of the bidomain model as excitability phenomena (see [11, 25, 32]).

The main difference between the bidomain model (1.1) and the monodomain model (1.4) is the fact that the first model is a system of two coupled parabolic equations, while the second is a system of parabolic-elliptic type. Therefore, from the control point of view, one could expect these two systems to have, at least a priori, different control properties. In this work we show that the properties of controllability and observability for the monodomain model can be seen as a limit process of the controllability properties of a family of coupled parabolic systems. Indeed, given $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon \leq 1$, we approximate the monodomain model by the following family of parabolic systems:

$$(1.5) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(\mathbf{M}_e(x) \nabla v^\varepsilon) + h(v^\varepsilon) = f^\varepsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div}(\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega. \end{cases}$$

In this paper we give a positive answer to the following question.

QUESTION 1.1. *If, for each $\varepsilon > 0$, there exists a control f^ε that drives the solution $(v^\varepsilon, u_e^\varepsilon)$ of (1.5) to zero at time $t = T$, i.e.,*

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0,$$

is it true that when $\varepsilon \rightarrow 0^+$, the control sequence $\{f^\varepsilon\}_{\varepsilon>0}$ converges to a function f which drives the associated solution (v, u_e) of (1.4) to zero at time $t = T$?

This question of approximating an equation by another having different physical properties has been used several times in the case of parabolic equations degenerating into hyperbolic ones (see, for example, [12, 14, 20]) and hyperbolic equations

degenerating into parabolic ones (see, for example, [27, 28]). However, as far as we know, this is the first time that controllability of parabolic systems degenerating into parabolic-elliptic systems has been studied. It is also important to mention that families of parabolic systems which degenerate into parabolic-elliptic ones arise in many areas, such as biology, chemistry, and astrophysics (see [9, 10, 24]).

As usual, in control theory, when dealing with the controllability of a nonlinear problem, we first consider the linearized version of (1.5):

$$(1.6) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div} (\mathbf{M}_e(x) \nabla v^\varepsilon) + a(t, x) v^\varepsilon = f^\varepsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div} (\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div} (\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

where a is a bounded function.

Given $\varepsilon > 0$, the first obstacle to answering, positively, Question 1.1 will be to drive $(v^\varepsilon, u_e^\varepsilon)$, solution of (1.6), to zero at time T by means of a control f^ε in such a way that the sequence of controls $\{f^\varepsilon\}_{\varepsilon>0}$ converges when $\varepsilon \rightarrow 0^+$. Once it is shown that such a convergent sequence of control, $\{f^\varepsilon\}_{\varepsilon>0}$, exists for the linear system (1.6), we employ a fixed point argument and conclude that the same is true for the nonlinear system (1.5).

Thus, we introduce the adjoint system of (1.6):

$$(1.7) \quad \begin{cases} -c_m \partial_t \varphi^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div} (M_e(x) \nabla \varphi^\varepsilon) + a(t, x) \varphi^\varepsilon = \operatorname{div} (M_i(x) \nabla \varphi_e^\varepsilon) & \text{in } Q, \\ -\varepsilon \partial_t \varphi_e^\varepsilon - \operatorname{div} (M(x) \nabla \varphi_e^\varepsilon) = 0 & \text{in } Q, \\ \varphi^\varepsilon = \varphi_e^\varepsilon = 0 & \text{on } \Sigma, \\ \varphi^\varepsilon(T) = \varphi_T, \quad \varphi_e^\varepsilon(T) = \varphi_{e,T} & \text{in } \Omega. \end{cases}$$

Using duality arguments, it is very easy to prove that the task of building such a convergent sequence of controls, $\{f^\varepsilon\}_{\varepsilon>0}$, for (1.6) is equivalent to proving the following (uniform) observability inequality for the solutions of (1.7):

$$(1.8) \quad \|\varphi^\varepsilon(0)\|_{L^2(\Omega)}^2 + \varepsilon \|\varphi_e^\varepsilon(0)\|_{L^2(\Omega)}^2 \leq C \iint_{Q_\omega} |\varphi^\varepsilon|^2 dx dt, \quad Q_\omega := \omega \times (0, T),$$

where $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$ and the constant $C = C(\varepsilon, \Omega, \omega, \|a\|_{L^\infty}, T)$ remains bounded when $\varepsilon \rightarrow 0^+$.

We prove inequality (1.8) as a consequence of an appropriate Carleman inequality for the solution $(\varphi^\varepsilon, \varphi_e^\varepsilon)$ of (1.7) (see section 3). We notice that due to the fact that the control is acting on the first equation of (1.6), in our Carleman inequality, we need to bound global integrals of φ^ε and φ_e^ε in terms of a local integral of φ^ε , uniformly with respect to ε . Two main difficulties appear: first, the coupling in the first equation of (1.7) is in $\operatorname{div} (M_i(x) \nabla \varphi_e^\varepsilon)$ and not in φ_e^ε ; second, we must show that the constant we get in our Carleman inequality does not blow up when $\varepsilon \rightarrow 0^+$.

The first difficulty is not so hard to overcome. Indeed, for each $\varepsilon > 0$ fixed, inequality (1.8) is known to be true for system (1.7) (see [18]). However, the main novelty here is the fact that we obtain the boundedness of the observability constant C with respect to ε . As we will see, Carleman inequalities alone are not enough for this task, and we need to combine sharp Carleman estimates, with respect to ε , and weighted energy inequalities.

As far as the controllability of nondegenerate coupled parabolic systems is concerned, the situation is, by now, fairly well understood. For instance, in [18], the controllability of a quite general linear coupled parabolic system is studied, and a null controllability result is obtained by means of Carleman inequalities. In [1], using a different strategy, the controllability of a nonlinear reaction-diffusion system of two coupled parabolic equations is analyzed, and the authors prove the null controllability for the linear system and the local null controllability of the nonlinear system. Another relevant work concerning the controllability of coupled systems is [16], in which the authors analyze the null controllability of a cascade system of m ($m > 1$) coupled parabolic equations and are able to obtain null controllability for the cascade system whenever they have a good coupling structure. It is also worth mentioning the works [2, 4, 17], where local and global controllability results for phase field systems were studied.

For a general discussion about the controllability of coupled parabolic systems, see the survey paper [3].

Concerning controllability results for the bidomain model, since in both equations the couplings are given by the time derivatives of the electrical potentials, it is very difficult to study controllability properties for such a model. To the best of our knowledge, for the bidomain model (1.1), the problems of null and approximate controllability are still open (even with two controls). Regarding the null controllability of the monodomain model (1.4), since the solution of the parabolic equation enters as a source term in the elliptic one, the following controllability result holds.

THEOREM 1.2.

1. If h is $C^1(\mathbb{R})$, globally Lipschitz and $h(0) = 0$, then for every $v_0 \in L^2(\Omega)$, there exists a control $f \in L^2(\omega \times (0, T))$ such that the solution (v, u_e) of (1.4) satisfies

$$v(T) = u_e(T) = 0.$$

Moreover, the control f satisfies the following estimate:

$$(1.9) \quad \|f1_\omega\|_{L^2(Q)}^2 \leq C \|v_0\|_{L^2(\Omega)}^2$$

for a constant $C = C(\Omega, \omega, T) > 0$.

2. If h is $C^1(\mathbb{R})$ and $h(0) = 0$, there exists $\gamma > 0$ such that for every $v_0 \in W^{2/3,6}(\Omega) \cap H_0^1(\Omega)$ with $\|v_0\|_{W^{2/3,6}(\Omega) \cap H_0^1(\Omega)} \leq \gamma$, there exists a control $f \in L^6(\omega \times (0, T))$ such that the solution (v, u_e) of (1.4) satisfies

$$v(T) = u_e(T) = 0.$$

Moreover, the control f satisfies the following estimate:

$$(1.10) \quad \|f1_\omega\|_{L^6(Q)} \leq C \|v_0\|_{L^2(\Omega)}$$

for a constant $C = C(\Omega, \omega, T) > 0$.

Theorem 1.2, case 1, follows from [13, Theorem 3.1], and case 2 follows from [19, Theorem 3.5] (see also [13, Theorem 4.2]).

This paper is organized as follows. In section 2, we state our main results. In section 3, we prove a uniform Carleman inequality for the adjoint system (1.7). Next, we show in section 4 the uniform null controllability of (1.6). In section 5, we deal with the uniform null controllability of the nonlinear system (1.5).

2. Main results. Throughout this paper we will assume that the matrices M_j , $j = i, e$, are C^∞ , bounded, symmetric, and positive semidefinite.

Our first main result is a uniform Carleman estimate for the adjoint system (1.7).

THEOREM 2.1. *Given any $0 < \epsilon \leq 1$, there exist positive constants $C = C(\Omega, \omega)$, $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$, and $s_0 = s_0(\Omega, \omega) \geq 1$ such that for any $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$ and any $a \in L^\infty(Q)$, the solution $(\varphi^\epsilon, \varphi_e^\epsilon)$ of (1.7) satisfies*

$$(2.1) \quad \begin{aligned} & \iint_Q e^{3s\alpha} |\rho^\epsilon|^2 dx dt + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi^\epsilon|^2 dx dt \\ & \leq C e^{6\lambda \|\psi\|} s^8 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi^\epsilon|^2 dx dt \end{aligned}$$

for every $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3})T^2)s_0$ and $\lambda \geq \lambda_0$, where $\rho^\epsilon(x, t) = \operatorname{div}(M(x)\nabla\varphi_e^\epsilon(x, t))$ and the weight functions ϕ and α are defined in (3.3) and (3.4), respectively.

The proof of Theorem 2.1 follows from a combination of Carleman inequalities for the heat equation, with a precise dependence on the degenerating parameter, and an energy inequality for the adjoint system (1.7). We prove Theorem 2.1 in section 3.

Remark 2.2. As a direct consequence of the Carleman inequality (2.1), we have the unique continuation property for the solutions $(\varphi^\epsilon, \varphi_e^\epsilon)$ of (1.7):

Given $\epsilon > 0$, if $\varphi^\epsilon = 0$ in $\omega \times (0, T)$, then $(\varphi^\epsilon, \varphi_e^\epsilon) \equiv (0, 0)$ in Q .

This unique continuation property for the adjoint system (1.7) implies, for each $\epsilon > 0$, the approximate controllability at time T of system (1.6), with a control acting only on the first equation.

The second main result of this paper gives the global null controllability of the linear system (1.6).

THEOREM 2.3. *For any $0 < \epsilon \leq 1$ and any $(v_0, u_{e,0}) \in L^2(\Omega)^2$, there exists a control $f^\epsilon \in L^2(\omega \times (0, T))$ such that the associated solution $(v^\epsilon, u_e^\epsilon)$ of (1.6) is driven to zero at time T . That is, the associated solution satisfies*

$$v^\epsilon(T) = 0, \quad u_e^\epsilon(T) = 0.$$

Moreover, the control f^ϵ satisfies the estimate

$$(2.2) \quad \|f^\epsilon 1_\omega\|_{L^2(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \epsilon \|u_{e,0}\|_{L^2(\Omega)}^2)$$

for a constant $C = C(\Omega, \omega, \|a\|_{L^\infty}, T) > 0$.

From Theorem 2.1, the proof of Theorem 2.3 is standard. However, for the sake of completeness, we prove Theorem 2.3 in section 4.

The third main result of this paper is concerned with the uniform null controllability of the nonlinear parabolic system (1.5).

THEOREM 2.4. *Given any $0 < \epsilon \leq 1$, we have the following:*

1. *If h is $C^1(\mathbb{R})$ and globally Lipschitz and $h(0) = 0$, then for every $(v_0, u_{e,0}) \in L^2(\Omega)^2$, there exists a control $f^\epsilon \in L^2(\omega \times (0, T))$ such that the solution $(v^\epsilon, u_e^\epsilon)$ of (1.5) satisfies*

$$v^\epsilon(T) = u_e^\epsilon(T) = 0.$$

Moreover, the control f^ϵ satisfies the estimate

$$(2.3) \quad \|f^\epsilon 1_\omega\|_{L^2(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \epsilon \|u_{e,0}\|_{L^2(\Omega)}^2)$$

for a constant $C = C(\Omega, \omega, T) > 0$.

2. If h is $C^1(\mathbb{R})$ and $h(0) = 0$, there exists $\gamma > 0$, not depending on ε , such that for every $(v_0, u_{e,0}) \in (W^{2/3,6}(\Omega) \cap H_0^1(\Omega))^2$ with $\|(v_0, u_{e,0})\|_{W^{2/3,6}(\Omega)} \leq \gamma$, there exists a control $f^\varepsilon \in L^6(\omega \times (0, T))$ such that the solution $(v^\varepsilon, u_e^\varepsilon)$ of (1.5) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Moreover, the control f^ε satisfies the estimate

$$(2.4) \quad \|f^\varepsilon 1_\omega\|_{L^6(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2)$$

for a constant $C = C(\Omega, \omega, T) > 0$.

The proof of Theorem 2.4 is achieved through fixed point arguments and will be given in section 5.

Remark 2.5. In this paper we restrict the dimension to $N = 2, 3$ because the bidomain model makes sense only in such dimensions. Nevertheless, from the mathematical point of view, systems (1.4)–(1.6) make sense for any $N \in \mathbb{N}$ (the one-dimensional case corresponding to the cable equation), and taking the initial data in the appropriate space, all the results of this paper can be extended to higher dimensions (see [6]).

3. Carleman inequality. In this section we prove Theorem 2.1.

To simplify the notation, we neglect the index ε , and since the only constant which matters in the analysis is ε , we assume that all the other constants are normalized to be the unity. In this case, the adjoint system (1.7) reads as

$$(3.1) \quad \begin{cases} -\partial_t \varphi - \operatorname{div}(M_e(x) \nabla \varphi) + a(x, t) \varphi = \operatorname{div}(M_i(x) \nabla \varphi_e) & \text{in } Q, \\ -\varepsilon \partial_t \varphi_e - \operatorname{div}(M(x) \nabla \varphi_e) = 0 & \text{in } Q, \\ \varphi = \varphi_e = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T, \quad \varphi_e(T) = \varphi_{e,T} & \text{in } \Omega. \end{cases}$$

We notice that if φ_T and $\varphi_{e,T}$ are regular enough, taking $\rho(x, t) = \operatorname{div}(M_i(x) \nabla \varphi_e(x, t))$, the pair (φ, ρ) satisfies

$$(3.2) \quad \begin{cases} -\partial_t \varphi - \operatorname{div}(M_e(x) \nabla \varphi) + a(x, t) \varphi = \rho & \text{in } Q, \\ -\varepsilon \partial_t \rho - \operatorname{div}(M(x) \nabla \rho) = 0 & \text{in } Q, \\ \varphi = \rho = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T, \quad \rho(T) = \rho_T & \text{in } \Omega. \end{cases}$$

We prove the Carleman inequality (2.1) using system (3.2).

Before starting the proof of the Carleman inequality, let us first define several weight functions which will be useful in what follows.

LEMMA 3.1. Let ω_0 be an arbitrary nonempty open set such that $\overline{\omega_0} \subset \omega \subset \Omega$. There exists a function $\psi \in C^2(\overline{\Omega})$ such that

$$\psi(x) > 0 \quad \forall x \in \Omega, \quad \psi \equiv 0 \quad \text{on } \partial\Omega, \quad |\nabla \psi(x)| > 0 \quad \forall x \in \overline{\Omega \setminus \omega_0}.$$

Proof. For the proof, see [13]. $\square$

Using Lemma 3.1, we introduce the weight functions

$$(3.3) \quad \phi(x, t) = \frac{e^{\lambda(\psi(x) + m|\psi|)}}{t(T-t)}; \quad \phi^*(t) = \min_{x \in \overline{\Omega}} \phi(x, t) = \frac{e^{\lambda m|\psi|}}{t(T-t)};$$

$$(3.4) \quad \alpha(x, t) = \frac{e^{\lambda(\psi(x)+m\|\psi\|)} - e^{2\lambda m\|\psi\|}}{t(T-t)}; \quad \alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(x, t) = \frac{e^{\lambda(m+1)\|\psi\|} - e^{2\lambda m\|\psi\|}}{t(T-t)}$$

for a parameter $\lambda > 0$ and a constant $m > 1$. Here,

$$\|\psi(x)\| = \max_{x \in \overline{\Omega}} |\psi(x)|.$$

Remark 3.2. From the definition of α and α^* it follows that for λ large enough, $3\alpha^* \leq 2\alpha$. Moreover,

$$\phi^*(t) \leq \phi(x, t) \leq e^{\lambda\|\psi\|} \phi^*(x, t)$$

and

$$|\partial_t \alpha^*| \leq e^{2\lambda\|\psi\|} T \phi^2.$$

Proof of Theorem 2.1. For better comprehension, we divide the proof into several steps.

Step 1. First estimate for the parabolic system. In this step we obtain a first Carleman estimate for the adjoint system (1.7). For that, we will apply sharp Carleman inequalities, with respect to ε , to the system and get a global estimate of φ and ρ in terms of a local integral of φ and another in ρ .

We consider a set ω_1 such that $\omega_0 \subset \subset \omega_1 \subset \subset \omega$ and apply the sharp Carleman inequality (A.2), with $\varepsilon = 1$, and (A.15) to φ and ρ , respectively. We get

$$(3.5) \quad \begin{aligned} & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\varphi_t|^2 dxdt + s^{-1} \iint_Q \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\ & + s^3 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |\varphi|^2 dxdt + s \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla \varphi|^2 dxdt \\ & \leq C \left(\iint_Q e^{2s\alpha} (|\rho|^2 + |\varphi|^2) dxdt + s^3 \lambda^4 \iint_{Q_{\omega_1}} \phi^3 e^{2s\alpha} |\varphi|^2 dxdt \right) \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} & \iint_Q e^{2s\alpha} |\partial_t \rho|^2 dxdt + \varepsilon^{-2} \iint_Q e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \rho \right|^2 dxdt \\ & + s^4 \lambda^4 \varepsilon^{-2} \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s^2 \lambda^2 \varepsilon^{-2} \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dxdt \\ & \leq C e^{\lambda\|\psi\|} s^4 \lambda^4 \varepsilon^{-2} \iint_{Q_{\omega_1}} \phi^4 e^{2s\alpha} |\rho|^2 dxdt \end{aligned}$$

for $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3})T^2)s_0$ and $\lambda \geq \lambda_0$.

Adding (3.5) and (3.6) and absorbing the lower order terms in the right-hand

side, we get

$$\begin{aligned}
 & \iint_Q \phi^{-1} e^{2s\alpha} |\varphi_t|^2 dxdt + \iint_Q \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\
 & + s^4 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |\varphi|^2 dxdt + s^2 \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla \varphi|^2 dxdt \\
 (3.7) \quad & + \varepsilon^2 \iint_Q e^{2s\alpha} |\partial_t \rho|^2 dxdt + \iint_Q e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \rho \right|^2 dxdt \\
 & + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dxdt \\
 & \leq C \left(e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_{\omega_1}} \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s^4 \lambda^4 \iint_{Q_{\omega_1}} \phi^3 e^{2s\alpha} |\varphi|^2 dxdt \right)
 \end{aligned}$$

for $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3})T^2)s_0$ and $\lambda \geq \lambda_0$.

Remark 3.3. If we were trying to drive the solution of (1.6) to zero by means of controls on both equations, inequality (3.7) would be sufficient.

Step 2. Estimate of the local integral of ρ . In this step we estimate the local integral on ρ in the right-hand side of (3.7). This will be done using (3.2)₁. Indeed, we consider a function ξ satisfying

$$\xi \in C_0^\infty(\omega), \quad 0 \leq \xi \leq 1, \quad \xi(x) = 1 \quad \forall x \in \omega_1$$

and write

$$\begin{aligned}
 (3.8) \quad & C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\rho|^2 \xi dxdt \\
 & = C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 \rho (-\varphi_t - \operatorname{div}(M_e \nabla \varphi) + a\varphi) \xi dxdt \\
 & := E + F + G.
 \end{aligned}$$

In what follows, we estimate each parcel in the expression above. First, we have

$$\begin{aligned}
 E & = C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} s \partial_t \alpha e^{2s\alpha} \phi^4 \rho \varphi \xi dxdt \\
 & + C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^3 \phi_t \rho \varphi \xi dxdt + C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 \partial_t \rho \varphi \xi dxdt \\
 & := E_1 + E_2 + E_3,
 \end{aligned}$$

and it is not difficult to see that

$$(3.9) \quad E_1 + E_2 \leq \frac{1}{10} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\rho|^2 dxdt + C e^{2\lambda \|\psi\|} s^8 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^8 |\varphi|^2 dxdt$$

and

$$E_3 \leq \frac{\varepsilon^2}{2} \iint_{Q_\omega} e^{2s\alpha} |\partial_t \rho|^2 dxdt + C e^{2\lambda \|\psi\|} \varepsilon^{-2} s^8 \lambda^8 \iint_{Q_\omega} e^{2s\alpha} \phi^8 |\varphi|^2 dxdt.$$

Next, integrating by parts, we get

$$\begin{aligned} e^{-\lambda\|\psi\|} s^{-4} \lambda^{-4} F &= \sum_{i,j=1}^N \iint_{Q_\omega} s \partial_{x_i} \alpha e^{2s\alpha} \phi^4 \rho (M_e^{ij} \partial_{x_j} \varphi) \xi dx dt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha} \phi^3 \partial_{x_i} \phi \rho (M_e^{ij} \partial_{x_j} \varphi) \xi dx dt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha} \phi^4 \partial_{x_i} \rho (M_e^{ij} \partial_{x_j} \varphi) \xi dx dt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha} \phi^4 \rho (M_e^{ij} \partial_{x_j} \varphi) \partial_{x_i} \xi dx dt, \end{aligned}$$

and we can show that

$$\begin{aligned} F &\leq \frac{1}{10} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\rho|^2 dx dt + \frac{1}{6} s^2 \lambda^2 \iint_{Q_\omega} e^{2s\alpha} \phi^2 |\nabla \rho|^2 dx dt \\ &\quad + C e^{2\lambda\|\psi\|} s^8 \lambda^8 \iint_{Q_\omega} e^{2s\alpha} \phi^8 |\varphi|^2 dx dt + \frac{1}{2} \iint_{Q_\omega} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \rho \right|^2 dx dt. \end{aligned}$$

Finally, we have

$$\begin{aligned} G &\leq \frac{1}{10} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\rho|^2 dx dt \\ &\quad + C e^{2\lambda\|\psi\|} s^4 \lambda^4 \|a\|_{L^\infty}^2 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\varphi|^2 dx dt. \end{aligned}$$

Putting E , F , and G together in (3.7), we obtain

(3.10)

$$\begin{aligned} &\iint_Q e^{2s\alpha} |\varphi_t|^2 dx dt + \iint_Q e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \varphi \right|^2 dx dt \\ &\quad + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\varphi|^2 dx dt + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \varphi|^2 dx dt \varepsilon^2 \iint_Q e^{2s\alpha} |\partial_t \rho|^2 dx dt \\ &\quad + \iint_Q e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \rho \right|^2 dx dt + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dx dt \\ &\quad + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dx dt \\ &\leq C e^{2\lambda\|\psi\|} \varepsilon^{-2} s^8 \lambda^8 \iint_{Q_\omega} e^{2s\alpha} \phi^8 |\varphi|^2 dx dt \end{aligned}$$

for $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3})T^2)s_0$ and $\lambda \geq \lambda_0$.

Using (3.10), we can prove that for every $\varepsilon > 0$, system (1.6) is null controllable. However, the sequence of controls obtained in this way will not be bounded when $\varepsilon \rightarrow 0^+$. Therefore, we need to go a step further and improve estimate (3.10). This is the goal of the next step.

Step 3. Weighted energy inequality. The reason we do not get a bounded sequence of controls out of Step 2 is because of the term ε^{-2} in the right-hand side of (3.10). In this step we prove a weighted energy inequality for $(3.2)_2$, which will be used to compensate this ε^{-2} term.

Let us introduce the function

$$y = e^{\frac{3}{2}s\alpha^*} \rho.$$

This new function satisfies

$$(3.11) \quad \begin{cases} \varepsilon \partial_t y - \operatorname{div}(M(x) \nabla y) = \varepsilon \frac{3}{2} s \partial_t \alpha^* e^{\frac{3}{2}s\alpha^*} \rho & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y(T) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying (3.11) by y and integrating over Ω , we get

$$\frac{\varepsilon}{2} \frac{d}{dt} \|y(t)\|_{L^2(\Omega)}^2 + C \|\nabla y(t)\|_{L^2(\Omega)}^2 \leq \varepsilon \frac{3}{2} \int_{\Omega} s \partial_t \alpha^*(t) e^{\frac{3}{2}s\alpha^*(t)} \rho(t) y(t) dx.$$

Integrating this last inequality from 0 to T and using Poincaré's and Young's inequalities, it is not difficult to see that

$$(3.12) \quad \iint_Q e^{3s\alpha^*} |\rho|^2 dx dt \leq C \varepsilon^2 e^{4\lambda \|\psi\|} \iint_Q s^4 \phi^4 e^{2s\alpha} |\rho|^2 dx dt.$$

Finally, from (3.10) and (3.12), we obtain

$$(3.13) \quad \iint_Q e^{3s\alpha^*} |\rho|^2 dx dt \leq C e^{6\lambda \|\psi\|} s^8 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dx dt.$$

This last estimate gives a global estimate of ρ in terms of a local integral of ϕ , with a constant C which is bounded with respect to ε .

Step 4. Last estimates and conclusion. In order to finish the proof of Theorem 2.1, we combine inequality (3.13) and a Carleman inequality that is slightly different from $(3.2)_1$. Indeed, the following Carleman inequality holds:

$$(3.14) \quad \begin{aligned} & \iint_Q s^{-1} \phi^{-1} e^{3s\alpha} |\varphi_t|^2 dx dt + s^{-1} \iint_Q \phi^{-1} e^{3s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 \varphi \right|^2 dx dt \\ & + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi|^2 dx dt + s \lambda^2 \iint_Q \phi e^{3s\alpha} |\nabla \varphi|^2 dx dt \\ & \leq C \left(\iint_Q e^{3s\alpha} |\rho|^2 dx dt + s^3 \lambda^4 \iint_{Q_\omega} \phi^3 e^{3s\alpha} |\varphi|^2 dx dt \right) \end{aligned}$$

for $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3}) T^2) s_0$ and $\lambda \geq \lambda_0$, where φ , together with ρ , is the solution of (3.2).

Notice that here we have just changed the weight $e^{2s\alpha}$ by $e^{3s\alpha}$. The proof of (3.14) is exactly the same as the proof of Theorem A.1, just taking the appropriate change of variable in (A.3).

Next, since $e^{3s\alpha} \leq e^{3s\alpha^*}$, we have

$$\iint_Q e^{3s\alpha} |\rho|^2 dx dt \leq \iint_Q e^{3s\alpha^*} |\rho|^2 dx dt,$$

and by (3.13), we have that

$$\iint_Q e^{3s\alpha} |\rho|^2 dx dt \leq C e^{6\lambda \|\psi\|} s^8 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dx dt.$$

From (3.13) and (3.14), it follows that

$$(3.15) \quad \iint_Q e^{3s\alpha} |\rho|^2 dx dt + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi|^2 dx dt \leq C e^{6\lambda \|\psi\|} s^8 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dx dt,$$

which is exactly (2.1).

By density, we can show that (3.15) remains true when we consider initial data in $L^2(\Omega)$. Therefore, the proof of Theorem 2.1 is finished. $\square$

4. Null controllability for the linearized system. This section is devoted to proving the null controllability of linearized equation (1.6). It will be done by showing the observability inequality (1.8) for the adjoint system (1.7), and solving a minimization problem. The arguments used here are classical in control theory for linear PDEs. Hence, we just give a sketch of the proof.

Proof of Theorem 2.3. Combining the standard energy inequalities for system (3.2) and the Carleman inequality given by Theorem 2.1, we can show the following observability inequality for the solutions of (3.2):

$$(4.1) \quad \|\varphi(0)\|_{L^2(\Omega)}^2 + \varepsilon \|\rho(0)\|_{L^2(\Omega)}^2 \leq e^{C(1+1/T+\|a\|_{L^\infty}^{2/3}+\|a\|_{L^\infty} T)} \iint_{Q_\omega} |\varphi|^2 dx dt,$$

where $C = C(\Omega, \omega)$ is a positive constant.

Next, since $\rho(x, t) = \operatorname{div}(M(x)\nabla\varphi_e(x, t))$ and $\varphi_e = 0$ on $\partial\Omega$, we have that

$$\|\varphi_e(t)\|_{H^2(\Omega)} \leq C \|\rho(t)\|_{L^2(\Omega)}$$

for all $t \in [0, T]$. Therefore, it follows from (4.1) that

$$(4.2) \quad \|\varphi(0)\|_{L^2(\Omega)}^2 + \varepsilon \|\varphi_e(0)\|_{L^2(\Omega)}^2 \leq e^{C(1+1/T+\|a\|_{L^\infty}^{2/3}+\|a\|_{L^\infty} T)} \iint_{Q_\omega} |\varphi|^2 dx dt,$$

which is the observability inequality (1.8).

From (4.2) and the density of smooth solutions in the space of solutions of (3.1) with initial data in $L^2(\Omega)$, we see that the above observability inequality is satisfied by all solutions of (1.7) with initial data in $L^2(\Omega)$.

Now, in order to obtain the null controllability for linear system (1.6), we solve, for any $\delta > 0$, the following minimization problem: Minimize $\mathcal{J}_\delta(\varphi_T, \varphi_{e,T})$, with

$$(4.3) \quad \mathcal{J}_\delta(\varphi_T, \varphi_{e,T}) = \left\{ \frac{1}{2} \int_0^T \int_\omega |\varphi^\varepsilon|^2 dx dt + \varepsilon(u_{e,0}, \varphi_e^\varepsilon(0)) \right. \\ \left. + (v_0, \varphi^\varepsilon(0)) + \delta(\|\varphi_T\|_{L^2(\Omega)} + \varepsilon^{1/2} \|\varphi_{e,T}\|_{L^2(\Omega)}) \right\},$$

where (φ, φ_e) is the solution of the adjoint problem (1.7) with initial data $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$.

It is an easy matter to check that $\mathcal{J}_\delta$ is strictly convex and continuous. So, in order to guarantee the existence of a minimizer, the only thing remaining to prove is the coercivity of $\mathcal{J}_\delta$.

Using the observability inequality (1.8) for the adjoint system (1.7), the coercivity of $\mathcal{J}_\delta$ is straightforward. Therefore, for each $\delta > 0$, there exists a unique minimizer $(\varphi_{e,T}^\delta, \varphi_T^\delta)$ of $\mathcal{J}_\delta$. Let us denote by $\varphi^{\varepsilon,\delta}$ the corresponding solution to (1.7) associated to this minimizer. Taking $f^{\varepsilon,\delta} = \varphi^{\varepsilon,\delta} 1_\omega$ as a control for (1.6), the duality between (1.6) and (1.7) gives

$$(4.4) \quad \|v^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} + \varepsilon^{1/2} \|u_e^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} \leq \delta,$$

where $(v^{\varepsilon,\delta}, u_e^{\varepsilon,\delta})$ is the solution of (1.6) associated to the control $f^{\varepsilon,\delta}$. It also gives

$$(4.5) \quad \|f^{\varepsilon,\delta} 1_\omega\|_{L^2(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2).$$

From (4.4) and (4.5), we get a control f^ε (the weak limit of a subsequence of $f^{\varepsilon,\delta} 1_\omega$ in $L^2(\omega \times (0, T))$) that drives the solution of (1.6) to zero at time T . From (4.5), we have the following estimate on the control f^ε :

$$(4.6) \quad \|f^\varepsilon 1_\omega\|_{L^2(Q)}^2 \leq C(\|v_0^\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}^\varepsilon\|_{L^2(\Omega)}^2).$$

This finishes the proof of Theorem 2.3. $\square$

5. The nonlinear system. In this section we prove Theorem 2.4. The proof is achieved through fixed point arguments.

Proof of Theorem 2.4 (case 1). We consider the following linearization of system (1.5):

$$(5.1) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(\mathbf{M}_e(x) \nabla v^\varepsilon) + g(z) v^\varepsilon = f^\varepsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div}(\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

where

$$(5.2) \quad g(s) = \begin{cases} \frac{h(s)}{s} & \text{if } |s| > 0, \\ h'(0) & \text{if } s = 0. \end{cases}$$

It follows from Theorem 2.3 that for each $(v_0, u_{e,0}) \in L^2(\Omega)^2$ and $z \in L^2(Q)$, there exists a control function $f^\varepsilon \in L^2(Q)$ such that the solution of (5.1) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

As we said before, the idea is to use a fixed point argument. For that, we need the following generalized version of Kakutani's fixed point theorem, due to Glicksberg [15].

THEOREM 5.1. *Let B be a nonempty convex, compact subset of a locally convex topological vector space X . If $\Lambda : B \rightarrow B$ is a convex set-valued mapping with closed graph and $\Lambda(B)$ is closed, then Λ has a fixed point.*

In order to apply Glicksberg's theorem, we define a mapping $\Lambda : B \rightarrow X$ as follows:

$$\Lambda(z) = \{v^\varepsilon; (v^\varepsilon, u_e^\varepsilon) \text{ is a solution of (5.1) such that } v^\varepsilon(T) = u_e^\varepsilon(T) = 0 \text{ for a control } f^\varepsilon \text{ satisfying (2.2)}\}.$$

Here, $X = L^2(Q)$, and B is the ball

$$B = \{z \in L^2(0, T; H_0^1(\Omega)), \partial_t z \in L^2(0, T; H^{-1}(\Omega)); \\ \|z\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t z\|_{L^2(0, T; H^{-1}(\Omega))}^2 \leq M\}.$$

It is easy to see that Λ is well defined and that B is a convex and compact subset of $L^2(Q)$.

Let us now prove that Λ is convex and compact and has closed graph.

• $\Lambda(B) \subset B$.

Let $z \in B$ and $v^\varepsilon \in \Lambda(z)$. Since v^ε satisfies (5.1)₁, the following inequality holds:

$$(5.3) \quad \|v^\varepsilon\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t v^\varepsilon\|_{L^2(0, T; H^{-1}(\Omega))}^2 \leq K_1.$$

In this way, if $z \in B$, then $\Lambda(z) \subset B$ if we take $M = K_1$.

• $\Lambda(z)$ is closed in $L^2(Q)$.

Let $z \in B$ fixed, and let $v_n^\varepsilon \in \Lambda(z)$ such that $v_n^\varepsilon \rightarrow v^\varepsilon$. Let us prove that $v^\varepsilon \in \Lambda(z)$. In fact, by definition we have that v_n^ε , together with a function $u_{e,n}^\varepsilon$ and a control f_n^ε , is the solution of (5.1), with $\|f_n^\varepsilon 1_\omega\|_{L^2(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2)$. Therefore, we can extract a subsequence of f_n^ε , denoted by the same index, such that

$$f_n^\varepsilon 1_\omega \rightarrow f^\varepsilon 1_\omega \text{ weakly in } L^2(Q).$$

Since f_n^ε is bounded, we can argue as in the previous section and show that

$$(5.4) \quad \|v_n^\varepsilon\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t v_n^\varepsilon\|_{L^2(0, T; H^{-1}(\Omega))}^2 \leq M.$$

Hence,

$$\begin{cases} v_n^\varepsilon \rightarrow v^\varepsilon \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ v_n^\varepsilon \rightarrow v^\varepsilon \text{ strongly in } L^2(Q), \\ \partial_t v_n^\varepsilon \rightarrow \partial_t v^\varepsilon \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{cases}$$

Using the convergences above and (5.1)₂, we see that there exists a function u_e^ε such that

$$\begin{cases} u_{e,n}^\varepsilon \rightarrow u_e^\varepsilon \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \\ u_{e,n}^\varepsilon \rightarrow u_e^\varepsilon \text{ strongly in } L^2(Q), \\ \partial_t u_{e,n}^\varepsilon \rightarrow \partial_t u_e^\varepsilon \text{ weakly in } L^2(0, T; H^{-1}(\Omega)). \end{cases}$$

It follows that $(v^\varepsilon, u_e^\varepsilon)$ is a controlled solution of (5.1) associated to the control f^ε . Hence, $v^\varepsilon \in \Lambda(z)$, and $\Lambda(z)$ is closed and compact in $L^2(Q)$.

• Λ has closed graph in $L^2(Q) \times L^2(Q)$.

We need to prove that if $z_n \rightarrow z$, $v_n^\varepsilon \rightarrow v^\varepsilon$ strongly in $L^2(Q)$ and $v_n^\varepsilon \in \Lambda(z_n)$, then $v^\varepsilon \in \Lambda(z)$. Using the two previous steps, it is easy to show that $v^\varepsilon \in \Lambda(z)$.

Therefore, we can apply Glicksberg's theorem to conclude that Λ has a fixed point. This proves Theorem 2.4 in the case where the nonlinearity is a C^1 globally Lipschitz function.

Proof of Theorem 2.4 (case 2). We consider the linear system

$$(5.5) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(\mathbf{M}_e(x) \nabla v^\varepsilon) + a(z) v^\varepsilon = f^\varepsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div}(\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

with $(v_0, u_{e,0}) \in (W^{2/3,6}(\Omega) \cap H_0^1(\Omega))^2$, $z \in L^\infty(Q)$, and

$$a(z) = \int_0^1 \frac{dh}{dz}(sz) ds.$$

Arguing as in the proof Theorem 2.3, we can show the null controllability of (5.5) with controls in $L^2(\omega \times (0, T))$. However, these L^2 controls are not sufficient to apply fixed point arguments and obtain the null controllability of the nonlinear system (1.5). For this reason, we modify the functional (4.3) a little, obtaining controls which will allow us to employ Schauder's fixed point theorem. Indeed, for any $\delta > 0$, we consider the following problem: Minimize $\mathcal{J}_\delta(\varphi_T, \varphi_{eT})$, with

$$(5.6) \quad \mathcal{J}_\delta(\varphi_T, \varphi_{eT}) = \left\{ \frac{1}{2} \int_0^T \int_\omega e^{2s\alpha} \phi^8 |\varphi^\varepsilon|^2 dx dt + \varepsilon(u_{e,0}, \varphi_e^\varepsilon(0)) \right. \\ \left. + (v_0, \varphi^\varepsilon(0)) + \delta \left(\|\varphi_T\|_{L^2(\Omega)} + \varepsilon^{1/2} \|\varphi_{e,T}\|_{L^2(\Omega)} \right) \right\},$$

where $(\varphi^\varepsilon, \varphi_e^\varepsilon)$ is the solution of the adjoint system (1.7) with initial data $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$.

As in section 4, we show that problem (5.6) has a unique minimizer $(\varphi^{\varepsilon,\delta}, \varphi_e^{\varepsilon,\delta})$. Defining $f^{\varepsilon,\delta} = e^{2s\alpha} \phi^8 \varphi^{\varepsilon,\delta}$ and using the fact that $\varphi^{\varepsilon,\delta}$, together with $\varphi_e^{\varepsilon,\delta}$, is the solution of (1.7), we see that $f^{\varepsilon,\delta}$ is a solution of a parabolic equation, with homogeneous Dirichlet boundary conditions, null initial data, and a right-hand side in $L^2(Q)$. Hence, $f^{\varepsilon,\delta} \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$, and in particular we have that $f^{\varepsilon,\delta} \in L^6(Q)$.

Arguing as in [1, Lemma 5], we can easily show that

$$(5.7) \quad \|f^{\varepsilon,\delta} 1_\omega\|_{L^6(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2),$$

where $C > 0$ is independent of ε and δ .

Moreover, the solution $(v^{\varepsilon,\delta}, u_e^{\varepsilon,\delta})$ of (5.5), associated to $f^{\varepsilon,\delta}$, satisfies

$$(5.8) \quad \|v^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} + \varepsilon^{1/2} \|u_e^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} \leq \delta.$$

Taking the limit when $\delta \rightarrow 0^+$, we get a control $f^\varepsilon \in L^6(Q)$ (the weak limit of a subsequence of $f^{\varepsilon,\delta}$) such that the associated solution $(v^\varepsilon, u_e^\varepsilon)$ of (5.5) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Next, we define a map $F : L^\infty(Q) \rightarrow L^\infty(Q)$ which for each $z \in L^\infty(Q)$ associates v^ε , where $(v^\varepsilon, u_e^\varepsilon)$ is the solution of (5.5) corresponding to z and the control f^ε built above. Note that this application is well defined since from the regularity theory for parabolic equations (see, for instance, [26]), we have that $v^\varepsilon \in X := L^6(0, T; W^{2,6}(\Omega)) \cap W^{1,6}(0, T; L^6(\Omega)) \cap C([0, T]; W^{2/3,6}(\Omega) \cap H_0^1(\Omega))$.

Let us now consider the set A defined as

$$A := \{z \in L^\infty(Q); \|z\|_{L^\infty(Q)} \leq 1\}.$$

It is clear that A is a convex closed subset of $L^\infty(Q)$. From (5.7), which still holds for f^ε , and the smallness assumption on the initial data, we can easily show that F is continuous and that $F(A) \subset A$. Finally, since the space X is compactly embedded

in $L^\infty(Q)$, we have that $F(A)$ is compact in $L^\infty(Q)$. Therefore, F has a fixed point, and the proof of case 2 of Theorem 2.4 is finished. $\square$

Appendix. Some technical results. In this section we prove the two sharp Carleman inequalities used in the proof of Theorem 2.1.

We consider the parabolic equation

$$(A.1) \quad \begin{cases} -\partial_t v(x, t) - \sum_{i,j=1}^N \partial_{x_i} (a_{ij}(x) \partial_{x_j} v(x, t)) = g(x, t) & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(T) = v_T & \text{in } \Omega, \end{cases}$$

where $v_T \in L^2(\Omega)$ and $g \in L^2(Q)$.

We assume that the matrix a_{ij} has the form

$$a_{ij} = \frac{M_{ij}}{\varepsilon},$$

where $(M_{ij})_{ij}$ is an elliptic matrix, i.e., there exists $\beta > 0$ such that $\sum_{i,j}^N M_{ij} \xi_j \xi_i \geq \beta |\xi|^2$ for all $\xi \in \mathbb{R}^N$.

A.1. A degenerating Carleman inequality. The first sharp Carleman inequality we prove is the following.

THEOREM A.1. *For any $0 < \epsilon \leq 1$, there exist $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$ and $s_0 = s_0(\Omega, \omega) \geq 1$ such that for every $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2)$, the solution v of (A.1) satisfies*

$$(A.2) \quad \begin{aligned} & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t v|^2 dx dt + s^{-1} \varepsilon^{-2} \iint_Q \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 v \right|^2 dx dt \\ & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 e^{2s\alpha} |v|^2 dx dt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi e^{2s\alpha} |\nabla v|^2 dx dt \\ & \leq C \left(\iint_Q e^{2s\alpha} |g|^2 dx dt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 e^{2s\alpha} |v|^2 dx dt \right), \end{aligned}$$

with $C > 0$ depending only on Ω , ω_0 , ψ , and β .

Proof. For $s > 0$ and $\lambda > 0$, we consider the change of variable

$$(A.3) \quad w(t, w) = e^{s\alpha} v(t, w),$$

which implies

$$w(T, x) = w(0, x) = 0.$$

Using the fact that v is the solution of (A.1), we write

$$(A.4) \quad L_1 w + L_2 w = g_s,$$

where

$$(A.5) \quad L_1 w = -\partial_t w + 2s\lambda \sum_{i,j=1}^N \phi a_{ij} \partial_{x_j} \psi \partial_{x_i} w + 2s\lambda^2 \sum_{i,j=1}^N \phi a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w,$$

$$(A.6) \quad L_2 w = - \sum_{i,j=1}^N \partial_{x_i} (a_{ij} \partial_{x_j} w) - s^2 \lambda^2 \sum_{i,j=1}^N \phi^2 a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w + s \partial_t \alpha w,$$

and

$$(A.7) \quad g_s = e^{s\alpha} g + s \lambda^2 \sum_{i,j=1}^N \phi a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w - s \lambda \sum_{i,j=1}^N \phi \partial_{x_i} (a_{ij} \partial_{x_j} \psi) w.$$

From (A.4), we have that

$$(A.8) \quad \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 + 2(L_1 w, L_2 w)_{L^2(Q)} = \|g_s\|_{L^2(Q)}^2.$$

The rest of the proof is devoted to analyzing the terms appearing in $(L_1 w, L_2 w)_{L^2(Q)}$.

First, we write

$$(L_1 w, L_2 w)_{L^2(Q)} = \sum_{i,j=1}^N I_{ij},$$

where I_{ij} is the inner product in $L^2(Q)$ of the i th term in the expression of $L_1 w$ and the j th term in $L_2 w$.

After a long, but straightforward, calculation, we can show that the following estimate holds:

$$(A.9) \quad \begin{aligned} 2(L_1 w, L_2 w)_{L^2(Q)} &\geq 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_Q \phi^3 |\nabla \psi|^4 |w|^2 dx dt + 2s \lambda^2 \beta^2 \varepsilon^{-2} \iint_Q \phi |\nabla \psi|^2 |\nabla w|^2 dx dt \\ &\quad - C \varepsilon^{-2} \left(T^2 s^2 \lambda^4 + T s^2 \lambda^2 + T^2 s + s^3 \lambda^3 + T s^2 \lambda \right) \iint_Q \phi^3 |w|^2 dx dt \\ &\quad - C \varepsilon^{-2} (s \lambda + \lambda^2) \iint_Q \phi |\nabla w|^2 dx dt. \end{aligned}$$

We take $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2)$, and it follows from Remark A.2 that

$$(A.10) \quad \begin{aligned} &2(L_1 w, L_2 w)_{L^2(Q)} + 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_{Q_{\omega_0}} \phi^3 |w|^2 dx dt \\ &\quad + 2s \lambda^2 \beta^2 \varepsilon^{-2} \iint_{Q_{\omega_0}} \phi |\nabla w|^2 dx dt \\ &\geq 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_Q \phi^3 |w|^2 dx dt + 2s \lambda^2 \beta^2 \varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dx dt. \end{aligned}$$

Remark A.2. Since $\overline{\Omega \setminus \omega_0}$ is compact and $|\nabla \psi| > 0$ on $\overline{\Omega \setminus \omega_0}$, there exists $\delta > 0$ such that

$$\beta |\nabla \psi| \geq \delta \text{ on } \overline{\Omega \setminus \omega_0}.$$

Putting (A.10) into (A.8), we get

$$\begin{aligned}
 & \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 + 2\beta^{-2}s^3\lambda^4\delta^4\varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt \\
 & + 2s\lambda^2\delta^2\varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
 (A.11) \quad & \leq \|g_s\|_{L^2(Q)}^2 + 2\beta^{-2}s^3\lambda^4\delta^4\varepsilon^{-2} \iint_{Q_{\omega_0}} \phi^3 |w|^2 dxdt \\
 & + 2s\lambda^2\delta^2\varepsilon^{-2} \iint_{Q_{\omega_0}} \phi |\nabla w|^2 dxdt.
 \end{aligned}$$

Now we deal with the local integral involving ∇w on the right-hand side of (A.11). To this end, we introduce a cut-off function ξ such that

$$\xi \in C_0^\infty(\omega), \quad 0 \leq \xi \leq 1, \quad \xi(x) = 1 \quad \forall x \in \omega_0.$$

Using the ellipticity condition on a_{ij} , we can prove that

$$\begin{aligned}
 & \beta\varepsilon^{-1} \iint_{Q_\omega} \phi \xi^2 |\nabla w|^2 dxdt \\
 & \leq C \left(\iint_Q L_2 w \phi \xi^2 w dxdt + (sT + \varepsilon^{-1}s^2\lambda^2) \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right. \\
 & \quad \left. + \lambda\varepsilon^{-1} \iint_{Q_\omega} \phi^{1/2} |\nabla w| \xi \phi^{1/2} w dxdt \right).
 \end{aligned}$$

Therefore, by Young's inequality, we have that

$$\begin{aligned}
 & s\lambda^2\delta^2\varepsilon^{-2} \iint_{Q_\omega} \phi \xi^2 |\nabla w|^2 dxdt \\
 & \leq \frac{1}{4} \iint_Q |L_2 w|^2 dxdt + C\beta^{-2}s^3\lambda^4(\delta^4 + \delta^2)\varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt.
 \end{aligned}$$

Thus, inequality (A.11) gives

$$\begin{aligned}
 & \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q_T)}^2 + \beta^{-2}s^3\lambda^4\varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt \\
 (A.12) \quad & + s\lambda^2\varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
 & \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + \beta^{-2}s^3\lambda^4\varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).
 \end{aligned}$$

Let us now use the first two terms in left-hand side of (A.12) to add the integrals of $|\Delta w|^2$ and $|w_t|^2$ to the left-hand side of (A.12). This is done using the expressions of $L_1 w$ and $L_2 w$. Indeed, from (A.5) and (A.6), we have

$$\begin{aligned}
 & \iint_Q s^{-1}\phi^{-1} |\partial_t w|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1}\phi^{-1} \sum_{i,j=1}^N |\partial_{x_i}(M_{ij}\partial_{x_j} w)|^2 dxdt \\
 (A.13) \quad & + s^3\lambda^4\varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt + s\lambda^2\varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
 & \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3\lambda^4\varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).
 \end{aligned}$$

Using the term in $|\partial_{x_i}(M_{ij}\partial_{x_j}w)|^2$ on the left-hand side of (A.13) and elliptic regularity, it is easy to show that

$$s^{-1}\varepsilon^{-2} \iint_Q \phi^{-1} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 w \right|^2 dxdt \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).$$

Estimate (A.13) then gives

$$\begin{aligned} & \iint_Q s^{-1} \phi^{-1} |\partial_t w|^2 dxdt + s^{-1} \varepsilon^{-2} \iint_Q \phi^{-1} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 w \right|^2 dxdt \\ & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\ (A.14) \quad & \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right). \end{aligned}$$

From (A.14) and the fact that $w = e^{s\alpha} v$, we finish the proof of Theorem A.1. $\square$

A.2. A slightly changed Carleman inequality. Our second sharp Carleman inequality is the following.

THEOREM A.3. *For any $0 < \epsilon \leq 1$, there exist $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$ and $s_0 = s_0(\Omega, \omega) \geq 1$ such that for every $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2)$, the solution v of (A.1) satisfies*

$$\begin{aligned} & \iint_Q e^{2s\alpha} |\partial_t v|^2 dxdt + \varepsilon^{-2} \iint_Q e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 v \right|^2 dxdt \\ (A.15) \quad & + s^4 \lambda^4 \varepsilon^{-2} \iint_Q \phi^4 e^{2s\alpha} |v|^2 dxdt + s^2 \lambda^2 \varepsilon^{-2} \iint_Q \phi^2 e^{2s\alpha} |\nabla v|^2 dxdt \\ & \leq C e^{\lambda \|\psi\|} \left(s \iint_Q \phi e^{2s\alpha} |g|^2 dxdt + s^4 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^4 e^{2s\alpha} |v|^2 dxdt \right), \end{aligned}$$

with $C > 0$ depending only on Ω , ω_0 , ψ , and β .

Proof. The starting point is the application of the Carleman inequality given in Theorem A.1 to equation (A.1). Indeed, we have

$$\begin{aligned} & \varepsilon^2 \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |v_t|^2 dxdt + \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 v \right|^2 dxdt \\ (A.16) \quad & + s^3 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |v|^2 dxdt + s \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla v|^2 dxdt \\ & \leq C \left(\iint_Q e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \iint_{Q_\omega} \phi^3 e^{2s\alpha} |v|^2 dxdt \right). \end{aligned}$$

Next, we introduce the function $y(x, t) = v(x, t)(\phi^*(t))^{\frac{1}{2}}$. This new function satisfies

$$(A.17) \quad \begin{cases} \varepsilon \partial_t y - \operatorname{div}(M(x) \nabla y) = -\varepsilon \frac{(T-2t)}{2} \phi^* y + \varepsilon (\phi^*(t))^{\frac{1}{2}} g & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases}$$

Applying again the Carleman inequality given by Theorem A.1, this time for y we obtain, for s large enough, that

$$\begin{aligned}
 (A.18) \quad & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t y|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 y \right|^2 dxdt \\
 & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 e^{2s\alpha} |y|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi e^{2s\alpha} |\nabla y|^2 dxdt \\
 & \leq C \left(\iint_Q \phi^* e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 e^{2s\alpha} |y|^2 dxdt \right).
 \end{aligned}$$

From the definition of y , it is easy to show that

$$(A.19) \quad \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |v_t(\phi^*)^{\frac{1}{2}}|^2 dxdt \leq \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t y|^2 dxdt + \iint_Q e^{2s\alpha} \phi |y|^2 dxdt.$$

Using (A.19), inequality (A.18) becomes

$$\begin{aligned}
 (A.20) \quad & \iint_Q s^{-1} \phi^{-1} \phi^* e^{2s\alpha} |v_t|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} \phi^* \phi^{-1} e^{2s\alpha} \sum_{i,j=1}^N \left| \partial_{x_i x_j}^2 v \right|^2 dxdt \\
 & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 \phi^* e^{2s\alpha} |v|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi \phi^* e^{2s\alpha} |\nabla v|^2 dxdt \\
 & \leq C \left(\iint_Q \phi^* e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 \phi^* e^{2s\alpha} |v|^2 dxdt \right).
 \end{aligned}$$

From Remark 3.2, the result follows. $\square$

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