

CONTROLLABILITY OF FAST DIFFUSION COUPLED PARABOLIC SYSTEMS

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ABSTRACT. In this work we are concerned with the null controllability of coupled parabolic systems depending on a parameter and converging to a parabolic-elliptic system. We show the uniform null controllability of the family of coupled parabolic systems with respect to the degenerating parameter.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough ($N \geq 1$). Let $T > 0$ and let ω_1 and ω_2 be two nonempty subsets of Ω , which will be referred to as *control domains*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

The main objective of this paper is to analyze the controllability of linear coupled parabolic systems in which one of the equations is degenerating into an elliptic equation.

In order to state our problem, we introduce the following system

$$\left\{ \begin{array}{ll} u_t - \Delta u = au + bv + f\chi_{\omega_1} & \text{in } Q, \\ \epsilon v_t - \Delta v = cu + dv + g\chi_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0; v(0) = v_0 & \text{in } \Omega. \end{array} \right. \quad (1)$$

For any $\epsilon > 0$, $a = a(x, t)$, $b = b(x, t)$, $c = c(x, t)$ and $d = d(x, t)$ in $L^\infty(Q)$, f and g in $L^2(Q)$ and u_0, v_0 in $L^2(\Omega)$, it is standard to show, from [13] for example, that (1) has a unique solution $(u, v) \in (L^2(0, T; H_0^1(\Omega) \cap C([0, T]; L^2(\Omega)))^2$. For the purpose of the present article we will consider a, b, c, d functions in $C^3(\overline{Q})$ and assume that $0 < \epsilon < 1/2$. In particular we want to study this problem when only one control is

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active, namely when $g \equiv 0$ or $f \equiv 0$ and analyze the dependence of the cost of the null controllability of system (1) with respect to the parameter ϵ .

Our interest in this problem comes from the fact that in many physical situations system (1) is formally approximated by the following parabolic-elliptic system

$$\begin{cases} u_t - \Delta u = au + bv + f\chi_{\omega_1} & \text{in } Q, \\ -\Delta v = cu + dv + g\chi_{\omega_2} & \text{in } Q, \\ u = v = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2)$$

This is the case for instance of biological systems modeling aggregation phenomena or chemical systems having two different concentrations, see [10] and [12] and references therein. However, even if this approximation is consistent with the existence and uniqueness point of view, it is not clear at all what can be done from a control theory point of view. The main reason for that arises from the fact that we are considering systems having different physical properties and therefore, at least a priori, different control properties.

If a, b, c, d are in $L^\infty(Q)$ and $d < \mu_1$, where μ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, as in the parabolic-parabolic case, it is standard to show that for every f and g in $L^2(Q)$ and u_0 in $L^2(\Omega)$ that (2) possesses a unique solution $u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ and $v \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$.

It is important to mention that this question of approximating an equation by another having different physical properties was already studied in the case of a hyperbolic equation degenerating into a parabolic one and vice-versa. In fact, it was proved in [11] that system

$$\begin{cases} \epsilon u_{tt} - \Delta u + u_t = f\chi_\omega & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0; \quad u_t(0) = u_1 & \text{in } \Omega, \end{cases} \quad (3)$$

is null controllable, for each ϵ fixed, and the controls remains bounded when $\epsilon \rightarrow 0$ if we impose some geometric condition on Ω . Furthermore, the control sequence converges, when $\epsilon \rightarrow 0$, to a control for the heat equation

$$\begin{cases} u_t - \Delta u = f\chi_\omega & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (4)$$

Another relevant work is [9], in which the authors consider the linear transport diffusion equation

$$\begin{cases} y_t - \epsilon \Delta y + M(x, t) \cdot \nabla y = f\chi_\omega & \text{in } \Omega \times (0, T), \\ y = 0 & \text{on } \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \quad (5)$$

and investigate the cost of the control in the vanishing viscosity limit $\epsilon \rightarrow 0^+$ and, in particular, they try to determine in which situation it is possible to obtain a control which remains bounded as $\epsilon \rightarrow 0^+$. In that paper the authors are able to prove boundedness of controls by assuming some conditions on the vector field M and the time T . See also [3] and [6] for the analysis of (5) in the 1-d case, with M constant.

Regarding the case of parabolic systems converging to parabolic-elliptic systems, as far as we know, the first time this problem was addressed was in [2] (see also [4]). There, the authors considered the case of a nonlinear parabolic-elliptic system

appearing in electrocardiology as a simplification of a coupled parabolic system modeling electrical activities in the heart. Combining Carleman estimates and weighted energy inequalities, the authors are able to prove that the control properties of the parabolic-elliptic system can be viewed as a limit process of the control properties of a family of nonlinear parabolic systems.

Let us denote by $(u(t; \epsilon, (u_0, v_0), f, g), v(t; \epsilon, (u_0, v_0), f, g))$ the solution of (1) at time $t \in [0, T]$ associated to $(u_0, v_0) \in L^2(\Omega)^2$ and $(f, g) \in L^2(Q)^2$.

Our first main result in this paper is given by the following Theorem.

Theorem 1.1. *Let $(u_0, v_0) \in L^2(\Omega)^2$ and a, b, c and d be $C^3(\overline{Q})$ functions. Then*

1. *If $c \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_1$ and $d < \mu_1$ (μ_1 being the first eigenvalue of $-\Delta$ with Dirichlet boundary condition), then system (1) is uniformly null controllable, with respect to ϵ , with control only in the first equation. More precisely, for each $0 < \epsilon < 1/2$, there exists $f = f(\epsilon) \in L^2(Q)$ such that*

$$(u(T; \epsilon, (u_0, v_0), f(\epsilon), 0), v(T; \epsilon, (u_0, v_0), f(\epsilon), 0)) = (0, 0). \quad (6)$$

Moreover, we have the following estimate on the control

$$\|f(\epsilon)\chi_{\omega_1}\|_{L^2(Q)}^2 \leq C(\|u_0\|_{L^2(\Omega)}^2 + \epsilon\|v_0\|_{L^2(\Omega)}^2), \quad (7)$$

where C is a constant that does not depend on ϵ , u_0 and v_0 .

2. *If $b \neq 0$ in $\overline{\omega}$, for some $\omega \subset\subset \omega_2$ and $d < \mu_1$, then system (1) is uniformly null controllable, with respect to ϵ , with control acting only in the second equation. More precisely, for each $0 < \epsilon < 1/2$, there exists $g = g(\epsilon) \in L^2(Q)$ such that*

$$(u(T; \epsilon, (u_0, v_0), 0, g(\epsilon)), v(T; \epsilon, (u_0, v_0), 0, g(\epsilon))) = (0, 0). \quad (8)$$

Moreover, we have the following estimate on the control

$$\|g(\epsilon)\chi_{\omega_2}\|_{L^2(Q)}^2 \leq C(\|u_0\|_{L^2(\Omega)}^2 + \epsilon\|v_0\|_{L^2(\Omega)}^2), \quad (9)$$

where C does not depend on ϵ , u_0 and v_0 .

Remark 1. We believe that the ideas given in this paper can be useful when dealing with the case where $d > \mu_1$ and d is not an eigenvalue of $-\Delta$ with Dirichlet boundary condition. The case in which d is an eigenvalue is an interesting open problem.

In order to prove Theorem 1.1 we are led to consider the adjoint system of (1),

$$\begin{cases} -\varphi_t - \Delta\varphi = a\varphi + c\xi & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T; \xi(T) = \xi_T & \text{in } \Omega, \end{cases} \quad (10)$$

with $(\varphi_T, \xi_T) \in L^2(\Omega)^2$.

It is well known that case 1 of Theorem 1.1 is equivalent to prove the existence of a universal constant C , which does not depend on ϵ , such that the observability inequality

$$\|\varphi(0)\|_{L^2(\Omega)}^2 + \epsilon\|\xi(0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega_1 \times (0, T)} |\varphi|^2 dx dt, \quad (11)$$

holds for all solutions (φ, ξ) of (10). Analogously, one can prove that case 2 of Theorem 1.1 is equivalent to show that

$$\|\varphi(0)\|_{L^2(\Omega)}^2 + \epsilon\|\xi(0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega_2 \times (0, T)} |\xi|^2 dx dt, \quad (12)$$

for all solutions (φ, ξ) of (10).

The study of the controllability of systems of parabolic equations has obtained a lot of attention in the recent years. For instance, in [1] the authors analyze the controllability of a reaction diffusion system consisting of two parabolic equations coupled by zero-order terms, obtaining the null controllability for the linear system and the local null controllability of the semilinear system. In [8] the controllability of a quite general system of two coupled linear parabolic equations is studied and, combining Carleman inequalities and some energy inequalities, null controllability is proved. See also [7] and the references therein.

Following [1] or [8] one can prove that, under the assumptions of case (2) in Theorem 1.1, the uniform null controllability with respect to ϵ can be obtained in the case of a control acting on the second equation of (1). On the other hand, following [1] or [8], if the control is only acting on the first equation of (1) one obtains a cost of null controllability of order ϵ^{-1} .

Thus, in this paper we obtain a uniform estimate on the cost of controllability of (1) in the case of a control acting only on the first equation, Theorem 1.1, case 1. Our proof can also be applied in order to obtain the boundedness of the cost of the null controllability of (1) when the control is acting on the second equation, see Theorem 2.3.

2. Carleman estimates and an extended adjoint system. In this section we deduce Carleman type estimates that will be used to prove observability inequalities (11) and (12). To this end we first define several weight functions which will be useful in the sequel.

The basic weight will be a function $\psi \in C^2(\overline{\Omega})$ verifying

$$\psi(x) > 0, \text{ in } \Omega, \quad \psi \equiv 0 \text{ on } \partial\Omega, \quad |\nabla\psi(x)| > 0, \quad \forall x \in \overline{\Omega \setminus \omega_0},$$

where $\omega_0 \subset \subset \omega_1$ ($\omega_0 \subset \subset \omega_2$ in the case 2) is a nonempty open set. The existence of such a function ψ is proved in [5]. Then, for a positive real number λ , we introduce:

$$\begin{aligned} \phi(x, t) &= \frac{e^{\lambda\psi(x)}}{t(T-t)}; \quad \alpha(x, t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}}{t(T-t)} \\ \hat{\phi}(t) &= \min_{x \in \overline{\Omega}} \phi(x, t); \quad \phi^*(t) = \max_{x \in \overline{\Omega}} \phi(x, t); \\ \hat{\alpha}(t) &= \min_{x \in \overline{\Omega}} \alpha(x, t); \quad \alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(x, t). \end{aligned} \tag{13}$$

The following Carleman inequality will be very important to our purposes:

Lemma 2.1. *Let $\beta \in \mathbb{R}$, $0 < \sigma \leq 1$ and $\omega_0 \subset \subset \omega$. There exists a constant $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$ such that for every $\lambda \geq \lambda_0$ there exists $s_0 = s_0(\Omega, \omega, \lambda)$ and $C = C(\Omega, \omega, \lambda)$ such that, for every $s \geq s_0(T + T^2)$, the following inequality holds:*

$$\begin{aligned} & s^{\beta-1} \iint_Q e^{2s\alpha} \phi^{\beta-1} (\sigma^2 |q_t|^2 + \sum_{i,j=1}^N |\frac{\partial^2 q}{\partial x_i \partial x_j}|^2) dx dt \\ & + s^{\beta+1} \iint_Q e^{2s\alpha} \phi^{\beta+1} |\nabla q|^2 dx dt + s^{\beta+3} \iint_Q e^{2s\alpha} \phi^{\beta+3} |q|^2 dx dt \\ & \leq C \left(s^\beta \iint_Q e^{2s\alpha} \phi^\beta |\sigma \partial_t q + \Delta q|^2 dx dt + s^{\beta+3} \iint_{\omega \times (0, T)} e^{2s\alpha} \phi^{\beta+3} |q|^2 dx dt \right), \end{aligned}$$

for all $q \in C^2(\overline{Q})$, with $q = 0$ on Σ .

Proof. See [2] or [5]. $\square$

Now we state the second main result of this paper, a Carleman type estimate for the adjoint system (10) given by the following Theorem:

Theorem 2.2. *Under the assumptions of case 1 in Theorem 1.1. There exists $\lambda_0 = \lambda_0(\Omega, \omega_1) \geq 1$ such that for every $\lambda \geq \lambda_0$, there exists $s_0 = s_0(\Omega, \omega_1, \lambda_0) > 0$ such that, for every $s > s_0(T + T^2)$, the solution (φ, ξ) of system (10) satisfies*

$$\begin{aligned} s^4 \iint_Q e^{2s\alpha} \phi^4 |\varphi|^2 dxdt + s^4 \iint_Q e^{2s\alpha} \phi^4 |\xi|^2 dxdt \\ \leq C s^{14} \iint_{\omega_1 \times (0, T)} e^{4s\alpha^* - 2s\hat{\alpha}} (\phi^*)^{14} |\varphi|^2 dxdt, \end{aligned} \quad (14)$$

with C depending on Ω , ω_1 and λ .

Remark 2. Observe that $4s\alpha^* - 2s\hat{\alpha} < 0$ for λ large enough and, therefore, that $e^{4s\alpha^* - 2s\hat{\alpha}} (\phi^*)^{14}$ is bounded.

With the purpose of proving Theorem 2.2, we extend our adjoint system to a system of 4 equations. We set the notation:

$$\mathcal{L}_{\gamma, \theta} = \gamma \partial_t - \Delta - \theta, \text{ with } \gamma \in \mathbb{R} \text{ and } \theta \in L^\infty(Q).$$

With this notation, we define the following function

$$w = \mathcal{L}_{-\epsilon, d} \varphi$$

and, if φ_T and ξ_T are smooth and (φ, ξ) is the solution of (10) associated to this initial data, a simple calculation gives

$$-w_t - \Delta w - aw = \varphi(cb + \mathcal{L}_{-\epsilon, 0}a - \mathcal{L}_{-1, 0}d) + \xi \mathcal{L}_{-\epsilon, 0}c - 2\nabla \xi \nabla c + 2\nabla \varphi(\nabla d - \nabla a).$$

Therefore, we can add two more equations to our adjoint system, going from a system of 2 equations to a system of 4 equations, namely

$$\begin{cases} \mathcal{L}_{-1, a} w = \varphi(cb + \mathcal{L}_{-\epsilon, 0}a - \mathcal{L}_{-1, 0}d) + \xi \mathcal{L}_{-\epsilon, 0}c - 2\nabla \xi \nabla c + 2\nabla \varphi(\nabla d - \nabla a) & \text{in } Q, \\ \mathcal{L}_{-\epsilon, d} \varphi = w & \text{in } Q, \\ \mathcal{L}_{-1, a} \varphi = c\xi & \text{in } Q, \\ \mathcal{L}_{-\epsilon, d} \xi = b\varphi & \text{in } Q, \\ \varphi = \xi = w = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T; \xi(T) = \xi_T; w(T) = -\epsilon \varphi_T - \Delta \varphi_T - d\varphi_T & \text{in } \Omega. \end{cases} \quad (15)$$

The plan of the proof of Theorem 2.2 contains five parts:

First part: We see equations of (15) as heat equations and apply the Carleman estimate for the heat equation given in Lemma 2.1. This yields a global estimate for φ , w and ξ in terms of local terms of φ , w and ξ .

Second part: Using the second equation in (15) we eliminate the local integral of w appearing in the Carleman estimate obtained in step 1.

Third part: We estimate a local integral of ξ in terms of a local integral of φ , a local integral of φ_t and some small order terms.

Fourth part: Using the extend adjoint system, we show that we can estimate φ_t locally in terms of a local integral of φ and some small order terms.

Fifth part: We gather the estimates of the previous steps and absorb the small order terms, obtaining our desired Carleman estimate.

Along the proof we will use the notation:

$$\begin{aligned} I_\beta(s, \sigma; \rho) = & s^{\beta+3} \iint_Q e^{2s\alpha} \phi^{\beta+3} |\rho|^2 dxdt + s^{\beta+1} \iint_Q e^{2s\alpha} \phi^{\beta+1} |\nabla \rho|^2 dxdt \\ & + s^{\beta-1} \iint_Q e^{2s\alpha} \phi^{\beta-1} (\sigma^2 |\rho_t|^2 + \sum_{i,j=1}^N |\frac{\partial^2 \rho}{\partial x_i \partial x_j}|^2) dxdt, \end{aligned} \quad (16)$$

where s, β and σ are real numbers and $\rho = \rho(x, t)$.

Proof of Theorem 2.2. For an easier comprehension, we divide the proof into several steps:

Step 1. First Carleman inequalities.

Let ω' be a nonempty set such that $\omega_0 \subset \subset \omega' \subset \subset \omega$, where ω is a subset of ω_1 where $c \neq 0$. We apply Lemma 2.1 to (15)₁, with $\beta = 2$, and to (15)₃ and (15)₄, with $\beta = 1$. Then

$$\begin{aligned} I_2(s, 1; w) \leq & C(s^5 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^5 |w|^2 dxdt \\ & + s^2 \iint_Q \phi^2 e^{2s\alpha} (|\varphi|^2 + |\nabla \varphi|^2 + |\xi|^2 + |\nabla \xi|^2) dxdt), \end{aligned} \quad (17)$$

$$I_1(s, 1; \varphi) \leq C(s^4 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^4 |\varphi|^2 dxdt + s \iint_Q e^{2s\alpha} \phi |\xi|^2 dxdt) \quad (18)$$

and

$$I_1(s, \epsilon; \xi) \leq C(s^4 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^4 |\xi|^2 dxdt + s \iint_Q e^{2s\alpha} \phi |\varphi|^2 dxdt). \quad (19)$$

Using (18) and (19) into (17), and absorbing the lower order terms, we get

$$\begin{aligned} & I_2(s, 1; w) + I_1(s, \epsilon; \xi) + I_1(s, 1; \varphi) \\ & \leq C(s^4 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^4 |\varphi|^2 dxdt + s^4 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^4 |\xi|^2 dxdt \\ & \quad + s^5 \iint_{\omega' \times (0, T)} e^{2s\alpha} \phi^5 |w|^2 dxdt). \end{aligned} \quad (20)$$

Step 2. Estimate of the local integral of w .

In this step we estimate the local integral of w in the right-hand side of (20) in terms of a local integral of φ and a small order term involving w . In order to do that, we introduce a cut-off function θ with

$$\theta \in C_0^\infty(\omega''), \text{ with } 0 \leq \theta \leq 1 \text{ and } \theta \equiv 1 \text{ on } \omega',$$

where $\omega' \subset \subset \omega'' \subset \subset \omega$.

We use (15)₂ to write

$$\begin{aligned} & s^5 \iint_{\omega'' \times (0, T)} e^{2s\alpha} \phi^5 |w|^2 \theta dxdt \\ & = s^5 \iint_{\omega'' \times (0, T)} e^{2s\alpha} \phi^5 \theta w (-\epsilon \varphi_t - \Delta \varphi - d\varphi) dxdt \\ & := M_1 + M_2 + M_3 \end{aligned} \quad (21)$$

and we estimate each term in the expression above.

For the first term, we integrate by parts to see that

$$\begin{aligned} M_1 &= 5\epsilon s^5 \iint_{\omega'' \times (0,T)} \phi^4 \phi_t e^{2s\alpha} \theta w \varphi dx dt \\ &\quad + 2\epsilon s^6 \iint_{\omega'' \times (0,T)} \alpha_t \phi^5 e^{2s\alpha} \theta w \varphi dx dt \\ &\quad + \epsilon s^5 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} \theta w_t \varphi dx dt \end{aligned} \quad (22)$$

and use Young's inequality to obtain

$$\begin{aligned} M_1 &\leq \epsilon^2 (s^5 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} |w|^2 dx dt + s \iint_{\omega'' \times (0,T)} \phi e^{2s\alpha} |w_t|^2 dx dt) \\ &\quad + C s^9 \iint_{\omega'' \times (0,T)} \phi^9 e^{2s\alpha} |\varphi|^2 dx dt. \end{aligned} \quad (23)$$

Here we have used that $|\alpha_t| \leq C\phi^2$.

Next, since

$$\begin{aligned} M_2 &= 5s^5 \iint_{\omega'' \times (0,T)} \phi^4 e^{2s\alpha} w \theta (\nabla \phi \cdot \nabla \varphi) dx dt \\ &\quad + 2s^6 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} w \theta (\nabla \alpha \cdot \nabla \varphi) dx dt \\ &\quad + s^5 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} w (\nabla \theta \cdot \nabla \varphi) dx dt \\ &\quad + s^5 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} \theta (\nabla w \cdot \nabla \varphi) dx dt, \end{aligned}$$

it is not difficult to see that, for any $\delta > 0$ there exists $C = C(\delta)$ such that

$$\begin{aligned} M_2 + M_3 &\leq C(s^7 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^7 |\nabla \varphi|^2 dx dt + s^5 \iint_{\omega'' \times (0,T)} \phi^5 e^{2s\alpha} |\varphi|^2 dx dt) \\ &\quad + \delta(s^5 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^5 |w|^2 dx dt + s^3 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^3 |\nabla w|^2 dx dt). \end{aligned} \quad (24)$$

Hence

$$\begin{aligned} &s^5 \iint_{\omega' \times (0,T)} e^{2s\alpha} \phi^5 |w|^2 dx dt \\ &\leq C(s^9 \iint_{\omega'' \times (0,T)} \phi^9 e^{2s\alpha} |\varphi|^2 dx dt + s^7 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^7 |\nabla \varphi|^2 dx dt) \\ &\quad + (\delta + \epsilon^2) I_2(s, 1, w). \end{aligned} \quad (25)$$

Now we eliminate the local integral of $\nabla \varphi$. For this, we consider a set ω''' with $\omega'' \subset \subset \omega''' \subset \subset \omega$ and a cut-off function $\theta_1 \in C_0^\infty(\omega''')$ satisfying

$$0 \leq \theta_1 \leq 1, \quad \theta_1 \equiv 1 \text{ on } \omega''.$$

Integration by parts gives

$$\begin{aligned} s^7 \iint_{\omega''' \times (0,T)} e^{2s\alpha} \theta_1 \phi^7 |\nabla \varphi|^2 dx dt &= -s^7 \iint_{\omega''' \times (0,T)} e^{2s\alpha} \theta_1 \phi^7 \Delta \varphi \varphi dx dt \\ &\quad - \frac{1}{2} s^7 \iint_{\omega''' \times (0,T)} \Delta(\theta_1 e^{2s\alpha} \phi^7) |\varphi|^2 dx dt. \end{aligned} \quad (26)$$

Using the fact that

$$|\Delta(\theta_1 e^{2s\alpha} \phi^7)| \leq C s^2 \phi^9 e^{2s\alpha} \quad \text{in } \omega''' \times (0, T),$$

together with Young's inequality, we see that, for any $\delta > 0$ there exists $C = C(\delta)$ such that

$$\begin{aligned} s^7 \iint_{\omega''' \times (0,T)} e^{2s\alpha} \phi^7 |\nabla \varphi|^2 dx dt &\leq \delta \iint_{\omega''' \times (0,T)} e^{2s\alpha} |\Delta \varphi|^2 dx dt \\ &\quad + C s^{14} \iint_{\omega''' \times (0,T)} e^{2s\alpha} \phi^{14} |\varphi|^2 dx dt, \end{aligned} \quad (27)$$

Therefore

$$\begin{aligned} s^5 \iint_{\omega' \times (0,T)} e^{2s\alpha} \phi^5 |w|^2 dx dt &\leq C s^{14} \iint_{\omega''' \times (0,T)} e^{2s\alpha} \phi^{14} |\varphi|^2 dx dt \\ &\quad + (\delta + \epsilon) I_2(s, 1; w) + \delta I_1(s, 1; \varphi), \end{aligned} \quad (28)$$

for any $\delta > 0$.

Combining (20) and (28), for δ small enough, we get

$$\begin{aligned} &I_2(s, 1; w) + I_1(s, \epsilon; \xi) + I_1(s, 1; \varphi) \\ &\leq C(s^{14} \iint_{\omega''' \times (0,T)} e^{2s\alpha} \phi^{14} |\varphi|^2 dx dt + s^4 \iint_{\omega' \times (0,T)} e^{2s\alpha} \phi^4 |\xi|^2 dx dt). \end{aligned} \quad (29)$$

Step 3. Estimate of the local integral of ξ .

In this step we estimate the local integral of ξ in the right-hand side of (29) in terms of a local integral of φ , a local integral of φ_t and some small order terms.

Using equation (15)₃, and the fact that $c \neq 0$ in $\bar{\omega}$, we see that

$$\begin{aligned} &s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} \theta(\phi^*)^4 |\xi|^2 dx dt \\ &= s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} \frac{\theta}{c} (\phi^*)^4 \xi (-\varphi_t - \Delta \varphi - a\varphi) dx dt \\ &:= M_4 + M_5 + M_6, \end{aligned} \quad (30)$$

where θ is the cut-off function introduced in step 2.

As in the previous step, we estimate each term in the expression above. We have

$$\begin{aligned} M_4 &\leq \frac{s^4}{2} \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} \theta(\phi^*)^4 |\xi|^2 dx dt \\ &\quad + \frac{s^4}{2} \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} \frac{\theta}{c^2} (\phi^*)^4 |\varphi_t|^2 dx dt. \end{aligned} \quad (31)$$

Integration by parts gives

$$M_5 = -s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} (\phi^*)^4 \left(\Delta \left(\frac{\theta}{c} \right) \xi + 2 \nabla \left(\frac{\theta}{c} \right) \nabla \xi + \left(\frac{\theta}{c} \right) \Delta \xi \right) \varphi dx dt.$$

Using this last equality, we can show that for any $\delta > 0$ there exists $C = C(\delta)$ such that

$$\begin{aligned} M_5 + M_6 &\leq \delta \left(s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^4 |\xi|^2 dx dt + s^2 \iint_{\omega'' \times (0,T)} e^{2s\alpha} \phi^2 |\nabla \xi|^2 dx dt \right. \\ &\quad \left. + \iint_{\omega'' \times (0,T)} e^{2s\alpha} |\Delta \xi|^2 dx dt \right) + C s^8 \iint_{\omega'' \times (0,T)} e^{4s\alpha^* - 2s\alpha} (\phi^*)^8 |\varphi|^2 dx dt. \end{aligned} \quad (32)$$

Hence

$$\begin{aligned} s^4 \iint_{\omega' \times (0,T)} e^{2s\alpha^*} \theta(\phi^*)^4 |\xi|^2 dx dt &\leq C \left(s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} (\phi^*)^4 |\varphi_t|^2 dx dt \right. \\ &\quad \left. + s^8 \iint_{\omega'' \times (0,T)} e^{4s\alpha^* - 2s\alpha} (\phi^*)^8 |\varphi|^2 dx dt \right) + \delta I_1(s, \epsilon; \xi). \end{aligned} \quad (33)$$

From (33), our objective is now reduced to estimate a local integral of φ_t in terms of a local integral of φ and small order terms. This will be done in the next steps.

Step 4. Estimate of the local integral of φ_t .

In this step we deal with the first term appearing in the right-hand side of (33).

First, we integrate by parts to get

$$\begin{aligned} s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} (\phi^*)^4 |\varphi_t|^2 dx dt &= -s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} (\phi^*)^4 \varphi_{tt} \varphi dx dt \\ &\quad + \frac{s^4}{2} \iint_{\omega'' \times (0,T)} (e^{2s\alpha^*} (\phi^*)^4)_{tt} |\varphi|^2 dx dt. \end{aligned} \quad (34)$$

Since

$$\begin{aligned} s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*} (\phi^*)^4 \varphi_{tt} \varphi dx dt &\leq \frac{s^{-6}}{2} \iint_{\omega'' \times (0,T)} e^{2s\hat{\alpha}} \hat{\phi}^{-5} |\varphi_{tt}|^2 dx dt \\ &\quad + \frac{s^{14}}{2} \iint_{\omega'' \times (0,T)} e^{4s\alpha^* - 2s\hat{\alpha}} (\phi^*)^8 \hat{\phi}^5 |\varphi|^2 dx dt, \end{aligned} \quad (35)$$

we just have to estimate the local integral of φ_{tt} in the right-hand side of (35). In order to do that, we use (15)₂ to see that

$$\begin{aligned} &-\epsilon(e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt})_t - \Delta(e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt}) \\ &= e^{s\hat{\alpha}} \hat{\phi}^{-5/2} w_{tt} - \epsilon(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t \varphi_{tt} + e^{s\hat{\alpha}} \hat{\phi}^{-5/2} (d_{tt} \varphi + 2d_t \varphi_t + d \varphi_{tt}) \end{aligned} \quad (36)$$

with $e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt} = 0$ in $\partial\Omega$ and $e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt}(T) = e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt}(0) = 0$.

Next, multiplying both sides of (36) by $e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt}$, integrating over Q and using Young's inequality, we get

$$\begin{aligned} &\iint_Q |\nabla(e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt})|^2 dx dt \\ &\leq C \left(\iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} w_{tt}|^2 dx dt + \epsilon^2 \iint_Q |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t \varphi_{tt}|^2 dx dt \right. \\ &\quad \left. + \iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_t|^2 dx dt + \iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi|^2 dx dt \right) \\ &\quad + \iint_Q d |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt}|^2 dx dt \end{aligned} \quad (37)$$

and, since $d < \mu_1$, we have

$$\begin{aligned} & s^{-6} \iint_Q |\nabla(e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_{tt})|^2 dx dt \\ & \leq C s^{-6} \left(\iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} w_{tt}|^2 dx dt + \iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi_t|^2 dx dt \right. \\ & \quad \left. + \iint_Q |e^{s\hat{\alpha}} \hat{\phi}^{-5/2} \varphi|^2 dx dt + \epsilon^2 \iint_Q |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t \varphi_{tt}|^2 dx dt \right). \end{aligned} \quad (38)$$

Let us now estimate the first and the last term in the right-hand side of (38).

Step 4.1. Estimate of the term in φ_{tt} .

Here, we estimate the last term in the right-hand side of (38). Using (15)₂ and (15)₃ we can show that

$$-\epsilon \varphi_{tt} = -\epsilon^2 \varphi_{tt} - \epsilon w_t - \epsilon d_t \varphi - \epsilon d \varphi_t + \epsilon a_t \varphi + \epsilon a \varphi_t + \epsilon c_t \xi + \epsilon c \xi_t, \quad (39)$$

from where we see that

$$\begin{aligned} \epsilon^2 |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t \varphi_{tt}|^2 & \leq \epsilon^4 |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |\varphi_{tt}|^2 + \epsilon^2 |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |w_t|^2 \\ & \quad + C \epsilon^2 (|(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |\varphi|^2 + |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |\varphi_t|^2 \\ & \quad + |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |\xi|^2 + |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t|^2 |\xi_t|^2). \end{aligned} \quad (40)$$

Since

$$|(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t| \leq C s^2 \hat{\phi}^{-1/2} e^{s\hat{\alpha}},$$

and $0 < \epsilon < 1/2$, inequality (40) implies

$$\begin{aligned} & \epsilon^2 s^{-6} \iint_Q |(e^{s\hat{\alpha}} \hat{\phi}^{-5/2})_t \varphi_{tt}|^2 dx dt \\ & \leq C \epsilon^2 s^{-2} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-1} (|\xi|^2 + |\xi_t|^2 + |\varphi_t|^2 + |\varphi|^2 + |w_t|^2) dx dt. \end{aligned} \quad (41)$$

Step 4.2. Estimate of the term in w_{tt} .

Here, we estimate the first term in the right-hand side of (38). From (15)₁ we have

$$\begin{aligned} & -w_{tt} - \Delta w_t - a_t w - a w_t \\ & = (cb)_t \varphi + cb \varphi_t - \epsilon c_{tt} \xi - \epsilon c_t \xi_t - \epsilon a_{tt} \varphi - \epsilon a_t \varphi_t - \xi_t \Delta c - \xi \Delta c_t \\ & \quad - 2 \nabla \xi_t \nabla c - 2 \nabla \xi \nabla c_t + \varphi_t \Delta(d - a) + \varphi \Delta(d - a)_t + 2 \nabla \varphi_t \nabla(d - a) \\ & \quad + 2 \nabla \varphi \nabla(d - a)_t + d_{tt} \varphi + d_t \varphi_t. \end{aligned} \quad (42)$$

We multiply both sides of (42) by $e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt}$ and integrate over Q , we obtain this way

$$\begin{aligned} & \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |w_{tt}|^2 dx dt \\ & = \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} \Delta w_t w_{tt} dx dt + \iint_Q a_t e^{2s\hat{\alpha}} \hat{\phi}^{-5} w w_{tt} dx dt \\ & \quad + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \varphi ((cb)_t - \epsilon a_{tt} + \Delta(d - a)_t + d_{tt}) dx dt \\ & \quad + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \varphi_t (cb - \epsilon a_t + \Delta(d - a) + d_t) dx dt + \iint_Q a e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_t w_{tt} dx dt \end{aligned}$$

$$\begin{aligned}
& + 2 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \nabla \varphi \nabla (d-a)_t dxdt + 2 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \nabla \varphi_t \nabla (d-a) dxdt \\
& + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \xi (-\epsilon c_{tt} - \Delta c_t) dxdt + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \xi_t (-\epsilon c_t - \Delta c) dxdt \\
& + 2 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \nabla \xi \nabla c_t dxdt + 2 \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} w_{tt} \nabla \xi_t \nabla c dxdt.
\end{aligned} \tag{43}$$

After a long, but straightforward, calculation, we can show that

$$\begin{aligned}
& s^{-6} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |w_{tt}|^2 dxdt \\
& \leq C s^{-6} \left(\iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} \Delta w_t w_{tt} dxdt + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} (|w|^2 + |w_t|^2) dxdt \right. \\
& \quad \left. + \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} (|\nabla \varphi|^2 + |\nabla \varphi_t|^2 + |\nabla \xi|^2 + |\nabla \xi_t|^2) dxdt \right).
\end{aligned} \tag{44}$$

The rest of the proof of this step is devoted to estimate the integrals $\iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} \Delta w_t w_{tt} dxdt$, $\iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |\nabla \varphi_t|^2 dxdt$ and $\iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |\nabla \xi_t|^2 dxdt$ appearing in the right-hand side of (44). This will be done in the next two substeps. The other terms in the right-hand side of (44) can be absorbed by the left-hand side of (29).

Step 4.2.1. Estimate of the term in φ_t .

We use (15)₂ to see that $-\Delta \varphi_t = w_t + \epsilon \varphi_{tt} + d_t \varphi + d \varphi_t$. From (39) and the fact that $-\Delta$ gives a norm in $H^2(\Omega) \cap H_0^1(\Omega)$, we get

$$\begin{aligned}
& s^{-6} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |\nabla \varphi_t|^2 dxdt \leq s^{-6} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} |\Delta \varphi_t|^2 dxdt \\
& \leq C s^{-6} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} (\epsilon^2 |\xi|^2 + \epsilon^2 |\xi_t|^2 + |\varphi_t|^2 + |\varphi|^2 + |w_t|^2) dxdt.
\end{aligned} \tag{45}$$

All the terms in the right-hand side of (45) can be absorbed by the left-hand side of (29).

Step 4.2.2. Estimate of the term in $\Delta w_t w_{tt}$.

We have

$$\begin{aligned}
s^{-6} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-5} \Delta w_t w_{tt} dxdt & = \frac{s^{-6}}{2} \iint_Q (e^{2s\hat{\alpha}} \hat{\phi}^{-5})_t |\nabla w_t|^2 dxdt \\
& \leq C s^{-4} \iint_Q e^{2s\hat{\alpha}} \hat{\phi}^{-3} |\nabla w_t|^2 dxdt,
\end{aligned} \tag{46}$$

since

$$|(e^{2s\hat{\alpha}} \hat{\phi}^{-5})_t| \leq C s^2 e^{2s\hat{\alpha}} \hat{\phi}^{-3}.$$

Next, we use (15)₁ to see that

$$\begin{aligned}
& -w_{tt} - \Delta w_t - a_t w - a w_t \\
& = (cb - \epsilon a_t + \Delta(d-a) + d_t) \varphi_t - (\epsilon c_{tt} + \Delta c_t) \xi - (\epsilon c_t + \Delta c) \xi_t \\
& \quad + ((cb)_t + \Delta(d-a)_t - \epsilon a_{tt} + d_{tt}) \varphi - 2 \nabla \xi_t \nabla c - 2 \nabla \xi \nabla c_t \\
& \quad + 2 \nabla \varphi_t \nabla (d-a) + 2 \nabla \varphi \nabla (d-a)_t.
\end{aligned} \tag{47}$$

Multiplying both sides of (47) by $s^{-4}e^{2s\hat{\alpha}}\hat{\phi}^{-3}w_t$, integrating by parts and using Young's inequality, we get

$$\begin{aligned} s^{-4} \iint_Q e^{2s\hat{\alpha}}\hat{\phi}^{-3}|\nabla w_t|^2 dxdt \\ \leq Cs^{-1} \left(\iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2 + |\nabla\xi|^2 + |\nabla\xi_t|^2) dxdt \right. \\ \left. + \iint_Q e^{2s\hat{\alpha}}(|w|^2 + |w_t|^2) dxdt \right), \end{aligned} \quad (48)$$

since

$$|(e^{2s\hat{\alpha}}\hat{\phi}^{-3})_t| \leq Cs^2e^{2s\hat{\alpha}}\hat{\phi}^{-1}$$

and

$$s^{-1}\varphi^{-1} \leq C. \quad (49)$$

Step 4.2.3. Estimate of the term in $\nabla\xi_t$.

We use (15)₃ to see that $-\epsilon\xi_{tt} - \Delta\xi_t = b_t\varphi + b\varphi_t + d_t\xi + d\xi_t$. Multiplying both sides by $e^{2s\hat{\alpha}}\xi_t$ and integrating over Q , we obtain

$$\begin{aligned} \iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2 dxdt &\leq \iint_Q de^{2s\hat{\alpha}}|\xi_t|^2 dxdt \\ &\quad + C \iint_Q e^{2s\hat{\alpha}}(|\xi|^2 + |\varphi|^2 + |\varphi_t|^2) dxdt, \end{aligned} \quad (50)$$

which gives, from the assumption on d ,

$$\iint_Q e^{2s\hat{\alpha}}|\nabla\xi_t|^2 dxdt \leq C \iint_Q e^{2s\hat{\alpha}}(|\xi|^2 + |\varphi|^2 + |\varphi_t|^2) dxdt. \quad (51)$$

Step 5. Last arrangements and conclusion.

From (34), (35), (38), (41), (44), (45), (48) and (51), we get

$$\begin{aligned} s^4 \iint_{\omega'' \times (0,T)} e^{2s\alpha^*}(\phi^*)^4|\varphi_t|^2 dxdt \\ \leq Cs^4 \iint_{\omega'' \times (0,T)} (e^{2s\alpha^*}(\phi^*)^4)_{tt}|\varphi|^2 dxdt \\ + Cs^{14} \iint_{\omega'' \times (0,T)} e^{4s\alpha^* - 2s\hat{\alpha}}(\phi^*)^8\hat{\phi}^5|\varphi|^2 dxdt, \\ + Cs^{-1} \iint_Q e^{2s\hat{\alpha}}(|\nabla\varphi|^2 + |w_t|^2 + |w|^2) dxdt \\ + \delta I_1(s, \epsilon; \xi), \end{aligned} \quad (52)$$

for any $\delta > 0$.

Putting (52) in (33), we obtain

$$\begin{aligned} s^4 \iint_{\omega' \times (0,T)} e^{2s\alpha^*}\theta(\phi^*)^4|\xi|^2 dxdt \\ \leq Cs^{14} \iint_{\omega'' \times (0,T)} e^{4s\alpha^* - 2s\hat{\alpha}}(\phi^*)^{14}|\varphi|^2 dxdt \end{aligned}$$

$$\begin{aligned}
& + Cs^{-1} \iint_Q e^{2s\hat{\alpha}} (|\nabla\varphi|^2 + |w_t|^2 + |w|^2) dxdt \\
& + \delta I_1(s, \epsilon; \xi),
\end{aligned} \tag{53}$$

since

$$|(e^{2s\alpha^*}(\phi^*)^4)_{tt}| \leq Cs^2 e^{2s\alpha^*}(\phi^*)^8 \quad \text{and} \quad e^{2s\alpha^*} \leq Ce^{4s\alpha^* - 2s\hat{\alpha}}. \tag{54}$$

Finally, choosing s large enough and δ small enough, we put (53) in (29) and absorb the small order terms, we obtain

$$\begin{aligned}
& I_2(s, 1; w) + I_1(s, \epsilon; \xi) + I_1(s, 1; \varphi) \\
& \leq Cs^{14} \iint_{\omega_1 \times (0, T)} e^{4s\alpha^* - 2s\hat{\alpha}} (\phi^*)^{14} |\varphi|^2 dxdt.
\end{aligned} \tag{55}$$

This finishes the proof of Theorem 2.2. $\square$

Observing that the system formed by the first two equations (15) has the same structure as the system formed by the third and fourth equation in (15) we can argue as in steps 1 and 2 above in order to prove the following result, which is the third main result of this paper,

Theorem 2.3. *Under the assumptions of case 2 in Theorem 1.1. There exists $\lambda_0 = \lambda_0(\Omega, \omega_2) \geq 1$ and $s_0 = s_0(\Omega, \omega_2, \lambda_0) > 0$ such that, for each $\lambda \geq \lambda_0$ and $s > s_0(T + T^2)$ the solution (φ, ξ) of system (10) satisfies*

$$\begin{aligned}
& s^3 \iint_Q e^{2s\alpha} \phi^3 |\xi|^2 dxdt + s \iint_Q e^{2s\alpha} \phi |\nabla \xi|^2 dxdt \\
& + s^{-1} \iint_Q e^{2s\alpha} \phi^{-1} (\epsilon^2 |\xi_t|^2 + |\Delta \xi|^2) dxdt \\
& + s^3 \iint_Q e^{2s\alpha} \phi^3 |\varphi|^2 dxdt + s \iint_Q e^{2s\alpha} \phi |\nabla \varphi|^2 dxdt \\
& + s^{-1} \iint_Q e^{2s\alpha} \phi^{-1} (|\varphi_t|^2 + |\Delta \varphi|^2) dxdt \\
& \leq Cs^7 \iint_{\omega_2 \times (0, T)} e^{2s\alpha} \phi^7 |\xi|^2 dxdt,
\end{aligned} \tag{56}$$

with C depending on Ω , ω_2 , ψ and λ_0 .

3. Proof of Theorem 1.1. Now we prove Theorem 1.1. As we said before, it is equivalent to prove an observability inequality for the adjoint system, inequality (11) in case 1 or inequality (12) in case 2.

Proof. As the proofs of (11) and (12) are similar, we just prove the first one. To do this, we first change the orientation in the adjoint system (10), i.e., instead of going from T to 0 the system will evolve from 0 to T . Changing t by $T - t$, we obtain the system

$$\begin{cases} \varphi_t - \Delta \varphi = a\varphi + c\xi & \text{in } Q, \\ \epsilon \xi_t - \Delta \xi = b\varphi + d\xi & \text{in } Q, \\ \varphi = \xi = 0 & \text{on } \Sigma, \\ \varphi(0) = \varphi_T; \quad \xi(0) = \xi_T & \text{in } \Omega. \end{cases} \tag{57}$$

Our desired observability inequality becomes

$$\|\varphi(T)\|^2 + \epsilon \|\xi(T)\|^2 \leq C \iint_{\omega_1 \times (0,T)} |\varphi|^2 dx dt, \quad (58)$$

where C is a constant which does not depend on ϵ .

In fact, multiplying (57)₁ by φ and (57)₂ by ξ and integrating over Ω we obtain,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi(t)\|^2 + \frac{\epsilon}{2} \frac{d}{dt} \|\xi(t)\|^2 + \|\nabla \varphi(t)\|^2 + \|\nabla \xi(t)\|^2 \\ &= \int_{\Omega} (a|\varphi(t)|^2 + d|\xi(t)|^2 + (b+c)\varphi(t)\xi(t)) dx. \end{aligned} \quad (59)$$

Using the assumption on d and Poincaré's inequality, we get

$$\frac{1}{2} \frac{d}{dt} \|\varphi(t)\|^2 + \frac{\epsilon}{2} \frac{d}{dt} \|\xi(t)\|^2 \leq C \|\varphi(t)\|^2. \quad (60)$$

Then, by Gronwall's inequality, we obtain

$$\frac{1}{2} \|\varphi(T)\|^2 + \frac{\epsilon}{2} \|\xi(T)\|^2 \leq C (\|\varphi(t)\|^2 + \epsilon \|\xi(t)\|^2), \quad (61)$$

where C does not depend on ϵ .

Integrating (61) from $T/4$ to $3T/4$ we get

$$\frac{1}{2} \|\varphi(T)\|^2 + \frac{\epsilon}{2} \|\xi(T)\|^2 \leq C \int_{T/4}^{3T/4} \int_{\Omega} (|\varphi(t)|^2 + \epsilon |\xi(t)|^2) dx dt. \quad (62)$$

Using the Carleman inequality given by Theorem 2.2 and Remark 2, we obtain the desired observability inequality

$$\frac{1}{2} \|\varphi(T)\|^2 + \frac{\epsilon}{2} \|\xi(T)\|^2 \leq C \iint_{\omega_1 \times (0,T)} |\varphi|^2 dx dt. \quad (63)$$

where C does not depend on ϵ .

Inequality (63) proves case (1) of Theorem 1.1. Using Theorem 2.3 we prove case (2) of Theorem 1.1. In this way, the proof of Theorem 1.1 is established. $\square$

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