

EXACT CONTROLLABILITY OF GALERKIN'S APPROXIMATIONS OF MICROPOLAR FLUIDS

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ABSTRACT. We consider the nonlinear model describing micropolar fluid in a bounded smooth region of $\mathbb{R}^N$ ($N = 2, 3$) with distributed controls supported in small subset of this domain. Under suitable assumptions on the Galerkin basis, we introduce Galerkin's approximations for the controllable micropolar fluid system. By using the Hilbert Uniqueness Method in combination with a fixed point argument, we prove the exact controllability result for this finite-dimensional system.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set whose boundary Γ is sufficiently regular. Let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset. For $T > 0$, we consider the cylindrical domain $Q = \Omega \times (0, T)$ of $\mathbb{R}^4$ with lateral boundary $\Sigma = \Gamma \times (0, T)$. By $\boldsymbol{\nu} = \boldsymbol{\nu}(x)$ we denote the outward unit normal vector to Ω at the point $x \in \Gamma$.

We adopt the notations $\mathbf{y}$, $\mathbf{w}$ and p for the velocity field, the angular velocity of rotation of the fluid particles, and the pressure distribution, respectively. Spaces of $\mathbb{R}^3$ -valued functions, as well as their elements, are represented by bold-face letters. The governing equations are the following:

$$(1.1) \quad \begin{cases} \mathbf{y}_t - (\mu + \mu_r)\Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = 2\mu_r \nabla \times \mathbf{w} + \mathbf{u}1_{\mathcal{O}} & \text{in } Q, \\ \mathbf{w}_t - \widehat{c}\Delta \mathbf{w} + (\mathbf{y} \cdot \nabla)\mathbf{w} - \overline{c}\nabla(\nabla \cdot \mathbf{w}) = 2\mu_r \nabla \times \mathbf{y} + \mathbf{v}1_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{w} = 0 & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}^0, \quad \mathbf{w}(0) = \mathbf{w}^0 & \text{in } \Omega, \end{cases}$$

where $\widehat{c} = c_a + c_d$ and $\overline{c} = c_0 + c_d - c_a$.

In (1.1), $\mathcal{O}$ is the control domain (which is supposed to be as small as necessary), $\mathbf{u}$ and $\mathbf{v}$ are the control functions which act over the system, $1_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$, and the positive constants μ , μ_r , c_0 , c_a and c_d characterize the physical properties of the fluid: μ is the usual *Newtonian viscosity* and μ_r , c_0 ,

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c_a, c_d are additional *viscosities* related to the lack of symmetry of the stress tensor and, consequently, to the fact that the internal rotation $\mathbf{w}$ does not vanish. These constants satisfy the physically relevant inequality $c_0 + c_d > c_a$.

For the model derivation and related physical discussion, see Condiff and Dalher [5], Eringen [9], [10], Lukaszewicz [24] and Petrosyan [26]. We observe that (1.1) includes as a particular case the classical Navier-Stokes equations, which have been widely studied (see, for instance, Ladyzhenskaya [19] or Temam [30] and the references therein). In this case $\mu_r = 0$ and equations (1.1)₁ and (1.1)₂ are decoupled. Several experiments show that solutions of the micropolar fluid model describe behavior of numerous real fluids better than corresponding solutions of the Navier-Stokes equations. In particular, this nonlinear coupled system also can be used to simulate the behavior of liquid crystals, polymeric fluids and blood in thin vessels (see [1], [4], [8], [17], [27], [29] and the references therein).

A good reference for the mathematical aspects of (1.1) is [24]; it contains important existence and uniqueness results. In [28] the existence of a strong solution is proved using the Galerkin method, and in [25] the regularity results for weak solutions is proved.

Due to the dissipative and nonreversible properties of the system, it is clear that one cannot expect the exact controllability for the micropolar fluid model with arbitrary target functions (in the same way as in Navier-Stokes equations). On the other hand, approximate controllability is an open question for this one. However, the notion of approximate controllability does not appear to be really meaningful. Indeed, even if we could reach an arbitrary neighborhood of a given target $\{\mathbf{y}^T, \mathbf{w}^T\}$ at time T by the action of a control, the question of what to do after time T to stay in the same neighborhood would remain open.

In the context of the Navier-Stokes equations, there are results of the local exact controllability to uncontrolled trajectories obtained in Fursikov-Imanuvilov [16], Imanuvilov [18] and Fernández-Cara et al. [12]. The global approximate controllability of the 2-D Navier-Stokes equations with slip boundary conditions was obtained by Coron [6]. Combining results on global approximate controllability and local controllability, the global exact controllability for the Navier-Stokes system on a 2-D manifold was analyzed in Coron-Fursikov [7]. In [21] and [22], Lions and Zuazua introduced the finite-dimensional Galerkin's approximations for the Navier-Stokes system, and they proved that these Galerkin's approximations are exactly controllable. Optimal control problems associated with the Navier-Stokes equations also have a wide and important range in applications. This issue has been studied, for instance, by Fursikov in [13] and by Gunzburger and Hou in [14] and [15].

Concerning control results for the micropolar fluids, Fernández-Cara and Guerrero in [11] studied the local exact controllability to the trajectories for system (1.1). Notice that this case involves a difficulty. The main reason is that $\mathbf{w}$ is a nonscalar variable and the equations satisfied by its components w_i are coupled through the second-order terms $\partial_i (\nabla \cdot \mathbf{w})$. In the present work, we study the exact controllability for the micropolar fluids approached by Galerkin's approximations, and we reach basically the same level of knowledge as in the case of the classical Navier-Stokes equations. The results given in our work can be applied to different models of fluid mechanics. Some of these models, such as generalized Boussinesq systems (see [2]) and equations describing liquid crystals (see [17]) are intensively studied.

This paper is organized as follows: In Section 2, we state the basic notations. In Section 3, we introduce Galerkin's approximations for (1.1) and, for this finite-dimensional system, we establish the exact controllability result. The proof is based on the Hilbert Uniqueness Method introduced by Lions (see, for instance, [20]) to study the exact controllability of linear systems and a fixed point technique.

2. PRELIMINARIES

The following vector spaces, usual in the context of incompressible fluids, will be used throughout the paper:

$$(2.1) \quad \mathbf{V} = \{\boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega); \nabla \cdot \boldsymbol{\varphi} = 0 \text{ in } \Omega\}$$

and

$$\mathbf{H} = \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega); \nabla \cdot \boldsymbol{\varphi} = 0 \text{ in } \Omega, \boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\}.$$

Since the constants μ , μ_r , c_a , c_d and c_0 are not relevant in the arguments and results, we set in (1.1) all coefficients equal to one.

Here and in what follows $(\cdot, \cdot)$ and $\|\cdot\|$ stand for the scalar product and the norm in $\mathbf{L}^2(\Omega)$, respectively, and we denote

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^3 \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx.$$

We will use C to denote a generic positive constant which may vary from line to line (unless otherwise stated).

Multiplying (1.1)₁ and (1.1)₂ by $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, respectively, and integrating formally in Ω , we obtain the following variational formulation for (1.1):

$$(2.2) \quad \begin{cases} (\mathbf{y}_t, \boldsymbol{\varphi}) + a(\mathbf{y}, \boldsymbol{\varphi}) + ((\mathbf{y} \cdot \nabla) \mathbf{y}, \boldsymbol{\varphi}) = (\nabla \times \mathbf{w}, \boldsymbol{\varphi}) + (\mathbf{u}1_{\mathcal{O}}, \boldsymbol{\varphi}), \\ (\mathbf{w}_t, \boldsymbol{\psi}) + a(\mathbf{w}, \boldsymbol{\psi}) - (\nabla(\nabla \cdot \mathbf{w}), \boldsymbol{\psi}) + ((\mathbf{y} \cdot \nabla) \mathbf{w}, \boldsymbol{\psi}) = (\nabla \times \mathbf{y}, \boldsymbol{\psi}) + (\mathbf{v}1_{\mathcal{O}}, \boldsymbol{\psi}), \\ \mathbf{y}(0) = \mathbf{y}^0 \in \mathbf{V}, \quad \mathbf{w}(0) = \mathbf{w}^0 \in \mathbf{H}_0^1(\Omega), \end{cases}$$

for all $\{\boldsymbol{\varphi}, \boldsymbol{\psi}\} \in \mathbf{V} \times \mathbf{H}_0^1(\Omega)$.

3. GALERKIN'S APPROXIMATIONS

Let us consider $\{\mathbf{e}_j\}_{j \geq 1}$ and $\{\mathbf{f}_j\}_{j \geq 1}$ bases of $\mathbf{V}$ and $\mathbf{H}_0^1(\Omega)$, respectively, such that

$$(3.1) \quad \{\mathbf{e}_j\}_{j \geq 1} \text{ and } \{\mathbf{f}_j\}_{j \geq 1} \text{ are linearly independent in } \mathbf{L}^2(\mathcal{O}).$$

The existence of those bases is guaranteed due to abstract results proved in [23].

We consider the finite-dimensional spaces

$$(3.2) \quad E = \text{span } [\mathbf{e}_1, \dots, \mathbf{e}_n] \quad \text{and} \quad F = \text{span } [\mathbf{f}_1, \dots, \mathbf{f}_n].$$

Notice that $E \subset \mathbf{V}$ and $F \subset \mathbf{H}_0^1(\Omega)$.

Let us introduce the Galerkin approximation of the variational formulation (2.2):

$$(3.3) \quad \begin{cases} (\mathbf{y}_t, \mathbf{e}) + a(\mathbf{y}, \mathbf{e}) + ((\mathbf{y} \cdot \nabla) \mathbf{y}, \mathbf{e}) = (\nabla \times \mathbf{w}, \mathbf{e}) + (\mathbf{u}1_{\mathcal{O}}, \mathbf{e}), \\ (\mathbf{w}_t, \mathbf{f}) + a(\mathbf{w}, \mathbf{f}) - (\nabla(\nabla \cdot \mathbf{w}), \mathbf{f}) + ((\mathbf{y} \cdot \nabla) \mathbf{w}, \mathbf{f}) = (\nabla \times \mathbf{y}, \mathbf{f}) + (\mathbf{v}1_{\mathcal{O}}, \mathbf{f}), \\ \mathbf{y}(0) = \mathbf{y}^0 \in E, \quad \mathbf{w}(0) = \mathbf{w}^0 \in F, \end{cases}$$

for all $\{\mathbf{e}, \mathbf{f}\} \in E \times F$.

System (3.3) has a unique solution $\{\mathbf{y}, \mathbf{w}\} \in C^0([0, T]; E) \times C^0([0, T]; F)$.

The Galerkin approximation (3.3) is said to be exactly controllable at time $T > 0$ if, given $\{\mathbf{y}^0, \mathbf{w}^0\}, \{\mathbf{y}^T, \mathbf{w}^T\} \in E \times F$, there exists a pair of controls $\{\mathbf{u}, \mathbf{v}\} \in \mathbf{L}^2(\mathcal{O} \times (0, T))^2$ such that the solution $\{\mathbf{y}, \mathbf{w}\}$ of (3.3) satisfies

$$(3.4) \quad \{\mathbf{y}(\cdot, T; \{\mathbf{u}, \mathbf{v}\}), \mathbf{w}(\cdot, T; \{\mathbf{u}, \mathbf{v}\})\} = \{\mathbf{y}^T, \mathbf{w}^T\}.$$

The cost to achieve (3.4) is given by

$$(3.5) \quad C\{\mathbf{u}, \mathbf{v}\} = \frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\mathbf{u}|^2 + |\mathbf{v}|^2) dx dt.$$

The main result of this work is the following:

Theorem 3.1. *Let $T > 0$ be given. Then the Galerkin approximation (3.3) is exactly controllable in the sense of (3.4). Furthermore, the cost of control given in (3.5) is bounded independently of the nonlinearity.*

Proof. Later on, to show and make explicit that the cost of control can be bounded independently of the nonlinearity, we introduce the family of systems

$$(3.6) \quad \begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + \alpha(\mathbf{y} \cdot \nabla) \mathbf{y} + \nabla p = \nabla \times \mathbf{w} + \mathbf{u}1_{\mathcal{O}} & \text{in } Q, \\ \mathbf{w}_t - \Delta \mathbf{w} + \beta(\mathbf{y} \cdot \nabla) \mathbf{w} - \nabla(\nabla \cdot \mathbf{w}) = \nabla \times \mathbf{y} + \mathbf{v}1_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{w} = 0 & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}^0, \quad \mathbf{w}(0) = \mathbf{w}^0 & \text{in } \Omega, \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$. We will prove some estimates independent of α and β .

We prove our main result for the system (3.6). First, we introduce its variational formulation. Namely,

$$(3.7) \quad \begin{cases} (\mathbf{y}_t, \mathbf{e}) + a(\mathbf{y}, \mathbf{e}) + \alpha((\mathbf{y} \cdot \nabla) \mathbf{y}, \mathbf{e}) = (\nabla \times \mathbf{w}, \mathbf{e}) + (\mathbf{u}1_{\mathcal{O}}, \mathbf{e}), \\ (\mathbf{w}_t, \mathbf{f}) + a(\mathbf{w}, \mathbf{f}) + \beta((\mathbf{y} \cdot \nabla) \mathbf{w}, \mathbf{f}) - (\nabla(\nabla \cdot \mathbf{w}), \mathbf{f}) = (\nabla \times \mathbf{y}, \mathbf{f}) + (\mathbf{v}1_{\mathcal{O}}, \mathbf{f}), \\ \mathbf{y}(0) = \mathbf{y}^0 \in E, \quad \mathbf{w}(0) = \mathbf{w}^0 \in F, \end{cases}$$

for all $\{\mathbf{e}, \mathbf{f}\} \in E \times F$.

We proceed with this proof in several steps.

Step 1 (Linear system). We take a function $\mathbf{h}$ such that

$$(3.8) \quad \mathbf{h} \in L^2(0, T; E),$$

and we consider the linear system

$$(3.9) \quad \begin{cases} (\mathbf{y}_t, \mathbf{e}) + a(\mathbf{y}, \mathbf{e}) + \alpha((\mathbf{h} \cdot \nabla) \mathbf{y}, \mathbf{e}) = (\nabla \times \mathbf{w}, \mathbf{e}) + (\mathbf{u}1_{\mathcal{O}}, \mathbf{e}), \\ (\mathbf{w}_t, \mathbf{f}) + a(\mathbf{w}, \mathbf{f}) - (\nabla(\nabla \cdot \mathbf{w}), \mathbf{f}) + \beta((\mathbf{h} \cdot \nabla) \mathbf{w}, \mathbf{f}) = (\nabla \times \mathbf{y}, \mathbf{f}) + (\mathbf{v}1_{\mathcal{O}}, \mathbf{f}), \\ \mathbf{y}(0) = 0, \quad \mathbf{w}(0) = 0, \end{cases}$$

for all $\{\mathbf{e}, \mathbf{f}\} \in E \times F$.

System (3.9) has a unique solution $\{\mathbf{y}, \mathbf{w}\} \in C^0([0, T]; E) \times C^0([0, T]; F)$.

Remark 3.2. Due to the linearity of system (3.9), we can assume null initial data. However, all results are valid as well if the initial data is not zero, that is, $\mathbf{y}(0) = \mathbf{y}^0 \in E$ and $\mathbf{w}(0) = \mathbf{w}^0 \in F$.

Let us check that system (3.9) is exactly controllable in any time $T > 0$, in the sense of (3.4). For this, it suffices to prove that if $\{\mathbf{g}_1, \mathbf{g}_2\} \in E \times F$ satisfies

$$(3.10) \quad (\{\mathbf{y}(\cdot, T; \{\mathbf{u}, \mathbf{v}\}), \mathbf{w}(\cdot, T; \{\mathbf{u}, \mathbf{v}\})\}, \{\mathbf{g}_1, \mathbf{g}_2\}) = 0, \quad \forall \{\mathbf{u}, \mathbf{v}\} \in \mathbf{L}^2(\mathcal{O} \times (0, T))^2,$$

then $\{\mathbf{g}_1, \mathbf{g}_2\} \equiv \{0, 0\}$.

Let us consider $\{\boldsymbol{\varphi}, \boldsymbol{\psi}\}$ as being a solution of the adjoint system

$$(3.11) \quad \begin{cases} (-\boldsymbol{\varphi}_t - \alpha(\mathbf{h} \cdot \nabla) \boldsymbol{\varphi}, \mathbf{e}) + a(\boldsymbol{\varphi}, \mathbf{e}) = (\nabla \times \boldsymbol{\psi}, \mathbf{e}), \quad \forall \mathbf{e} \in E, \\ (-\boldsymbol{\psi}_t - \beta(\mathbf{h} \cdot \nabla) \boldsymbol{\psi}, \mathbf{f}) + a(\boldsymbol{\psi}, \mathbf{f}) - (\nabla(\nabla \cdot \boldsymbol{\psi}), \mathbf{f}) = (\nabla \times \boldsymbol{\varphi}, \mathbf{f}), \quad \forall \mathbf{f} \in F, \\ \boldsymbol{\varphi}(T) = \mathbf{g}_1 \in E, \quad \boldsymbol{\psi}(T) = \mathbf{g}_2 \in F. \end{cases}$$

System (3.11) has a unique solution $\{\boldsymbol{\varphi}, \boldsymbol{\psi}\} \in C^0([0, T]; E) \times C^0([0, T]; F)$.

Taking $\{\mathbf{e}, \mathbf{f}\} = \{\mathbf{y}(t), \mathbf{w}(t)\}$ in (3.11) and observing that

$$((\mathbf{h}(t) \cdot \nabla) \boldsymbol{\varphi}(t), \mathbf{y}(t)) = -((\mathbf{h}(t) \cdot \nabla) \mathbf{y}(t), \boldsymbol{\varphi}(t)), \quad \forall t \in [0, T]$$

(similarly for the term $((\mathbf{h} \cdot \nabla) \boldsymbol{\psi}, \mathbf{w}(t))$), we get, after integration by parts in t , that

$$(3.12) \quad -(\mathbf{y}(T), \boldsymbol{\varphi}(T)) + \int_0^T [(\mathbf{y}_t, \boldsymbol{\varphi}) + a(\mathbf{y}, \boldsymbol{\varphi}) + \alpha((\mathbf{h} \cdot \nabla) \mathbf{y}, \boldsymbol{\varphi})] dt = \int_0^T (\nabla \times \boldsymbol{\psi}, \mathbf{y}) dt$$

and

$$(3.13) \quad \begin{aligned} & -(\mathbf{w}(T), \boldsymbol{\psi}(T)) + \int_0^T [(\mathbf{w}_t, \boldsymbol{\psi}) + a(\mathbf{w}, \boldsymbol{\psi}) + \beta((\mathbf{h} \cdot \nabla) \mathbf{w}, \boldsymbol{\psi}) - (\nabla(\nabla \cdot \mathbf{w}), \boldsymbol{\psi})] dt \\ & = \int_0^T (\nabla \times \boldsymbol{\varphi}, \mathbf{w}) dt. \end{aligned}$$

Adding the equations (3.12) and (3.13) and taking into account

$$(\nabla \times \boldsymbol{\psi}, \mathbf{e}) = (\boldsymbol{\psi}, \nabla \times \mathbf{e}) \quad \text{and} \quad (\nabla \times \boldsymbol{\varphi}, \mathbf{f}) = (\boldsymbol{\varphi}, \nabla \times \mathbf{f}),$$

we obtain from (3.9) that

$$(3.14) \quad (\{\mathbf{y}(T), \mathbf{w}(T)\}, \{\mathbf{g}_1, \mathbf{g}_2\}) = \int_0^T (\{\mathbf{u}1_{\mathcal{O}}, \mathbf{v}1_{\mathcal{O}}\}, \{\boldsymbol{\varphi}, \boldsymbol{\psi}\}) dt.$$

If (3.10) holds true, then (3.14) implies that

$$(3.15) \quad \int_0^T (\{\mathbf{u}1_{\mathcal{O}}, \mathbf{v}1_{\mathcal{O}}\}, \{\boldsymbol{\varphi}, \boldsymbol{\psi}\}) dt = 0, \quad \forall \{\mathbf{u}, \mathbf{v}\} \in \mathbf{L}^2(\mathcal{O} \times (0, T))^2.$$

So, one assures

$$(3.16) \quad \{\boldsymbol{\varphi}, \boldsymbol{\psi}\} = \{0, 0\} \quad \text{in} \quad \mathcal{O} \times (0, T).$$

Since $\{\boldsymbol{\varphi}, \boldsymbol{\psi}\} = \sum_{i=1}^N \{\varphi_i(t) \mathbf{e}_i, \psi_i(t) \mathbf{f}_i\}$ and thanks to (3.1), we can guarantee by (3.16) that $\{\varphi_i(t), \psi_i(t)\} = \{0, 0\}$, for $i = 1, \dots, n$. Hence, $\{\boldsymbol{\varphi}, \boldsymbol{\psi}\} \equiv \{0, 0\}$ and, therefore, $\{\mathbf{g}_1, \mathbf{g}_2\} \equiv \{0, 0\}$. Thus, system (3.9) is exactly controllable.

Step 2 (Estimates). Thanks to the results obtained in Step 1, one can define the functional $M : L^2(0, T; E) \rightarrow \mathbb{R}$ by

$$(3.17) \quad M(\mathbf{h}) = \inf_{\{\mathbf{u}, \mathbf{v}\} \in \mathcal{U}_{ad}} \frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\mathbf{u}|^2 + |\mathbf{v}|^2) dx dt,$$

where $\mathcal{U}_{ad}$ is the set of admissible controls

$$\mathcal{U}_{ad} = \{\{\mathbf{u}, \mathbf{v}\} \in L^2(\mathcal{O} \times (0, T))^2; \{\mathbf{y}, \mathbf{w}\} \text{ solution of (3.9) satisfying (3.4)}\}.$$

We are interested in proving that

$$(3.18) \quad M(\mathbf{h}) \leq \text{constant independent of } \mathbf{h}, \alpha \text{ and } \beta.$$

For this, we use a duality argument.

We consider the continuous linear operator $L : L^2(\mathcal{O} \times (0, T))^2 \rightarrow E \times F$ defined by

$$L\{\mathbf{u}, \mathbf{v}\} = \{\mathbf{y}(\cdot, T; \{\mathbf{u}, \mathbf{v}\}), \mathbf{w}(\cdot, T; \{\mathbf{u}, \mathbf{v}\})\},$$

and we introduce the functionals

$$F_1\{\mathbf{u}, \mathbf{v}\} = \frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\mathbf{u}|^2 + |\mathbf{v}|^2) dx dt$$

and

$$F_2\{\mathbf{g}_1, \mathbf{g}_2\} = \begin{cases} 0, & \text{if } \{\mathbf{g}_1, \mathbf{g}_2\} = \{\mathbf{y}^T, \mathbf{w}^T\}, \\ \infty, & \text{otherwise.} \end{cases}$$

In this way, we can rewrite the functional M as follows:

$$(3.19) \quad M(\mathbf{h}) = \inf_{\{\mathbf{u}, \mathbf{v}\} \in L^2(\mathcal{O} \times (0, T))^2} [F_1\{\mathbf{u}, \mathbf{v}\} + F_2(L\{\mathbf{u}, \mathbf{v}\})].$$

From the duality theorem of Fenchel and Rockafellar (see, for instance, [3]), we have

$$(3.20) \quad -M(\mathbf{h}) = \inf_{\{\mathbf{g}_1, \mathbf{g}_2\} \in E \times F} [F_1^*(L^*\{\mathbf{g}_1, \mathbf{g}_2\}) + F_2^*(-\{\mathbf{g}_1, \mathbf{g}_2\})],$$

where $L^* : E \times F \rightarrow L^2(\mathcal{O} \times (0, T))^2$ is the adjoint of L .

Using (3.14), one sees that

$$(3.21) \quad L^*\{\mathbf{g}_1, \mathbf{g}_2\} = \{\boldsymbol{\varphi}, \boldsymbol{\psi}\} \quad \text{in } \mathcal{O} \times (0, T).$$

Since

$$F_1^*\{\boldsymbol{\varphi}, \boldsymbol{\psi}\} = \frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\boldsymbol{\varphi}|^2 + |\boldsymbol{\psi}|^2) dx dt$$

and

$$F_2^*(-\{\mathbf{g}_1, \mathbf{g}_2\}) = -(\{\mathbf{g}_1, \mathbf{g}_2\}, \{\mathbf{y}^T, \mathbf{w}^T\}),$$

then

$$(3.22) \quad -M(\mathbf{h}) = \inf_{\{\mathbf{g}_1, \mathbf{g}_2\} \in E \times F} \left[\frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\boldsymbol{\varphi}|^2 + |\boldsymbol{\psi}|^2) dx dt - (\{\mathbf{g}_1, \mathbf{g}_2\}, \{\mathbf{y}^T, \mathbf{w}^T\}) \right].$$

In view of the hypothesis on the basis of E and F , we have that $\int_{\mathcal{O}} (|\mathbf{e}|^2 + |\mathbf{f}|^2) dx$ is a norm on $E \times F$, so that

$$C\|\{\mathbf{e}, \mathbf{f}\}\|^2 \geq \int_{\mathcal{O}} (|\mathbf{e}|^2 + |\mathbf{f}|^2) dx \geq c\|\{\mathbf{e}, \mathbf{f}\}\|^2, \quad \forall \{\mathbf{e}, \mathbf{f}\} \in E \times F,$$

with c and C positive constants that only depend on E and F , where

$$\|\{\mathbf{e}, \mathbf{f}\}\|^2 = \int_{\Omega} (|\mathbf{e}|^2 + |\mathbf{f}|^2) dx, \quad \forall \{\mathbf{e}, \mathbf{f}\} \in E \times F.$$

Thus, (3.22) gives

$$(3.23) \quad -M(\mathbf{h}) \geq \inf_{\{\mathbf{g}_1, \mathbf{g}_2\} \in E \times F} \left[\frac{c}{2} \int_Q (|\boldsymbol{\varphi}|^2 + |\boldsymbol{\psi}|^2) dx dt - (\{\mathbf{g}_1, \mathbf{g}_2\}, \{\mathbf{y}^T, \mathbf{w}^T\}) \right].$$

Now, we take $\{\mathbf{e}, \mathbf{f}\} = \{\boldsymbol{\varphi}(t), \boldsymbol{\psi}(t)\}$ in (3.11) and integrate from t to T . Then the terms containing $\mathbf{h}$ drop out and we obtain

$$\frac{1}{2}\|\boldsymbol{\varphi}(t)\|^2 + \int_t^T a(\boldsymbol{\varphi}, \boldsymbol{\varphi})ds - \int_t^T (\nabla \times \boldsymbol{\psi}, \boldsymbol{\varphi})ds = \frac{1}{2}\|\mathbf{g}_1\|^2$$

and

$$\frac{1}{2}\|\boldsymbol{\psi}(t)\|^2 + \int_t^T a(\boldsymbol{\psi}, \boldsymbol{\psi})ds - \int_t^T (\nabla \times \boldsymbol{\psi}, \boldsymbol{\varphi})ds + \int_t^T \int_{\Omega} (\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\psi})dxds = \frac{1}{2}\|\mathbf{g}_2\|^2.$$

Adding the last two equations and integrating in $(0, T)$, we get

$$(3.24) \quad \begin{aligned} & \frac{1}{2} \int_0^T (\|\boldsymbol{\varphi}(t)\|^2 + \|\boldsymbol{\psi}(t)\|^2)dt + \int_0^T t[a(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + a(\boldsymbol{\psi}, \boldsymbol{\psi})]dt - 2 \int_0^T t(\nabla \times \boldsymbol{\psi}, \boldsymbol{\varphi})dt \\ & - \int_0^T t \int_{\Omega} (\nabla \cdot \boldsymbol{\psi})(\nabla \cdot \boldsymbol{\psi})dxdt = \frac{T}{2}(\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2). \end{aligned}$$

Notice that

$$a(\boldsymbol{\varphi}(t), \boldsymbol{\varphi}(t)) + a(\boldsymbol{\psi}(t), \boldsymbol{\psi}(t)) \leq C\|\{\boldsymbol{\varphi}(t), \boldsymbol{\psi}(t)\}\|^2$$

and

$$\int_{\Omega} (\nabla \cdot \boldsymbol{\psi}(t))(\nabla \cdot \boldsymbol{\psi}(t))dx \leq C\|\boldsymbol{\psi}(t)\|^2$$

for some $C > 0$ that depends only on E and F because E and F are finite dimensional.

So

$$(3.25) \quad \begin{aligned} & \frac{T}{2}(\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2) \leq \left(\frac{1}{2} + CT\right) \int_0^T (\|\boldsymbol{\varphi}(t)\|^2 + \|\boldsymbol{\psi}(t)\|^2)dt \\ & + T \int_0^T (a(\boldsymbol{\psi}, \boldsymbol{\psi}) + \|\boldsymbol{\varphi}(t)\|^2)dt + CT \int_0^T \|\boldsymbol{\psi}(t)\|^2 dt, \end{aligned}$$

which implies

$$(3.26) \quad \frac{T}{2}(\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2) \leq \left(\frac{1}{2} + CT\right) \int_0^T (\|\boldsymbol{\varphi}(t)\|^2 + \|\boldsymbol{\psi}(t)\|^2)dt$$

for a suitable $C > 0$ that depends only on E and F .

In view of (3.23) we get

$$(3.27) \quad -M(\mathbf{h}) \geq \inf_{(\mathbf{g}_1, \mathbf{g}_2) \in E \times F} \left[\frac{cT}{(1 + 2CT)} (\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2) - (\{\mathbf{g}_1, \mathbf{g}_2\}, \{\mathbf{y}^T, \mathbf{w}^T\}) \right].$$

Hence

$$(3.28) \quad M(\mathbf{h}) \leq \frac{1 + 2CT}{cT} (\|\mathbf{y}^T\|^2 + \|\mathbf{w}^T\|^2),$$

which implies (3.18).

Step 3 (Nonlinear system). Let $\mathbf{h} \in L^2(0, T; E)$ be given. We choose for $\{\mathbf{u}, \mathbf{v}\}$ the unique element such that

$$(3.29) \quad \frac{1}{2} \int_{\mathcal{O} \times (0, T)} (|\mathbf{u}|^2 + |\mathbf{v}|^2) dxdt = M(\mathbf{h}).$$

We define a continuous mapping $\mathbf{h} \mapsto \{\mathbf{u}, \mathbf{v}\}$ from $L^2(0, T; E)$ in $\mathbf{L}^2(\mathcal{O} \times (0, T))^2$.

We denote by $\{\mathbf{y}(\mathbf{h}), \mathbf{w}(\mathbf{h})\}$ the solution of (3.9) with the control $\{\mathbf{u}, \mathbf{v}\} = \{\mathbf{u}(\mathbf{h}), \mathbf{v}(\mathbf{h})\}$.

Taking $\{\mathbf{e}, \mathbf{f}\} = \{\mathbf{y}(t), \mathbf{w}(t)\}$ in (3.9) we obtain

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{y}(t)\|^2 + a(\mathbf{y}(t), \mathbf{y}(t)) = (\nabla \times \mathbf{w}(t), \mathbf{y}(t)) + (\mathbf{u}1_{\mathcal{O}}, \mathbf{y}(t))$$

and

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}(t)\|^2 + a(\mathbf{w}(t), \mathbf{w}(t)) + \int_{\Omega} (\nabla \cdot \mathbf{w})(\nabla \cdot \mathbf{w}) dx = (\nabla \times \mathbf{y}(t), \mathbf{w}(t)) + (\mathbf{v}1_{\mathcal{O}}, \mathbf{w}(t)).$$

In this way, integrating the last two equations in $(0, t)$ and adding them, we have

$$\begin{aligned} (3.30) \quad & \frac{1}{2} (\|\mathbf{y}(t)\|^2 + \|\mathbf{w}(t)\|^2) + \int_0^t [a(\mathbf{y}(s), \mathbf{y}(s)) + a(\mathbf{w}(s), \mathbf{w}(s))] ds \\ & \leq - \int_0^t \int_{\Omega} (\nabla \cdot \mathbf{w})(\nabla \cdot \mathbf{w}) dx ds + 2 \int_0^t (\nabla \times \mathbf{w}(s), \mathbf{y}(s)) ds \\ & \quad + \|\mathbf{u}\|_{L^2(\mathcal{O} \times (0, T))} \|\mathbf{y}\|_{L^2(\mathcal{O} \times (0, t))} + \|\mathbf{v}\|_{L^2(\mathcal{O} \times (0, T))} \|\mathbf{w}\|_{L^2(\mathcal{O} \times (0, t))}; \end{aligned}$$

that is,

$$\begin{aligned} (3.31) \quad & \frac{1}{2} (\|\mathbf{y}(t)\|^2 + \|\mathbf{w}(t)\|^2) \leq C (\|\mathbf{u}\|_{L^2(\mathcal{O} \times (0, T))}^2 + \|\mathbf{v}\|_{L^2(\mathcal{O} \times (0, T))}^2) \\ & \quad + C \int_0^t (\|\mathbf{y}(s)\|^2 + \|\mathbf{w}(s)\|^2) ds. \end{aligned}$$

From Gronwall's inequality, we have by (3.31) that

$$(3.32) \quad (\|\mathbf{y}(t)\|^2 + \|\mathbf{w}(t)\|^2) \leq C (\|\mathbf{u}\|_{L^2(\mathcal{O} \times (0, T))}^2 + \|\mathbf{v}\|_{L^2(\mathcal{O} \times (0, T))}^2).$$

In view of (3.18) we obtain that, when $\mathbf{h}$ varies in $L^2(0, T; E)$,

$$(3.33) \quad \{\mathbf{y}, \mathbf{w}\} \text{ remains a bounded subset } K_1 \times K_2 \subset L^2(0, T; E) \times L^2(0, T; F).$$

We claim that

$$(3.34) \quad \text{the map } \mathbf{h} \mapsto \mathbf{y}(\mathbf{h}) \text{ admits a fixed point in } K_1.$$

Remark 3.3. The application in (3.34) can be seen as being the composition of the applications

$$\mathbf{h} \mapsto \{\mathbf{y}(\mathbf{h}), \mathbf{w}(\mathbf{h})\}$$

and

$$\{\mathbf{y}(\mathbf{h}), \mathbf{w}(\mathbf{h})\} \mapsto \mathbf{y}(\mathbf{h}).$$

Assuming for a moment that (3.34) holds, we can conclude the proof of the theorem. In fact, if $\mathbf{h}$ is a fixed point, then, since system (3.9) is exactly controllable in $T > 0$, we have the exact controllability of (3.7). Moreover, for any h we have the uniform estimate (3.18), so that the control $\{\mathbf{u}(\mathbf{h}), \mathbf{v}(\mathbf{h})\}$ satisfies the conditions of the theorem.

It remains to prove (3.34).

According to Schauder's fixed point theorem, it suffices to show that the range of $\mathbf{y}(\mathbf{h})$ when $\mathbf{h}$ spans K_1 is relatively compact in K_1 , which follows from the following estimate:

$$(3.35) \quad \mathbf{y}_t \text{ remains in a bounded set of } L^2(0, T; E) \text{ when } h \text{ describes } K_1.$$

To prove (3.35), we observe that, from (3.9), the following holds:

$$\begin{aligned} |(\mathbf{y}_t, \mathbf{e})| & \leq \alpha \|\mathbf{h}(t)\| \|\nabla \mathbf{y}(t)\| \|\mathbf{e}\| + \|\nabla \mathbf{y}(t)\| \|\nabla \mathbf{e}\| + \|\nabla \mathbf{w}(t)\| \|\mathbf{e}\| + \|\mathbf{u}(t)\|_{L^2(\mathcal{O})} \|\mathbf{e}\|_{L^2(\mathcal{O})} \\ & \leq C (\alpha \|\mathbf{h}(t)\| \|\nabla \mathbf{y}(t)\| + \|\nabla \mathbf{y}(t)\| + \|\nabla \mathbf{w}(t)\| + \|\mathbf{u}(t)\|_{L^2(\mathcal{O})}) \|\mathbf{e}\|, \quad \forall \mathbf{e} \in E, \end{aligned}$$

since on the finite-dimensional space E all the norms are equivalent. Therefore

$$(3.36) \quad \|\mathbf{y}_t(t)\| \leq C(\alpha\|\mathbf{h}(t)\| \|\mathbf{y}(t)\| + \|\mathbf{y}(t)\| + \|\mathbf{w}(t)\| + \|\mathbf{u}(t)\|_{\mathbf{L}^2(\mathcal{O})}),$$

which implies (3.35). $\square$

Remark 3.4. In the bidimensional case, the nonlinear system which describes the behavior of micropolar fluids (with distributed controls in small sets) reads

$$(3.37) \quad \begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla) \mathbf{y} + \nabla p = \nabla \times w + \mathbf{u} \mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ w_t - \Delta w + \mathbf{y} \cdot \nabla w = \nabla \times \mathbf{y} + \mathbf{v} \mathbf{1}_{\mathcal{O}} & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = 0, \quad w = 0 & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}^0, \quad w(0) = w^0 & \text{in } \Omega. \end{cases}$$

In this case, the angular velocity w is a scalar variable and the quantities $\nabla \times w$ and $\nabla \times \mathbf{y}$ are defined respectively by

$$\nabla \times w = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right) \quad \text{and} \quad \nabla \times \mathbf{y} = \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}.$$

Using analogous arguments, the Galerkin approximation of (3.37) is also exactly controllable.

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