Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós Graduação em Matemática
Doutorado em Matemática

Rigidity, nonexistence and sharp integral inequalities for linear Weingarten submanifolds

por

Lucas Siebra Rocha

Campina Grande – PB
Fevereiro de 2024

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sob orientação do

Prof. Dr. Henrique Fernandes de Lima

Tese apresentada ao Corpo Docente do Programa Associado de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Área de Concentração: Geometria

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Dedicatória

Aos meus pais.

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Resumo

Este trabalho está dividido em duas partes. Na primeira parte, estamos interessados em obter resultados de rigidez e de não existência baseados em princípios do máximo relacionados a subvariedades Weingarten lineares imersas em variedades Riemannianas ou Lorentzianas. Usamos fortemente a noção de convergência para zero no infinito e de crescimento de volume polinomial. A segunda parte é dedicada também ao estudo de subvariedades Weingarten lineares, dessa vez fechadas, em variedades Riemannianas ou Lorentzianas imersas com curvatura média normalizada paralela. Precisamente, estabelecemos desigualdades integrais a partir de uma estimativa inferior adequada do operador de Cheng-Yau agindo sobre a norma ao quadrada da segunda forma fundamental sem traço e a usamos para caracterizar subvariedades totalmente umbílicas.

Abstract

This work is divided into two parts. In the first part, we are interested to get rigidity and nonexistence results based on suitable maximum principles related to linear Weingarten submanifolds immersed into Riemannian or Lorentzian manifolds. We strongly use the notion of convergence to zero at infinity and of polynomial volume growth. The second part is dedicated also to study linear Weingarten submanifolds into Riemannian or Lorentzian manifolds, but this time closed and immersed with parallel normalized mean curvature. Precisely, we establish sharp integral inequalities from a suitable lower estimate of the Cheng-Yau operator acting on the squared norm of the traceless second fundamental form and we use it to characterize totally umbilical submanifolds.

Contents

Introduction

This thesis is divided into two main parts as follows:

Part I: Rigidity and nonexistence results for complete submanifolds.

The study of the rigidity of n-dimensional submanifolds immersed into a Riemannian or a Lorentzian space constitutes an important thematic in Diferential Geometry and, in particular, into the theory of isometric immersions. It is still profuse and many researchers have extensively explored this area.

An analytical tool that has become fruitful for this research branch is a self-adjoint differential operator acting on smooth functions defined on a Riemannian or on a Lorentzian manifold, known as *Cheng-Yau operator*, which was introduced by Cheng and Yau in their remarkable paper [31]. In this work, they used the square operator to classify n-dimensional compact (without boundary) hypersurfaces with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in a space form $\mathbb{Q}_c^{n+1}$. Posteriorly, Li [62] extended the results of Cheng and Yau in terms of the squared norm of the second fundamental form of the hypersurface. Next, Li [61] studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed in a unit Euclidean sphere $\mathbb{S}^{n+1}$ under the assumption that the scalar and mean curvatures are proportional.

Over the past few decades, significant advancements have been made in the field described above. In 2009, for instance, Li, Suh and Wei [63] extended the results of [31] and [61] by examining the concept of *linear Weingarten* (LW) hypersurfaces immersed in $\mathbb{S}^{n+1}$ whose normalized scalar curvature R and mean curvature H satisfy a linear relation of the type $R = aH + b$, for some constants $a, b \in \mathbb{R}$. Subsequently, Shu [81] contributed to the field by presenting some rigidity theorems concerning LW hypersurfaces with two distinct principal curvatures immersed in $\mathbb{Q}_c^{n+1}$. Also working in this context, de Lima, Velásquez and Aquino [18] extended the results of [63] to complete LW hypersurfaces immersed in $\mathbb{Q}_c^{n+1}$ resorting to a suitable Cheng-Yau modified operator.

Regarding immersed submanifolds with (possibly) high codimension $p \geq 1$ and whose normalized mean curvature vector field is parallel as a section of the normal bundle, we also have in the current literature several works addressing characterization results. In this setting, we can highlight the papers of Cheng [34] and Guo and Li [59]. In the first one, the author applied the generalized maximum principle of Omori-Yau [74, 87] to show that the totally umbilical sphere $\mathbb{S}^n(r)$, the totally geodesic Euclidean space $\mathbb{R}^n$ and the generalized cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$ are the only n-dimensional complete submanifolds with constant scalar curvature and parallel normalized mean curvature vector in the Euclidean space $\mathbb{R}^{n+p}$ satisfying a suitable constraint on the norm of the second fundamental form. In the second one, the authors investigated the problem of generalize the previous results of [62]. So, they proved that the only n-dimensional compact (without boundary) submanifolds immersed in $\mathbb{S}^{n+p}$ with constant scalar curvature, parallel normalized mean curvature vector and such that the second fundamental form satisfies an appropriate boundedness are the totally umbilical spheres $\mathbb{S}^n(r)$ and the Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, where $r > 0$ stands for the positive radius.

More recently, Araújo, de Lima, dos Santos and Velásquez [25] obtained an Omori-type maximum principle for the Cheng-Yau operator and applied it to establish an extension of the results of [34, 59] for n-dimensional complete submanifolds immersed with parallel normalized mean curvature vector field in $\mathbb{Q}_c^{n+p}$, with constant normalized scalar curvature. Next, these same authors [44] used the Hopf strong maximum principle and a maximum principle at infinity due to Caminha [30] to obtain versions of the results of [25, 34, 59] for the context of n-dimensional complete LW submanifolds immersed with parallel normalized mean curvature vector field in $\mathbb{Q}_c^{n+p}$. In [23], Velásquez and Araújo established a version of the classical Liebmann's rigidity theorem showing that a compact LW surface immersed with flat normal bundle and parallel normalized mean curvature vector with nonnegative Gaussian curvature in $\mathbb{Q}_c^{2+p}$ must be isometric to a totally umbilical round sphere. They also obtained in [22] another version of this Liebmann's result assuming that the ambient is the hyperbolic space (for other characterizations concerning complete LW submanifolds in the hyperbolic space we refer the reader to [16, 20, 21, 42]).

In a higher codimension, Liu [67] showed that the totally umbilical round spheres are the only n-dimensional compact (without boundary) spacelike LW submanifolds of $\mathbb{S}_p^{n+p}$ with nonnegative sectional curvature and flat normal bundle. Later on, Yang and Hou [85] applied the Omori-Yau's generalized maximum principle to show that a spacelike LW submanifold in $\mathbb{S}_p^{n+p}$, with $a > 0$, $b < 1$, having parallel normalized mean curvature vector and such that the squared norm of its second fundamental form satisfies a suitable boundedness, must be either totally umbilical or isometric to a certain hyperbolic cylinder. Afterwards, Liu and Zhang [66] used the classical strong maximum principle of Hopf to obtain other classifications for complete spacelike LW submanifolds in $\mathbb{S}_p^{n+p}$ having parallel normalized mean curvature.

In [24], raújo, de Lima, dos Santos and Velásquez obtained other characterization results related to complete spacelike LW submanifolds with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$ under suitable constraints on the values of the mean curvature and of the norm of the traceless part of the second fundamental form, now through an extension of Hopf's maximum principle for complete Riemannian manifolds. Next, de Lima and Velásquez jointly with Barboza and de Lima [26] studied the umbilicity of n-dimensional complete spacelike LW submanifolds immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, with index $p > 1$. They applied a generalized maximum principle for a modified Cheng-Yau operator $\mathcal{L}$ to show that such a spacelike LW submanifold must be either totally umbilical or isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors M_i are totally umbilical submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections. More recently, imposing some restrictions on the values of the mean curvature function H, Antonia and de Lima in [40] established a parabolicity criterion related to the operator $\mathcal{L}$ and used it to obtain sufficient conditions which guarantee that a spacelike LW submanifold immersed in $\mathbb{S}_p^{n+p}$ must be either totally umbilical or isometric to certain hyperbolic cylinders.

Lately, in [17], when the ambient space is a Lorentzian space form, de Lima jointly with Aquino characterized constant mean curvature spacelike hypersurfaces, whose support functions are linearly related. Continuing the study of the geometry of spacelike hypersurfaces, now with da Silva, de Lima established in [39] a parabolicity criterion related to a suitable Cheng-Yau modified operator $\mathcal{L}$ and used it to obtain sufficient conditions which guarantee that spacelike hypersurfaces immersed in a more general Lorentz space, a locally symmetric *Einstein spacetime* (that is, a Lorentz space whose metric and Ricci tensors are homothetic) obeying standard curvature constraints must be either totally umbilical or isometric to an isoparametric spacelike hypersurface with two distinct principal curvatures, one of which is simple.

Motivated by this research history, in this part of the work, the main aim is to establish new rigidity and nonexistence results concerning n-dimensional submanifolds immersed in a variety of spaces via certain maximum principles. In Chapter 1, we will briefly introduce these maximum principles. Among them, the suitable maximum principles due to Alías, Caminha and do Nascimento [8,9], involving the concept of convergence to zero at infinity and polynomial volume growth.

In Chapter 2, we are going to establish new rigidity theorems for n-dimensional spacelike linear Weingarten submanifolds immersed with parallel normalized mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_p^{n+p}$ of index p. In one of them, we present a new version of [85, Theorem 1.4] and of [66, Theorem 1.1].

Theorem 0.0.1 (Theorem 2.1.8)**.** *Let M^n be a complete noncompact spacelike LW submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$. If $|A|^2 \leq 2\sqrt{n-1}$ and $|\Phi|$ converges to zero at infinity, then M^n is isometric to either the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$.*

We also are going to use the one-parameter family of real functions (see (2.43)) given by

$$P_{t,p}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}tx - n(t^2 - 1)$$

to study it behavior based on a limitation of the mean curvature H, as we can see below:

Theorem 0.0.2 (Theorem 2.1.20)**.** *Let M^n be a complete spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$ and $H + \frac{a}{2} \geq \beta$, for some positive constant $\beta \in \mathbb{R}$. Suppose that* $\sup_M |\Phi| <$

$\vartheta(n,p)$ and that

$$H^2 \leq \frac{4(n-1)}{Q(p)},$$

where $Q(p) = p(n-2)^2 + 4(n-1)$, and $\vartheta(n,p)$ is the real root of $P_{H,p}$. If M^n has polynomial volume growth, then M^n is isometric to either the Euclidean space $\mathbb{R}^n$ or the Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.

The second section of this chapter is devoted to investigate the nonexistence and umbilicity of n-dimensional ($n \geq 3$) spacelike submanifolds immersed with parallel mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$. Also we prove (see Theorem 2.2.6) that the only n-dimensional stochastically complete spacelike submanifold immersed in $\mathbb{S}_q^{n+p}$, which are maximal and having locally timelike second fundamental form, are the totally geodesic ones.

In Chapter 3, we will study linear Weingarten submanifolds immersed in an $(n+p)$-dimensional Riemannian space form $\mathbb{Q}_c^{n+p}$ with constant sectional curvature $c \in \{-1, 0, 1\}$. Between the results obtained in this chapter, we can mention

Theorem 0.0.3 (Theorem 3.2.5). *Let M^n be a complete LW submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \frac{n-2}{n}$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Assume in addition that $|\nabla\Phi|$ is bounded and such that $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in (3.16). If M^n has polynomial volume growth and $\inf_R(Q_R(\gamma)) > 0$, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Concerning the hyperbolic space, we cite

Theorem 0.0.4 (Theorem 3.2.8). *Let M^n be a complete noncompact LW submanifold immersed with parallel normalized mean curvature vector field into the hyperbolic space $\mathbb{H}^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b > -1$. Suppose that $\left(H - \frac{a}{2}\right) \geq 0$ on M^n and that $R \geq 0$. Assume in addition that $|\Phi| \leq x_R^*$, for x_R^* defined in (3.16). If $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Reaching Chapter 4, we will study the geometry of linear Weingarten spacelike hypersurfaces immersed in an Einstein space obeying some standard curvature conditions. Considering the hypersurface immersed in the Lorentz space $\mathbb{L}_1^{n+1}$, we are going to assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of $\mathbb{L}_1^{n+1}$ satisfies the two constraints (see 4.3 and 4.4)

$$\overline{K}(u,\eta) = -\frac{c_1}{n},$$

for any $u \in TM$ and $\eta \in TM^{\perp}$, and

$$\overline{K}(u,v) \geq c_2,$$

for any tangent vectors $u, v \in TM$. By doing this, we can use the modified Cheng-Yau operator $\mathcal{L}$ to obtain some results concerning complete LW spacelike hypersurfaces immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$:

Theorem 0.0.5 (Theorem 4.1.8). *Let M^n be a complete LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with $b \leq \overline{\mathcal{R}} < b + c$, where $c = 2c_2 + \dfrac{c_1}{n} > 0$, and $b \leq R$. In the case where $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change sign. Then*

(i) either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical hypersurface,

(ii) or

$$\sup_M |\Phi|^2 \geq \alpha(n, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, a, b, c, \overline{\mathcal{R}})$ is a positive constante depending on n, a, b, c and $\overline{\mathcal{R}}$.

In particular, if $b < \overline{\mathcal{R}}$, the equality $\sup_M |\Phi|^2 = \alpha(n, a, b, c, \overline{\mathcal{R}})$ holds and this supremum is attained at some point of M^n, then M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

To the context of a hypersurface immersed in a Riemannian manifold $\overline{M}^{n+1}$, we will also assume the existence constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of $\overline{M}^{n+1}$ satisfies the following two constraints (see 4.70 and 4.71)

$$\overline{K}(u, \eta) = \frac{c_1}{n},$$

for any tangent vector $u \in TM$ and normal vector $\eta \in TM^{\perp}$; and

$$\overline{K}(u, v) \geq c_2,$$

for any tangent vectors $u, v \in TM$. Involving these constraints and the concept of polynomial volume growth, we state the following:

Theorem 0.0.6 (Theorem 4.2.4). *Let M^n be a complete LW hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature constraints* (4.70) *and* (4.71) *with $n \geq 3$, such that $R = aH + b$ with $b \geq \overline{\mathcal{R}}$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \overline{\mathcal{R}} - \frac{2}{n}c$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. Assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in* (3.16)*. If M^n has polynomial volume growth and $\inf\limits_R(Q_R(\gamma)) > 0$, then M^n is a totally umbilical hypersurface.*

In the results of Chapter 5, we discuss about LW submanifolds in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$ with second fundamental form locally timelike. For example, revisiting [89, Theorem 2], we replace the assumption that the mean curvature function attains a global maximum to arrive at:

Theorem 0.0.7 (5.2.1)**.** *Let M^n be a complete LW spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{N}_q^{n+p}(c)$, such that $R = aH + b$ with $b \leq c$. Suppose that the second fundamental form of M^n is locally timelike and that it has nonnegative sectional curvature. If H is bounded and, for some reference point $o \in M^n$ and $\delta > 0$,*

$$\int_{\delta}^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty,$$

where B_t is the geodesic ball of radius t in M^n centered at the reference point o, then M^n is either totally umbilical or a product $M_1 \times M_2 \times \cdots \times M_k$, where the factors M_i, mutually perpendicular along their intersections, are totally umbilical submanifolds of $\mathbb{N}_q^{n+p}(c)$.

Part II: Sharp integral inequalities for closed submanifolds.

Let us first describe about the Riemannian context. In 1977, Cheng and Yau [31] studied the rigidity problem for hypersurfaces M^n with constant scalar curvature in a space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c, introducing a self-adjoint second order differential operator, the so-called squared operator. By using Cheng-Yau's technique, Li [62] studied the pinching problem on the square norm of the second fundamental form for complete hypersurfaces with constant scalar curvature. Afterwards, Li [61] also studied the rigidity of oriented and without boundary compact hypersurfaces with nonnegative sectional curvature in a unit sphere $\mathbb{S}^{n+1}$ with scalar curvature proportional to the mean curvature.

Later on, Wei [84] investigated compact rotational hypersurfaces in $\mathbb{S}^{n+1}$, obtaining suitable integral formulas and applying them to characterize Clifford tori $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, $0 < r < 1$, under the assumption that some higher order mean curvature vanishes identically. Next, de Lima jointly with Aquino and Velásquez [16, 19] established another characterization results related to complete linear Weingarten hypersurfaces in $\mathbb{Q}_c^{n+1}$, under appropriated restriction on the norm of the traceless part of the second fundamental form.

In [12], de Lima, dos Santos, Alías and Meléndez extended these results for the context of complete linear Weingarten hypersurfaces in a locally symmetric Riemannian manifold obeying some standard curvature conditions (in particular, in a Riemannian space with constant sectional curvature). Under appropriated constrains on the scalar curvature function, they proved that such a hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface with two distinct principal curvatures, one of them being simple.

More recently, Alías and Meléndez [5] studied the rigidity of compact hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$. In particular, they established a sharp Simons type integral inequality for the behavior of the norm of the traceless second fundamental form, with the equality characterizing the totally umbilical hypersurfaces and the Clifford tori $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$.

Towards the context of the Lorentzian geometry, Aiyama [3] studied closed (compact without boundary) spacelike submanifolds M^n in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector field and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. In the same work [3], she proved that closed spacelike submanifolds in $\mathbb{S}^{n+p}_p$ with parallel mean curvature vector field and nonnegative sectional curvatures must be also totally umbilical. Meanwhile, Alías and Romero [6] introduced a new method to study n-dimensional closed spacelike submanifolds in de Sitter space $\mathbb{S}^{n+p}_q$ of index q $(1 \leq q \leq p)$ by means of certain integral formulas which have a very clear geometric meaning. In particular, they got a uniqueness result for closed spacelike surfaces in $\mathbb{S}^{2+p}_q$ with parallel mean curvature vector field. Next, Li [61] showed that Montiel's result [70] still holds for higher codimensional spacelike submanifolds in $\mathbb{S}^{n+p}_p$.

Later on, Camargo, Chaves and Sousa [29] studied complete spacelike submanifolds with parallel normalized mean curvature vector field and constant scalar curvature immersed in a semi-Riemannian space form $\mathbb{Q}^{n+p}_p(c)$ of constant sectional curvature c and index p. In particular, they obtained characterizations results concerning totaly umbilical spacelike submanifolds and hyperbolic cylinders of $\mathbb{S}^{n+p}_p$, under certain constraints on both the squared norm of the second fundamental form and on the mean curvature. Another outcome in this regard is attributed to López [68], who proved that compact spacelike surfaces with constant mean curvature in the 3-dimensional Lorentz-Minkowski spacetime $\mathbb{R}^3_1$ with boundary on a plane can reach at most a height of $\frac{|H|A}{2\pi}$, where A is the area of the region of the surface above the plane containing its boundary. Later on, Montiel [72] obtained height estimates of compact spacelike graphs in the steady state spacetime and he applied them to prove some existence and uniqueness theorems for complete spacelike hypersurfaces in the de Sitter spacetime with constant mean curvature $H > 1$ and prescribed asymptotic future boundary. Also, de Lima studied height estimates and obtained a sharp estimate of compact spacelike hypersurfaces with some constant higher order mean curvature in the Lorentz-Minkowski spacetime $\mathbb{R}^{n+1}_1$ and with boundary contained into a spacelike hyperplane (see [41]).

This part of the thesis is devoted to use the ideas and techniques of [5] to establish a sharp integral inequality related to closed linear Weingarten submanifolds and apply it to get rigidity results based on a suitable lower estimate of the Cheng-Yau operator acting on the square norm of the traceless second fundamental form of such a spacelike submanifold.

To be more precisely, in Chapter 6 we establish a sharp integral inequality for n-dimensional closed spacelike submanifolds with constant scalar curvature immersed with parallel normalized mean curvature vector field in the de Sitter space $\mathbb{S}^{n+p}_p$ of index p, and we use it to characterize totally umbilical round spheres $\mathbb{S}^n(r)$, with $r > 1$, of $\mathbb{S}^{n+1}_1 \hookrightarrow \mathbb{S}^{n+p}_p$. We set the following:

Theorem 0.0.8 (Theorem 6.0.3). *Let M^n be a closed spacelike submanifold immersed in $\mathbb{S}^{n+p}_p$ with parallel normalized mean curvature vector field and constant normalized scalar curvature $R \leq 1$. Then*

$$\int_M |\Phi|^{q+2} Q_{R,n,p}(|\Phi|)dM \leq 0,$$

for every real number $q \geq 2$, where the real function $Q_{R,n,p}(x)$ is

$$Q_{R,n,p}(x) = \frac{(n-p-1)}{p}x^2 - (n-2)x\sqrt{x^2 + n(n-1)(1-R)} + n(n-1)R. \tag{1}$$

Moreover, assuming in addition that $0 < R < 1$*, the equality holds if, and only if,* M^n *is a totally umbilical round sphere* $\mathbb{S}^n(r)$*, with* $r = \frac{1}{R} > 1$*, immersed in* $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$*.*

Later on, in Chapter 7 we prove a sharp Simons type integral inequality for n-dimensional closed linear Weingarten hypersurfaces immersed in an Einstein manifold $\mathcal{E}^{n+1}$ and we use it to characterize totally umbilical hypersurfaces and isoparametric hypersurfaces with two distinct principal curvatures, one which is simple, in such an ambient space, as stated bellow:

Theorem 0.0.9 (Theorem 7.1.1)**.** *Let* M^n *be a closed linear Weingarten hypersurface immersed in an Einstein manifold* $\mathcal{E}^{n+1}$ *satisfying curvature conditions* (4.3) *and* (4.4)*, with* $R = aH + b$ *such that* $b \geq \overline{\mathcal{R}}$*. In the case where* $b = \overline{\mathcal{R}}$*, suppose that* $a > 0$*. If its totally umbilical tensor* Φ *satisfies* (4.101)*, for some* $1 \leq p \leq \frac{n-\sqrt{n}}{2}$*, then*

$$\int_M |\Phi|^{q+2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) dM \leq 0,$$

for every $q \geq 2$*, where the real function* $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}$ *is defined in* (4.116)*. Moreover, assuming* $b > \overline{\mathcal{R}}$*, the equality holds if and only if*

(i) either M^n *is a totally umbilical hypersurface,*

(ii) or

$$|\Phi|^2 = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0) > 0,$$

where $\alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$ *is a positive constant depending only on* $a, b, n, p, \overline{\mathcal{R}}$ *and* c_0*, and* M^n *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* p *and* $n - p$*.*

To close this part, in Chapter 8 we study complete LW spacelike submanifolds immersed with parallel normalized mean curvature vector and second fundamental form locally timelike in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$. In other words, we will state the next theorem:

Theorem 0.0.10 (Theorem 8.0.2)**.** *Let* M^n *be a closed LW spacelike submanifold immersed with parallel normalized mean curvature vector in a semi-Riemannian space form* $\mathbb{N}_q^{n+p}(c)$*, such that* $R = aH + b$ *with* $b \leq c$ *and* $a \geq 0$ *(suppose* $a > 0$ *when* $b = c$*). If the second fundamental form of* M^n *is locally timelike, then*

$$\int_M |\Phi|^{t+2} \varphi_{a,b,c,q,n}(|\Phi|) \mathrm{dM} \leq 0,$$

for every real number $t > 2$*, where the real function* $\varphi_{a,b,c,q,n}$ *is defined in* (8.1)*. Moreover, assuming in addition that either* $0 < b < c$ *or* $-\frac{a^2}{4} < b \leq c \leq 0$*, the equality holds if, and only if,* M^n *is a totally umbilical submanifold of* $\mathbb{N}_q^{n+p}(c)$*.*

We point out that, with the intention of all chapters being self-contained, we will place the necessary preliminaries in each of them, seeking to avoid as much as possible the search for results outside of them.

Part I

Rigidity and nonexistence results for complete submanifolds

Chapter 1

A brief comment about some maximum principles

For clarity, this chapter will present a number of important definitions and notations that will be used consistently in the remainder of this thesis. For instance, we recall briefly a generalized version of the Omori-Yau's maximum principle for trace type differential operators proved in [15] as well as the well known Omori-Yau's maximum principle for the Laplacian operator.

Let M^n be a Riemannian manifold and let $\mathcal{L} = \mathrm{tr}(\mathcal{P} \circ \mathrm{hess})$ be a semi-elliptic operator, where $\mathcal{P} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a positive semi-definite symmetric tensor. Following the terminology introduced by Pigola et al. [77], we say that the Omori-Yau maximum principle holds on M^n for the operator $\mathcal{L}$ if, for any function $u \in C^2(M)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$.

In this sense, the classical result given by Omori and Yau in [74, 87] states that the Omori-Yau maximum principle holds for the Laplacian on every complete Riemannian manifold with Ricci curvature bounded from below, that is:

Lemma 1.0.1. *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $u \in C^2(M)$ satisfying $u^* < +\infty$. Then, there exists a sequence of points $\{p_j\} \subset M^n$ such that*

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

Conversely, as observed also by Pigola et al. [77], the validity of Omori-Yau's maximum principle on M^n does not depend on curvatures bounds as much as one would expect. For instance, the Omori-Yau maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [77], Example 1.14). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

Following the terminology introduced in [77], the weak Omori-Yau maximum principle is said, more broadly, to hold on a (not necessarily complete) n-dimensional Riemannian manifold M^n if, for any smooth function $u \in C^2(M)$ with $u^* < +\infty$ there exists a sequence of points $\{p_j\} \subset M^n$ with the properties

$$u(p_j) > u^* - \frac{1}{j} \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

Proceeding, we say that a (non necessarily complete) Riemannian manifold M^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x,t) \in M^n \times (0, +\infty)$, the heat kernel $p(x, y, t)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_M p(x, y, t) d\mu(y) = 1. \tag{1.1}$$

From the probabilistic viewpoint, stochastic completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (1.1) means that the total probability of the particle to be found in the state space is constantly equal to one (see [56–58, 82]).

On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (see [76, Theorem 1.1] and [77, Theorem 3.1]), as is expressed below.

Lemma 1.0.2. *A Riemannian manifold M^n is stochastically complete if, and only if, for every $u \in C^2(M)$ satisfying $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\} \subset M^n$ such that*

$$\lim_{k\to\infty} u(p_k) = \sup_M u \quad \text{and} \quad \limsup_{k\to\infty} \Delta u(p_k) \leq 0.$$

We also note that stochastic completeness of Riemannian manifold M^n is equivalent (among others conditions) to the fact that for every $\lambda > 0$, the only nonnegative bounded smooth solution u of $\Delta u \geq \lambda u$ on M^n is the constant $u = 0$. Moreover, it is a direct consequence of Lemma 1.0.2 jointly with the Omori-Yau maximum principle [74, 87] that complete Riemannian manifolds having Ricci curvature bounded from below are stochastically complete.

Let us recall that a Riemannian manifold M^n is said to be parabolic if the constant functions are the only subharmonic functions on M^n which are bounded from above, that is, for a function $u \in \mathcal{C}^2(M)$ with

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{imply} \quad u = \text{constant}.$$

We observe that every parabolic Riemannian manifold is stochastically complete. As a consequence, the weak maximum principle holds on every parabolic Riemannian manifold (see Corollary 6.4 of [57]). Obviously, every closed Riemannian manifold M^n is parabolic, where by closed we mean compact and without boundary. Moreover, there are several interesting geometric conditions which imply the parabolicity of a Riemannian manifold M^n. For instance, in dimension $n = 2$ parabolicity is strongly related to the behavior of the Gaussian curvature;

for instance, from a classical result by Ahlfors [2] and Huber [60] it is well known that every complete Riemannian surface with nonnegative Gaussian curvature is parabolic. More generally, every complete Riemannian surface with finite total curvature is parabolic (see Section 10 of [64]).

Recently, many authors have explored new variations of the Omori-Yau maximum principle in order to extend the investigation to a wider array of differential operators containing the Laplacian operator. For a thorough understanding of this subject, we refer to the interested reader the comprehensive book [15] due to Alias, Mastrolia and Rigoli.

Now, let $(M^n, \langle\,,\rangle)$ be a connected, oriented, complete Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p with radius t. Given a polynomial function $\sigma : (0,+\infty) \to (0,+\infty)$, we say that M^n has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in M^n$ such that

$$\mathrm{vol}(B(p,t)) = \mathcal{O}(\sigma(t)),$$

as $t \to +\infty$, where vol denotes the standard Riemannian volume. As it was already observed in the beginning of Section 2 of [9], if $p, q \in M^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

Consequently, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that M^n has polynomial volume growth.

Keeping in mind this previous digression and denoting by $\mathrm{div}X$ the divergence of a smooth vector field $X \in \mathfrak{X}(M)$ in the metric $\langle\,,\rangle$, we quote the following key lemma which corresponds to a particular case of a new maximum principle due to Alías, Caminha and do Nascimento (see [9, Theorem 2.1]).

Lemma 1.0.3. *Let $(M^n, \langle\,,\rangle)$ be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a bounded smooth vector field on M^n. Assume that $f \in C^\infty(M)$ is a smooth function on M^n such that $\langle \nabla f, X\rangle \geq 0$ and $\mathrm{div}X \geq \alpha f$, for some positive constant α. If M^n has polynomial volume growth, then $f \leq 0$ on M^n.*

To finish this section, let us see the notion of convergence to zero at infinity established in [8, Section 2]: If M^n is a connected, complete noncompact Riemannian manifold, we let $d(\cdot, o) : M \to [0,+\infty)$ stand for the Riemannian distance of M^n, measured from a fixed point $o \in M^n$. Thus, if $f \in C^0(M)$ satisfies

$$\lim_{d(x,o)\to+\infty} f(x) = 0,$$

we say that f converges to zero at infinity. In this context, we have the following maximum principle which can be found in [8, Theorem 2.2(a)].

Lemma 1.0.4. *Let $(M^n, \langle\,,\rangle)$ be a connected, oriented, complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field on M^n. Assume that there exists a nonnegative,* *non-identically vanishing function* $f \in C^\infty(M)$ *which converges to zero at infinity and such that* $\langle \nabla f, X \rangle \geq 0$*. If* $\mathrm{div} X \geq 0$ *on* M^n*, then* $\langle \nabla f, X \rangle \equiv 0$ *on* M^n*.*

Chapter 2

Submanifolds in the de Sitter space $\mathbb{S}_p^{n+p}$

In this chapter we establish new rigidity and nonexistence theorems for n-dimensional spacelike submanifolds based on the maximum principles 1.0.3 and 1.0.4 due to Alías, Caminha and do Nascimento [8, 9] related to complete noncompact Riemannian manifolds. Here we present the results of [50, 51].

2.1 Spacelike LW submanifolds

The main intention of this section is to establish new rigidity theorems for n-dimensional spacelike linear Weingarten (LW) submanifolds immersed with parallel normalized mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_p^{n+p}$ of index p.

The starting point is to prove that under suitable assumption that the norm of the total umbilicity tensor converges to zero at infinity, a complete noncompact spacelike LW submanifold of $\mathbb{S}_p^{n+p}$ must be either isometric to the Euclidian space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$. Afterwards, under the assumption that such a complete spacelike LW submanifold of $\mathbb{S}_p^{n+p}$ has polynomial volume growth, we prove that it must be either isometric to the Euclidean space $\mathbb{R}^n$ or a Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$.

2.1.1 Preliminaries

Let us consider the semi-Euclidean space $\mathbb{R}_p^{n+p+1}$, that is, the $(n+p+1)$-dimensional real vector space $\mathbb{R}^{n+p+1}$ endowed with the inner product of index p given by

$$\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n+p+1} x_j y_j,$$

where $x = (x_1, x_2, \ldots, x_{n+p+1})$ is the natural coordinate of $\mathbb{R}^{n+p+1}$. The $(n+p)$-dimensional *de Sitter space* $\mathbb{S}_p^{n+p}$ is defined as being the following hyperquadric of $\mathbb{R}_p^{n+p+1}$

$$\mathbb{S}_p^{n+p} = \left\{(x_1, x_2, \ldots, x_{n+p+1}) \in \mathbb{R}_p^{n+p+1} : \langle x, x\rangle = 1\right\}.$$

It is not difficult to verify that the induced metric $\langle\, ,\rangle$ makes $\mathbb{S}_p^{n+p}$ a semi-Riemannian manifold of index p with constant sectional curvature equal to 1.

We also recall that an n-dimensional submanifold M^n of $\mathbb{S}_p^{n+p}$ is said to be *spacelike* if the induced metric on M^n from that of the ambient space $\mathbb{S}_p^{n+p}$ is positive definite. So, we choose a local orthonormal frame $e_1, \ldots, e_{n+p}$ in $\mathbb{S}_p^{n+p}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n. Using the standard convention of indices

$$1 \le A, B, C, \ldots \le n+p, \quad 1 \le i, j, k, \ldots \le n \quad \text{and} \quad n+1 \le \alpha, \beta, \gamma, \ldots \le n+p,$$

and taking the corresponding dual coframe $\omega_1, \ldots, \omega_{n+p}$, the semi-Riemannian metric of $\mathbb{S}_p^{n+p}$ is given by $ds^2 = \sum_A \epsilon_A\, \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. So, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{S}_p^{n+p}$, we have that the structure equations of $\mathbb{S}_p^{n+p}$ are given by

$$d\omega_A = \sum_B \epsilon_B\, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

and

$$d\omega_{AB} = \sum_C \epsilon_C\, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D K_{ABCD}\, \omega_C \wedge \omega_D, \tag{2.2}$$

where $K_{ABCD} = \epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$.

Restricting these forms to M^n, we note that $\omega_\alpha = 0$ and, hence, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $\displaystyle\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0$, from Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.3}$$

This gives the second fundamental form of M^n, $A = \displaystyle\sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$ and its squared norm $|A|^2 = \sum_{\alpha,i,j}(h_{ij}^\alpha)^2$. Moreover, the mean curvature vector field and the mean curvature function on M^n are defined, respectively, by

$$h := \frac{1}{n}\sum_\alpha \left(\sum_i h_{ii}^\alpha\right) e_\alpha \qquad \text{and} \qquad H := |h| = \frac{1}{n}\sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha\right)^2}.$$

From (2.1) and (2.2), the structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \ \text{ and } \ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.4}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Hence, from (2.4) we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}). \tag{2.5}$$

The components of the Ricci curvature R_{ij} and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = (n-1)\delta_{ij} - \sum_{\alpha}\left(\sum_{k} h^{\alpha}_{kk}\right) h^{\alpha}_{ij} + \sum_{\alpha,k} h^{\alpha}_{ik}h^{\alpha}_{kj} \quad \text{and} \quad R = \frac{1}{n(n-1)}\sum_{i} R_{ii}. \tag{2.6}$$

From (2.6) we obtain that

$$|A|^2 = n^2H^2 + n(n-1)(R-1). \tag{2.7}$$

We also have the structure equations of the normal bundle of M^n given by

$$d\omega_{\alpha} = -\sum_{\beta}\omega_{\alpha\beta}\wedge\omega_{\beta}, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \text{ and } d\omega_{\alpha\beta} = -\sum_{\gamma}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k\wedge\omega_l,$$

where $R_{\alpha\beta jk}$ satisfies the Ricci equation

$$R_{\alpha\beta ij} = \sum_{l}\left(h^{\alpha}_{il}h^{\beta}_{lj} - h^{\alpha}_{jl}h^{\beta}_{li}\right). \tag{2.8}$$

From (2.3) we obtain the Codazzi equation

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} = h^{\alpha}_{kij}, \tag{2.9}$$

where h^{α}_{ijk} are the components of the covariant derivative ∇A, which satisfy

$$\sum_{k} h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} + \sum_{k} h^{\alpha}_{ik}\omega_{kj} + \sum_{k} h^{\alpha}_{jk}\omega_{ki} - \sum_{\beta} h^{\beta}_{ij}\omega_{\beta\alpha}. \tag{2.10}$$

Taking the exterior derivative in (2.10) we obtain the following Ricci formula for the second fundamental form

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_{m} h^{\alpha}_{mj}R_{mikl} + \sum_{m} h^{\alpha}_{im}R_{mjkl} + \sum_{k,\beta} h^{\beta}_{ik}R_{\alpha\beta jk}. \tag{2.11}$$

From equations (2.9) and (2.11), we get

$$\Delta h^{\alpha}_{ij} = \sum_{k} h^{\alpha}_{kkij} + \sum_{k,l} h^{\alpha}_{kl}R_{lijk} + \sum_{k,l} h^{\alpha}_{li}R_{lkjk} + \sum_{k,\beta} h^{\beta}_{ik}R_{\alpha\beta jk}. \tag{2.12}$$

Considering $H > 0$, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ of $T\mathbb{S}^{n+p}_p$ such that $e_{n+1} = \frac{h}{H}$. Consequently, we get

$$H^{n+1} := \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} := \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2, \tag{2.13}$$

where h^{α} denotes the matrix (h^{α}_{ij}). From equations (2.5), (2.8), (2.12) and (2.13) we obtain the following Simons type formula

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 = &\sum_{\alpha,i,j,k}(h^{\alpha}_{ijk})^2 + n\sum_{\alpha,i,j} h^{\alpha}_{ij}H^{\alpha}_{ij} + n(|A|^2 - nH^2) + \sum_{\alpha,\beta}\big(\mathrm{tr}(h^{\alpha}h^{\beta})\big)^2 \\ &-nH\sum_{\alpha}\mathrm{tr}\left(h^{n+1}(h^{\alpha})^2\right) + \sum_{\alpha,\beta}N\big(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}\big),\end{aligned} \tag{2.14}$$

where $N(B) = \mathrm{tr}(BB^t)$, for all matrix $B = (b_{ij})$.

In what follows, we will also consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j}\Phi^{\alpha}_{ij}\omega_i \otimes \omega_j e_{\alpha}, \tag{2.15}$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij},$$

for $n+2 \leq \alpha \leq n+p$. So, let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^{\alpha}_{ij})^2$ be the square of the length of Φ. It is not difficult to check that Φ is traceless with

$$|\Phi|^2 = |A|^2 - nH^2. \tag{2.16}$$

In addition, from (2.7) we obtain

$$|\Phi|^2 = n(n-1)H^2 + n(n-1)(R-1), \tag{2.17}$$

We recall that a submanifold is *linear Weingarten* (LW) when its mean and normalized scalar curvatures are linearly related, that is, when they satisfy the following relation

$$R = aH + b, \tag{2.18}$$

for constants $a, b \in \mathbb{R}$. We observe that when $a = 0$, (2.18) reduces to R constant. Moreover, equation (2.17) becomes

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)aH + n(n-1)(b-1). \tag{2.19}$$

For a LW submanifold M^n satisfying (2.18) we introduce the second-order linear differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2} a\Delta, \tag{2.20}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M)$ denotes the Cheng-Yau's operator, which is given by

$$Lu = \operatorname{tr}(P \circ \operatorname{Hess}(u)), \tag{2.21}$$

for every $u \in C^\infty(M)$, where Hess is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n which is given by $P = nHI - A$. So, from (2.20) and (2.21), we have that

$$\mathcal{L}u = \operatorname{tr}(\mathcal{P} \circ \operatorname{Hess}(u)), \tag{2.22}$$

with

$$\mathcal{P} = \left(nH + \frac{n-1}{2} a \right) I - A, \tag{2.23}$$

and it is verifies that $\mathcal{L}$ can be rewritten in the following divergence form (see, for instance, [78, Section 4])

$$\mathcal{L}u = \operatorname{div}(\mathcal{P}(\nabla u)). \tag{2.24}$$

Returnig to (2.14), we are going to obtain the following equality, reasoning as [24]:

Remark 2.1.1. *Taking $u = nH$, we get*

$$L(nH) = nH\Delta(nH) - n \sum_{i,j} h_{ij}^{n+1} H_{ij}. \tag{2.25}$$

Given this, from (2.7) and (2.25), we have

$$L(nH) = \frac{1}{2}\Delta|A|^2 - \frac{n(n-1)}{2}\Delta R - n^2|\nabla H|^2 - n \sum_{i,j} h_{ij}^{n+1} H_{ij}. \tag{2.26}$$

Since $R = aH + b$, from (2.14) and (2.26),

$$\begin{aligned} L(nH) \;\; = \;\; & \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + n \sum_{i,j} h_{ij}^{n+1} H_{ij}^{n+1} - n \sum_{ij} h_{ij}^{n+1} H_{ij} - n^2|\nabla H|^2 \\ & + \sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) + n(|A|^2 - nH^2) + n \sum_{\alpha > n+1} \sum_{i,j} h_{ij}^\alpha H_{ij}^\alpha \\ & + \sum_{\alpha\beta} (tr(h^\alpha h^\beta))^2 - \frac{n-1}{2} a\Delta(nH). \end{aligned} \tag{2.27}$$

In the theorems of this chapter, we will deal with spacelike submanifolds M^n having parallel normalized mean curvature vector field. In this context, we are going to rewrite (2.27). For this, we choose a local orthonormal frame $\{e_i\}$ such that $e_{n+1} = \frac{h}{H}$. Since e_{n+1} is parallel, denoting *by $\nabla^\perp$ the normal connection of M^n, it follows that*

$$0 = \nabla^\perp e_{n+1} = \sum_\alpha \omega_{\alpha n+1} e_\alpha.$$

Thus,

$$\omega_{\alpha n+1} = 0 \quad \textit{for all} \quad \alpha > n+1. \tag{2.28}$$

On the other hand, taking into account equation (2.10)*, we have*

$$\sum_{i,k} h^\alpha_{ijk}\omega_k = \sum_i dh^\alpha_{ii} + 2\sum_{i,k} h^\alpha_{ik}\omega_{ki} - \sum_{i,\beta} h^\beta_{ii}\omega_{\beta\alpha}. \tag{2.29}$$

Considering $\alpha = n+1$, from (2.28) *and* (2.29)*, it follows that*

$$\sum_k H^{n+1}_k \omega_k = dH. \tag{2.30}$$

Besides that, given a smooth function f on M^n, the first and second derivatives are given by

$$df = \sum_i f_i\omega_i \quad \textit{and} \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}.$$

So, from (2.30)*, we get $H_k = H^{n+1}_k$. When $\alpha > n+1$, from* (2.28) *and* (2.29)*, we have that $\sum_k H^\alpha_k\omega_k = -H\omega_{n+1\alpha} = 0$ and, hence, $H^\alpha_k = 0$. Making an analysis of the covariant derivative H^α_{kl}, from* (2.28)*, we also have*

$$\sum_l H^{n+1}_{kl}\omega_l = dH_k + \sum_l H_l\omega_{lk}$$

and we obtain $H_{kl} = H^{n+1}_{kl}$. Moreover, in the case that $\alpha > n+1$, from (2.28)*, we have $\sum_l H^\alpha_{kl}\omega_l = -H_k\omega_{n+1\alpha} = 0$. Hence, $H^\alpha_{kl} = 0$.*

As a by-product of the previous digression, replacing $H_{kl} = H^{n+1}_{kl}$ and $H^\alpha_{kl} = 0$, for $\alpha > n+1$ in (2.27)*, we conclude that*

$$\begin{aligned} L(nH) \;\; = \;\; & \sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2 - n^2|\nabla H|^2 - nH\sum_\alpha \operatorname{tr}(h^{n+1}(h^\alpha)^2) + \sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) \\ & + \sum_{\alpha\beta} (\operatorname{tr}(h^\alpha h^\beta))^2 + n(|A|^2 - nH^2) - \frac{n-1}{2}a\Delta(nH). \end{aligned} \tag{2.31}$$

We can now establish the following proposition that gives a sufficient criteria for the ellipticity of the operator $\mathcal{L}$, whose proof can be found in [85, Proposition 2.1] for $a \neq 0$ and [10, Lemma 1] for $a = 0$.

Proposition 2.1.2. *Let M^n be a n-dimensional spacelike linear Weingarten submanifold in the de Sitter space $\mathbb{S}^{n+p}_p$ with $R = aH + b$. If $b < 1$, then $\mathcal{L}$ is elliptic.*

Next, we introduce the following propositon, whose proof can be found in [85, Proposition

2.2], which gives us an inequality between the covariant derivative of the second fundamental form and the gradient of the mean curvature.

Proposition 2.1.3. *Let M^n be an n-dimensional spacelike linear Weingarten submanifold in the de Sitter space $\mathbb{S}_p^{n+p}$ with $R = aH + b$. If $(n-1)a^2 + 4n(1-b) \geq 0$, then*

$$\sum_{\alpha,i,j,k} (h_{ijk}^{\alpha}) \geq n^2 |\nabla H|^2.$$

Moreover, suppose that the equality holds, then H is constant on M^n.

Besides that, we will also need of the next key lemma, which is due to Barros et al. (see [28, Lemma 1]).

Lemma 2.1.4. *Let M^n be a Riemannian manifold isometrically immersed into a Riemannian manifold N^{n+p}. Consider $\Psi = \sum_{\alpha,i,j} \Psi_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_{\alpha}$ a traceless symmetric tensor satisfying the Codazzi equation. Then the following inequality holds*

$$|\nabla |\Psi|^2|^2 \leq \frac{4n}{n+2} |\Psi|^2 |\nabla \Psi|^2,$$

where $|\Psi|^2 = \sum_{\alpha,i,j} (\Psi_{ij}^{\alpha})^2$ and $|\nabla \Psi|^2 = \sum_{\alpha,i,j,k} (\Psi_{ijk}^{\alpha})^2$. In particular, the conclusion holds for the tensor Φ defined in (2.15).

Lastly, we also need the next algebraic lemma presented in [79, Lemma 2.6].

Lemma 2.1.5. *Let $C, D : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $[C, D] = CD - DC = 0$ and $trC = trD = 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}} (trC^2)(trD^2)^{\frac{1}{2}} \leq trC^2 D \leq \frac{n-2}{\sqrt{n(n-1)}} (trC^2)(trD^2)^{\frac{1}{2}},$$

and the equality holds on the right hand side if and only if $n-1$ of the eigenvalues x_i of C and the corresponding eigenvalues y_i of D satisfy

$$|x_i| = \frac{(trC^2)^{\frac{1}{2}}}{\sqrt{n(n-1)}}, \quad x_i x_j \geq 0, \quad y_i = \frac{(trD^2)^{\frac{1}{2}}}{\sqrt{n(n-1)}}.$$

2.1.2 Rigidity results for complete spacelike LW submanifolds with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$

In this subsection we will establish our initial rigidity results. Here, it is the first one:

Theorem 2.1.6. *Let M^n $(n \geq 3)$ be a complete noncompact spacelike LW submanifold having nonnegative sectional curvature and immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$. If $|\Phi|$ converges to zero at infinity, then M^n is isometric to the Euclidean space $\mathbb{R}^n$.*

Proof. Let us suppose that such a spacelike LW submanifold M^n is not a totally umbilical submanifold, and we consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, f is a non-identically vanishing function which converges to zero at infinity. Moreover, Proposition 2.1.2 gives that $\mathcal{P}$ is positive definite for $b < 1$. Thus

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0. \tag{2.32}$$

In order to apply Lemma 1.0.4, we claim that $\mathrm{div}X \geq 0$. Indeed, applying $\mathcal{L}$ in (2.19) we have that

$$\begin{aligned}\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) + \frac{a}{2}\mathcal{L}(nH)\\ &= H\mathcal{L}(nH) + n\langle \mathcal{P}\nabla H, \nabla H\rangle + \frac{a}{2}\mathcal{L}(nH).\end{aligned} \tag{2.33}$$

In particular, since $\mathcal{P}$ is positive definite, from (2.33) we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H + \frac{a}{2})\mathcal{L}(nH). \tag{2.34}$$

From Ricci equation (2.8) we can verify that

$$\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk} = \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha). \tag{2.35}$$

Thus, since we are assuming that that the normalized mean curvature vector is parallel, from (2.12), (2.20), (2.31) and (2.35), we get

$$\begin{aligned}\mathcal{L}(nH) &= \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 - n^2|\nabla H|^2 + \sum_{i,j,k,m}\big(h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk}\big)\\ &\quad + \frac{1}{2}\sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big).\end{aligned} \tag{2.36}$$

For each fixed α, considering a local orthonormal frame $\{e_i\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, we have

$$\sum_{i,j,k,m} h^\alpha_{ij}h^\alpha_{km}R_{mijk} + \sum_{i,j,k,m} h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} \geq \frac{1}{2}\sum_{i,j}(\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij}. \tag{2.37}$$

Moreover, it is not difficult to verify that

$$N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) = N\big(\Phi^\alpha\Phi^\beta - \Phi^\beta\Phi^\alpha\big) \geq 0. \tag{2.38}$$

On the other hand, using once more that $b < 1$, from Proposition 2.1.3 we obtain

$$|\nabla A|^2 = \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 \geq n^2|\nabla H|^2. \tag{2.39}$$

Hence, taking into account that the sectional curvature of M^n is nonnegative, from (2.36), (2.37), (2.38) and (2.39) we reach at

$$\mathcal{L}(nH) \geq \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha - \lambda_j^\alpha)^2 R_{ijij} \geq 0. \tag{2.40}$$

At this point, we observe that in [11] it was verified that $(H + \frac{a}{2}) \geq 0$. Thus, from (2.34) and (2.40), we have

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)\mathcal{L}(nH) \geq 0.$$

Hence, we can apply Lemma 1.0.4 to get that

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle = 0.$$

Therefore, since the operator $\mathcal{P}$ is positive definite, we have that $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant. But f converges to zero at infinity, so it must be identically zero, leading us to a contradiction since we are supposing that M^n is not a totally umbilical submanifold.

Now, taking into account (2.13), we get

$$h^\alpha = \langle H, e_\alpha\rangle I = H^\alpha I = 0,$$

for all $\alpha > n+1$. Thus, we have that the first normal subspace,

$$N_1 = \left\{e_\alpha \in \mathfrak{X}^\perp(M^n); h^\alpha = 0\right\}^\perp,$$

is parallel and it has dimension 1. Therefore, we can apply [38, Proposition 4.1] to reduce the codimension of M^n to 1. Thus, from the characterizations of the totally umbilical spacelike hypersurfaces of the de Sitter space (see, for instance, [71]), we conclude that M^n must be isometric to $\mathbb{R}^n$, since M^n is a complete noncompact submanifold with nonnegative sectional curvature. □

We obtain the following consequence of Theorem 2.1.6.

Corollary 2.1.7. *Let M^n be a complete noncompact spacelike submanifold having nonnegative sectional curvature and constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$. If $|\Phi|$ converges to zero at infinity, then M^n is isometric to the Euclidean space $\mathbb{R}^n$.*

In the next theorem, we are going to establish a new version of [85, Theorem 1.4] and of [66, Theorem 1.1].

Theorem 2.1.8. *Let M^n be a complete noncompact spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$*

with $b < 1$. If $|A|^2 \leq 2\sqrt{n-1}$ and $|\Phi|$ converges to zero at infinity, then M^n is isometric to either the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$.

Proof. From (2.34) we have

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H + \frac{a}{2})\mathcal{L}(nH). \tag{2.41}$$

On the other hand, from inequality (3.19) of [85] jointly with relation (2.16) we get

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(n - \frac{n}{2\sqrt{(n-1)}}|A|^2\right). \tag{2.42}$$

Since we are assuming $|A|^2 \leq 2\sqrt{n-1}$, from (2.42) we have that $\mathcal{L}(nH) \geq 0$. Then, from (2.41) we obtain, for $X = \mathcal{P}(\nabla|\Phi|^2)$,

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)\mathcal{L}(nH) \geq 0.$$

At this point, we can reason as in the last part of the proof of Theorem 2.1.6 to conclude that M^n is a totally umbilical submanifold of $\mathbb{S}_p^{n+p}$ and, reducing the codimension of M^n to 1, we infer that M^n must be isometric to either the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$. $\square$

When the spacelike submanifold has constant normalized scalar curvature, Theorem 2.1.8 reads as follows.

Corollary 2.1.9. *Let M^n be a complete noncompact spacelike submanifold with constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$. If $|A|^2 \leq 2\sqrt{n-1}$ and $|\Phi|$ converges to zero at infinity, then M^n is isometric to either the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $\mathbb{H}^n$.*

Before presenting our next results, in the remark below we collect some properties related to a suitable polynomial function, which will appear in their proofs.

Remark 2.1.10. Let it be the following one-parameter family of real functions given by

$$P_{t,p}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}tx - n(t^2-1) \tag{2.43}$$

where $t \in \mathbb{R}$ corresponds to the real parameter, while n and p are real constants. When $t^2 < \frac{4(n-1)}{Q(p)}$, where

$$Q(p) = p(n-2)^2 + 4(n-1), \tag{2.44}$$

we have $P_{t,p}(x) > 0$ for all $x \in \mathbb{R}$. In the case $t^2 = \frac{4(n-1)}{Q(p)}$, we can write $t = \frac{2\sqrt{n-1}}{\sqrt{Q(p)}}$ and $P_{t,p}(x)$ has only one real root, namely

$$\vartheta(n,p) = \frac{p(n-2)\sqrt{n}}{\sqrt{Q(p)}}. \tag{2.45}$$

In this case, $P_{t,p}(x)$ is strictly decreasing for all $x \leq \vartheta(n,p)$ and

$$P_{t,p}(x) = \left(\frac{x}{\sqrt{p}} - \frac{(n-2)\sqrt{np}}{\sqrt{Q(p)}} \right)^2 \geq 0,$$

for all $x \in \mathbb{R}$.

Assuming $t^2 \leq \frac{4(n-1)}{Q(p)}$, from (2.17) we also have that

$$R = \frac{|\Phi|^2}{n(n-1)} - t^2 + 1 \geq \frac{-4(n-1)}{Q(p)} + 1 \geq 0. \tag{2.46}$$

When $t^2 > \dfrac{4(n-1)}{Q(p)}$, $P_t(x)$ has two distinct real roots, which are given by

$$\vartheta^{\pm}(n,p,t) = \frac{\sqrt{n}}{2\sqrt{n-1}} \left(p(n-2)t \pm \sqrt{pQ(p)t^2 - 4p(n-1)} \right). \tag{2.47}$$

We observe that $\vartheta^{+}(n,p,t)$ is always positive and $\vartheta^{-}(n,p,t)$ is positive if, and only if,

$$\frac{4(n-1)}{Q(p)} \leq t^2 < 1.$$

Taking into account the discussion made in Remark 2.1.10, we will prove our next result.

Theorem 2.1.11. *There does not exist a complete noncompact spacelike LW submanifold M^n with $n \geq 3$ immersed with parallel normalized mean curvature vector field in $\mathbb{S}^{n+p}_p$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$, where $|\Phi|$ converges to zero at infinity and*

$$H^2 \leq \frac{4(n-1)}{Q(p)} \tag{2.48}$$

on M^n, where Q is defined in (2.44).

Proof. Let us assume by contradiction that there exists such a submanifold and let us consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. Suppose that M^n is not a umbilical submanifold. So, f is non-identically vanishing function which converges to zero at infinity. Moreover, we already know from (2.32) that

$$\langle \nabla f, X \rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2) \rangle \geq 0.$$

Taking into account the Ricci equation (2.8) we can verify that $h^{\alpha}h^{n+1} = h^{n+1}h^{\alpha}$ for all α, that is, h^{n+1} commutes with all the matrices h^{α}. Thus, since

$$\Phi^{n+1} = h^{n+1} - H^{n+1}I \quad \text{and} \quad \Phi^{\alpha} = h^{\alpha} \quad \text{for all} \quad \alpha > n+1,$$

we also have that Φ^{n+1} commutes with all the matrices Φ^{α}. Thus, taking into account that the matrices Φ^{α} are symmetric and traceless, we can use Lemma 2.1.5 for Φ^{α} and Φ^{n+1} in order to obtain

$$\left|\mathrm{tr}((\Phi^\alpha)^2\Phi^{n+1})\right| \leq \frac{n-2}{\sqrt{n(n-1)}} N(\Phi^\alpha)\sqrt{N(\Phi^{n+1})}. \tag{2.49}$$

On the other hand, using the Cauchy-Schwarz inequality we get that

$$p\sum_{\alpha,\beta}[\mathrm{tr}(\Phi^\alpha\Phi^\beta)]^2 \geq p\sum_\alpha[\mathrm{tr}(\Phi^\alpha)^2]^2 = p\sum_\alpha[N(\Phi^\alpha)]^2 \geq \left(\sum_\alpha N(\Phi^\alpha)\right)^2 = |\Phi|^4. \tag{2.50}$$

Furthermore, from (2.31), it follows that

$$\begin{aligned}\mathcal{L}(nH) = &\sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 - n^2|\nabla H|^2 - nH\sum_\alpha \mathrm{tr}\left(h^{n+1}(h^\alpha)^2\right) + \sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) \\ &+ \sum_{\alpha,\beta}\left(\mathrm{tr}(h^\alpha h^\beta)\right)^2 + n(|A|^2 - nH^2).\end{aligned} \tag{2.51}$$

Hence, since

$$N(\Phi^{n+1}) = \mathrm{tr}(\Phi^{n+1})^2 \leq |\Phi|^2 \ \text{ and } \ \sum_\alpha N(\Phi^\alpha) = |\Phi|^2,$$

from (2.20), (2.49), (2.50) and (2.51) we obtain

$$\mathcal{L}(nH) \geq |\Phi|^2 P_{H,p}(|\Phi|), \tag{2.52}$$

where $P_{H,p}(x)$ is defined in (2.43). Consequently, since $H^2 \leq \dfrac{4(n-1)}{Q(p)}$, from (2.52) and Remark 2.1.10, we get that

$$\mathcal{L}(nH) \geq |\Phi|^2 P_{H,p}(|\Phi|) \geq 0. \tag{2.53}$$

Since $(H + \frac{a}{2}) > 0$, using (2.34) jointly with (2.53) we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)|\Phi|^2 P_{H,p}(|\Phi|) \geq 0.$$

Consequently, we can apply Lemma 1.0.4 to get that

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2\rangle = 0,$$

which implies $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant and, since f converges to zero at infinity, it must be identically zero, leading us to a contradiction and M^n must be a totally umbilical submanifold of $\mathbb{S}^{n+p}_p$.

Now, taking into account (2.13), we get

$$h^\alpha = \langle H, e_\alpha\rangle I = H^\alpha I = 0,$$

for all $\alpha > n+1$. Thus, we have that the first normal subspace

$$N_1 = \left\{e_\alpha \in \mathfrak{X}^\perp(M^n); h^\alpha = 0\right\}^\perp$$

is parallel and it has dimension 1. Therefore, we can apply [38, Proposition 4.1] to reduce the codimension of M^n to 1. Thus, from the characterizations of the totally umbilical hypersurfaces of the de Sitter space, M^n is isometric to either the Euclidean space $\mathbb{R}^n$ with $H = 1$ or the hyperbolic space $\mathbb{H}^n$ with $H \in (1, \infty)$, what cannot occur since $H^2 \leq \dfrac{4(n-1)}{Q(p)} < 1$ for $n \geq 3$. $\square$

From Theorem 2.1.11 we obtain the following.

Corollary 2.1.12. *There does not exist a complete noncompact spacelike submanifold M^n with $n \geq 3$ and constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$ such that inequality* (2.48) *holds and $|\Phi|$ converges to zero at infinity.*

We can study the case $n = 2$ and proceed as the proof of the Theorem 2.1.11 to conclude that the submanifold M^2 can be isometric to the Euclidean space and establish the following theorem.

Theorem 2.1.13. *Let M^2 be a complete noncompact spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{2+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$ and suppose that $H^2 \leq 1$. If $|\Phi|$ converges to zero at infinity, then M^2 is isometric to the Euclidean space $\mathbb{R}^2$.*

We also get a version of Corollary 2.1.12 for $n = 2$.

Corollary 2.1.14. *Let M^2 be a complete noncompact spacelike LW submanifold immersed with constant normalized scalar curvature $R < 1$ in $\mathbb{S}_p^{2+p}$, such that $H^2 \leq 1$. If $|\Phi|$ converges to zero at infinity, then M^2 is isometric to the Euclidean space $\mathbb{R}^2$.*

Moving foward, we present the following non-existence result.

Theorem 2.1.15. *There does not exist a complete noncompact spacelike LW submanifold M^n immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$, where $|\Phi|$ converges to zero at infinity and*

$$\frac{4(n-1)}{Q(p)} < H^2 < 1 \quad \textit{and} \quad |\Phi| \leq \vartheta^-(n, p, H), \tag{2.54}$$

in which ϑ^- is the real root of P_H given by (2.47).

Proof. Let us assume for contradiction that there exists such a submanifold. As it was observed in Remark 2.1.10, hypotheses (2.54) guarantees that $P_H(x)$ defined in (2.43) has two distinct real roots, which are given by (2.47). Moreover, from our constraint on H, we also get that $\vartheta^-(n, p, H)$ is positive. Consequently, we conclude that $P_H(|\Phi|) \geq 0$ for $|\Phi| \leq \vartheta^-(n, p, H)$. So, using this fact jointly with (2.34) and (2.52), and taking into account that $(H + \frac{a}{2}) \geq 0$, we obtain

$$\text{div}X = \text{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)|\Phi|^2 P_H(|\Phi|) \geq 0.$$

Therefore, we can reason as in the proof of Theorem 2.1.8 to conclude that there does not exist such manifolf M^n since $H^2 < 1$. $\square$

Considering the case of constant normalized scalar curvature, we get the following corollary from Theorem 2.1.15.

Corollary 2.1.16. *There does not exist a complete noncompact spacelike submanifold M^n with constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$ such that inequalities* (2.54) *hold and $|\Phi|$ converges to zero at infinity.*

Before we continue, let us look at the following:

Remark 2.1.17. *For a linear Weingarten submanifold with $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$, we have from* (2.19) *that*

$$nH(nH + (n-1)a) = |A|^2 + n(n-1)(1-b) \geq n(n-1)(1-b) > 0.$$

In particular, H is far away from 0 and so $nH + (n-1)a \geq 0$, if $a \geq 0$. Thus,

$$(H + \frac{a}{2}) \geq \beta,$$

for a constant $\beta > 0$. If $a < 0$, we have

$$H + \frac{a}{2} > H + \frac{n-1}{n}a = \frac{nH + (n-1)a}{n} \geq 0$$

and we also obtain $(H + \frac{a}{2}) \geq \beta > 0$.

Now, we are in position to establish our next rigidity result.

Theorem 2.1.18. *Let M^n be a complete spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$. Suppose that $|\nabla\Phi|$ is bounded and that $\sup_M |A|^2 < 2\sqrt{n-1}$. If M^n has polynomial volume growth, then M^n is isometric to the Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proof. Taking the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$, we claim that the required conditions to apply Lemma 1.0.3 are satisfied. Indeed, since H and $|A|$ are bounded (see (2.19)), from definition (2.23) we get

$$|X| = |\mathcal{P}(\nabla|\Phi|^2)| \leq |\mathcal{P}||\nabla|\Phi|^2| \leq k|\nabla|\Phi|^2|,$$

for some positive constant k. Besides that, the boundedness of H and $|A|$ also assure the boundedness of $|\Phi|$ by equation (2.19). So, since we are supposing that $|\nabla\Phi|$ is bounded, Lemma 2.1.4 guarantees that $\nabla|\Phi|^2$ is also bounded and, consequently,

$$|X| \leq C < +\infty,$$

for some positive constant C.

On the other hand, from (2.32) we have that the condition

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0$$

is also verified.

Now, we must obtain $\mathrm{div}X \geq \alpha f$ on M^n, for some positive constant $\alpha \in \mathbb{R}$. For this, from inequality (3.19) of [85] jointly with relation (2.19) we get

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(n - \frac{n}{2\sqrt{(n-1)}}|A|^2 \right). \tag{2.55}$$

Thus, using (2.34), from (2.24) and (2.55), we have

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla H)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)\left(n - \frac{n}{2\sqrt{(n-1)}}|A|^2 \right)|\Phi|^2. \tag{2.56}$$

Since we are assuming $\sup(|A|^2) < 2\sqrt{n-1}$ and, by Remark 2.1.17, $(H + \frac{a}{2}) \geq \beta$, for some positive constant $\beta \in \mathbb{R}$, from (2.56) we obtain

$$\mathrm{div}(X) \geq \alpha|\Phi|^2,$$

where $\alpha = 2(n-1)\beta\left(n - \frac{n}{2\sqrt{(n-1)}}\sup(|A|^2) \right) > 0$. Therefore, in case that M^n is a complete noncompact submanifold, we are able to apply Lemma 1.0.3 to obtain that $|\Phi|^2 \leq 0$ and, hence, $|\Phi|^2 = 0$ on M^n, guaranteeing that M^n is totally umbilical.

In the case that M^n is a compact submanifold, we can integrate both sides of (2.55) and use Divergence Theorem to get that

$$\int_M |\Phi|^2 \left(n - \frac{n}{2\sqrt{(n-1)}}|A|^2 \right) \mathrm{dM} \leq \int_M \mathcal{L}(nH)\, \mathrm{dM} = 0,$$

since the operator $\mathcal{L}$ is a divergence type as it was observed in (2.24). Hence, as we are assuming that $\sup(|A|^2) < 2\sqrt{n-1}$, we must have $|\Phi| = 0$ on M^n.

In both cases, we conclude that M^n is a totally umbilical submanifold of $\mathbb{S}_p^{n+p}$. Taking into account (2.13), we get

$$h^\alpha = \langle H, e_\alpha\rangle I = H^\alpha I = 0,$$

for all $\alpha > n+1$. Thus, we have that the first normal subspace,

$$N_1 = \left\{ e_\alpha \in \mathfrak{X}^\perp(M^n); h^\alpha = 0 \right\}^\perp,$$

is parallel and it has dimension 1. Therefore, we can apply once more [38, Proposition 4.1] to reduce the codimension of M^n to 1. Thus, from the characterizations of the totally umbilical hypersurfaces of the de Sitter space, we conclude that M^n is isometric to either the Euclidean space $\mathbb{R}^n$, in the noncompact case, or the Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$, in compact case. $\square$

The next corollary is derived from Theorem 2.1.18.

Corollary 2.1.19. *Let M^n be a complete spacelike submanifold with constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$. Suppose that $|\nabla\Phi|$ is bounded and that $\sup_M |A|^2 < 2\sqrt{n-1}$. If M^n has polynomial volume growth, then M^n is isometric to either the Euclidean space $\mathbb{R}^n$ or the Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proceeding, we obtain the following rigidity result.

Theorem 2.1.20. *Let M^n be a complete spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$. Suppose that $\sup_M |\Phi| < \vartheta(n,p)$ and that*

$$H^2 \leq \frac{4(n-1)}{Q(p)}, \tag{2.57}$$

where Q and $\vartheta(n,p)$ are defined in (2.44) and (2.45), respectively. If M^n has polynomial volume growth, then M^n is isometric to the Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$ when $n \geq 3$, or isometric to either the Euclidean space $\mathbb{R}^2$ or the Euclidean sphere $\mathbb{S}^2(r)$ with radius $r > 0$, when $n = 2$.

Proof. Reasoning as in the proof of Theorem 2.1.18, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, we have that

$$|X| \leq C, \tag{2.58}$$

for some positive constant $C \in \mathbb{R}$, and

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0. \tag{2.59}$$

Moreover, from (2.34) and (2.52) we get

$$\operatorname{div}X = \operatorname{div}(\mathcal{P}(\nabla H)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right) P_H(|\Phi|)|\Phi|^2. \tag{2.60}$$

Now, let us take $\gamma := \dfrac{4(n-1)}{Q(p)}$. As we are supposing (2.57), we have

$$P_H(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n\left(H^2 - 1\right) \geq \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}\sqrt{\gamma}x - n\left(\gamma - 1\right) = P_\gamma(x). \tag{2.61}$$

Besides that, taking into account (2.61) and the behavior of $P_{H,p}(x)$ described in Remark 2.1.10, for $\sup_M(|\Phi|) < \vartheta(n,p)$ we have that

$$P_H(|\Phi|) \geq P_\gamma(|\Phi|) \geq P_\gamma(\sup_M |\Phi|) > P_\gamma(\vartheta(n,p)) = 0. \tag{2.62}$$

Hence, since $H+\frac{a}{2} \geq \beta > 0$ from Remark 2.1.17, for some positive constant $\beta \in \mathbb{R}$, from (2.60) and (2.62) we obtain

$$\operatorname{div} X \geq 2(n-1)\left(H + \frac{a}{2}\right) P_H(|\Phi|)|\Phi|^2 \geq \alpha f, \tag{2.63}$$

where $\alpha = 2(n-1)\beta P_\gamma(\sup_M(|\Phi|)) > 0$.

Supposing that M^n is a complete noncompact submanifold, since (2.58), (2.59) and (2.63) were verified and M^n has polynomial volume growth, we are able to apply Lemma 1.0.3 to obtain that $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$ and M^n is totally umbilical submanifold. In case M^n is a compact submanifold, we can apply once more Divergence Theorem to infer that M^n is also totally umbilical submanifold.

Therefore, reasoning as in the last part of the proof of Theorem 2.1.18, we can reduce the codimension of M^n to 1 and conclude that it must be isometric to the Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$ since $H^2 \leq \dfrac{4(n-1)}{Q(p)} < 1$ for $n \geq 3$. Moreover, as we have $\dfrac{4(n-1)}{Q(p)} = 1$ for $n = 2$, we get $H^2 \leq 1$ and, in this case, M^2 is isometric to either the Euclidean space $\mathbb{R}^2$ or the Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$. □

Theorem 2.1.20 gives the following particular case.

Corollary 2.1.21. *Let M^n be a complete spacelike submanifold with constant normalized scalar curvature $R < 1$, immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that H is bounded away from zero. Suppose that $\sup_M |\Phi| < \vartheta(n,p)$, where $\vartheta(n,p)$ is defined in* (2.45)*, and that inequality* (2.57) *is satisfied. If M^n has polynomial volume growth, then M^n is isometric to the Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$ when $n \geq 3$, or isometric to either the Euclidean space $\mathbb{R}^2$ or the Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$ when $n = 2$.*

In our last rigidity result of this section, we will present a new characterization of Theorem 2.1.15, dealing with complete spacelike LW submanifolds immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$.

Theorem 2.1.22. *Let M^n be a complete spacelike LW submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < 1$. Suppose that*

$$\frac{4(n-1)}{Q(p)} < H^2 < 1 \quad \textit{and} \quad \sup(|\Phi|) < \vartheta^-(n,p,H), \tag{2.64}$$

where ϑ^- is the real root of P_H given by (2.47)*. If M^n has polynomial volume growth, then M^n is isometric to the Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proof. As in the proof of Theorem 2.1.20, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, we have that $|X| \leq C$, for some positive constant $C \in \mathbb{R}$,

$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0$ and

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right) P_H(|\Phi|)|\Phi|^2. \tag{2.65}$$

From hypothesis (2.64), we saw in Remark 2.1.10 that the polynomial $P_H(x)$ defined in (2.43) has two distinct real roots, which are positive and given by (2.47). So, taking $\gamma := \sup_M(H)$, we have

$$P_H(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n\left(H^2-1\right) \geq \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}\sqrt{\gamma}x - n\left(\gamma-1\right) = P_\gamma(x)$$

and, since $\sup_M(|\Phi|) < \vartheta^-(n,p,H)$ and $P_{H,p}(x)$ is strictly decreasing for $x \leq \vartheta^-(n,p,H)$, we have that

$$P_H(|\Phi|) \geq P_\gamma(|\Phi|) \geq P_\gamma(\sup_M|\Phi|) > P_\gamma(\vartheta^-(n,p,\gamma)) = 0.$$

Using this fact jointly with $(H+\frac{a}{2}) \geq \beta > 0$, for some positive constant $\beta \in \mathbb{R}$, and (2.65), we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right) P_H(|\Phi|)|\Phi|^2 \geq \alpha|\Phi|^2,$$

where $\alpha = 2(n-1)\beta P_\gamma(\sup_M|\Phi|) > 0$.

Therefore, proceeding as the proof of Theorem 2.1.20 we conclude that M^n is isometric to the Euclidean sphere $\mathbb{S}^n(r)$ with radius $r > 0$. □

2.2 Spacelike submanifolds with second fundamental form locally timelike

The aim of this section is to investigate the nonexistence and umbilicity of n-dimensional ($n \geq 3$) spacelike submanifolds immersed with parallel mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$.

In the first part of this section, we show that there does not exist an n-dimensional complete spacelike submanifold M^n immersed with parallel mean curvature vector, whose the second fundamental form is locally timelike in $\mathbb{S}_q^{n+p}$ and the mean curvature H satisfies $\frac{4(n-1)}{Q(p)} < H^2 < 1$, where $Q(x) = (n-2)^2x + 4(n-1)$, such that either $|\nabla\Phi|$ is bounded and M^n has polynomial volume growth or M^n is noncompact and $|\Phi|$ converges to zero at infinity (see Theorem 2.2.2). Afterwards, we show that a complete noncompact submanifold of $\mathbb{S}_p^{n+p}$ with $H^2 = \frac{4(n-1)}{Q(p)}$ and such that $|\Phi|$ converges to zero at infinity, must be a totally umbilical submanifold (see Theorem 2.2.3). Next, we suppose that the spacelike submanifold M^n is stochastically complete in order to show that if $H^2 < 1$, then either M^n is totally umbilical or $\sup_M|\Phi| \geq \vartheta_H^+$, where ϑ_H^+ is the positive root of the polynomial $P_{H,q}(x)$ defined in (2.43) (see Theorem 2.2.4). Finally, we prove that the only n-dimensional stochastically complete spacelike submanifold immersed in $\mathbb{S}^{n+p}_q$, which are maximal and having locally timelike second fundamental form, are the totally geodesic ones (see Theorem 2.2.6). Our approach is based on a Simon's type inequality involving the norm of the total umbilicity tensor, obtained by Mariano in [69].

2.2.1 Preliminaries

Let M^n be an n-dimensional (connected) spacelike submanifold isometrically immersed into the de Sitter space $\mathbb{S}^{n+p}_q$ of index $1 \leq q \leq p$, meaning that the induced metric on M^n via immersion is a Riemannian metric. In this setting, we choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{S}^{n+p}_q$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n. We use the following convention of indices:

$$1 \leq A, B, C, \dots \leq n+p, \quad 1 \leq i, j, k, \dots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

Let $\{\omega_1, \dots, \omega_{n+p}\}$ be the dual frame of $\{e_1, \dots, e_{n+p}\}$, so that the semi-Riemannian metric of $\mathbb{S}^{n+p}_q$ is given by $ds^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_A = 1$, if $1 \leq A \leq n+p-q$, and $\epsilon_A = -1$, if $n+p-q+1 \leq A \leq n+p$. Denoting by $\{\omega_{AB}\}$ the connection 1-forms of $\mathbb{S}^{n+p}_q$, we have that the structure equations of $\mathbb{S}^{n+p}_q$ are given by

$$d\omega_A = -\sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \epsilon_B \omega_{AB} + \epsilon_A \omega_{BA} = 0, \quad \text{for all} \quad 1 \leq A, B \leq n+p, \tag{2.66}$$

and

$$d\omega_{AB} = -\sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \, \omega_C \wedge \omega_D, \tag{2.67}$$

where $K_{ABCD} = \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$.

Restricting those forms to M^n, we note that $\omega_\alpha = 0$ for $n+1 \leq \alpha \leq n+p$ and, hence, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0$, from Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.68}$$

This gives the second fundamental form of M^n, $A = \sum_{\alpha,i,j} \epsilon_\alpha h^\alpha_{ij} \omega_i \omega_j e_\alpha$ and the square of its length $|A|^2 = \left| \sum_\alpha \epsilon_\alpha \sum_{i,j} (h^\alpha_{ij})^2 \right|$. Moreover, we define the mean curvature vector field and the mean curvature function on M^n, respectively, by

$$h := \frac{1}{n} \sum_\alpha \left(\sum_i h^\alpha_{ii} \right) e_\alpha \qquad \text{and} \qquad H := |h| = \frac{1}{n} \sqrt{\sum_\alpha \epsilon_\alpha \left(\sum_i h^\alpha_{ii} \right)^2}.$$

In particular, M^n is called maximal when its mean curvature vector h vanishes identically.

From (2.66) and (2.67), we get the structure equations of M^n

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \text{ and } d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.69}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Therefore, from (2.69) we obtain the Gauss equation

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha \epsilon_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}).$$

The components of the Ricci curvature R_{ij} and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = (n-1)\delta_{ij} + \sum_\alpha \epsilon_\alpha \left\{ \left(\sum_k h^\alpha_{kk} \right) h^\alpha_{ij} - \sum_{\alpha,k} h^\alpha_{ik}h^\alpha_{kj} \right\}$$

and

$$R = n(n-1) + \sum_\alpha \epsilon_\alpha \left(\sum_i h^\alpha_{ii} \right)^2 - \sum_\alpha \sum_{i,j} \epsilon_\alpha (h^\alpha_{ij})^2.$$

We also have the structure equations of the normal bundle of M^n given by

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \text{ and } d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k \wedge \omega_l,$$

where the components $R_{\alpha\beta jk}$ satisfy the Ricci equation

$$R_{\alpha\beta ij} = \sum_l \left(h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li} \right).$$

Moreover, from (2.68) we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{kij}, \tag{2.70}$$

where h^α_{ijk} are the components of the covariant derivative ∇A, which satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_k h^\alpha_{jk}\omega_{ki} + \sum_\beta \epsilon_\beta\epsilon_\alpha h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.71}$$

Taking the exterior derivative in (2.71) we obtain the following Ricci formula for the second fundamental form

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{mj}R_{mikl} - \sum_m h^\alpha_{im}R_{mjkl} + \sum_{k,\beta} \epsilon_\beta\epsilon_\alpha h^\beta_{ik}R_{\alpha\beta jk}. \tag{2.72}$$

The Laplacian Δh_{ij}^{α} of the components h_{ij}^{α} of second fundamental form is defined by

$$\Delta h_{ij}^{\alpha} := \sum_{k} h_{ijkk}^{\alpha}.$$

Therefore, from equations (2.70) and (2.72) we get the following formula

$$\Delta h_{ij}^{\alpha} = \sum_{k} h_{kkij}^{\alpha} - \sum_{k,l} h_{kl}^{\alpha} R_{lijk} - \sum_{k,l} h_{li}^{\alpha} R_{lkjk} + \sum_{k,\beta} \epsilon_{\beta}\epsilon_{\alpha} h_{ik}^{\beta} R_{\alpha\beta jk}.$$

We will assume that the mean curvature vector field h is parallel as a section of the normal bundle of M^n, which means that $\nabla^{\perp} h = 0$, where $\nabla^{\perp}$ is the normal connection of M^n. Considering $H > 0$, we can assume that the orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{S}_q^{n+p}$ is such that $e_{n+p-q+1} = \frac{h}{H}$. Consequently, we get

$$H^{n+p-q+1} := \frac{1}{n}\mathrm{tr}(h^{n+p-q+1}) = H \quad \text{and} \quad H^{\alpha} := \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \neq n+p-q+1,$$

where h^{α} denotes the matrix (h_{ij}^{α}). Furthermore, we will also consider the total umbilicity tensor

$$\Phi = \sum_{i,j,\alpha \geq n+p-q+1} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha}, \tag{2.73}$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$. We have that

$$\Phi_{ij}^{n+p-q+1} = h_{ij}^{n+p-q+1} - H\delta_{ij} \quad \text{and} \quad \Phi_{ij}^{\alpha} = h_{ij}^{\alpha},$$

for $\alpha \neq n+p-q+1$. Since $|\Phi|^2 = \sum_{\alpha,i,j} (\Phi_{ij}^{\alpha})^2$ is the square of the length of Φ, it is not difficult to verify that Φ is traceless with

$$|\Phi|^2 = |A|^2 - nH^2.$$

Besides, we observe that $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.

To establish some results, we will need the following Simon's type inequality involving the norm of the total umbilicity tensor, which is deduced in [69, Lemma 3.2]. At this point, we draw attention that in the proof of this inequality it is not necessary to assume the hypothesis of the spacelike submanifold be complete.

Lemma 2.2.1. *Let M^n be a spacelike submanifold immersed with parallel mean curvature vector in $\mathbb{S}_q^{n+p}(c)$, $1 \leq q \leq p$, and such that its second fundamental form is locally timelike. Then the following inequality holds:*

$$\frac{1}{2}\Delta|\Phi|^2 \geq |\Phi|^2 P_{H,q}(|\Phi|),$$

where $P_{H,q}(x)$ is the polynomial defined in (2.43).

2.2.2 Nonexistence results for complete spacelike submanifolds immersed with parallel mean curvature vector in $\mathbb{S}_p^{n+p}$

For the context of spacelike submanifolds immersed in the de Sitter space, we start this section obtaining the following nonexistence result.

Theorem 2.2.2. *There does not exist an n-dimensional ($n \geq 3$) complete noncompact spacelike submanifold M^n immersed with parallel mean curvature vector in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$, such that the second fundamental form is locally timelike, $\frac{4(n-1)}{Q(q)} < H^2 < 1$, where $Q(x) = (n-2)^2 x + 4(n-1)$, and $|\Phi|$ converges to zero at infinity.*

Proof. Let us suppose by contradiction the existence of such a submanifold M^n. So, we take the smooth vector field $X = \nabla|\Phi|^2$ and the smooth function $f = |\Phi|^2$. Thus, we have that

$$\langle \nabla f, X \rangle = |\nabla|\Phi|^2|^2 \geq 0 \tag{2.74}$$

is verified.

Assuming that M^n is noncompact and $|\Phi|$ converges to zero at infinity, since $P_{H,q}(|\Phi|) \geq 0$ for $|\Phi| \leq \vartheta_H^-$ from Remark 2.1.10, Lemma 2.2.1 gives

$$\mathrm{div} X = \mathrm{div}(\nabla|\Phi|^2) = \Delta|\Phi|^2 \geq P_{H,q}(|\Phi|)|\Phi|^2 \geq 0.$$

Consequently, we can apply Lemma 1.0.4 to get that

$$\langle \nabla f, X \rangle = |\nabla|\Phi|^2|^2 = 0$$

and conclude that $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant and, since f converges to zero at infinity, it must be identically zero and M^n must be a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.

However, from the proof of item (d) of [69, Theorem 1.1], our constraint on the value of the mean curvature imply that $\vartheta_H^- \leq \sup_M |\Phi| \leq \vartheta_H^+$ with $\vartheta_H^- > 0$, leading us to a contradiction. □

2.2.3 Rigidity of complete noncompact and stochastically complete spacelike submanifolds in $\mathbb{S}_p^{n+p}$

Proceeding, we obtain a characterization for totally umbilical spacelike submanifolds of $\mathbb{S}_q^{n+p}$.

Theorem 2.2.3. *Let M^n be an n-dimensional ($n \geq 3$) complete noncompact spacelike submanifold immersed with parallel mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$, such that the second fundamental form is locally timelike and $H^2 = \frac{4(n-1)}{Q(q)}$, where $Q(x) = (n-2)^2 x + 4(n-1)$. If $|\Phi|$ converges to zero at infinity, then M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.*

Proof. Let us consider once more the smooth vector field $X = \nabla|\Phi|^2$ and the smooth function $f = |\Phi|^2$. So,

$$\langle \nabla f, X\rangle = |\nabla|\Phi|^2|^2 \geq 0. \tag{2.75}$$

Let us suppose that M^n is not a umbilical submanifold. So, we have that f is a non-identically vanishing function which converges to zero at infinity. Moreover, since $H^2 = \dfrac{4(n-1)}{Q(q)}$, we have from Remark 2.1.10 that $P_{H,q} \geq 0$. Hence, we can apply Lemma 1.0.4 to get that

$$\langle \nabla f, X\rangle = |\nabla|\Phi|^2|^2 = 0,$$

which implies that $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant and, since f converges to zero at infinity, it must be identically zero, leading us to a contradiction. Therefore, M^n must be a totally umbilical submanifold of $\mathbb{S}^{n+p}_q$. $\square$

Considering stochastically complete spacelike submanifolds with parallel mean curvature vector field, we obtain the following result.

Theorem 2.2.4. *Let M^n be an n-dimensional ($n \geq 3$) stochastically complete spacelike submanifold immersed with parallel mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}^{n+p}_q$ of index $1 \leq q \leq p$, such that the second fundamental form is locally timelike. If $H^2 < 1$, then either M^n is totally umbilical or* $\sup_M |\Phi| \geq \vartheta^-_H$.

Proof. From Remark 2.1.10, if $H^2 \leq \dfrac{4(n-1)}{Q(q)}$, then $P_{H,q} \geq 0$. Also, $\vartheta^-_H > 0$ if, and only if $\dfrac{4(n-1)}{Q(q)} \leq H^2 < 1$. Hence, we have $P_{H,q}(|\Phi|) \geq 0$ for $|\Phi| \leq \vartheta^-_H$. Thus, from Lemma 2.2.1, we obtain

$$\Delta|\Phi|^2 \geq P_{H,q}(|\Phi|)|\Phi|^2 \geq 0, \tag{2.76}$$

for $H^2 < 1$.

If $\sup_M |\Phi|^2 = +\infty$, then it is immediate that $\sup_M |\Phi| \geq \vartheta^+_H$. So, let us suppose that $\sup_M |\Phi|^2 < +\infty$. Thus, Lemma 1.0.2 guarantees that there exists a sequence of points $\{p_k\} \subset M^n$ such that

$$\lim_{k\to\infty} |\Phi|^2(p_k) = \sup_M |\Phi|^2 \quad \text{and} \quad \limsup_{k\to\infty} \Delta|\Phi|^2(p_k) \leq 0.$$

Consequently, taking into account the continuity of the polynomial $P_{H,q}(x)$, from (2.76) we have

$$\begin{aligned} 0 \geq \frac{1}{2}\limsup_{k\to\infty} \Delta|\Phi|^2(p_k) &\geq \limsup_{k\to\infty}(|\Phi|^2 P_{H,q}(|\Phi|))(p_k) = \lim_{k\to\infty}(|\Phi|^2 P_{H,q}(|\Phi|))(p_k) \\ &= \lim_{k\to\infty} |\Phi|^2(u_k) P_{H,q}(\lim_{k\to\infty} |\Phi|(p_k)) = \sup_M |\Phi|^2 P_{H,q}(\sup_M |\Phi|). \end{aligned}$$

Hence, we obtain

$$\sup_M |\Phi|^2 P_{H,q}(\sup_M |\Phi|) \leq 0. \tag{2.77}$$

Thus, either $\sup_M |\Phi| > 0$ and then

$$P_{H,q}(\sup_M |\Phi|) \leq 0,$$

which implies that $\sup_M |\Phi| \geq \vartheta_H^-$, or $\sup_M |\Phi| = 0$, which means that $|\Phi| \equiv 0$ and M^n must be totally umbilical. □

We recall that a Riemannian manifold without boundary M^n is said to be *parabolic* when the only superharmonic functions on M^n bounded from below are the constant ones. Taking into account that every parabolic Riemannian manifold is stochastically complete, we obtain the following consequence of Theorem 2.2.4.

Corollary 2.2.5. *Let M^n be an n-dimensional ($n \geq 3$) parabolic spacelike submanifold immersed with parallel mean curvature vector field in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$, such that the second fundamental form is locally timelike. If $H^2 < 1$, then either M^n is a totally umbilical submanifold or $\sup_M |\Phi| \geq \vartheta_H^-$.*

We close this section extending the case $c > 0$ in [69, Theorem 1.2] for the context of stochastically complete spacelike submanifolds.

Theorem 2.2.6. *The only n-dimensional ($n \geq 3$) stochastically complete spacelike submanifold immersed in the $(n+p)$-dimensional de Sitter space $\mathbb{S}_q^{n+p}$ of index $1 \leq q \leq p$, which are maximal and having locally timelike second fundamental form, are the totally geodesic ones.*

Proof. Let M^n be such a spacelike submanifold of $\mathbb{S}_q^{n+p}$. Since H is identically zero, we obtain from (2.43) that

$$P_{0,q}(\sup_M |\Phi|) = \frac{(\sup_M |\Phi|)^2}{q} + n > 0.$$

Hence, inequality (2.77) allows us to conclude that $\sup_M |\Phi| = 0$. Therefore, $|\Phi| = 0$ and M^n must be totally geodesic. □

Chapter 3

LW submanifolds in Riemannian space forms $\mathbb{Q}_c^{n+1}$

This chapter is dedicated to establish new rigidity results concerning n-dimensional linear Weingarten (LW) submanifolds immersed in an $(n+p)$-dimensional Riemannian space form $\mathbb{Q}_c^{n+p}$ with constant sectional curvature $c \in \{-1, 0, 1\}$. Under the assumption that a complete LW submanifold has polynomial volume growth, we prove that it must be isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. When the ambient space is the hyperbolic space $\mathbb{H}^{n+p}$, we suppose that the norm of the total umbilicity tensor converges to zero at infinity in order to show that a complete noncompact LW submanifold of $\mathbb{H}^{n+p}$ must be isometric to a horosphere of $\mathbb{H}^{n+1}$. In this chapter we include the results of [49].

3.1 Preliminaries

Let us denote by $\mathbb{Q}_c^{n+p}$ the standard model of an $(n+p)$-dimensional Riemannian space form with constant sectional curvature $c \in \{0, 1, -1\}$. Actually, $\mathbb{Q}_c^{n+p}$ denotes the Euclidean $(n+p)$-space $\mathbb{R}^{n+p}$ when $c = 0$, the $(n+p)$-dimensional Euclidean sphere $\mathbb{S}^{n+p}$ when $c = 1$ and the $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}$ when $c = -1$. We also denote by $\langle , \rangle$ the corresponding Riemannian metric induced on $\mathbb{Q}_c^{n+p} \hookrightarrow \mathbb{R}^{n+p+1}$.

Let M^n be an n-dimensional connected submanifold immersed in $\mathbb{Q}_c^{n+p}$. We choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{Q}_c^{n+p}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$ such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and $e_n, \ldots, e_{n+p}$ are normal to M^n. Moreover, let $\{\omega_{BC}\}$ denote the connection 1-forms on $\mathbb{Q}_c^{n+p}$. In what follows, we will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

The second fundamental form A, the curvature tensor R and the normal curvature tensor $R^{\perp}$ of M^n are given by

$$\omega_{i\alpha} = \sum_j h_{ij}^{\alpha} \omega_j, \quad A = \sum_{i,j,\alpha} h_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha},$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{k,l} R^{\perp}_{\alpha\beta kl}\omega_k \wedge \omega_l.$$

It is not difficult to see that the components $h^{\alpha}_{ijk}\omega_k$ of the covariant derivate ∇A satisfy

$$\sum_k h^{\alpha}_{ijk}\omega_k = dh^{\alpha}_{ij} + \sum_k h^{\alpha}_{ki}\omega_{kj} + \sum_k h^{\alpha}_{kj}\omega_{ki} + \sum_\beta h^{\beta}_{kj}\omega_{ki}. \tag{3.1}$$

Moreover, the Gauss equation is given by

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}).$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^{\alpha}h^{\alpha}_{ik} - \sum_{\alpha,j} h^{\alpha}_{ij}h^{\alpha}_{jk} \tag{3.2}$$

and

$$R = \frac{1}{n-1}\sum_i R_{ii}. \tag{3.3}$$

From (3.2) and (3.3), we get the following relation

$$n(n-1)R = n(n-1)c + n^2H^2 - |A|^2, \tag{3.4}$$

where $|A|^2 = \sum_{\alpha,i,j}(h^{\alpha}_{ij})^2$ is the squared norm of the second fundamental form A and $H = |\mathbf{H}|$ is the mean curvature function related to the mean curvature vector field $\mathbf{H} = \sum_\alpha H^{\alpha}e_\alpha = \frac{1}{n}\sum_\alpha(\sum_k h^{\alpha}_{kk})e_\alpha$ of M^n.

Furthermore, the Codazzi equation is given by

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj} = h^{\alpha}_{jik}. \tag{3.5}$$

We will also consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^{\alpha}_{ij}\omega_i \otimes \omega_j e_\alpha, \tag{3.6}$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij},$$

for $n+2 \leq \alpha \leq n+p$.

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^{\alpha}_{ij})^2$ be the squared norm of Φ. It is not difficult to check that Φ is traceless with

$$|\Phi|^2 = |A|^2 - nH^2. \tag{3.7}$$

In addition, from (3.4) we obtain

$$n(n-1)R = n(n-1)(c + H^2) - |\Phi|^2. \tag{3.8}$$

We recall once more that a submanifold is said to be *linear Weingarten* (LW) when its mean and normalized scalar curvatures are linearly related, that is, when they satisfy the following relation

$$R = aH + b, \tag{3.9}$$

for constants $a, b \in \mathbb{R}$. We observe that when $a = 0$, (3.9) reduces to R constant.

In this setting, equation (3.7) becomes

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 - n(n-1)aH - n(n-1)(b-c). \tag{3.10}$$

For a LW submanifold M^n satisfying (3.9) we consider again the second-order linear differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L} = L - \frac{n-1}{2}a\Delta, \tag{3.11}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M)$ denotes the Cheng-Yau operator, which is given by

$$Lu = \mathrm{tr}(P \circ \mathrm{Hess}\,(u)), \tag{3.12}$$

for every $u \in C^\infty(M)$, where Hess is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n which is given by $P = nHI - A$. So, from (3.11) and (3.12), we have that

$$\mathcal{L}u = \mathrm{tr}(\mathcal{P} \circ \mathrm{Hess}\,(u)),$$

with

$$\mathcal{P} = \left(nH - \frac{n-1}{2}a\right) I - A \tag{3.13}$$

and we can rewrite $\mathcal{L}$ in the following divergence form

$$\mathcal{L}u = \mathrm{div}(\mathcal{P}(\nabla u)). \tag{3.14}$$

In order to establish our main results, we present the next auxiliary propositions, which can be found in [44, Lemma 4.1, Lemma 4.4].

Proposition 3.1.1. *Let M^n be a linear Weingarten submanifold immersed in a Riemannian space form $\mathbb{Q}_c^{n+p}$ such that $R = aH + b$. If $b > c$ ($b \geq c$), then $\mathcal{L}$ is elliptic (semi-elliptic).*

Proposition 3.1.2. *Let M^n be a linear Weingarten submanifold immersed in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that $(n-1)a^2 + 4n(b-c) \geq 0$. Then*

$$|\nabla A|^2 \geq n^2 |\nabla H|^2.$$

Moreover, the equality holds on M^n if, and only if, M^n is an isoparametric submanifold of $\mathbb{Q}_c^{n+p}$.

We close this subsection recalling a classic algebraic lemma due to Okumura in [73], which was completed with the equality case by Alencar and do Carmo in [4].

Lemma 3.1.3. *Let $\kappa_1, \ldots, \kappa_n$ be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, with $\beta \geq 0$. Then,*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \kappa_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds if and only if at least $(n-1)$ of the numbers κ_i are equals.

3.2 Rigidity results for LW hypersurfaces immersed in $\mathbb{Q}_c^{n+1}$

Before to present our results, we need to collect some properties related to the following one-parameter family of real functions

$$Q_t(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(t-c)} + n(n-1)t, \tag{3.15}$$

where $t \in \mathbb{R}$ corresponds to the real parameter, while n and c are real constants. We note that Alías, García-Martínez and Rigoli introduced in [13] the definition of the function $Q_R(x)$ when they were studying hypersurfaces with constant normalized scalar curvature R in an $(n+1)$-dimensional Riemannian space form of constant sectional curvature c.

For each nonnegative (positive) parameter t, we have that $Q_t(0) = n(n-1)t$ is also nonnegative (positive). When $n \geq 3$, each function Q_t is (strictly) decreasing for $x \geq 0$, with $Q_t(x_t^*) = 0$ only at

$$x_t^* = t\sqrt{\frac{n(n-1)}{(n-2)(nt-(n-2)c)}}. \tag{3.16}$$

Moreover, in the case $n = 2$, we have that $Q_t(x) = 2t$.

Now, we are in position to present the next rigidity result concerning a complete LW hypersurface M^n immersed in $\mathbb{Q}_c^{n+1}$.

Theorem 3.2.1. *Let M^n be a complete LW hypersurface immersed into a Riemannian space form $\mathbb{Q}_c^{n+1}$ with $n \geq 3$, such that $R = aH + b$ with $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \frac{n-2}{n}$ for $c = 1$ and $R > 0$ for $c = 0$ or $c = -1$. Assume in addition that $|\nabla \Phi|$ is bounded and $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ, and x_R^* defined* *in* (3.16). *If M^n has polynomial volume growth and* $\inf_R(Q_R(\gamma)) > 0$*, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Proof. Taking the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$, it will fulfill the required conditions to apply Lemma 1.0.3. Indeed, by hypothesis we have that $|\Phi|$ is bounded on M^n and, by equation (3.10), $|A|$ is also bounded on M^n. Consequently, from definition (3.13), we get

$$|X| = |\mathcal{P}(\nabla|\Phi|^2)| \leq |\mathcal{P}||\nabla|\Phi|^2| \leq k|\nabla|\Phi|^2|,$$

for some positive constant k. But, since we are supposing the boundedness of $|\Phi|$ and $|\nabla\Phi|$, Lemma 2.1.4 guarantees that $\nabla|\Phi|^2$ is also bounded. Thus, we have that

$$|X| \leq C < +\infty,$$

for some positive constant C.

On the other hand, the condition

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0$$

is also verified because Proposition 3.1.1 gives that $\mathcal{P}$ is positive semi-definite for $b \geq c$.

Now, we must obtain $\operatorname{div} X \geq \alpha f$ on M^n, for some positive constant α. For this, we will find a suitable lower bound for $\mathcal{L}(|\Phi|^2)$. Applying $\mathcal{L}$ in (3.10), we get that

$$\begin{aligned} \frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) - \frac{a}{2}\mathcal{L}(nH) \\ &= H\mathcal{L}(nH) + n\langle \mathcal{P}\nabla H, \nabla H\rangle - \frac{a}{2}\mathcal{L}(nH). \end{aligned} \tag{3.17}$$

In particular, since $\mathcal{P}$ is positive semi-definite, from (3.17) we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})\mathcal{L}(nH). \tag{3.18}$$

Let us choose a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Since $R = aH + b$, from [19, Equation (2.19)] jointly with the definition of $\mathcal{L}$ and with $R_{ijij} = \lambda_i\lambda_j + c$, we get

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + nc(|A|^2 - nH^2) - |A|^4 + nH\sum_i \lambda_i^3. \tag{3.19}$$

Moreover, we have $\Phi_{i,j} = \mu_i\lambda_{ij}$ and, with straightforward computation, we verify that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \textit{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{3.20}$$

Thus, using the Gauss equation jointly with (3.19) and (3.20), we get

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \tag{3.21}$$

We can apply Proposition 3.1.2 jointly with Lemma 3.1.3 for $n \geq 3$, to obtain from (3.21) that

$$\mathcal{L}(nH) \geq |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 + nc\right). \tag{3.22}$$

Furthermore, from (3.8) we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R-c). \tag{3.23}$$

Thus, from (3.22) and (3.23) we achieve in

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{3.24}$$

where Q_R is defined in (3.15). Hence, using (3.18) jointly with (3.24), from (3.14) we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right)Q_R(|\Phi|)|\Phi|^2. \tag{3.25}$$

Since we have $\left(H - \frac{a}{2}\right) \geq \beta > 0$ by hypothesis and, from the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq \sup_M |\Phi| \leq \gamma < x_R^*$, we have that

$$Q_R(|\Phi|) \geq Q_R(\gamma) > \inf_R(Q_R(\gamma)) > 0. \tag{3.26}$$

Then, from (3.25) and (3.26) we obtain

$$\mathrm{div}X \geq 2\left(H - \frac{a}{2}\right)Q_R(|\Phi|)|\Phi|^2 \geq \alpha|\Phi|^2, \tag{3.27}$$

and $\mathrm{div}X \geq \alpha f$ for $\alpha = 2\beta \inf_R(Q_R(\gamma)) > 0$.

Consequently, supposing that M^n is noncompact and with polynomial volume growth, we are able to apply Lemma 1.0.3 obtaining that $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$, which means that M^n is a totally umbilical hypersurface. But, from the characterizations of the totally umbilical hypersurfaces of the Riemannian space forms, we conclude that M^n must be isometric to $\mathbb{R}^n$, which corresponds to a contradiction with the hypothesis that $R > 0$.

Thus, M^n must be compact. So, we can integrate both sides of (3.27) and use Divergence Theorem to get that

$$\int_M |\Phi|^2 \mathrm{dM} = 0.$$

Therefore, we have that $|\Phi| \equiv 0$ and, hence, M^n is a compact totally umbilical hypersurface of $\mathbb{Q}_c^{n+1}$. So, M^n must be isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. □

Revisiting the proof of Theorem 3.2.1, we observe that if $n = 2$, then $\sum_i \mu_i^3 = 0$. Consequently, from (3.21) we get

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(-|\Phi|^2 + 2H^2 + 2c\right),$$

and (3.22) is still true in this case. Hence, it is not difficult to verify that we also have the following rigidity result.

Theorem 3.2.2. *Let M^2 be a complete LW surface immersed into a Riemannian space form $\mathbb{Q}_c^3$, such that $R = aH + b$ with $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^2, for some positive constant β, and that $\inf_M R > 0$. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is isometric to an Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$.*

Observing that, when $R > 0$ is constant, the hypothesis $\inf_R(Q_R(\gamma)) > 0$ is automatically satisfied, from Theorems 3.2.1 and 3.2.2 we obtain, respectively, the following consequences:

Corollary 3.2.3. *Let M^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{Q}_c^{n+1}$ with $n \geq 3$, with constant normalized scalar curvature $R \geq 1$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Suppose that $H \geq \beta$ on M^n, for some positive constant β. Assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| < x_R^*$, for x_R^* defined in (3.16). If M^n has polynomial volume growth, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$.*

Corollary 3.2.4. *Let M^2 be a complete surface immersed into a Riemannian space form $\mathbb{Q}_c^3$, with constant normalized scalar curvature $R \geq 1$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Suppose that $H \geq \beta$ on M^2, for some positive constant β. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is isometric to an Euclidean sphere $\mathbb{S}^2(r)$, with radius $r > 0$.*

Proceeding, we will deal with LW submanifolds M^n of $\mathbb{Q}_c^{n+p}$ having parallel normalized mean curvature vector field $\mathbf{H}$, which means that the mean curvature function H is positive and that the corresponding normalized mean curvature vector field $\frac{\mathbf{H}}{H}$ is parallel as a section of the normal bundle. In this context, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{\mathbf{H}}{H}$. Consequently, we have

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}\mathrm{tr}(h^\alpha) = 0, \quad \alpha \geq n+2. \tag{3.28}$$

Considering this previous context, we can state a version of Theorem 3.2.1 for higher codimension.

Theorem 3.2.5. *Let M^n be a complete LW submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b \geq c$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \frac{n-2}{n}$ for $c = 1$ and $R > 0$ when $c = -1$ or $c = 0$. Assume in addition that $|\nabla\Phi|$ is bounded and such that $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in (3.16). If M^n*

has polynomial volume growth and $\inf_R(Q_R(\gamma)) > 0$*, then* M^n *is isometric to an Euclidean sphere* $\mathbb{S}^n(r)$*, with radius* $r > 0$*.*

Proof. Reasoning as in the proof of Theorem 3.2.1, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, we have that

$$|X| \leq C, \tag{3.29}$$

for some positive constant C, and

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0. \tag{3.30}$$

Moreover,

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})\mathcal{L}(nH). \tag{3.31}$$

On the other hand, following the same initial steps of the proof of [44, Theorem 5.1], we can achieve in [44, Inequality (5.16)] which is given by

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) + (|\Phi| - |\Phi^{n+1}|)(\frac{n-2}{n-1} - \frac{16}{27})|\Phi|.$$

Thus, since we are also assuming that $n \geq 4$, we get

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|). \tag{3.32}$$

So, using (3.31) jointly with (3.32), we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2. \tag{3.33}$$

But, since $\left(H - \frac{a}{2}\right) \geq \beta > 0$, taking into account once more the behavior of $Q_R(x)$, for $0 \leq |\Phi| \leq \sup_M |\Phi| < \gamma < x^*$, we have that

$$Q_R(|\Phi|) \geq Q_R(\gamma) > \inf_R(Q_R(\gamma)) > 0.$$

Hence, from (3.33) we obtain

$$\mathrm{div}X \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq \alpha f, \tag{3.34}$$

where $\alpha = 2\beta \inf_R(Q_R(\gamma)) > 0$.

Supposing that M^n is a noncompact submanifold, since (3.29), (3.30) and (3.34) were verified and M^n has polynomial volume growth, we are able to apply Lemma 1.0.3 to obtain that $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$ and M^n is totally umbilical submanifold. Consequently, taking into account (3.28), we get

$$h^\alpha = \langle H, e_\alpha\rangle I = H^\alpha I = 0,$$

for all $\alpha > n+1$. Thus, we have that the first normal subspace,

$$N_1 = \left\{e_\alpha \in \mathfrak{X}^\perp(M^n); h^\alpha = 0\right\}^\perp,$$

is parallel and it has dimension 1. Therefore, we can apply [38, Proposition 4.1] to reduce the codimension of M^n to 1. So, since M^n is, in fact, a totally umbilical noncompact hypersurface with polynomial volume growth, we infer that it is isometric to $\mathbb{R}^n$, which corresponds to a contradiction with the hypothesis $R > 0$.

At this point, we can reason as in the last part of the proof of Theorem 3.2.1 to conclude, reducing the codimension of M^n again, that M^n must be isometric to a totally umbilical Euclidean sphere $\mathbb{S}^n(r)$, with radius $r > 0$. □

In what follows we will apply Lemma 1.0.4 to get further rigidity results concerning complete noncompact LW submanifolds in the hyperbolic space. So, we state and prove our first one.

Theorem 3.2.6. *Let M^n be a complete noncompact LW hypersurface immersed into the hyperbolic space $\mathbb{H}^{n+1}$ with $n \geq 3$, such that $R = aH + b$ with $b > -1$. Suppose that $R \geq 0$ and that $|\Phi| \leq x_R^*$, for x_R^* defined in* (3.16)*. If $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof. Let us consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$ and let us suppose that M^n is not a umbilical hypersurface. So, f is non-identically vanishing function which converges to zero at infinity. Moreover, we have that

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0.$$

We claim that $\mathrm{div}X \geq 0$. Indeed, we already know that

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})\mathcal{L}(nH) \quad \text{and} \quad \mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{3.35}$$

where Q_R is the one-parameter family of real functions given by (3.15). Thus, since $\left(H - \frac{a}{2}\right) \geq 0$, from (3.35) jointly with the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq x_R^*$, we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq 0.$$

Hence, we can apply Proposition 1.0.4 to get that

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2\rangle \equiv 0.$$

Consequently, since Lemma 3.1.1 gives that $\mathcal{P}$ is positive definite, we have that $\nabla|\Phi|^2 \equiv 0$. Thus, $f = |\Phi|^2$ is constant. But, since f converges to zero at infinity, it must be identically zero, leading us to a contradiction. Therefore, M^n is a complete noncompact totally umbilical hypersurface of $\mathbb{H}^{n+1}$ with $R \geq 0$, which means that M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$. □

In the case $n = 2$, reasoning as in the proof of Theorem 3.2.6 we also obtain the following.

Theorem 3.2.7. *Let M^2 be a complete noncompact LW surface immersed into the hyperbolic space $\mathbb{H}^3$, such that $R = aH + b$ with $b > -1$. Suppose that $R \geq 0$. If $|\Phi|$ converges to zero at infinity, then M^2 is isometric to a horosphere of $\mathbb{H}^3$.*

Applying again a codimension reduction process, we obtain our next rigidity result.

Theorem 3.2.8. *Let M^n be a complete noncompact LW submanifold immersed with parallel normalized mean curvature vector field into the hyperbolic space $\mathbb{H}^{n+p}$ with $n \geq 4$, such that $R = aH + b$ with $a \geq 0$ and $b > -1$. Suppose that $R \geq 0$ and that $|\Phi| \leq x_R^*$, for x_R^* defined in* (3.16). *If $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof. It is not difficult to verify that, using inequality (3.33) and following similar steps of the proof of Theorem 3.2.6, we can achieve in $\nabla|\Phi|^2 \equiv 0$. So, taking into account (3.28), we get

$$h^\alpha = \langle H, e_\alpha \rangle I = H^\alpha I = 0,$$

for every $\alpha > n + 1$. This implies that the first normal subspace,

$$N_1 = \left\{ e_\alpha \in \mathfrak{X}^\perp(M^n); h^\alpha = 0 \right\}^\perp,$$

is parallel and has dimension 1. Therefore, we are in position to apply once more [38, Proposition 4.1], reducing the codimension of M^n to 1 and concluding that it is a totally umbilical noncompact hypersurface of $\mathbb{H}^{n+1}$ with $R \geq 0$. Consequently, M^n must be a horosphere of $\mathbb{H}^{n+1}$. □

In our last rigidity result of this section, we will deal with complete noncompact LW submanifolds having nonnegative sectional curvature, which are immersed with globally flat normal bundle in $\mathbb{H}^{n+p}$.

Theorem 3.2.9. *Let M^n be a complete noncompact LW submanifold with nonnegative sectional curvature immersed into the hyperbolic space $\mathbb{H}^{n+p}$, $n \geq 2$ with globally flat normal bundle and parallel normalized mean curvature vector field, such that $R = aH + b$ with $b > -1$. If the total umbilicity tensor of the immersion $|\Phi|$ converges to zero at infinity, then M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$.*

Proof. As before, we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. Supposing that M^n is not a totally umbilical submanifold, reasoning as in the proof of Theorem 3.2.6 we obtain that f is non-identically vanishing function which converges to zero at infinity and such that $\langle \nabla f, X \rangle \geq 0$.

Now, let us verify that $\mathrm{div} X \geq 0$. Indeed, we have

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2. \tag{3.36}$$

Using Codazzi equation (3.5) into (3.36), we get

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j,k,\alpha} h_{ij}^{\alpha} h_{kijk}^{\alpha}. \tag{3.37}$$

On the other hand, by exterior differentiation of (3.1) and assuming that M^n has globally flat normal bundle (that is, $R^{\perp} = 0$), we obtain the following Ricci identity

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m h_{mj}^{\alpha} R_{mikl} + \sum_m h_{im}^{\alpha} R_{mjkl}. \tag{3.38}$$

Thus, from (3.28), (3.37) and (3.38) we reach at

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i,j} nH_{ij}^{n+1} h_{ij}^{n+1} + \sum_{i,j,m,k,\alpha} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} + \sum_{i,j,k,m,\alpha} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk}. \tag{3.39}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$, for every α, from (3.39) we obtain the following Simons-type formula

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_i \lambda_i^{n+1}(nH)_{ii} + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2. \tag{3.40}$$

Moreover, using the definition (3.12), we obtain

$$\begin{aligned} L(nH) &= nH\Delta(nH) - \sum_i \lambda_i^{n+1}(nH)_{ii} \\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta|A|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i^{n+1}(nH)_{ii}. \end{aligned} \tag{3.41}$$

Thus, inserting (3.40) into (3.41) we get

$$L(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2. \tag{3.42}$$

Provided that $R = aH + b$, from (3.11) and (3.42) we have

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2. \tag{3.43}$$

Hence, since M^n is supposed to have nonnegative sectional curvature and using Proposition 3.1.2, from (3.43) we get $\mathcal{L}(nH) \geq 0$. Thus, since $(H - \frac{a}{2}) \geq 0$, from (3.35) we finally deduce that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H - \frac{a}{2}\right)\mathcal{L}(nH) \geq 0.$$

Now, applying Lemma 1.0.4 we obtain

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2\rangle \equiv 0.$$

So, since Proposition 3.1.1 guarantees that $\mathcal{P}$ is positive definite, we get that $\nabla|\Phi|^2 \equiv 0$. Thus, as in the last part of the proof of Theorem 3.2.6, we will have that $f = |\Phi|^2$ is identically zero, leading us to a contradiction. Therefore, M^n must be totally umbilical and, reducing the codimension of M^n to 1, we conclude that M^n is isometric to a horosphere of $\mathbb{H}^{n+1}$. $\square$

Chapter 4

LW hypersurfaces in Einstein manifolds

In this chapter we study the geometry of linear Weingarten (LW) spacelike hypersurfaces immersed in an Einstein space obeying some standard curvature conditions.The results that will be present in this chapter make part of [36, 47, 52, 54]

4.1 Rigidity results for closed LW hypersurfaces in an Einstein spacetime $\mathcal{E}_1^{n+1}$

4.1.1 Preliminaries

In this section, we will consider complete spacelike hypersurfaces M^n immersed in a Lorentz space $\mathbb{L}_1^{n+1}$. We choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in $\mathbb{L}_1^{n+1}$ such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. Using the following convention of indices

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n,$$

and taking the corresponding dual coframe $\omega_1, \ldots, \omega_{n+1}$, the semi-Riemannian metric of $\mathbb{L}_1^{n+1}$ is given by $ds^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$. So, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{L}_1^{n+1}$, we have that the structure equations of $\mathbb{L}_1^{n+1}$ are given by

$$d\omega_A = -\sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0 \tag{4.1}$$

and

$$d\omega_{AB} = -\sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D \overline{R}_{ABCD} \, \omega_C \wedge \omega_D, \tag{4.2}$$

where $\overline{R}_{ABCD}$ denotes the components of the curvature tensor of $\mathbb{L}_1^{n+1}$.

In this setting, denoting by $\overline{R}_{CD}$ and $\overline{R}$ the components of the Ricci tensor and the scalar curvature of the Lorentz space $\mathbb{L}_1^{n+1}$, respectively, we also have

$$\overline{R}_{CD} = \sum_B \epsilon_B \overline{R}_{BCDB}, \quad \overline{R} = \sum_A \epsilon_A \overline{R}_{AA}.$$

We are going to assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of the Lorentz manifold $\mathbb{L}_1^{n+1}$ satisfies the following two constraints

$$\overline{K}(u,\eta) = -\frac{c_1}{n}, \tag{4.3}$$

for any $u \in TM$ and $\eta \in TM^{\perp}$, and

$$\overline{K}(u,v) \geq c_2, \tag{4.4}$$

for any tangent vectors $u, v \in TM$. In particular, the Lorentzian space forms $\mathbb{L}_1^{n+1}(c)$ of constant sectional curvature c satisfy curvature conditions (4.3) and (4.4) for any spacelike hypersurface M^n immersed in $\mathbb{L}_1^{n+1}(c)$ and $-\frac{c_1}{n} = c_2 = c$.

Moreover, there are several examples of Lorentz spaces which are not Lorentz space forms and satisfy (4.3) and (4.4). For instance, Lorentz product manifolds $\mathbb{H}_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, where $c_1 > 0$, and $\mathbb{R}_1^k \times \mathbb{S}^{n+1-k}$, where we are considering the spacelike hypersurface M^n as being a slice of the ambient space. In particular, $\mathbb{R}_1^1 \times \mathbb{S}^n$ is a so-called *Einstein Static Universe*. Also the so-called *Robertson-Walker spacetime* $N(c,f) = I \times_f N^3(c)$ is another general example of Lorentz space, where I denotes an open interval of $\mathbb{R}_1^1$, f is a positive smooth function defined on the interval I and $N^3(c)$ is a 3-dimensional Riemannian manifold of constant curvature c. $N(c,f)$ also satisfies curvature conditions (4.3) and (4.4) for an appropriate choice of the function f and $M^3 = \{t_0\} \times N^3$ for some $t_0 \in I$ (for more details, see [35, 83]).

We also observe that denoting by $\overline{R}_{AB}$ the components of the Ricci tensor of a manifold $\overline{M}^{n+1}$ satisfying curvature condition (4.3), the scalar curvature $\overline{R}$ of $\overline{M}^{n+1}$ is given by

$$\overline{R} = \sum_{A=1}^{n+1} \varepsilon_A \overline{R}_{AA} = \sum_{i,j=1}^{n} \overline{R}_{ijji} - 2\sum_{i=1}^{n} \overline{R}_{(n+1)ii(n+1)} = \sum_{i,j=1}^{n} \overline{R}_{ijji} + 2c_1. \tag{4.5}$$

Consequently, if $(\mathcal{E}^{n+1}, \overline{g})$ is an Einstein manifold, the components of its Ricci tensor satisfy $\overline{R}_{CD} = \lambda \overline{g}_{CD}$, for some constant $\lambda \in \mathbb{R}$. In particular, the scalar curvature $\overline{R}$ is constant and, from (4.5), we conclude that $\sum_{i,j=1}^{n} \overline{R}_{ijji}$ is also constant. So, for sake of simplicity, along this chapter we will denote the constant $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ by $\overline{\mathcal{R}}$.

The components $\overline{R}_{ABCD;E}$ of the covariant derivative of the curvature tensor of $\mathbb{L}_1^{n+1}$ are defined by

$$\sum_E \epsilon_E \overline{R}_{ABCD;E}\omega_E = d\overline{R}_{ABCD} - \sum_E \epsilon_E \left(\overline{R}_{EBCD}\omega_{EA} + \overline{R}_{AECD}\omega_{EB} + \overline{R}_{ABED}\omega_{EC} + \overline{R}_{ABCE}\omega_{ED}\right).$$

Furthermore, restricting all the tensors to the spacelike hypersurface M^n, since $\omega_{n+1} = 0$ on M^n, we get $\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0$. So, from Cartan's Lemma we obtain

$$\omega_{(n+1)i} = \sum_j h_{ij}\omega_j \quad \text{and} \quad h_{ij} = h_{ij}. \tag{4.6}$$

This gives the second fundamental form of M^n, $A = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$, and its squared length $|A|^2 = \sum_{i,j} h_{ij}^2$. Beyond that, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

From (4.1) and (4.2) we deduce that the connection forms $\{\omega_{ij}\}$ of M^n are characterized by the following structure equations

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \quad \text{and} \quad d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{4.7}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Hence, from (4.7) we obtain the Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} - (h_{ik}h_{jl} - h_{il}h_{jk}).$$

The components R_{ij} of the Ricci tensor and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{kijk} - nHh_{ij} + \sum_k h_{ik}h_{kj}$$

and

$$|A|^2 = n^2H^2 + n(n-1)R - \sum_{i,j} \overline{R}_{ijij}. \tag{4.8}$$

Moreover, the first covariant derivatives h_{ijk} of h_{ij} satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}. \tag{4.9}$$

Then, by exterior differentiation of (4.6) we obtain the Codazzi equation

$$h_{ijk} - h_{ikj} = \overline{R}_{(n+1)ijk}. \tag{4.10}$$

The second covariant derivatives h_{ijkl} of h_{ij} are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Taking the exterior derivative in (4.9) we obtain the following Ricci formula

$$h_{ijkl} - h_{ijlk} = -\sum_m h_{im}R_{mjkl} - \sum_m h_{jm}R_{mikl}. \tag{4.11}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ on M^n, we get

$$\overline{R}_{(n+1)ijk;l} = \overline{R}_{(n+1)ijkl} + \overline{R}_{(n+1)i(n+1)k}h_{jl} + \overline{R}_{(n+1)ij(n+1)}h_{kl} + \sum_m \overline{R}_{mijk}h_{ml}, \tag{4.12}$$

where $\overline{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\overline{R}_{(n+1)ijk}$ as a tensor on M^n so that

$$\sum_l \overline{R}_{(n+1)ijkl}\omega_l = d\overline{R}_{(n+1)ijk} - \sum_l \overline{R}_{(n+1)ljk}\omega_{li} - \sum_l \overline{R}_{(n+1)ilk}\omega_{lj} - \sum_l \overline{R}_{(n+1)ijl}\omega_{lk}.$$

Since the Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} := \sum_k h_{ijkk}$. From (4.10), (4.11) and (4.12), we obtain

$$\begin{aligned}\Delta h_{ij} &= (nH)_{ij} - nH\sum_l h_{il}h_{lj} + |A|^2 h_{ij} + \sum_k \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) \\ &\quad - \sum_k (h_{kk}\overline{R}_{(n+1)ij(n+1)} + h_{ij}\overline{R}_{(n+1)k(n+1)k}) - \sum_{k,l}(2h_{kl}\overline{R}_{lijk} + h_{jl}\overline{R}_{lkik} + h_{il}\overline{R}_{lkjk}).\end{aligned} \tag{4.13}$$

Thus, since $\Delta|A|^2 = 2\left(\sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij}\right)$, from (4.13) we get

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= (|A|^2)^2 + \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j}(nH)_{ij}h_{ij} + \sum_{i,j,k}\left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}\right) h_{ij} \\ &\quad - \sum_{i,j} nHh_{ij}\overline{R}_{(n+1)ij(n+1)} + |A|^2\sum_k \overline{R}_{(n+1)k(n+1)k} \\ &\quad -2\sum_{i,j,k,l}(h_{kl}h_{ij}\overline{R}_{lijk} + h_{il}h_{ij}\overline{R}_{lkjk}) - nH\sum_{i,j,l} h_{il}h_{lj}h_{ij}.\end{aligned} \tag{4.14}$$

Again, we will work with the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j, \tag{4.15}$$

where $\Phi_{ij} = h_{ij} - H\delta_{ij}$. Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ, we can check that Φ is traceless and

$$|\Phi|^2 = |A|^2 - nH^2. \tag{4.16}$$

Moreover, considering that M^n is a linear Weingarten with $R = aH + b$, it holds for an Einstein spacetime the following algebraic relations from equations (4.8) and (4.16):

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)aH + n(n-1)(b - \overline{\mathcal{R}}), \tag{4.17}$$

where $\overline{\mathcal{R}} = \frac{1}{n(n-1)}\sum_{i,j}\overline{R}_{ijij}$. In the case that $b < \overline{\mathcal{R}}$, it follows from (4.17) that $H(p) \neq 0$ for every $p \in M^n$. In this case, we choose on M^n the orientation such that $H > 0$.

Following Cheng-Yau [31], we introduce the Cheng-Yau operator $L : C^\infty(M^n) \to C^\infty(M^n)$ associated to ϕ acting on any smooth function f by

$$Ł(f) = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{4.18}$$

Hence, setting $f = nH$ in (4.18) and taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (4.8) we obtain the following

$$\begin{aligned} L(nH) &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta|A|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{ii} \\ &+ \frac{1}{2}\Delta\left(\sum_{i,j} \overline{R}_{ijji} - n(n-1)R\right). \end{aligned} \tag{4.19}$$

In what follows, we will quote some key lemmas in order to prove the results of the next section. The first one corresponds to [43, Lemma 3.2].

Lemma 4.1.1. *Let M^n be a complete LW spacelike hypersurface immersed in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with*

$$(n-1)a^2 - 4n(b - \overline{\mathcal{R}}) \geq 0. \tag{4.20}$$

Then,

$$|\nabla H|^2 = \sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2. \tag{4.21}$$

Moreover, if the inequality (4.20) *is strict and equality occurs in* (4.21)*, then H is constant on M^n.*

For a LW spacelike hypersurface M^n, we can also introduce the second-order linear differential operator $\mathcal{L} : C^\infty(M^n) \to C^\infty(M^n)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2} a\Delta, \tag{4.22}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M^n) \to C^\infty(M^n)$ denotes the Cheng-Yau operator defined in (4.18), which is given by

$$Lu = \operatorname{tr}(P \circ \operatorname{Hess}(u)), \tag{4.23}$$

for every $u \in C^\infty(M^n)$, where Hess is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ denotes the first Newton transformation of M^n which is given by $P = nHI - A$. So, from (4.22) and (4.23), we have that

$$\mathcal{L}u = \operatorname{tr}(\mathcal{P} \circ \operatorname{Hess}(u)),$$

with

$$\mathcal{P} = \left(nH + \frac{n-1}{2}a\right) I - A. \tag{4.24}$$

Thus, by using the standard notation $\langle , \rangle$ for the (induced) metric of M^n, we get

$$Lu = \sum_i \langle P(\nabla_{e_i} \nabla u), e_1 \rangle,$$

where $\{e_1, \cdots, e_n\}$ is a (local) orthonormal frame on M^n. Consequently, we obtain

$$\begin{aligned} \mathrm{div}(P(\nabla u)) &= \sum_i \langle (\nabla_{e_i} P)(\nabla u), e_i \rangle + \sum_i \langle P(\nabla_{e_i} \nabla u), e_1 \rangle \\ &= \langle \mathrm{div} P, \nabla u \rangle + L(u). \end{aligned} \tag{4.25}$$

Since we are assuming that $\mathcal{E}_1^{n+1}$ is an Einstein manifold, there exist a constant λ such that $\overline{\mathrm{Ric}} = \lambda \langle , \rangle$, in which $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\mathcal{E}_1^{n+1}$. Thus, from [7, Lemma 3.1]

$$\langle \mathrm{div} P, \nabla u \rangle = \sum_i \langle \overline{R}(N, e_i) e_i, \nabla u \rangle = -\overline{\mathrm{Ric}}(N, \nabla u) = -\lambda \langle N, \nabla u \rangle = 0,$$

where N stands for the Gauss mapping of M^n. Hence, from equation (4.25), we conclude that

$$Lu = \mathrm{div}(P(\nabla u)). \tag{4.26}$$

Thus, from (4.22) and (4.26), we can verify that $\mathcal{L}$ can be rewritten in the following divergence form

$$\mathcal{L}u = \mathrm{div}(\mathcal{P}(\nabla u)). \tag{4.27}$$

In our next result, we establish a sufficient criteria of ellipticity for the operator $\mathcal{L}$(see [43, Lemma 3.3]).

Lemma 4.1.2. *Let M^n be a LW spacelike hypersurface immersed in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature condition* (4.3)*, such that $R = aH + b$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator $\mathcal{P}$ defined in* (4.24) *at every point $p \in M^n$.*

If $b < \overline{\mathcal{R}}$, then the operator $\mathcal{L}$ defined in (4.22) *is elliptic, with*

$$\mu_- > 0 \quad \textit{and} \quad \mu_+ < 2nH + (n-1)a.$$

In the case where $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change sign and $b \leq R$. Then the operator $\mathcal{L}$ is semi-elliptic, with

$$\mu_- \geq 0 \quad \textit{and} \quad \mu_+ \leq 2nH + (n-1)a,$$

unless M^n is totally geodesic. Moreover, in the case where $b < \overline{\mathcal{R}}$ on M^n, the above inequalities are strict and the operator $\mathcal{L}$ is elliptic.

Remark 4.1.3. Regarding the ellipticity of $\mathcal{L}$, observe that when M^n is totally geodesic then the operator $\mathcal{L}$ reduces to $\mathcal{L} = \frac{n-1}{2} a\Delta$, which is elliptic if and only if $a > 0$. For that reason, in order to keep the validity of Lemma 4.1.2 when $b = \overline{\mathcal{R}}$ even in the totally geodesic case we choose a to be any positive constant.

The following lower boundedness for the operator $\mathcal{L}$ acting on the squared length of the traceless operator Φ of a LW spacelike hypersurface will be essential to establish our characterization results.

Proposition 4.1.4. *Let M^n $(n \geq 3)$ be a complete LW spacelike hypersurface immersed with parallel normalized mean curvature vector in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with $b \leq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, assume that the mean curvature function H does not change sign and $b \leq R$. Then,*

$$\mathcal{L}(|\Phi|^2) \geq 2(n-1)|\Phi|^2 \varphi_{a,b}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}},$$

where

$$\begin{aligned}\varphi_{a,b}(x) = {} & \frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}} \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}}\frac{a}{2}x - n\left(\overline{\mathcal{R}} - b - c + \frac{a^2}{2}\right)\end{aligned} \tag{4.28}$$

and $c = 2c_2 + \dfrac{c_1}{n}$.

Proof. Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$ and $\Phi_{ij} = \kappa_i \delta_{ij}$. Taking into account equations (4.14), (4.19), we get from (4.22) that

$$\begin{aligned}\mathcal{L}(nH) = {} & (|A|^2)^2 - nH\sum_i \lambda_i^3 + \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 \\ & -2\sum_{i,k}(\lambda_i\lambda_k\overline{R}_{kiik} + \lambda_i^2\overline{R}_{ikik}) + \sum_{i,k}\lambda_i(\overline{R}_{(n+1)iik;k} + \overline{R}_{(n+1)kik;i}) \\ & - \left(nH\sum_i \lambda_i \overline{R}_{(n+1)ii(n+1)} + |A|^2\sum_k \overline{R}_{(n+1)k(n+1)k}\right).\end{aligned} \tag{4.29}$$

On the other hand, since $(\mathcal{E}_1^{n+1}, \overline{g})$ is an Einstein spacetime, the components of its Ricci tensor satisfy $\overline{R}_{CD} = \lambda \overline{g}_{CD}$, for some constant $\lambda \in \mathbb{R}$. Moreover, we can consider $\{e_1, \ldots, e_n\}$ a local orthonormal frame on M^n such that $h_{ij} = \lambda_i \delta_{ij}$. So, proceeding as in [65], from the differential Bianchi identity and the fact that $\overline{g}_{AB;C} \equiv 0$ we get

$$\begin{aligned}\sum_{i,k}\lambda_i \overline{R}_{(n+1)iik;k} &= -\sum_{i,k}\lambda_i\left(\overline{R}_{ikik;(n+1)} + \overline{R}_{k(n+1)ik;i}\right) \\ &= -\sum_i \lambda_i\left(\overline{R}_{ii;(n+1)} - \overline{R}_{(n+1)i;i}\right) \\ &= -\sum_i \lambda_i\left(\lambda\overline{g}_{ii;(n+1)} - \lambda\overline{g}_{(n+1)i;i}\right) = 0\end{aligned} \tag{4.30}$$

and

$$\sum_{i,k} \lambda_i \overline{R}_{(n+1)kik;i} = \sum_i \lambda_i \overline{R}_{(n+1)i;i} = \sum_i \lambda_i \lambda \overline{g}_{(n+1)i;i} = 0, \tag{4.31}$$

where $\overline{R}_{ijkl;m}$ are the covariant derivatives of $\overline{R}_{ijkl}$ on $\mathcal{E}_1^{n+1}$. Consequently, from (4.30) and (4.31) we obtain

$$\sum_{i,j,k} \left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)iki;j} \right) h_{ij} = 0. \tag{4.32}$$

Still, since we are assuming that $b \leq \overline{\mathcal{R}}$, we have that the relation (4.20) holds, and hence we can apply Lemma 4.1.1 to guarantee that

$$\sum_{i,j,k} h_{ijk}^2 - n^2 |\nabla H|^2 \geq 0. \tag{4.33}$$

Thus, from (4.29), (4.32) and (4.33) we have

$$\begin{aligned} \mathcal{L}(nH) \geq & (|A|^2)^2 - nH \sum_i \lambda_i^3 - 2 \sum_{i,k} (\lambda_i \lambda_k \overline{R}_{kiik} + \lambda_i^2 \overline{R}_{ikik}) \\ & - \left(nH \sum_i \lambda_i \overline{R}_{(n+1)ii(n+1)} + |A|^2 \sum_k \overline{R}_{(n+1)k(n+1)k} \right). \end{aligned} \tag{4.34}$$

Moreover, it is not difficult to verify the following algebraic relations

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \kappa_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{4.35}$$

Hence, from equations (4.17) and (4.35), we have

$$\begin{aligned} (|A|^2)^2 - nH \sum_i \lambda_i^3 &= (|\Phi|^2 + nH^2)^2 - nH \sum_i \kappa_i^3 - 3nH^2|\Phi|^2 - n^2 H^4 \\ &= |\Phi|^4 - nH^2|\Phi|^2 - nH \sum_i \kappa_i^3. \end{aligned}$$

At this point, we observe that, when $n = 2$, since Φ is traceless, we have $\sum_i \kappa_i^3 = 0$ and so,

$$(|A|^2)^2 - 2H \sum_i \lambda_i^3 = |\Phi|^2 \left(|\Phi|^2 - 2H^2 \right).$$

On the other hand, when $n \geq 3$, it follows from Lemma 3.1.3 that

$$(|A|^2)^2 - nH \sum_i \lambda_i^3 \geq |\Phi|^2 \left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - nH^2 \right). \tag{4.36}$$

Consequently, inequality (4.36) holds for all $n \geq 2$.

By using the curvature conditions (4.3) and (4.4), after straightforward computations we get

$$-\left(\sum_i nH\lambda_i\overline{R}_{(n+1)ii(n+1)} + |A|^2\sum_k \overline{R}_{(n+1)k(n+1)k}\right) = c_1(|A|^2 - nH^2) \tag{4.37}$$

and

$$\begin{aligned}-2\sum_{i,k}(\lambda_i\lambda_k\overline{R}_{kiik} + \lambda_i^2\overline{R}_{ikik}) &\geq c_2\sum_{i,k}(\lambda_i - \lambda_k)^2\\ &= 2nc_2(|A|^2 - nH^2).\end{aligned} \tag{4.38}$$

Therefore, inserting (4.36), (4.37) and (4.38) in (4.34), we have

$$\begin{aligned}\mathcal{L}(nH) &\geq |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - nH^2\right) + c_1|\Phi|^2 + 2nc_2|\Phi|^2\\ &= |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2 - c)\right),\end{aligned} \tag{4.39}$$

where $c = \frac{c_1}{n} + 2c_2$.

But from (4.17), we obtain

$$\frac{1}{n-1}|\Phi|^2 = nH^2 + naH + n(b - \overline{\mathcal{R}}). \tag{4.40}$$

Since $\mathcal{E}_1^{n+1}$ satisfies curvature condition (4.3), it follows that $\overline{\mathcal{R}}$ is a constant. If M^n is totally geodesic, then the operator $\mathcal{L}$ reduces to $\mathcal{L} = \frac{n-1}{2}a\Delta$ where $a > 0$ is any positive constant (see Remark 4.1.3). In this case $|\Phi|^2 \equiv 0$ and the inequality in Proposition 4.1.4 holds trivially. On the other hand, if M^n is not totally geodesic then Lemma 4.1.2 guarantees that the operator $\mathcal{P}$ is positive definite if $b < \overline{\mathcal{R}}$, and $\mathcal{P}$ is positive semi-definite if $b = \overline{\mathcal{R}}$. In any case, from (4.40) we have

$$\begin{aligned}\frac{1}{n-1}\mathcal{L}(|\Phi|^2) &= 2H\mathcal{L}(nH) + 2n\langle\mathcal{P}(\nabla H), \nabla H\rangle + a\mathcal{L}(nH)\\ &\geq 2\left(H + \frac{a}{2}\right)\mathcal{L}(nH),\end{aligned} \tag{4.41}$$

since (4.22) gives that $\mathcal{L}(u^2) = 2u\mathcal{L}(u) + 2\langle\mathcal{P}(\nabla u), \nabla u\rangle$ for every $u \in \mathcal{C}^2(M)$.

Therefore, from (4.39) and (4.41) we get

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H + \frac{a}{2}\right)|\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2 - c)\right). \tag{4.42}$$

Besides, from (4.23) we have

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + \overline{\mathcal{R}} - aH - b. \tag{4.43}$$

Consequently, taking into account that $(H + \frac{a}{2}) \geq 0$, we can write

$$H + \frac{a}{2} = \sqrt{\frac{|\Phi|^2}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}}. \tag{4.44}$$

From (4.39), after a straightforward computation, we have

$$|\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - n(H^2 - c) = \varphi_{a,b}(|\Phi|), \tag{4.45}$$

where $\varphi_{a,b}(x)$ is the function defined in (4.28). Therefore, replacing (4.45) and (4.44) in (4.42), we obtain the desired inequality. $\square$

4.1.2 Complete LW spacelike hypersurfaces immersed in $\mathcal{E}_1^{n+1}$ satisfying standard curvature constraints

Having seen this, we are in position to establish our rigidity results using the two suitable maximum principles due to [8, 9] jointly with the modified Cheng-Yau's operator $\mathcal{L}$ concerning complete LW spacelike hypersurfaces immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$.

Theorem 4.1.5. *Let M^n be a complete noncompact LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ for some constants $a, b \in \mathbb{R}$ with $b < \overline{\mathcal{R}}$. Suppose that $|A|^2 \leq 2\sqrt{n-1}c$ with $c = \frac{c_1}{n} + 2c_2 > 0$. If $|\Phi|$ converges to zero at infinity, then M^n is a totally umbilical hypersurface.*

Proof. Let us suppose by contradiction that such a LW spacelike hypersurface M^n is not totally umbilical. We consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$ and we claim that the required conditions to apply Lemma 1.0.4 are satisfied. Surely, f is non-identically vanishing function which converges to zero at infinity. Moreover, we can obtain from Lemma 4.1.2 that $\mathcal{P}$ is positive definite for $b < \overline{\mathcal{R}}$ and then

$$\langle \nabla f, X \rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2) \rangle \geq 0. \tag{4.46}$$

We affirm that $\mathrm{div} X \geq 0$. Indeed, we have from equation (4.41) that

$$\frac{1}{2(n-1)} \mathcal{L}(|\Phi|^2) \geq (H + \frac{a}{2}) \mathcal{L}(nH). \tag{4.47}$$

Besides that, from (4.17) we can verify that

$$\begin{aligned} |A|^2 - 2nH^2 \;\; = \;\; & \frac{1}{2\sqrt{n-1}} \left(\left(\sqrt{n-1} + 1\right) |\Phi| - \left(\sqrt{n-1} - 1\right) \sqrt{n} H \right)^2 \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| - \frac{n}{2\sqrt{n-1}} |A|^2. \end{aligned} \tag{4.48}$$

Thus, from (4.39) and (4.48) we obtain

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(nc - \frac{n}{2\sqrt{n-1}}|A|^2 \right) \geq 0, \tag{4.49}$$

from the assumption that $|A|^2 \leq 2\sqrt{n-1}c$.

As $\left(H + \frac{a}{2}\right) \geq 0$, from (4.27), (4.47) and (4.49), we have

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)\mathcal{L}(nH) \geq 0.$$

Hence, we can apply Lemma 1.0.4 to get that

$$\langle \nabla f, X \rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2) \rangle = 0.$$

Therefore, since the operator $\mathcal{P}$ is positive definite, we conclude that $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant. But f converges to zero at infinity, so it must be identically zero, leading us to a contradiction since we are supposing that M^n is not a totally umbilical hypersurface. $\square$

The next result involves polynomial volume growth.

Theorem 4.1.6. *Let M^n be a complete noncompact LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions (4.3) and (4.4), such that $R = aH + b$ for some $a, b \in \mathbb{R}$ with $b < \overline{\mathcal{R}}$. Suppose that $|\nabla A|$ is bounded and $\sup_M |A|^2 < 2\sqrt{n-1}c$ with $c = \frac{c_1}{n} + 2c_2 > 0$. If M^n has polynomial volume growth, then M^n is is a totally umbilical hypersurface.*

Proof. Let us take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$. By hypothesis, we have that $|A|$ is bounded on M^n and, consequently, from definition (4.24), we get

$$|X| = |\mathcal{P}(\nabla|\Phi|^2)| \leq |\mathcal{P}||\nabla|\Phi|^2| \leq k|\nabla|\Phi|^2|,$$

for some positive constant k. Besides that, by equation (4.17), since $|A|$ is bounded, $|\Phi|$ is also bounded on M^n and as we are supposing the boundedness of $|\nabla A|$, it follows the boundedness of $|\nabla\Phi|$. Thereby, Kato's inequality guarantees that

$$|X| \leq k|\nabla|\Phi|^2| = 2k|\Phi||\nabla|\Phi|| \leq 2k|\Phi||\nabla\Phi| \leq C < +\infty, \tag{4.50}$$

for some positive constant C.

Now, we must have $\mathrm{div}X \geq \alpha f$ on M^n, for some positive constant $\alpha \in \mathbb{R}$. For this, we can combine (4.27) and (4.49) with (4.47) to obtain

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2(n-1)\left(H + \frac{a}{2}\right)\left(nc - \frac{n}{2\sqrt{(n-1)}}|A|^2\right)|\Phi|^2. \tag{4.51}$$

Since we are assuming $\sup_M(|A|^2) < 2\sqrt{n-1}c$ and $(H + \frac{a}{2}) \geq \beta$, for some positive constant

$\beta \in \mathbb{R}$ (see Remark 2.1.17), from (4.51) we get

$$\operatorname{div}(X) \geq \alpha|\Phi|^2, \tag{4.52}$$

where $\alpha = 2(n-1)\beta\left(nc - \frac{n}{2\sqrt{(n-1)}}\sup_M(|A|^2)\right) > 0$. Therefore, in case that M^n is a complete noncompact hypersurface, conditions (4.46), (4.50) and (4.52) are verified and we are able to apply Lemma 1.0.3 to obtain that $|\Phi|^2 \leq 0$. Hence, $|\Phi| = 0$ on M^n, guaranteeing that M^n is a totally umbilical hypersurface.

In the case that M^n is a compact hypersurface, we can integrate both sides of (4.49) and use the Divergence Theorem to get that

$$\int_M |\Phi|^2 \left(nc - \frac{n}{2\sqrt{(n-1)}}|A|^2\right) \mathrm{dM} \leq \int_M \mathcal{L}(nH)\,\mathrm{dM} = 0,$$

since the operator $\mathcal{L}$ is a divergence type as it was observed in (4.27). Consequently, as we are assuming that $\sup_M(|A|^2) < 2\sqrt{n-1}c$, we must have $|\Phi| = 0$ on M^n and M^n is also a totally umbilical hypersurface. □

As an application of [14, Lemma 4.2] (see also [15, Theorem 6.13]), we establish the following Omori-Yau maximum principle which will be our analytical key tool for the proof of our characterization result of LW spacelike hypersurfaces in an Einstein spacetime $\mathcal{E}_1^{n+1}$.

Proposition 4.1.7. *Let M^n be a complete noncompact LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with $b \leq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, assume that the mean curvature function H does not change sign and $b \leq R$. If $\sup_M |\Phi|^2 < +\infty$, then the Omori-Yau maximum principle holds on M^n for the operator $\mathcal{L}$ defined in* (4.22).

Proof. Since $\mathcal{E}_1^{n+1}$ satisfies curvature condition (4.3), we have that $\overline{\mathcal{R}}$ is constant. Now, taking into account (4.17) we get

$$|\Phi|^2 = n(n-1)\left(H^2 + aH\right) + n(n-1)\left(b - \overline{\mathcal{R}}\right). \tag{4.53}$$

Since we are assuming $\sup_M |\Phi|^2 < +\infty$, from (4.53) we conclude that $\sup_M H < +\infty$. Thus, from (4.24) we have

$$\operatorname{tr}(P) = n(n-1)H + \frac{n(n-1)}{2}a$$

and, hence,

$$\sup_M \operatorname{tr}(P) < +\infty. \tag{4.54}$$

On the other hand, from (4.17) and curvature condition (4.4) we see that the sectional curvatures of M^n satisfy

$$R_{ijij} \geq c_2 - \left(nH + \frac{n-1}{2}a\right)^2 > -\infty. \tag{4.55}$$

Furthermore, Lemma 4.1.2 guarantees that the operator $\mathcal{L}$ is semi-elliptic. Therefore, taking into account (4.22), (4.54) and (4.55), we can apply [14, Lemma 4.2] to reach the desired result. $\square$

Now, applying Proposition 4.1.4, we are able to prove the following result for LW spacelike hypersurfaces immersed in an Einstein spacetime $\mathcal{E}_1^{n+1}$.

Theorem 4.1.8. *Let M^n be a complete LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions (4.3) and (4.4), such that $R = aH + b$ with $b \leq \overline{\mathcal{R}} < b + c$, where $c = 2c_2 + \frac{c_1}{n} > 0$, and $b \leq R$. In the case where $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change sign. Then*

(i) either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical hypersurface,

(ii) or

$$\sup_M |\Phi|^2 \geq \alpha(n, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, a, b, c, \overline{\mathcal{R}})$ is a positive constante depending on n, a, b, c and $\overline{\mathcal{R}}$.

In particular, if $b < \overline{\mathcal{R}}$, the equality $\sup_M |\Phi|^2 = \alpha(n, a, b, c, \overline{\mathcal{R}})$ holds and this supremum is attained at some point of M^n, then M^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Proof. If $\sup_M |\Phi|^2 = 0$, then M^n is totally umbilical and, hence, item (i) holds. If $\sup_M |\Phi|^2 = +\infty$, then (ii) is trivially satisfied. So, let us suppose that $0 < \sup_M |\Phi|^2 < +\infty$ and let us take $u = |\Phi|^2$. Then, from Proposition 4.1.4 we get

$$\mathcal{L}(u) \geq f(u), \tag{4.56}$$

where

$$f(u) = 2(n-1)u\varphi_{a,b}(\sqrt{u})\sqrt{\frac{u}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}}$$

and $\varphi_{a,b}(x)$ is given by (4.28).

If M^n is compact, there exists a point $p_0 \in M^n$ such that $u(p_0) = u^* = \sup u$. Consequently, $\nabla u(p_0) = 0$ and $\mathcal{L}u(p_0) \leq 0$. Therefore, from (4.56) we get $f(u^*) \leq 0$. Now, assume that M^n is complete and non-compact. Since $u^* < +\infty$, Proposition 4.1.7 guarantees that there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ satisfying

$$u(p_k) > u^* - \frac{1}{k} \quad \text{and} \quad \mathcal{L}u(p_k) < \frac{1}{k}, \tag{4.57}$$

for every $k \in \mathbb{N}$. Therefore from (4.56) and (4.57), we get

$$\frac{1}{k} > \mathcal{L}u(p_k) \geq f(u(p_k)). \tag{4.58}$$

Taking into (4.58) the limit when $k \to +\infty$, by continuity, we have

$$f(u^*) = 2(n-1)u^*\varphi_{a,b}(\sqrt{u^*})\sqrt{\frac{u^*}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}} \leq 0.$$

Since $u^* > 0$ and $b \leq \overline{\mathcal{R}}$, we obtain

$$\varphi_{a,b}(\sqrt{u^*}) \leq 0. \tag{4.59}$$

Note that the hypotheses $b \leq \overline{\mathcal{R}} < b + c$ and $b \leq R$ guarantee us that

$$\begin{aligned}
\varphi_{a,b}(0) &= na\sqrt{\overline{\mathcal{R}} - b + \frac{a^2}{4}} - n\left(\frac{a^2}{2} + \overline{\mathcal{R}} - b - c\right) \\
&\geq na\sqrt{\frac{a^2}{4}} - n\left(\frac{a^2}{2} + \overline{\mathcal{R}} - b - c\right) = -n\left(\overline{\mathcal{R}} - b - c\right) > 0.
\end{aligned}$$

On the other hand, it is not difficult to verify that the function $\varphi_{a,b}(x)$ is strictly decreasing for $x \geq 0$. Thus, by the continuity of $\varphi_{a,b}(x)$, we may assume the equation $\varphi_{a,b}(x) = 0$ attains its positive root at $x_0 = \sqrt{\alpha(n,a,b,c,\overline{\mathcal{R}})} > 0$. Therefore, (4.59) implies

$$u^* \geq x_0^2 = \alpha(n,a,b,c,\overline{\mathcal{R}}),$$

that is,

$$\sup_M |\Phi|^2 \geq \alpha(n,a,b,c,\overline{\mathcal{R}}).$$

This proves the inequality in (ii).

Moreover, equality $\sup_M |\Phi|^2 = \alpha(n,a,b,c,\overline{\mathcal{R}})$ holds if and only if $\sqrt{u^*} = x_0$. Thus $\varphi_{a,b}(\sqrt{u}) \geq 0$ on M^n, which jointly with (4.56) implies that

$$\mathcal{L}(u) \geq 0 \quad \text{on} \quad M^n.$$

Now, suppose that $b < \overline{\mathcal{R}}$. Hence, Lemma 4.1.2 assures that the operator $\mathcal{L}$ is elliptic. Therefore, if there exists a point $p_0 \in M^n$ such that $|\Phi(p_0)| = \sup_M |\Phi|$, from the maximum principle the function $u = |\Phi|^2$ must be constant and, consequently, $|\Phi| \equiv x_0$. Thus,

$$0 = \mathcal{L}(|\Phi|^2) \geq 2(n-1)|\Phi|^2\varphi_{a,b}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + b - \overline{\mathcal{R}} + \frac{a^2}{4}}.$$

Hence, all the inequalities along the proof of Proposition 4.1.4 must be equalities. In particular, since $\mathcal{L}$ is elliptic if and only if $\mathcal{P}$ is positive defined, returning to (4.41) we obtain that H is constant. Moreover, it also occurs equality in (4.33) or, equivalently,

$$|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0.$$

So, it follows that λ_i is constant for every $i = 1, \ldots, n$, that is, M^n is an isoparametric hypersurface. Finally, (4.59) must be also an equality, which guarantees that occurs the equality in Lemma 3.1.3. This implies that the hypersurface has exactly two distinct principal curvatures one of which is simple. □

Remark 4.1.9. Considering $b = 0$ in Theorem 4.1.8, we have the case when the mean and scalar curvatures of the spacelike hypersurface are proportional to each other, that is, $R = aH$ for a nonzero constant $a \in \mathbb{R}$. So, Theorem 4.1.8 can be regarded as a sort of extension of similar characterization results obtained by Li in [61] and Shu in [80] when the ambient space is the de Sitter space $\mathbb{S}_1^{n+1}$.

From Theorem 4.1.8, we use a classical result of congruence due to Abe, Koike and Yamaguchi (cf. [1, Theorem 5.1]) to obtain the following

Corollary 4.1.10. *Let M^n be a complete LW spacelike hypersurface immersed in de Sitter space $\mathbb{S}_1^{n+1}$, such that $R = aH + b$ with $0 < b \leq 1$ and $R \geq 1$. In the case where $b = 1$, assume further that the mean curvature function H does not change sign. Then*

(i) either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical hypersurface,

(ii) or

$$\sup_M |\Phi|^2 \geq \alpha(n, a, b, 1) > 0,$$

In particular, if $b < 1$, the equality $\sup_M |\Phi|^2 = \alpha(n, a, b, 1)$ holds and this supremum is attained at some point of M^n, then M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1 + r^2})$ of radius $r > 0$.

Recall that a Riemannian manifold M^n is said to be parabolic if the constant functions are the only subharmonic functions on M^n which are bounded from above, that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

So, considering the Cheng-Yau modified operator $\mathcal{L}$, we say that M^n is $\mathcal{L}$-parabolic if the only solutions of the inequality $\mathcal{L}(u) \geq 0$ which are bounded from above are the constant functions. In this setting, and motivated by Theorem 3 in [14] we have the following result.

Theorem 4.1.11. *Let M^n be a complete LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature conditions (4.3) and (4.4), such that $R = aH + b$ with $b \leq \overline{\mathcal{R}} < b + c$, where $c = 2c_2 + \dfrac{c_1}{n} > 0$ and $b \leq R$. In the case $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change sign. Suppose that M^n is not totally umbilical. If M^n is $\mathcal{L}$-parabolic, then*

$$\sup_M |\Phi|^2 \geq \alpha(n, a, b, c, \overline{\mathcal{R}}) > 0. \tag{4.60}$$

Moreover, if the equality occurs in (4.60), then M^n is a isoparametric hypersurface with two distinct principal curvatures one of which is simple.

Proof. If $\sup_M |\Phi|^2 = +\infty$ then there is nothing to prove. On the other hand, in the case that $0 < \sup_M |\Phi|^2 < +\infty$, reasoning as in the first part of the proof of Theorem 4.1.8, we guarantee that $\sup_M |\Phi|^2 \geq \alpha(n,a,b,c,\overline{\mathcal{R}})$. Moreover, if equality holds in (4.60), then we have $\varphi_{a,b}(|\Phi|) \geq 0$ and, consequently, $\mathcal{L}(|\Phi|^2) \geq 0$ on M^n. Therefore, from the $\mathcal{L}$-parabolicity of M^n we conclude that the function $u = |\Phi|^2$ must be constant and equal to $\alpha(n,a,b,c,\overline{\mathcal{R}})$. At this point, we can reason as in the proof of the previous theorem. □

It is not difficult to verify that from Theorem 4.1.11 jointly with [10, Corollary 2] we get

Corollary 4.1.12. *Let M^n be a complete LW spacelike hypersurface immersed in de Sitter space $\mathbb{S}_1^{n+1}$, such that $R = aH + b$ with $0 < b \leq 1$ and $R \geq 1$. In the case where $b = 1$, assume further that the mean curvature function H does not change sign. Suppose that M^n is not totally umbilical. If M^n is $\mathcal{L}$-parabolic, then*

$$\sup_M |\Phi|^2 \geq \alpha(n,a,b,1) > 0,$$

with equality if and only if M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$.

We are also in a position to establish the following $\mathcal{L}$-parabolicity criterium.

Proposition 4.1.13. *Let M^n be a complete LW spacelike hypersurface immersed with parallel normalized mean curvature vector field in an Einstein spacetime $\mathcal{E}_1^{n+1}$ satisfying curvature condition* (4.3)*, such that $R = aH + b$ and $b \leq \overline{\mathcal{R}}$. In the case $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change sign and that $b \leq R$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,*

$$\int_0^{+\infty} \frac{dr}{\operatorname{vol}(\partial B_r)} = +\infty, \tag{4.61}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Proof. We consider on M^n the symmetric $(0,2)$ tensor field $\mathcal{T}$ given by

$$\mathcal{T}(X,Y) = \langle \mathcal{P}X, Y \rangle,$$

or, equivalently,

$$\mathcal{T}(\nabla u, \cdot)^\sharp = \mathcal{P}(\nabla u),$$

for every $u \in \mathcal{C}^2(M)$, where $\sharp : T^*M \to TM$ denotes the musical isomorphism. Thus, from (4.27) we obtain

$$\mathcal{L}(u) = \operatorname{div}\left(\mathcal{T}(\nabla u, \cdot)^\sharp\right).$$

On the other hand, as $\sup_M |\Phi|^2 < +\infty$, from equation (4.53), we have $\sup_M H < +\infty$. So, we can define a positive continuous function ξ_+ on $[0, +\infty)$, by

$$\xi_+(r) = 2n \sup_{\partial B_r} H + (n-1)a. \tag{4.62}$$

Thus, from (4.62) we have

$$\xi_+(r) = 2n \sup_{\partial B_r} H + (n-1)a \leq 2n \sup_{M} H + (n-1)a < +\infty. \tag{4.63}$$

Hence, from (4.61) and (4.63) we get

$$\int_0^{+\infty} \frac{dr}{\xi_+(r)\mathrm{vol}(\partial \mathrm{B_r})} = +\infty.$$

Therefore, we can apply [75, Theorem 2.6] to conclude the proof. □

4.2 Rigidity results for closed LW hypersurfaces in an Einstein manifold $\mathcal{E}^{n+1}$

Our purpose here is to study the umbilicity of LW hypersurfaces immersed in an Einstein manifold satisfying standard curvature constraints which, in particular, are verified by a Riemannian space with constant sectional curvature (see Remark 4.2.1). Our approach is based on the maximum principles established in [8, 9].

4.2.1 Preliminaries

Along this section, we will consider an n-dimensional, orientable and connected hypersurface M^n immersed into a Riemannian manifold $\overline{M}^{n+1}$. We choose a local orthonormal frame $(e_1, \cdots, e_{n+1})$ in $\overline{M}^{n+1}$ with dual coframe $(\omega_1, \ldots, \omega_{n+1})$ such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. So, we will use the following convention of indices

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n.$$

In this setting, $\overline{R}_{ABCD}$, $\overline{R}_{CD}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of $\overline{M}^{n+1}$. We have

$$\overline{R}_{CD} = \sum_B \overline{R}_{BCDB} \quad \text{and} \quad \overline{R} = \sum_A \overline{R}_{AA}.$$

Furthermore, restricting all these tensors to M^n, since $\omega_{n+1} = 0$ on M^n, we get

$$-\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

Thus, from Cartan's Lemma we obtain

$$\omega_{(n+1)i} = \sum_j h_{ij}\omega_j \quad \text{and} \quad h_{ij} = h_{ij}. \tag{4.64}$$

This gives the second fundamental form of M^n, $A = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$, and its squared length $|A|^2 = \sum_{i,j} h_{ij}^2$. Beyond that, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

As it is well-known, the *Gauss equation* is given by

$$R_{ijkl} = \overline{R}_{ijkl} + \left(h_{ik}h_{jl} - h_{il}h_{jk}\right), \tag{4.65}$$

where R_{ijkl} are the components of the curvature tensor of M^n. Moreover, the first covariant derivatives h_{ijk} of h_{ij} satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\omega_{kj} - \sum_k h_{jk}\omega_{ki}. \tag{4.66}$$

By exterior differentiation of (4.64) we obtain the *Codazzi equation*

$$h_{ijk} - h_{ikj} = -\overline{R}_{(n+1)ijk}.$$

The second covariant derivatives h_{ijkl} of h_{ij} are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}.$$

Taking the exterior derivative in (4.66) we obtain the following *Ricci formula*

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{jm}R_{mikl}.$$

From (4.65), the Ricci curvature R_{ik} and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{ikjk} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{4.67}$$

and

$$R = \frac{1}{n-1}\sum_i R_{ii}. \tag{4.68}$$

Hence, from (4.67) and (4.68), we get the following relation

$$|A|^2 = n^2H^2 + n(n-1)R - \sum_{i,j} \overline{R}_{ijij}. \tag{4.69}$$

Returning to the context of a hypersurface immersed in a Riemannian manifold $\overline{M}^{n+1}$, we will assume the existence constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of $\overline{M}^{n+1}$ satisfies the following two constraints

$$\overline{K}(u,\eta) = \frac{c_1}{n}, \tag{4.70}$$

for any tangent vector $u \in TM$ and normal vector $\eta \in TM^{\perp}$; and

$$\overline{K}(u, v) \geq c_2, \tag{4.71}$$

for any tangent vectors $u, v \in TM$.

Remark 4.2.1. We note that, when the ambient Riemannian manifold $\overline{M}^{n+1}$ has constant sectional curvature c, it satisfies curvature constraints (4.70) and (4.71) for any hypersurface M^n immersed in $\overline{M}^{n+1}(c)$ with $\frac{c_1}{n} = c_2 = c$. But, we can also find examples of Einstein manifolds which do not have constant sectional curvature. Indeed, inspired by [65, Example 1.1], the product space $\mathbb{R} \times M^n$, where M^n is a Ricci flat Riemannian manifold (which is not flat; for instance, the Schwarszchild space), is an Einstein manifold. Moreover, supposing that the sectional curvature K_M of M^n is such that $K_M(u, v) \geq c_2$ for any $u, v \in TM$ and some constant c_2 we can verify that the curvature constraint (4.71) is satisfied. Besides, we see that curvature constraint (4.70) is verified for $c_1 = 0$.

We remember that a hypersurface is *linear Weingarten* (LW) when its mean and normalized scalar curvatures are linearly related, that is, when they satisfy the following linear relation

$$R = aH + b, \tag{4.72}$$

for constants $a, b \in \mathbb{R}$. We observe that when $a = 0$, (4.72) reduces to R constant.

For a LW hypersurface Σ^n satisfying (4.72) we introduce the second-order linear differential operator $\mathcal{L} : C^{\infty}(\Sigma) \to C^{\infty}(\Sigma)$ defined by

$$\mathcal{L} = L - \frac{n-1}{2} a\Delta, \tag{4.73}$$

where Δ is the Laplacian operator on Σ^n and $L : C^{\infty}(\Sigma) \to C^{\infty}(\Sigma)$ denotes the Cheng-Yau operator, which is given by

$$Lu = \mathrm{tr}(P \circ \mathrm{Hess}\,(u)), \tag{4.74}$$

for every $u \in C^{\infty}(\Sigma)$, where Hess is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the first Newton transformation of Σ^n which is given by $P = nHI - A$. So, from (4.73) and (4.74), we have that

$$\mathcal{L}u = \mathrm{tr}(\mathcal{P} \circ \mathrm{Hess}\,(u)),$$

with

$$\mathcal{P} = \left(nH - \frac{n-1}{2} a\right) I - A. \tag{4.75}$$

Thus, by using the standard notation $\langle , \rangle$ for the (induced) metric of Σ^n, we get

$$Lu = \sum_i \langle P(\nabla_{e_i} \nabla u), e_1 \rangle,$$

where $\{e_1, \cdots, e_n\}$ is a (local) orthonormal frame on Σ^n. Consequently, we obtain

$$\begin{aligned} \operatorname{div}(P(\nabla u)) &= \sum_i \langle (\nabla_{e_i} P)(\nabla u), e_i\rangle + \sum_i \langle P(\nabla_{e_i}\nabla u), e_1\rangle \\ &= \langle \operatorname{div} P, \nabla u\rangle + Lu. \end{aligned} \tag{4.76}$$

Let us assume that $\overline{M}^{n+1}$ is an Einstein manifold, so that there exists a constant λ satisfying $\overline{\mathrm{Ric}} = \lambda\langle , \rangle$, where $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\overline{M}^{n+1}$. Thus, from [7, Lemma 3.1]

$$\langle \operatorname{div} P, \nabla u\rangle = \sum_i \langle \overline{R}(N, e_i)e_i, \nabla u\rangle = -\overline{\mathrm{Ric}}(N, \nabla u) = -\lambda\langle N, \nabla u\rangle = 0,$$

where N stands for the Gauss mapping of Σ^n. Hence, from equation (4.76), we conclude that

$$Lu = \operatorname{div}(P(\nabla u)). \tag{4.77}$$

Thus, from (4.73) and (4.77), we can verify that $\mathcal{L}$ can be rewritten in the following divergence form

$$\mathcal{L}u = \operatorname{div}(\mathcal{P}(\nabla u)). \tag{4.78}$$

We can state the versions of Lemmas 4.2.2 and 4.2.3 for the context that we are working supposing that the hypersurface is immersed in an Einstein manifold $\mathcal{E}^{n+1}$ (see [12, Lemma 3.2, Lemma 3.4]).

Lemma 4.2.2. *Let M^n be a complete LW spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature conditions* (4.70) *and* (4.71)*, such that $R = aH + b$ with*

$$(n-1)a^2 + 4n(b - \overline{\mathcal{R}}) \geq 0. \tag{4.79}$$

Then,

$$|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2. \tag{4.80}$$

Moreover, if the inequality (4.79) *is strict and equality occurs in* (4.80)*, then H is constant on M^n.*

Lemma 4.2.3. *Let M^n be a LW spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature condition* (4.70)*, such that $R = aH + b$. Let μ_- and μ_+ be, respectively, the minimum and the maximum of the eigenvalues of the operator $\mathcal{P}$ defined in* (4.75) *at every point $p \in M^n$.*

If $b > \overline{\mathcal{R}}$, then the operator $\mathcal{L}$ defined in (4.73) *is elliptic, with*

$$\mu_- > 0 \quad \text{and} \quad \mu_+ < 2nH + (n-1)a.$$

In the case where $b = \overline{\mathcal{R}}$, assume further that the mean curvature function H does not change *sign and $b \geq R$. Then the operator $\mathcal{L}$ is semi-elliptic, with*

$$\mu_{-} \geq 0 \quad \text{and} \quad \mu_{+} \leq 2nH + (n-1)a,$$

unless M^n is totally geodesic. Moreover, in the case where $b > \overline{\mathcal{R}}$ on M^n, the above inequalities are strict and the operator $\mathcal{L}$ is elliptic.

In what follows, we will also work with the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij} \omega_i \otimes \omega_j,$$

where $\Phi_{ij} = h_{ij} - H\delta_{ij}$. Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ, we can check that Φ is traceless and

$$|\Phi|^2 = |A|^2 - nH^2. \tag{4.81}$$

Moreover, it holds for an Einstein manifold, from equations (4.69) and (4.81), the following algebraic relation :

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 - n(n-1)aH - n(n-1)(b - \overline{\mathcal{R}}), \tag{4.82}$$

where $\overline{\mathcal{R}} = \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$.

4.2.2 Complete LW hypersurfaces immersed in $\mathcal{E}^{n+1}$ satisfying standard curvature constraints

Now, we are in position to use the modified Cheng-Yau's operator $\mathcal{L}$ jointly with the lemmas quoted in the previous section to establish our umbilicity results concerning LW hypersurfaces M^n immersed in an Einstein manifold $\mathcal{E}^{n+1}$.

Theorem 4.2.4. *Let M^n be a complete LW hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature constraints* (4.70) *and* (4.71) *with $n \geq 3$, such that $R = aH + b$ with $b \geq \overline{\mathcal{R}}$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^n, for some positive constant β, and that $R > \overline{\mathcal{R}} - \frac{2}{n}c$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. Assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| \leq \gamma < x_R^*$, for some constant γ and x_R^* defined in* (3.16). *If M^n has polynomial volume growth and $\inf_R(Q_R(\gamma)) > 0$, then M^n is a totally umbilical hypersurface.*

Proof. Let us see that if we take the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$, it will fulfill the required conditions to apply Lemma 1.0.3. Indeed, by hypothesis we have that $|\Phi|$ is bounded on Σ^n and, by equation (4.82), $|A|$ is also bounded on M^n. Consequently, from definition (4.75), we get

$$|X| = |\mathcal{P}(\nabla|\Phi|^2)| \leq |\mathcal{P}||\nabla|\Phi|^2| \leq k|\nabla|\Phi|^2|,$$

for some positive constant k. Besides that, as we are supposing the boundedness of $|\Phi|$ and of $|\nabla\Phi|$, Kato's inequality guarantees that

$$|X| \leq k|\nabla|\Phi|^2| = 2k|\Phi||\nabla|\Phi|| \leq 2k|\Phi||\nabla\Phi| \leq C < +\infty, \tag{4.83}$$

for some positive constant C.

Also the condition

$$\langle \nabla f, X\rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2)\rangle \geq 0 \tag{4.84}$$

is verified because Lemma 4.2.3 gives that $\mathcal{P}$ is positive semi-definite for $b \geq \overline{\mathcal{R}}$.

Applying $\mathcal{L}$ in (4.82), we get that

$$\begin{aligned} \frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) - \frac{a}{2}\mathcal{L}(nH) \\ &= H\mathcal{L}(nH) + n\langle \mathcal{P}\nabla H, \nabla H\rangle - \frac{a}{2}\mathcal{L}(nH) \\ &\geq (H - \frac{a}{2})\mathcal{L}(nH). \end{aligned} \tag{4.85}$$

On the other hand, since $(\mathcal{E}^{n+1}, \overline{g})$ is an Einstein manifold, the components of its Ricci tensor satisfy $\overline{R}_{CD} = \lambda\overline{g}_{CD}$, for some constant $\lambda \in \mathbb{R}$. Moreover, we can consider a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $h_{ij} = \lambda_i\delta_{ij}$. So, proceeding as in [65], from the differential Bianchi identity and the fact that $\overline{g}_{AB;C} \equiv 0$ we get

$$\begin{aligned} \sum_{i,k}\lambda_i\overline{R}_{(n+1)iik;k} &= -\sum_{i,k}\lambda_i\left(\overline{R}_{ikik;(n+1)} + \overline{R}_{k(n+1)ik;i}\right) \\ &= -\sum_{i}\lambda_i\left(\overline{R}_{ii;(n+1)} - \overline{R}_{(n+1)i;i}\right) \\ &= -\sum_{i}\lambda_i\left(\lambda\overline{g}_{ii;(n+1)} - \lambda\overline{g}_{(n+1)i;i}\right) = 0 \end{aligned} \tag{4.86}$$

and

$$\sum_{i,k}\lambda_i\overline{R}_{(n+1)kik;i} = \sum_{i}\lambda_i\overline{R}_{(n+1)i;i} = \sum_{i}\lambda_i\lambda\overline{g}_{(n+1)i;i} = 0, \tag{4.87}$$

where $\overline{R}_{ijkl;m}$ are the covariant derivatives of $\overline{R}_{ijkl}$ on $\mathcal{E}^{n+1}$. Consequently, from (4.86) and (4.87) we obtain

$$\sum_{i,j,k}\left(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)iki;j}\right)h_{ij} = 0. \tag{4.88}$$

On the other hand, let us choose a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Since $R = aH + b$, from [12, Equation (2.10)] jointly with (4.88) and the definition of $\mathcal{L}$, we get

$$\begin{aligned} \mathcal{L}(nH) &= \sum_{i,j,k}h_{ijk}^2 - n^2|\nabla H|^2 + nH\sum_i\lambda_i^3 - |A|^4 \\ &\quad + \sum_i\overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j}(\lambda_i - \lambda_j)^2\overline{R}_{ijij}. \end{aligned} \tag{4.89}$$

Moreover, we have $\Phi_{i,j} = \mu_i \lambda_{ij}$ and, with straightforward computation, we verify that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \tag{4.90}$$

Besides that, from curvature constraints (4.70) and (4.71), we get

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) = c_1(nH^2 - |A|^2) = -c_1|\Phi|^2 \tag{4.91}$$

and

$$\begin{aligned}\sum_{i,j}(\lambda_i - \lambda_j)^2 \overline{R}_{ijij} &\geq c_2 \sum_{i,j}(\lambda_i - \lambda_j)^2 \\ &= 2nc_2(|A|^2 - nH^2) = 2nc_2|\Phi|^2.\end{aligned} \tag{4.92}$$

Since $c = 2c_2 - \frac{c_1}{n}$, from the Gauss equation jointly with (4.89), (4.90), (4.91) and (4.92), we obtain

$$\mathcal{L}(nH) = |\nabla A|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \tag{4.93}$$

Thereafter, as Lemma 4.2.2 is true as we are supposing $b \geq \overline{\mathcal{R}}$, we can use Lemma 3.1.3 for $n \geq 3$ to get from (4.93) that

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + nH^2 + nc \right). \tag{4.94}$$

Furthermore, from (4.17) we get

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R - \overline{\mathcal{R}}). \tag{4.95}$$

Thus, from (4.94) and (4.95) we achieve in

$$\mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{4.96}$$

where Q_R is defined in (3.15). Hence, using (4.85) jointly with (4.96), from (4.78) we conclude that

$$\operatorname{div} X = \operatorname{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2. \tag{4.97}$$

Since we have $\left(H - \frac{a}{2}\right) \geq \beta > 0$ by hypothesis and from the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq \sup_M |\Phi| \leq \gamma < x_R^*$, we have that

$$Q_R(|\Phi|) \geq Q_R(\gamma) > \inf_R(Q_R(\gamma)) > 0. \tag{4.98}$$

Then, from (4.97) and (4.98) we obtain

$$\mathrm{div}X \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq \alpha|\Phi|^2 \tag{4.99}$$

and $\mathrm{div}X \geq \alpha f$ for $\alpha = 2\beta \inf_R (Q_R(\gamma)) > 0$.

If M^n is a noncompact hypersurface with polynomial volume growth, we are able to apply Lemma 1.0.3 to obtain $|\Phi|^2 \leq 0$ on M^n. Then, $|\Phi| \equiv 0$, which means that M^n is a totally umbilical hypersurface.

If M^n is a compact hypersurface, we can integrate both sides of (4.99) and use the Divergence Theorem to get that

$$\int_M |\Phi|^2 \mathrm{dM} = 0.$$

Therefore, we have $|\Phi| \equiv 0$ and hence, M^n must also be a totally umbilical hypersurface. $\square$

Revisiting the proof of Theorem 4.2.4, we observe that if $n = 2$, then $\sum_i \mu_i^3 = 0$. Consequently, from (4.93) we get

$$\mathcal{L}(2H) \geq |\Phi|^2 \left(-|\Phi|^2 + 2H^2 + 2c\right)$$

and (4.94) is still true in this case. Hence, we also have the following umbilicity result.

Theorem 4.2.5. *Let M^2 be a complete LW surface immersed into an Einstein manifold $\mathcal{E}^3$ satisfying curvature constraints* (4.70) *and* (4.71)*, such that $R = aH + b$ with $b \geq \overline{\mathcal{R}}$. Suppose that $\left(H - \frac{a}{2}\right) \geq \beta$ on M^2, for some positive constant β, that $\inf_M R > 0$ and that $R > \overline{\mathcal{R}} - c$. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is a totally umbilical surface.*

Noting that when $R > 0$ is constant, the hypothesis $\inf_R (Q_R(\gamma)) > 0$ is automatically satisfied, from Theorems 4.2.4 and 4.2.5 we obtain, respectively, the following consequences:

Corollary 4.2.6. *Let M^n be a complete hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature constraints* (4.70) *and* (4.71) *with $n \geq 3$ and constant normalized scalar curvature $R \geq \overline{\mathcal{R}}$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. Suppose that $H \geq \beta$ on M^n, for some positive constant β and assume in addition that $|\nabla\Phi|$ is bounded and $\sup_M |\Phi| < x_R^*$, for x_R^* defined in* (3.16)*. If M^n has polynomial volume growth, then M^n is a totally umbilical hypersurface.*

Corollary 4.2.7. *Let M^2 be a complete surface immersed into an Einstein manifold $\mathcal{E}^3$ satisfying curvature constraints* (4.70) *and* (4.71)*, with constant normalized scalar curvature $R \geq \overline{\mathcal{R}}$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. Suppose that $H \geq \beta$ on M^2, for some positive constant β. Assume in addition that $|\Phi|$ and $|\nabla\Phi|$ are bounded. If M^2 has polynomial volume growth, then M^2 is a totally umbilical surface.*

In what follows we will apply Lemma 1.0.4 to get further umbilicity results concerning complete noncompact LW hypersurface in an Einstein manifold. So, we state and prove the following theorem.

Theorem 4.2.8. *Let M^n be a complete noncompact LW hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature constraints* (4.70) *and* (4.71) *with $n \geq 3$, such that $R = aH+b$ with $b > \overline{\mathcal{R}}$. Suppose that $R > \overline{\mathcal{R}} - \frac{2}{n}c$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. Assume in addition that $|\Phi| \leq x_R^*$, for x_R^* defined in* (3.16)*. If $|\Phi|$ converges to zero at infinity, then M^n is a totally umbilical hypersurface.*

Proof. Let us consider the smooth vector field $X = \mathcal{P}(\nabla|\Phi|^2)$ and the smooth function $f = |\Phi|^2$ and let us suppose that M^n is not a umbilical hypersurface. So, f is non-identically vanishing function which converges to zero at infinity. Moreover, we have that

$$\langle \nabla f, X \rangle = \langle \nabla|\Phi|^2, \mathcal{P}(\nabla|\Phi|^2) \rangle \geq 0.$$

We claim that $\mathrm{div}X \geq 0$. Indeed, we already know that

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})\mathcal{L}(nH) \quad \text{and} \quad \mathcal{L}(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \tag{4.100}$$

where Q_R is the function given by (3.15). Thus, since $\left(H - \frac{a}{2}\right) \geq 0$, from (4.100) jointly with the behavior of $Q_R(x)$ for $0 \leq |\Phi| \leq x_R^*$, we conclude that

$$\mathrm{div}X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \geq 2\left(H - \frac{a}{2}\right) Q_R(|\Phi|)|\Phi|^2 \geq 0.$$

Hence, we can apply Lemma 1.0.4 to get

$$\langle \nabla f, X \rangle = \langle \mathcal{P}(\nabla|\Phi|^2), \nabla|\Phi|^2 \rangle \equiv 0.$$

Consequently, since Lemma 4.2.3 gives that $\mathcal{P}$ is positive definite, we have that $\nabla|\Phi|^2 \equiv 0$. Thus, $f = |\Phi|^2$ is constant. But, since f converges to zero at infinity, it must be identically zero, leading us to a contradiction. Therefore, M^n is a complete noncompact totally umbilical hypersurface of $\mathcal{E}^{n+1}$. □

In the case $n = 2$, reasoning as in the proof of Theorem 4.2.8 we also obtain the following result.

Theorem 4.2.9. *Let M^2 be a complete noncompact LW surface immersed into an Einstein manifold $\mathcal{E}^3$ satisfying curvature constraints* (4.70) *and* (4.71)*, such that $R = aH + b$ with $b \geq \overline{\mathcal{R}}$. Suppose that $R > \overline{\mathcal{R}} - c$ for $c > 0$ and $R > \overline{\mathcal{R}} - c$ for $c \leq 0$. If $|\Phi|$ converges to zero at infinity, then M^2 is a totally umbilical surface.*

Taking into account the setup described in the previous section, we obtain the following rigidity result:

Theorem 4.2.10. *Let M^n be a complete LW hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature constraints* (4.70) *and* (4.71) *with $R = aH + b$ such that $a \geq 0$ and*

$b > \max\{\overline{\mathcal{R}} - c_0, \overline{\mathcal{R}}\}$. *If its total umbilicity tensor* Φ *satisfies*

$$\mathrm{tr}(\Phi^3) \geq -\frac{(n-2p)}{\sqrt{np(n-p)}}|\Phi|^3, \tag{4.101}$$

for some $1 \leq p \leq \frac{n-\sqrt{n}}{2}$, *then*

(i) either $\sup|\Phi| = 0$ *and* M^n *is a totally umbilical hypersurface,*

(ii) or

$$\sup_M |\Phi| \geq \alpha(a,b,n,p,\overline{\mathcal{R}},c_0) > 0,$$

where $\alpha(a,b,n,p,\overline{\mathcal{R}},c_0)$ *is a positive constant depending only on* a, b, n, p, $\overline{\mathcal{R}}$ *and* c_0. *Moreover, if the equality* $\sup_M |\Phi| = \alpha(a,b,p,n,\overline{\mathcal{R}},c_0)$ *holds and this supremum is attained at some point of* M^n, *then* M^n *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* p *and* $n-p$.

Proof. Initially we must obtain a suitable lower bound for the operator $\mathcal{L}$ acting on the squared norm of the total umbilicity tensor Φ of M^n. Since $\overline{\mathcal{R}}$ is constant, we get from (4.82) that

$$\begin{aligned} \frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) - \frac{a}{2}\mathcal{L}(nH) \\ &= H\mathcal{L}(nH) + n\langle \mathcal{P}\nabla H, \nabla H\rangle - \frac{a}{2}\mathcal{L}(nH). \end{aligned} \tag{4.102}$$

By using Lemma 4.2.3, we have that the operator $\mathcal{P}$ is positive definite. In particular, from (4.102) we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})\mathcal{L}(nH). \tag{4.103}$$

Without loss of generality we can choose the orientation of M^n such that $H > 0$, occurring the strict inequality because of $b > \overline{\mathcal{R}}$. From this, we claim that $H - \frac{a}{2} > 0$. Indeed, it is enough to see that we can rewrite equation (4.82) as

$$nH(nH-(n-1)a) = |A|^2 + n(n-1)(b - \overline{\mathcal{R}}) > 0. \tag{4.104}$$

So, formula (4.89) and inequality (4.103) jointly with Lemma 4.2.2 give

$$\begin{aligned} \frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \;\geq\; & (H-\frac{a}{2})(|\nabla A|^2 - n^2|\nabla H|^2 + nH\mathrm{tr}(A^3) - |A|^4) \\ & + (H-\frac{a}{2})\left(\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j}(\lambda_i-\lambda_j)^2\overline{R}_{ijij}\right) \\ \geq\; & (H-\frac{a}{2})(nH\mathrm{tr}(A^3) - |A|^4) \\ & + (H-\frac{a}{2})\left(\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j}(\lambda_i-\lambda_j)^2\overline{R}_{ijij}\right). \end{aligned} \tag{4.105}$$

The curvatures constraint (4.70) and (4.71) yield

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) = -c_1|\Phi|^2 \tag{4.106}$$

and

$$\sum_{i,j}(\lambda_i - \lambda_j)^2\overline{R}_{ijij} \geq 2nc_2|\Phi|^2. \tag{4.107}$$

Thus, plugging (4.106) and (4.107) into (4.105), we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})(nH\mathrm{tr}(A^3) - |A|^4 + nc_0|\Phi|^2). \tag{4.108}$$

On the other hand, it is not difficult to see that

$$\mathrm{tr}(A^3) = \mathrm{tr}(\Phi^3) + 3H|\Phi|^2 + nH^3. \tag{4.109}$$

Putting (4.109) into (4.108) we find

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})(-|\Phi|^4 + nH\mathrm{tr}(\Phi^3) + n(H^2 + c_0)|\Phi|^2). \tag{4.110}$$

Now, taking into account the Okumura type inequality (4.101), from (4.110) we get

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq (H - \frac{a}{2})|\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}H|\Phi| + n(H^2 + c_0)\right). \tag{4.111}$$

Since $H - \frac{a}{2} > 0$, we observe that equation (4.82) implies that the mean curvature can be written as

$$H - \frac{a}{2} = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(\frac{a^2}{4} + b - \overline{\mathcal{R}})}. \tag{4.112}$$

Thus, substituting (4.112) into (4.111) we get

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|\Phi|^2) \geq & \frac{(n-1)}{\sqrt{n(n-1)}}|\Phi|^2\Big\{-|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}\left(\frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(\frac{a^2}{4} + b - \overline{\mathcal{R}})} + \frac{a}{2}\right)\\ & |\Phi| + n\left[\left(\frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(\frac{a^2}{4} + b - \overline{\mathcal{R}})} + \frac{a}{2}\right)^2 + c_0\right]\Big\}\cdot\\ & \cdot\sqrt{|\Phi|^2 + n(n-1)(\frac{a^2}{4} + b - \overline{\mathcal{R}})}.\end{aligned} \tag{4.113}$$

After some straightforward computations, inequality (4.113) gives us the next one

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|\Phi|^2)\geq &\frac{1}{\sqrt{n(n-1)}}|\Phi|^2\Big\{-(n-1)|\Phi|^2-\frac{n(n-1)(n-2p)a}{2\sqrt{np(n-p)}}|\Phi|-\frac{(n-1)(n-2p)}{\sqrt{(n-1)p(n-p)}}|\Phi|\cdot\\ &\cdot\sqrt{|\Phi|^2+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})}+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})+|\Phi|^2 \qquad (4.114)\\ &+a\sqrt{n(n-1)}\sqrt{|\Phi|^2+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})}+n(n-1)\frac{a^2}{4}+n(n-1)c_0\Big\}.\end{aligned}$$

So, inequality (4.114) lead us to obtain the following estimate

$$\frac{1}{2}\mathcal{L}(|\Phi|^2)\geq\frac{1}{\sqrt{n(n-1)}}|\Phi|^2Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|)\sqrt{|\Phi|^2+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})}, \qquad (4.115)$$

where the function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}$ is given by

$$\begin{aligned}Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(x) \;=\;& -(n-2)x^2-\frac{n(n-1)(n-2p)a}{2\sqrt{np(n-p)}}x\\ &-\left((n-2p)\frac{\sqrt{n-1}}{\sqrt{p(n-p)}}x-a\sqrt{n(n-1)}\right)\sqrt{x^2+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})}\\ &+n(n-1)(\frac{a^2}{2}+b-\overline{\mathcal{R}}+c_0). \qquad (4.116)\end{aligned}$$

Now, we are going to finish the proof by applying the Omori-Yau's maximum principle to the operator $\mathcal{L}$ acting on the function $|\Phi|^2$. We note that if $\sup_M|\Phi|=+\infty$, then the claim (ii) of Theorem 4.2.10 trivially holds and there is nothing to prove. Otherwise, if $\sup_M|\Phi|<+\infty$, then the Omori-Yau maximum principle holds on M^n for the operator $\mathcal{L}$ and there exists a sequence of points $\{p_j\}$ in M^n such that

$$\lim|\Phi|(p_j)=\sup|\Phi| \quad \text{and} \quad \mathcal{L}(|\Phi|^2)(p_j)<\frac{1}{j}.$$

Hence, estimate (4.115) implies that

$$\frac{1}{j}>\mathcal{L}(|\Phi|^2)(p_j)\geq\frac{2}{\sqrt{n(n-1)}}|\Phi|^2(p_j)Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|(p_j))\sqrt{|\Phi|^2(p_j)+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})},$$

and, taking the limit as $j\to+\infty$, we infer

$$\left(\sup_M|\Phi|\right)^2Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sup_M|\Phi|)\sqrt{(\sup_M|\Phi|)^2+n(n-1)(\frac{a^2}{4}+b-\overline{\mathcal{R}})}\leq 0.$$

It follows that either $\sup_M|\Phi|=0$, which means that $|\Phi|\equiv 0$ and the hypersurface is totally umbilical, or $\sup_M|\Phi|>0$ and then $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sup_M|\Phi|)\leq 0$. In the latter case, since

$b > \max\{\overline{\mathcal{R}} - c_0, \overline{\mathcal{R}}\}$, we have that

$$Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(0) = n(n-1)a\sqrt{\frac{a^2}{4} + b - \overline{\mathcal{R}}} + n(n-1)(\frac{a^2}{2} + b - \overline{\mathcal{R}} + c_0) > 0.$$

Moreover, since $1 \leq p \leq \frac{n-\sqrt{n}}{2}$, the function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(x)$ is strictly decreasing for $x \geq 0$.

Hence, we guarantee the existence of a unique positive real number $\alpha(a,b,n,p,\overline{\mathcal{R}},c_0) > 0$, depending only on $a, b, n, p, \overline{\mathcal{R}}$ and c_0, such that $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\alpha(a,b,n,p,\overline{\mathcal{R}},c_0)) = 0$. Therefore, we must have

$$\sup_M |\Phi| \geq \alpha(a,b,n,p,\overline{\mathcal{R}},c_0) > 0,$$

concluding the proof of the first part of Theorem 4.2.10.

Finally, let us assume that equality $\sup_M |\Phi| = \alpha(a,b,n,p,\overline{\mathcal{R}},c_0)$ holds. In particular, we get

$$Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) \geq 0$$

on M^n and then (4.115) assures that $|\Phi|^2$ is a $\mathcal{L}$-subharmonic function on M^n, that is,

$$\mathcal{L}(|\Phi|^2) \geq 0 \quad \text{on} \quad M^n. \tag{4.117}$$

Furthermore, since $b > \overline{\mathcal{R}}$, Lemma 4.2.3 asserts that the operator $\mathcal{L}$ is elliptic. Thus, since M^n is complete and taking into account that we are assuming the existence of a point $p \in M^n$ such that $|\Phi(p)| = \alpha(a,b,n,p,\overline{\mathcal{R}},c_0) = \sup_M |\Phi|$, from (4.117) we can apply Hopf's strong maximum principle for the elliptic operator $\mathcal{L}$ acting on the function $|\Phi|^2$ to conclude that it must be constant, that is, $|\Phi| = \alpha(a,b,n,p,\overline{\mathcal{R}},c_0)$. Hence, it holds the equality in (4.115), namely,

$$\frac{1}{2}\mathcal{L}(|\Phi|^2) = 0 = \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(\frac{a^2}{4} + b - \overline{\mathcal{R}})}.$$

Consequently, all the inequalities along the proof of (4.115) must be, in fact, equalities. In particular, we obtain that equation (4.103) must be an equality, which jointly with the positiveness of the operator $\mathcal{P}$ imply that the mean curvature H is constant. Moreover, it also occurs equality in (4.105), that is,

$$|\nabla A|^2 = n^2|\nabla H|^2 = 0.$$

Therefore, the principal curvatures of M^n must be constant and M^n is an isoparametric hypersurface. Besides, equation (4.111) is also an equality, which implies by Lemma 3.1.3 that M^n has exactly two distinct constant principal curvatures, with multiplicities p and $n-p$. This finishes the proof. $\square$

Remark 4.2.11. When $\overline{M}^{n+1} = \mathbb{Q}_c^{n+1}$ is a Riemannian space form of constant sectional curvature c, the constants $\overline{\mathcal{R}}$ and c_0 in Theorem 4.2.10 just agree with c. For this reason, Theorem 4.2.10 can be regarded as a natural generalization of [53, Theorem 1].

Chapter 5

LW submanifolds in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$ with second fundamental form locally timelike

We are going to present the results of [27].

5.1 Preliminaries

Let M^n be an n-dimensional complete spacelike submanifold immersed in the $(n+p)$-dimensional semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$ of index $1 \leq q \leq p$ and constant curvature c. We choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{N}_q^{n+p}(c)$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

In this setting, the metric of $\mathbb{N}_q^{n+p}(c)$ is defined by $ds^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_A = 1$, $1 \leq A \leq n+p-q$ and $\epsilon_A = -1$, $n+p-q+1 \leq A \leq n+p$. Denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{N}_q^{n+p}(c)$, we have that the structure equations of $\mathbb{N}_q^{n+p}(c)$ are given by:

$$d\omega_A = -\sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \epsilon_B \omega_{AB} + \epsilon_A \omega_{BA} = 0, \tag{5.1}$$

$$d\omega_{AB} = -\sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \, \omega_C \wedge \omega_D, \tag{5.2}$$

where $K_{ABCD} = c\epsilon_A \epsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD})$.

Restricting all the tensors to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p.$$

Thus, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0$, by Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

This gives the second fundamental form of M^n, $A = \epsilon_\alpha \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j e_\alpha$. Furthermore, we define the mean curvature vector field h and the mean curvature function H of M^n, respectively, by

$$h = \frac{1}{n} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha \quad \text{and} \quad H = |h| = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}.$$

From (5.1) and (5.2), we get the structure equations of M^n

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad \text{and} \quad d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad (5.3)$$

where R_{ijkl} are the components of the curvature tensor of M^n. Therefore, from (5.3) we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_\alpha \epsilon_\alpha (h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha).$$

The components of the Ricci curvature R_{ij} and the normalized scalar curvature R of M^n are given, respectively, by

$$R_{ij} = (n-1)c\delta_{ij} + \sum_\alpha \epsilon_\alpha \left\{ \left(\sum_k h_{kk}^\alpha \right) h_{ij}^\alpha - \sum_{\alpha,k} h_{ik}^\alpha h_{kj}^\alpha \right\} \quad (5.4)$$

and

$$R = n(n-1)c + \sum_\alpha \epsilon_\alpha \left(\sum_i h_{ii}^\alpha \right)^2 - \sum_\alpha \sum_{i,j} \epsilon_\alpha (h_{ij}^\alpha)^2. \quad (5.5)$$

From (5.4) and (5.5) we obtain

$$|A|^2 = n^2 H^2 + n(n-1)(R-c), \quad (5.6)$$

where $|A|^2 = \sum_{\alpha,i,j} \epsilon_\alpha (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form A of M^n.

We also have the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 \text{ and } d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl} \omega_k \wedge \omega_l,$$

where $R_{\alpha\beta jk}$ satisfy Ricci equation

$$R_{\alpha\beta ij} = \sum_l \left(h_{il}^\alpha h_{lj}^\beta - h_{jl}^\alpha h_{li}^\beta \right). \quad (5.7)$$

Furthermore, we will consider the symmetric tensor

$$\Phi = \sum_{i,j,\alpha \geq n+p-q+1} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$. We have that

$$\Phi_{ij}^{n+p-q+1} = h_{ij}^{n+p-q+1} - H\delta_{ij} \quad \text{and} \quad \Phi_{ij}^{\alpha} = h_{ij}^{\alpha},$$

for $\alpha \neq n+p-q+1$. Since $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi_{ij}^{\alpha})^2$ is the square of the length of Φ, it is not difficult to verify that Φ is traceless and, using (5.6),

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)(R-c). \tag{5.8}$$

Besides, we observe that $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical submanifold of $\mathbb{N}_q^{n+p}(c)$. For this reason, Φ is usually called the total umbilicity tensor of M^n.

Throughout this chapter, we will assume that the mean curvature vector field h is parallel as a section of the normal bundle of M^n, which means that $\nabla^{\perp}h = 0$, where $\nabla^{\perp}$ is the normal connection of M^n. We will also consider that the second fundamental form is locally timelike. Then, we have

$$|A|^2 = \sum_{\alpha=n+p-q+1}^{n+p} \sum_{i,j}(h_{ij}^{\alpha})^2$$

and h is timelike. So, considering $H > 0$, we can assume that the orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{N}_q^{n+p}(c)$ is such that $e_{n+p-q+1} = \frac{h}{H}$. Consequently, we get

$$H^{n+1} := \frac{1}{n}\mathrm{tr}(h^{n+p-q+1}) = H \quad \text{and} \quad H^{\alpha} := \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \neq n+p-q+1,$$

where h^{α} denotes the matrix (h_{ij}^{α}).

We recall that a spacelike submanifold M^n is said to be *linear Weingarten* (LW) when its mean curvature H and normalized scalar curvature R satisfy the following linear relation

$$R = aH + b, \tag{5.9}$$

for some constants $a, b \in \mathbb{R}$. We observe that when $a = 0$, (5.9) reduces to R constant. In this context, equation (5.8) yields to

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 + n(n-1)aH + n(n-1)(b-c). \tag{5.10}$$

For a LW spacelike submanifold M^n satisfying (5.9), we introduce the second-order linear differential operator $\mathcal{L} : C^{\infty}(M^n) \to C^{\infty}(M^n)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2}a\Delta, \tag{5.11}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M^n)$ denotes the Cheng-Yau operator, which is given by

$$Lu = \mathrm{tr}(P \circ \mathrm{Hess}\,(u)), \tag{5.12}$$

for every $u \in C^\infty(M^n)$. Here, Hess stands for the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ denotes the first Newton transformation of M^n which is given by $P = nHI - A$. So, from (5.11) and (5.12), we have that

$$\mathcal{L}u = \mathrm{tr}(\mathcal{P} \circ \mathrm{Hess}\,(u)),$$

with

$$\mathcal{P} = \left(nH + \frac{n-1}{2}a\right) I - A. \tag{5.13}$$

We can verify that $\mathcal{L}$ can be rewritten as

$$\mathcal{L}u = \mathrm{div}(\mathcal{P}(\nabla u)). \tag{5.14}$$

5.2 Rigidity results for complete LW spacelike submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{N}^{n+p}_q(c)$

In our next result, we revisit [89, Theorem 2] replacing the assumption that the mean curvature function attains a global maximum by hypothesis (5.15) that guarantees the $\mathcal{L}$-parabolicity of a complete LW spacelike submanifold.

Theorem 5.2.1. *Let M^n be a complete LW spacelike submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{N}^{n+p}_q(c)$, such that $R = aH + b$ with $b \le c$. Suppose that the second fundamental form of M^n is locally timelike and that it has nonnegative sectional curvature. If H is bounded and, for some reference point $o \in M^n$ and $\delta > 0$,*

$$\int_\delta^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{5.15}$$

where B_t is the geodesic ball of radius t in M^n centered at the reference point o, then M^n is either totally umbilical or a product $M_1 \times M_2 \times \cdots \times M_k$, where the factors M_i, mutually perpendicular along their intersections, are totally umbilical submanifolds of $\mathbb{N}^{n+p}_q(c)$.

Proof. We can verify from Ricci equation (5.7) that

$$\sum_{\alpha,\beta,i,j,k} h^\alpha_{ij} h^\beta_{ki} R_{\alpha\beta jk} = \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha) \ge 0, \tag{5.16}$$

since $N(A) = \mathrm{tr}(AA^t)$ for all matrix $A = (a_{ij})$ and $(AA^t)_{ij} = \sum_k a_{ik}a^t_{kj}$ gives us $N(A) = \mathrm{tr}(AA^t) = \sum_{i,k} a_{ik}a^t_{ki} = \sum_{i,k} (a^t_{ki})^2 \ge 0$.

On the other hand, for each fixed α, considering a local orthonormal frame $\{e_i\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, we have

$$\sum_{i,j,k,m} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \geq \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha - \lambda_j^\alpha)^2 R_{ijij}. \tag{5.17}$$

Moreover, since we are also supposing that $b \leq c$, from [86, Lemma 3.3] we have

$$|\nabla A|^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 \geq n^2 |\nabla H|^2. \tag{5.18}$$

Hence, taking into account that the sectional curvature of M^n is nonnegative, as we are assuming that the normalized mean curvature vector field is parallel, from [89, Equation 30], (5.16), (5.17) and (5.18), we obtain

$$\begin{aligned} \mathcal{L}(nH) &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + \sum_{i,j,k,m} \big(h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}\big) \\ &\quad + \frac{1}{2}\sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) \\ &\geq \frac{1}{2}\sum_{i,j}(\lambda_i^\alpha - \lambda_j^\alpha)^2 R_{ijij} \geq 0. \end{aligned} \tag{5.19}$$

Now, we consider on M^n the symmetric $(0,2)$-tensor field ξ given by

$$\xi(X,Y) := \langle \mathcal{P}X, Y\rangle,$$

for all $X, Y \in TM$ or, equivalently,

$$\xi(\nabla f, \cdot)^\sharp = \mathcal{P}(\nabla f),$$

where $\sharp : T^*M \to TM$ denotes the musical isomorphism, for all smooth function $f : M^n \to \mathbb{R}$, and $\mathcal{P}$ is defined in (5.13), in which $\mathcal{P}$ is positive semi-definite since $b \leq c$ and, from (5.14), is true $\mathcal{L}f = \operatorname{div}\big(\xi(\nabla f, \cdot)^\sharp\big)$. Taking a local orthonormal frame $\{e_i\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$, we get

$$\sum_{i,j} \left(h_{ij}^{n+1}\right)^2 \leq \sum_{\alpha,i,j} \left(h_{ij}^\alpha\right)^2 = |A|^2$$

and thereafter from (5.10),

$$n^2H^2 \geq \left(\lambda_i^{n+1}\right)^2 - n(n-1)aH,$$

for all $i = 1, \cdots, n$. Furthermore, whereas

$$\left(\lambda_i^{n+1}\right)^2 \leq n^2H^2 + n(n-1)aH \leq \left(nH + \frac{n-1}{2}a\right)^2$$

and the normalized mean curvature vector field is parallel on $TM^{\perp}$, we have

$$-nH-\frac{n-1}{2}a\leq\lambda_i^{n+1}\leq nH+\frac{n-1}{2}a,\quad i=1,\cdots,n.$$

So, for all $i\in\{1,\cdots,n\}$, we obtain

$$0\leq\Sigma_i\leq 2nH+(n-1)a,$$

where $\Sigma_i:=nH+\dfrac{n-1}{2}a-\lambda_i^{n+1}$ are the eigenvalues of the operator $\mathcal{P}$. Thus, we can define a positive continuous function ξ_+ on $[0,+\infty)$ by

$$\xi_+(t):=2n\sup_{\partial B_t}H+(n-1)a.$$

From the boundedness of H, it follows that

$$\xi_+(t)\leq 2n\sup_M H+(n-1)a<+\infty.$$

Hence, we reach at the following estimate

$$\int_\delta^{+\infty}\frac{dt}{\xi_+(t)\mathrm{vol}(\partial B_t)}\geq\left(2n\sup_M H+(n-1)a\right)^{-1}\int_\delta^{+\infty}\frac{dt}{\mathrm{vol}(\partial B_t)}.$$

Thus, from (5.15) we achieve in

$$\int_\delta^{+\infty}\frac{dt}{\xi_+(t)\mathrm{vol}(\partial B_t)}=+\infty$$

and we are in position to apply [75, Theorem 2.6] to conclude that M^n is $\mathcal{L}$-parabolic. Consequently, since H is bounded, from (5.19) we get that H is constant. Therefore, H attain the maximum on M^n and we apply [89, Theorem 2] to finish the proof. □

The next auxiliary lemma corresponds to [30, Proposition 2.1]. In what follows, $\mathbb{L}^1(M^n)$ stands for the space of Lebesgue integrable functions on M^n.

Lemma 5.2.2. *Let X be a smooth vector field on a complete oriented Riemannian manifold M^n, such that* $\mathrm{div}X$ *does not change sign on M^n. If $|X|\in\mathbb{L}^1(M)$, then* $\mathrm{div}X=0$.

Returning to the context of complete LW spacelike submanifolds of a semi-Riemannian space form, we apply the previous lemma to obtain the following result.

Theorem 5.2.3. *Let M^n be a complete LW spacelike submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{N}_q^{n+p}(c)$, such that $R=aH+b$ with $b<c$. Suppose that the second fundamental form of M^n is locally timelike and that it has nonnegative sectional curvature. If H is bounded and $|\nabla H|\in\mathbb{L}^1(M)$, then M^n is either totally umbilical or a product $M_1\times M_2\times\cdots\times M_k$, where the factors M_i, mutually perpendicular along their intersections, are totally umbilical submanifolds of $\mathbb{N}_q^{n+p}(c)$.*

Proof. Since H is bounded on M^n, from equation (5.10) A is also bounded and so the operator $\mathcal{P}$ by (5.13). That is, there exists $C_1 > 0$ such that $|\mathcal{P}| \le C_1$. Since we are assuming $|\nabla H| \in \mathbb{L}^1(M)$, we have

$$|\mathcal{P}(\nabla H)| \le |\mathcal{P}||\nabla H| \le C_1 |\nabla H| \in \mathbb{L}^1(M). \tag{5.20}$$

Following the same steps of Theorem 5.2.1, we can prove that $\frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha) \ge 0$ and then, from equation (5.19) and from [86, Lemma 3.3], we obtain

$$\begin{aligned} \mathcal{L}(nH) &= \sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2 - n^2|\nabla H|^2 + \sum_{i,j,k,m} \big(h^\alpha_{ij} h^\alpha_{km} R_{mijk} + h^\alpha_{ij} h^\alpha_{mi} R_{mkjk}\big) \\ &\quad + \frac{1}{2}\sum_{\alpha,\beta} N\big(h^\alpha h^\beta - h^\beta h^\alpha\big) \\ &\ge \sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \ge 0. \end{aligned} \tag{5.21}$$

Thus, from (5.14), (5.20) and (5.21), we can apply Lemma 5.2.2 for $X = \mathcal{P}(\nabla H)$ to conclude that $\mathcal{L}(nH) = 0$ on M^n. Consequently, the inequality in (5.21) is, in fact, an equality and it follows that

$$\sum_{\alpha,i,j,k} (h^\alpha_{ijk})^2 = n^2 |\nabla H|^2. \tag{5.22}$$

Using again [86, Lemma 3.3], we obtain that H is constant on M^n. Thus, H attain the maximum on M^n and we can apply [89, Theorem 2] to conclude the proof. □

We close the Part I with the following rigidity result.

Theorem 5.2.4. *Let M^n be a complete noncompact LW spacelike submanifold immersed with parallel normalized mean curvature vector field in $\mathbb{N}^{n+p}_q(c)$, such that $R = aH + b$ with $b < c$. Suppose that the second fundamental form of M^n is locally timelike and that it has nonnegative sectional curvature. If $|\Phi|$ converges to zero at infinity, then M^n is a totally umbilical submanifold of $\mathbb{N}^{n+p}_q(c)$.*

Proof. Let us suppose by contradiction that such a LW spacelike submanifold M^n is not totally umbilical. We consider the smooth vector field $X = \mathcal{P}(\nabla |\Phi|^2)$ and the smooth function $f = |\Phi|^2$. So, f is a nonnegative, non-identically vanishing function which converges to zero at infinity. Moreover, we have that $\langle \nabla f, X\rangle \ge 0$ and that

$$\frac{1}{2(n-1)} \mathcal{L}(|\Phi|^2) > (H + \frac{a}{2}) \mathcal{L}(nH). \tag{5.23}$$

Hence, taking into account that $(H + \frac{a}{2}) \ge 0$ and using equations (5.19) and (5.23), we reach at

$$\mathrm{div} X = \mathrm{div}(\mathcal{P}(\nabla|\Phi|^2)) = \mathcal{L}(|\Phi|^2) \ge 2(n-1)\left(H + \frac{a}{2}\right)\mathcal{L}(nH) \ge 0.$$

Then, we are in position to apply Lemma 1.0.4 and get that

$$\langle \nabla f, X\rangle = \langle \mathcal{P}(\nabla |\Phi|^2), \nabla |\Phi|^2\rangle = 0.$$

Therefore, since the operator $\mathcal{P}$ is positive definite, we have that $\nabla|\Phi| \equiv 0$. Thus, $f = |\Phi|$ is constant. But f converges to zero at infinity, so it must be identically zero, leading us to a contradiction since we are supposing that M^n is not a totally umbilical submanifold. $\square$

Part II

Sharp integral inequalities for closed submanifolds

Chapter 6

LW submanifolds immersed in the de Sitter space

In this chapter, we extend the techniques of [5, 10] in order to establish the results of [46, 55], a sharp integral inequality for a closed (compact without boundary) spacelike submanifold M^n with constant scalar curvature immersed with parallel normalized mean curvature vector field in the de Sitter space $\mathbb{S}_p^{n+p}$, and we use it to characterize totally umbilical round spheres $\mathbb{S}^n(r)$ of $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$.

We begin stating that a complete spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector and constant scalar curvature $0 < R \leq 1$ must be either totally umbilical or it holds a sharp estimate for the norm of its total umbilicity tensor $|\Phi|^2$, with equality if and only if the submanifold is isometric to a hyperbolic cylinder of the ambient space.

Proposition 6.0.1 (Proposition 1 of [10])**.** *Let M^n be a complete spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector field and constant scalar curvature $0 < R \leq 1$. Then*

1. *either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical submanifold,*

2. *or*
$$\sup_M |\Phi|^2 \geq \alpha(n, p, R) > 0,$$
where $\alpha(n, p, R)$ is a positive constant depending only on (n, p, R).

Moreover, the equality $\sup_M |\Phi|^2 = \alpha(n, p, R)$ holds and this supremum is attained at some point of M^n if and only if $p = 1$, $n \geq 3$ and M^n is isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1 + r^2})$ of radius $r > 0$.

In order to prove the main result of this chapter, we present a version of Proposition 4.1.4 in the next lemma, a suitable lower estimate for the operator $\mathcal{L}$ acting on the square of the norm of the total umbilicity tensor of a spacelike submanifold (see [10, Proposition 1]).

Lemma 6.0.2. *Let M^n be a spacelike submanifold in $\mathbb{S}_p^{n+p}$, with parallel normalized mean curvature vector field and constant scalar curvature $R \leq 1$. Then*

$$\frac{1}{2}\mathcal{L}(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)},$$

where the real function $Q_{R,n,p}(x)$ is defined by

$$Q_{R,n,p}(x) = \frac{(n-p-1)}{p}x^2 - (n-2)x\sqrt{x^2 + n(n-1)(1-R)} + n(n-1)R. \tag{6.1}$$

Once this is established, we ca obtain a sharp integral inequality involving the total umbilicity tensor Φ apply it to characterize totally umbilical round spheres. More precisely,

Theorem 6.0.3. *Let M^n be a closed spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector field and constant normalized scalar curvature $R \leq 1$. Then*

$$\int_M |\Phi|^{q+2} Q_{R,n,p}(|\Phi|)dM \leq 0, \tag{6.2}$$

for every real number $q \geq 2$, where the real function $Q_{R,n,p}(x)$ is defined in (6.1). *Moreover, assuming in addition that $0 < R < 1$, the equality holds in* (6.2) *if, and only if, M^n is a totally umbilical round sphere $\mathbb{S}^n(r)$, with $r = \frac{1}{R} > 1$, immersed in $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$.*

Proof. From Lemma 6.0.2 we have that

$$\mathcal{L}(|\Phi|^2) \geq \frac{2}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(1-R)}. \tag{6.3}$$

Now, let us take $u = |\Phi|^2$. So, (6.3) can be rewritten as follows

$$\mathcal{L}(u) \geq \frac{2}{\sqrt{n(n-1)}} u Q_{R,n,p}(\sqrt{u})\sqrt{u + n(n-1)(1-R)}. \tag{6.4}$$

Taking into account that $u \geq 0$, $R \leq 1$ and observing that when $R = 1$ (2.19) guarantees that $u > 0$, from (6.4) we get

$$u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u}) \leq \frac{\sqrt{n(n-1)}}{2} \frac{u^{\frac{q}{2}}}{\sqrt{u + n(n-1)(1-R)}}\mathcal{L}(u), \tag{6.5}$$

for every real number q. As M^n is closed, we can integrate both sides of (6.5) in order to obtain

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leq \frac{\sqrt{n(n-1)}}{2} \int_M \frac{u^{\frac{q}{2}}}{\sqrt{u + n(n-1)(1-R)}}\mathcal{L}(u)dM. \tag{6.6}$$

But, from (2.24) we have

$$f(u)\mathcal{L}(u) = \operatorname{div}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle, \tag{6.7}$$

for every smooth function $f \in \mathcal{C}^1(\mathbb{R})$. Integrating both sides of (6.7) and using the Divergence Theorem, we deduce that

$$\int_M f(u)\mathcal{L}(u)dM = -\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle dM, \tag{6.8}$$

for every smooth function f. In our case, for every real number $q \geq 2$, we choose

$$f(t) = \frac{t^{q/2}}{\sqrt{t + n(n-1)(1-R)}}, \quad \text{for } t \geq 0.$$

Hence, assuming $R \leq 1$ and that $R = 1$ only for $t > 0$, we get

$$f'(t) = \frac{(q-1)t^{q/2} + n(n-1)(1-R)qt^{\frac{q-2}{2}}}{2\left(t + n(n-1)(1-R)\right)^{3/2}} \geq 0, \tag{6.9}$$

for every real number $q \geq 2$. Using (6.8) and (6.9) into (6.6), we can estimate

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leq -\frac{\sqrt{n(n-1)}}{2}\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle dM \leq 0, \tag{6.10}$$

since we know that the operator $\mathcal{P}$ is positive semi-definite. Therefore, we conclude

$$\int_M u^{\frac{q+2}{2}} Q_{R,n,p}(\sqrt{u})dM \leq 0. \tag{6.11}$$

This proves inequality (6.2).

Furthermore, if the equality holds in (6.2), from (6.10) we get

$$\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle dM = 0. \tag{6.12}$$

But, since $q \geq 2$ and assuming that $R < 1$, from (6.9) we have

$$f'(u) = \frac{(q-1)u^{q/2} + n(n-1)(1-R)qu^{\frac{q-2}{2}}}{2\left(u + n(n-1)(1-R)\right)^{3/2}}. \tag{6.13}$$

Observe that $f'(u) \geq 0$ with equality if and only if $q > 2$ and $u = 0$. Consequently, taking into account [10, Lemma 1], (6.12) and (6.13) imply

$$\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0. \tag{6.14}$$

Thus, since $\mathcal{P}$ is positive definite, from (6.14) we get that $\nabla u = 0$ on M^n. Hence, the function $u = |\Phi|^2$ must be constant.

In the case that $|\Phi| = 0$, we can reason as in the last part of the proof of Theorem 1.3 of [59] to conclude that M^n must be a totally umbilical round sphere $\mathbb{S}^n(r)$, with $r = \frac{1}{R} > 1$, immersed in a totally geodesic de Sitter space $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{S}_p^{n+p}$. Indeed, let N_1 be the sub-bundle spanned by $\{e_{n+2}, \cdots, e_{n+p}\}$. Then, from our assumption $\nabla^{\perp} e_{n+1} = 0$ it follows that N_1 is parallel in the normal bundle. Besides, from (2.13) and (2.15) we get that $|\Phi^{\alpha}|^2 = \sum_{i,j} (\Phi^{\alpha}_{ij})^2 = 0$ for each $n+2 \leq \alpha \leq n+p$, which means M^n is totally geodesic with respect to N_1. Hence, from [88, Theorem 1] we obtain the desired conclusion.

Finally, let us consider the case that $|\Phi|$ is a positive constant. As in the last part of the proof of [5, Theorem 1.2], we have that $|\Phi| = u_0$ is such that $Q_{R,n,p}(u_0) = 0$ because of the equality in (6.2). Consequently, we can apply Proposition 6.0.1 obtaining that $p = 1$, $n \geq 3$ and that M^n should be isometric to a hyperbolic cylinder $\mathbb{H}^1(r) \times \mathbb{S}^{n-1}(\sqrt{1+r^2})$ of radius $r > 0$. Therefore, since we are assuming that M^n is closed, we conclude that this second case cannot occur.

□

Proceeding with this picture, we can obtain the same result for linear Weingarten hypersurfaces through a similar process, starting from the following lemma(see [11, Proposition 6]):

Lemma 6.0.4. *Let M^n be a linear Weingarten spacelike hypersurface immersed in $\mathbb{S}^{n+1}_1$, $n \geq 2$, such that $R = aH + b$ with $b \leq 1$. In the case where $b = 1$, assume that the mean curvature function H does not change sign and $R \geq 1$. Then,*

$$\mathcal{L}(|\Phi|^2) \geq 2(n-1)|\Phi|^2 \varphi_{a,b}(|\Phi|) \sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + 1 - b},$$

where the real function $\varphi_{a,b}(x)$ is given by

$$\begin{aligned} \varphi_{a,b}(x) = {} & \frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x \right) \sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} - b + 1} \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}} \frac{a}{2} x - n \left(\frac{a^2}{2} - b \right). \end{aligned} \tag{6.15}$$

Therefore, involving the total umbilicity tensor Φ of closed linear Weingarten spacelike hypersurfaces immersed in the de Sitter space $\mathbb{S}^{n+1}_1$, we can apply it to obtain our next characterization result.

Theorem 6.0.5. *Let M^n be a closed linear Weingarten spacelike hypersurface isometrically immersed in the de Sitter space $\mathbb{S}^{n+1}_1$, $n \geq 2$, such that $R = aH + b$ with $b \leq 1$. In the case where $b = 1$, suppose that $a > 0$. Then*

$$\int_M |\Phi|^{q+2} \varphi_{a,b}(|\Phi|) dM \leq 0, \tag{6.16}$$

for every real number $q > 2$, where the real function $\varphi_{a,b}(x)$ is defined in (6.15).

Moreover, assuming in addition that $n \geq 3$ and $0 < b \leq R < \frac{n-2}{n}$, the equality holds in (6.16) *if, and only if, M^n is a totally umbilical round sphere $\mathbb{S}^n(r) \hookrightarrow \mathbb{S}^{n+1}_1$, with $r = \frac{1}{R} > 1$.*

Taking into account that when $n = 2$, $a \geq 0$ and $0 < b \leq 1$ we have that $\varphi_{a,b}(x) > 0$ for all $x \geq 0$, and noting that $R = K$ is the Gaussian curvature of M^2, it is not difficult to verify that from the integral inequality (6.16) we get the following consequence.

Corollary 6.0.6. *Let M^2 be a closed linear Weingarten spacelike surface isometrically immersed in the de Sitter space $\mathbb{S}_1^3$, such that $R = aH + b$.*

(i) If $0 < b < 1$ and $a \geq 0$, then M^2 is a totally umbilical round sphere $\mathbb{S}^2(r) \hookrightarrow \mathbb{S}_1^3$, with $r = \frac{1}{K} > 1$.

(ii) If $b = 1$ and $a > 0$, then M^2 is a totally geodesic unit round sphere $\mathbb{S}^2 \hookrightarrow \mathbb{S}_1^3$.

Chapter 7

LW hypersurfaces in Einstein manifolds

In this chapter, we prove a sharp Simons type integral inequality for n-dimensional closed linear Weingarten hypersurfaces immersed in an Einstein manifold $\mathcal{E}^{n+1}$ and we use it to characterize totally umbilical hypersurfaces and isoparametric hypersurfaces with two distinct principal curvatures, one which is simple, in such an ambient space. Our approach is based on a suitable lower estimate of a Cheng-Yau modified operator acting on the square norm of the traceless second fundamental form of such a submanifold. The results presented in this chapter make part of [45, 47, 48, 54]

7.1 A sharp Simons type integral inequality for closed LW hypersurfaces in an Einstein manifold $\mathcal{E}^{n+1}$

In this first section, we will establish a sharp integral inequality concerning closed linear Weingarten hypersurface when the ambient space is an Einstein manifold.

Theorem 7.1.1. *Let M^n be a closed linear Weingarten hypersurface immersed in an Einstein manifold $\mathcal{E}^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, with $R = aH + b$ such that $b \geq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, suppose that $a > 0$. If its totally umbilical tensor Φ satisfies* (4.101)*, for some $1 \leq p \leq \frac{n-\sqrt{n}}{2}$, then*

$$\int_M |\Phi|^{q+2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|)dM \leq 0, \tag{7.1}$$

for every $q \geq 2$, where the real function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}$ is defined in (4.116)*. Moreover, assuming $b > \overline{\mathcal{R}}$, the equality holds in* (7.1) *if and only if*

(i) either M^n is a totally umbilical hypersurface,

(ii) or

$$|\Phi|^2 = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0) > 0,$$

where $\alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$ is a positive constant depending only on $a, b, n, p, \overline{\mathcal{R}}$ and c_0, and M^n is an isoparametric hypersurface with two distinct principal curvatures of multiplicities

p and $n-p$.

Proof. Taking $u = |\Phi|^2$, we can rewrite inequality (4.115) as follows

$$\mathcal{L}(u) \geq \frac{1}{\sqrt{n(n-1)}} u Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u})\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}.$$

Since $u \geq 0$ and $a > 0$ when $b = \overline{\mathcal{R}}$, we obtain

$$u^{\frac{q+2}{2}} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u}) \leq \sqrt{n(n-1)} \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}} \mathcal{L}(u),$$

for every real number q. Besides that, M^n being closed guarantees us that we can integrate both sides of the previous equation getting

$$\int_M u^{\frac{q+2}{2}} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u}) dM \leq \sqrt{n(n-1)} \int_M \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}} \mathcal{L}(u) dM. \quad (7.2)$$

But, from (4.27) we deduce that

$$f(u)\mathcal{L}(u) = \operatorname{div}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle,$$

for every smooth function $f \in C^1(\mathbb{R})$. So, integrating both sides and using the Divergence Theorem, we reach at

$$\int_M f(u)\mathcal{L}(u) dM = -\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle dM,$$

for every smooth function f. In our case, we choose

$$f(t) = \frac{t^{\frac{q}{2}}}{\sqrt{4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}, \quad \text{for} \quad t \geq 0. \quad (7.3)$$

With this choice, we achieve in

$$f'(t) = \frac{(q-1)4t^{\frac{q}{2}} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2) q t^{\frac{q-2}{2}}}{2(4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{\frac{3}{2}}} \geq 0, \quad (7.4)$$

for every real number $q \geq 2$ and $t \geq 0$. Putting (7.3) and (7.4) into (7.2), we obtain

$$\int_M u^{\frac{q+2}{2}} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u}) dM \leq -\sqrt{n(n-1)} \int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle dM \leq 0, \quad (7.5)$$

since $\mathcal{P}$ is positive semi-defined by Lemma 4.2.3. Therefore,

$$\int_M u^{\frac{q+2}{2}} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u}) dM \leq 0, \quad (7.6)$$

proving inequality (7.1).

For the second part of Theorem 7.1.1, assuming that the equality holds in (7.1), from (7.5) we obtain

$$\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle dM = 0. \tag{7.7}$$

Consequently, we get from (7.4) that

$$f'(u) = \frac{(q-1)4u^{\frac{q}{2}} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)qu^{\frac{q-2}{2}}}{2(4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{\frac{3}{2}}} \geq 0,$$

with equality if and only if $u = 0$ and $q \geq 2$. Moreover, since $b > \overline{\mathcal{R}}$, we know from Lemma 4.2.3 that

$$\langle \mathcal{P}(\nabla u), \nabla u\rangle \geq 0,$$

with equality if and only if $\nabla u = 0$. Thus, from (7.7) we have

$$f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0.$$

Hence, the function $u = |\Phi|^2$ must be constant, either $u = 0$ or $\nabla u = 0$. In the case that $|\Phi|^2 = 0$, M^n must be totally umbilical. Otherwise, $|\Phi|^2$ is a positive constant and the equality in (7.1) implies $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) = 0$. Therefore, we can reason as in the last part of the proof of Theorem 4.2.10 to conclude that M^n is an isoparametric hypersurface with two distinct principal curvatures of multiplicities p and $n-p$. $\square$

Remark 7.1.2. With the same argumentation made in Remark 4.2.11, we conclude that Theorem 7.1.1 corresponds to an extension of [5, Theorem 4.1].

Here, reasoning as we did in Theorem 7.1.1, it is going to be natural make the study for closed linear Weingarten submanifolds immersed in a space form. In this first theorem, the ambient space is the unit Euclidean sphere.

Theorem 7.1.3. *Let M^n be a closed linear Weingarten submanifold immersed with parallel normalized mean curvature vector field into the unit Euclidean sphere $\mathbb{S}^{n+p}(n \geq 4)$, such that $R = aH + b$ with $a \geq 0$ and $b > 1$. Then*

$$\int_M |\Phi|^{q+2}\varphi_{a,b,1}(|\Phi|)\mathrm{dM} \geq 0, \tag{7.8}$$

for every real number $q > 2$, where the real function $\varphi_{a,b,1}$ is obtained making $c = 1$ in

$$\varphi_{a,b,c}(x) = \frac{n-2}{n-1}x^2 - \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} + b - c} + \frac{n(n-2)}{\sqrt{n(n-1)}}\frac{a}{2}x - n\left(\frac{a^2}{2} + b - c\right). \tag{7.9}$$

Moreover, the equality holds in (7.8) if, and only if, *i. either M^n is a totally umbilical sphere $\mathbb{S}^n(r)$, with $0 < r < 1$,*

ii. or

$$|\Phi|^2 = \alpha(n, a, b) > 0,$$

where $\alpha(n, a, b)$ is a positive constant depending on n, a, b and M^n is isometric to a Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, with $r = \sqrt{\frac{(n-2)}{nR}} > 0$.

Proceeding, we consider the case that the ambient space $\mathbb{Q}$ is either the hyperbolic space ($c = 1$) or the Eclidean space ($c = 0$). We finish this section addressing the Euclidean and hyperbolic cases.

Theorem 7.1.4. *Let M^n be a closed linear Weingarten submanifold immersed with parallel normalized mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ ($c \in -1, 0$ and $n \geq 4$), such that $R = aH + b$ with $a \geq 0$ and $b > c$. Then*

$$\int_M |\Phi|^{q+2} \varphi_{a,b,c}(|\Phi|) \mathrm{dM} \geq 0, \tag{7.10}$$

for every real number $q > 2$, where the real function $\varphi_{a,b,c}$ is defined in (7.9)*. Moreover, the equality holds in* (7.10) *if, and only if, M^n is a totally umbilical sphere $\mathbb{S}^n(r)$, with $r > 0$.*

7.2 A sharp integral inequality for closed LW hypersurfaces in an Einstein spacetime $\mathcal{E}_1^{n+1}$

For the Lorentzian context, we establish a sharp integral inequality concerning closed LW spacelike hypersurfaces immersed in an Einstein manifold $\mathcal{E}_1^{n+1}$, which follows closely the ideas and techniques of section 7.1. For that, let us enunciate the lower boundedness for the operator $\mathcal{L}$ for this case.

Proposition 7.2.1. *Let M^n be a LW spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with $b \leq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, assume that the mean curvature function H does not change sign and $b \leq R$. Then,*

$$\mathcal{L}(|\Phi|^2) \geq 2(n-1)|\Phi|^2 \varphi_{a,b}(|\Phi|) \sqrt{\frac{|\Phi|^2}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}},$$

where

$$\begin{aligned} \varphi_{a,b}(x) = & \frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x \right) \sqrt{\frac{x^2}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}} \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}} \frac{a}{2} x - n \left(\overline{\mathcal{R}} - b - c + \frac{a^2}{2} \right) \end{aligned} \tag{7.11}$$

and $c = 2c_2 + \dfrac{c_1}{n}$.

Now, we are in a position to establish our rigidity result.

Theorem 7.2.2. *Let M^n be a closed LW spacelike hypersurface immersed in an Einstein manifold $\mathcal{E}_1^{n+1}$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $R = aH + b$ with $b \leq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, suppose that $a > 0$. Then,*

$$\int_M |\Phi|^{q+2} \varphi_{a,b}(|\Phi|) dM \leq 0, \tag{7.12}$$

for every $q \geq 2$, where the real function $\varphi_{a,b}$ is defined in (7.11)*. Moreover, assuming $b < \overline{\mathcal{R}}$, equality holds in* (7.12) *if and only if*

(i) either M^n is a totally umbilical hypersurface,

(ii) or

$$|\Phi|^2 = \alpha(n, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, a, b, c, \overline{\mathcal{R}})$ is a positive constant depending on $n, a, b, c, \overline{\mathcal{R}}$, and M^n is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

Proof. Let $u = |\Phi|^2$. So, we can rewrite the equation from Proposition 7.2.1 as

$$\mathcal{L}(u) \geq 2(n-1) u \varphi_{a,b}(\sqrt{u}) \sqrt{\frac{u}{n(n-1)} + \overline{\mathcal{R}} - b + \frac{a^2}{4}}.$$

Since $u \geq 0$ and $a > 0$ when $b = \overline{\mathcal{R}}$, we obtain

$$u^{\frac{q+2}{2}} \varphi_{a,b}(\sqrt{u}) \leq \sqrt{\frac{n}{n-1}} \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(\overline{\mathcal{R}} - b) + a^2)}} \mathcal{L}(u),$$

for every real number q. Besides that, the fact of M^n be closed guarantees that we can integrate both sides of the previous equation and get

$$\int_M u^{\frac{q+2}{2}} \varphi_{a,b}(\sqrt{u}) dM \leq \sqrt{\frac{n}{n-1}} \int_M \frac{u^{\frac{q}{2}}}{\sqrt{4u + n(n-1)(4(\overline{\mathcal{R}} - b) + a^2)}} \mathcal{L}(u) dM. \tag{7.13}$$

But, from (4.27), we gain

$$f(u)\mathcal{L}(u) = \operatorname{div}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle$$

for every smooth function $f \in C^1(\mathbb{R})$. We can integrate both sides and use the Stokes' Theorem, yielding

$$\int_M f(u)\mathcal{L}(u) dM = -\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u \rangle dM,$$

for every smooth function f. In our case, we choose

$$f(t)=\frac{t^{\frac{q}{2}}}{\sqrt{4t+n(n-1)(4(\overline{\mathcal{R}}-b)+a^2)}},\quad \text{for}\quad t\geq 0. \tag{7.14}$$

For this reason, we achieve in

$$f'(t)=\frac{(q-1)4t^{\frac{q}{2}}+n(n-1)(4(\overline{\mathcal{R}}-b)+a^2)qt^{\frac{q-2}{2}}}{2(4t+n(n-1)(4(\overline{\mathcal{R}}-b)+a^2))^{\frac{3}{2}}}\geq 0, \tag{7.15}$$

for every real number $q\geq 2$ and $t\geq 0$. Putting (7.14) and (7.15) into (7.13), we can estimate

$$\int_M u^{\frac{q+2}{2}}\varphi_{a,b}(\sqrt{u})dM\leq -\sqrt{\frac{n}{n-1}}\int_M f'(u)\langle P(\nabla u),\nabla u\rangle dM\leq 0, \tag{7.16}$$

since $\mathcal{P}$ is positive semi-definite by Lemma 4.1.2. Therefore,

$$\int_M u^{\frac{q+2}{2}}\varphi_{a,b}(\sqrt{u})dM\leq 0. \tag{7.17}$$

This proves inequality (7.12). For the second part of our theorem, if the equality holds in (7.17), from (7.16) we obtain

$$\int_M f'(u)\langle\mathcal{P}(\nabla u),\nabla u\rangle dM=0. \tag{7.18}$$

Consequently, we get from (7.15) that

$$f'(u)=\frac{(q-1)4u^{\frac{q}{2}}+n(n-1)(4(\overline{\mathcal{R}}-b)+a^2)qu^{\frac{q-2}{2}}}{2(4u+n(n-1)(4(\overline{\mathcal{R}}-b)+a^2))^{\frac{3}{2}}}\geq 0,$$

with equality if and only if $u=0$ and $q>2$. Besides that, since $b<\overline{\mathcal{R}}$, we know from Lemma 4.1.2 that

$$\langle\mathcal{P}(\nabla u),\nabla u\rangle\geq 0,$$

with equality if and only if $\nabla u=0$. Well, from (7.18), we get

$$f'(u)\langle\mathcal{P}(\nabla u),\nabla u\rangle=0,$$

Thus, the function $u=|\Phi|^2$ must be constant, either $u=0$ or $\nabla u=0$. In the case that $|\Phi|^2=0$, M^n must be totally umbilical. Otherwise, $|\Phi|^2$ is a positive constant and the equality in (7.12) implies $\varphi_{a,b}(|\Phi|)=0$. Hence, $|\Phi|^2=\alpha>0$ and the proof follows as in Theorem 4.1.8. $\square$

In case $n=2$, we have for $a\geq 0$ that $\varphi_{a,b}(x)>0$ for all $x\geq 0$. Noting that $R=K$ is the Gaussian curvature of M^2, we get the following consequence of Theorem 7.2.2.

Corollary 7.2.3. *Let M^2 be a closed LW spacelike surface immersed in an Einstein manifold $\mathcal{E}_1^3$ satisfying curvature conditions* (4.3) *and* (4.4)*, such that $K = aH + b$ where $b \leq \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, suppose that $a > 0$. Then, M^2 must be totally umbilical.*

Finally, from Theorem 7.2.2 we obtain

Corollary 7.2.4. *Let M^n be a closed LW spacelike hypersurface immersed in the de Sitter space $\mathbb{S}_1^{n+1}$, such that $R = aH + b$ with $b \leq 1$. In the case where $b = 1$, suppose that $a > 0$. Then*

$$\int_M |\Phi|^{q+2} \varphi_{a,b}(|\Phi|) dM \leq 0, \tag{7.19}$$

for every $q \geq 2$, where the real function $\varphi_{a,b}$ is given by

$$\begin{aligned} \varphi_{a,b}(x) = \frac{n-2}{n-1}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x \right) \sqrt{\frac{x^2}{n(n-1)} + 1 - b + \frac{a^2}{4}} \\ + \frac{n(n-2)}{\sqrt{n(n-1)}} \frac{a}{2} x - n \left(\frac{a^2}{2} - b \right). \end{aligned}$$

Moreover, assuming $0 < b < R < \frac{n-2}{n} < 1$, equality holds in (7.19) *if and only if M^n is a totally umbilical round sphere $\mathbb{S}(r) \hookrightarrow \mathbb{S}_1^{n+1}$, with $r = \frac{1}{R} > 1$.*

Chapter 8

LW submanifolds in $\mathbb{N}_q^{n+p}(c)$ with second fundamental form locally timelike

This last chapter is devoted to state and prove our characterization results concerning complete LW spacelike submanifolds immersed with parallel normalized mean curvature vector field and second fundamental form locally timelike in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$. For this, in the next lemma we quote a lower bound of the operator $\mathcal{L}$ acting on the square norm of Φ, which is derived from [89, Inequality (26)].

Lemma 8.0.1. *Let M^n be a LW spacelike submanifold immersed in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$, such that $R = aH + b$ with $b \leq c$ and $a \geq 0$. Suppose that the second fundamental form is locally timelike and the normalized mean curvature vector field is parallel in $\mathbb{N}_q^{n+p}(c)$. Then,*

$$\mathcal{L}(|\Phi|^2) \geq 2(n-1)|\Phi|^2 \varphi_{a,b,c,q,n}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + c - b},$$

where the real function $\varphi_{a,b,c,q,n}$ is given by

$$\begin{aligned} \varphi_{a,b,c,q,n}(x) = & \frac{n-q-1}{q(n-1)}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} - b + c} \\ & + \frac{n(n-2)}{\sqrt{n(n-1)}}\frac{a}{2}x - n\left(\frac{a^2}{2} - b\right). \end{aligned} \tag{8.1}$$

As we did in the last chapters, we are going to use the lower bound of the operator $\mathcal{L}$ acting on the squared norm of Φ to obtain the following sharp integral inequality involving the norm of the umbilicity tensor of closed spacelike submanifolds of a semi-Riemannian space form.

Theorem 8.0.2. *Let M^n be a closed LW spacelike submanifold immersed with parallel normalized mean curvature vector field in a semi-Riemannian space form $\mathbb{N}_q^{n+p}(c)$, such that $R = aH + b$ with $b \leq c$ and $a \geq 0$ (suppose $a > 0$ when $b = c$). If the second fundamental form of M^n is locally timelike, then*

$$\int_M |\Phi|^{t+2}\varphi_{a,b,c,q,n}(|\Phi|)\mathrm{dM} \leq 0, \tag{8.2}$$

for every real number $t > 2$, where the real function $\varphi_{a,b,c,q,n}$ is defined in 8.1. Moreover, assuming in addition that either $0 < b < c$ or $-\frac{a^2}{4} < b \le c \le 0$, the equality holds in (8.2) if, and only if, M^n is a totally umbilical submanifold of $\mathbb{N}_q^{n+p}(c)$.

Proof. We begin making $u = |\Phi|^2$ in Lemma 8.0.1 for simplicity. Thus, we have

$$\mathcal{L}(u) \ge 2(n-1)u\varphi_{a,b,c,q,n}(\sqrt{u})\sqrt{\frac{u}{n(n-1)} + \frac{a^2}{4} + c - b}. \tag{8.3}$$

Since $u \ge 0$ and $b \le c$, with $b = c$ only for $a > 0$, equation (8.3) becomes

$$u^{\frac{t+2}{2}}\varphi_{a,b,c,q,n}(\sqrt{u}) \le \sqrt{\frac{n}{n-1}}\frac{u^{\frac{t}{2}}}{\sqrt{4u + n(n-1)(a^2 + 4(c-b))}}\mathcal{L}(u), \tag{8.4}$$

for every real number t. As M^n is closed, we can integrate both sides of (8.4) in order to obtain

$$\int_M u^{\frac{t+2}{2}}\varphi_{a,b,c,q,n}(\sqrt{u})\mathrm{dM} \le \sqrt{\frac{n}{n-1}}\int_M \frac{u^{\frac{t}{2}}}{\sqrt{4u + n(n-1)(a^2 + 4(c-b))}}\mathcal{L}(u)\mathrm{dM}. \tag{8.5}$$

But, from (5.14) we have

$$f(u)\mathcal{L}(u) = \mathrm{div}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle\mathcal{P}(\nabla u), \nabla u\rangle, \tag{8.6}$$

for every smooth function $f \in \mathcal{C}^1(\mathbb{R})$. So, integrating both sides of (8.6) and using Stokes' theorem it follows that

$$\int_M f(u)\mathcal{L}(u)\mathrm{dM} = -\int_M f'(u)\langle\mathcal{P}(\nabla u), \nabla u\rangle\mathrm{dM}, \tag{8.7}$$

for every smooth function f. In our case, substituting (8.7) into (8.5) we can estimate

$$\int_M u^{\frac{t+2}{2}}\varphi_{a,b,c,q,n}(\sqrt{u})\mathrm{dM} \le -\sqrt{\frac{n}{n-1}}\int_M f'(u)\langle\mathcal{P}(\nabla u), \nabla u\rangle\mathrm{dM}, \tag{8.8}$$

where

$$f(u) = \frac{u^{t/2}}{\sqrt{4u + n(n-1)(a^2 + 4(1-b))}},$$

with

$$f'(u) = \frac{4(r-1)u^{t/2} + n(n-1)(a^2 + 4(c-b))tu^{\frac{t-2}{2}}}{2\left(4u + n(n-1)(a^2 + 4(c-b))\right)^{3/2}} \ge 0, \tag{8.9}$$

for every real number $t > 2$, occurring equality if and only if $u = 0$. Therefore, since [89, Lemma 1] assures that the operator P is positive semi-definite for $b \le c$, we conclude from (8.8) and (8.9) that

$$\int_M u^{\frac{t+2}{2}}\varphi_{a,b,c,q,n}(\sqrt{u})\mathrm{dM} \le 0. \tag{8.10}$$

This proves inequality (8.2).

Now, we proceed supposing that the equality holds in (8.2). From (8.8) we get

$$\int_M f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle \mathrm{dM} = 0. \tag{8.11}$$

We can return to equation (8.9) and use the fact that $\mathcal{P}$ is positive definite when $b < c$ to obtain

$$f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0.$$

In the case that $f'(u) = 0$, we have $u = 0$. In other words, $|\Phi| = 0$. In the case that $\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0$, since $\mathcal{P}$ is positive definite, we get that $\nabla u = 0$ on M^n. Hence, the function $u = |\Phi|^2$ must be constant.

If $|\Phi| = 0$, we conclude that M^n is a totally umbilical submanifold. Otherwise, $|\Phi|$ is a positive constant and so H and R by equation (5.10) and, consequently, the mean curvature vector field h and e_{n+p} are parallel in $T^{\perp}(M^n)$. Therefore, reasoning as in the last part of the proof of [89, Theorem 1], we can apply [88, Theorem 1] to obtain that M^n lies in a totally geodesic submanifold $\mathbb{N}_1^{n+1}(c)$ of $\mathbb{N}_q^{n+p}(c)$. Since we are assuming in addition that either $0 < b < c$ or $-\frac{a^2}{4} < b \le c \le 0$, we conclude that M^n should be isometric to $\mathbb{H}^1(c - c\coth^2 r) \times \mathbb{S}^{n-1}(c - c\tanh^r)$, when $c = 0$, to $\mathbb{H}^1(-\coth^2 r) \times \mathbb{R}^{n-1}$, when $c > 0$, and to $\mathbb{H}^1(c + c\coth^2 r) \times \mathbb{H}^{n-1}(c + c\tan^2 r)$, when $c < 0$, for some radius $r > 0$. However, since we are supposing that M^n is closed, these situations cannot occur. Therefore, M^n must be a totally umbilical submanifold of $\mathbb{N}_q^{n+p}(c)$. $\square$

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