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Doutorado em Matemática

Equigenerated Gorenstein ideals of codimension 3

With a chapter on general forms

Dayane Santos de Lira

João Pessoa - PB

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Autora: **Dayane Santos de Lira** †

Orientador: **Prof. Dr. Aron Simis**

Coorientador: **Prof. Dr. Zaqueu Alves Ramos**

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Prof. Dr. Rafael Heraclio Villarreal Rodríguez (CINVESTAV/IPN)

Prof. Dr. Víctor Hugo Jorge Perez (ICMC/USP)

Prof. Dr. Ricardo Burity Croccia Macedo (UFPB)

Profa. Dra. Jacqueline Fabiola Rojas Arancibia (UFPB)

Prof. Dr. Zaqueu Alves Ramos (UFS)

Coorientador

Prof. Dr. Aron Simis (UFPE-UFPB-UFCG)

Orientador

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Resumo

Esta tese versa sobre ideais de Gorenstein equigerados de co-comprimento finito em um anel de polinômios graduado standard $R = \mathbb{k}[x_1, \ldots, x_n]$ sobre um corpo infinito $\mathbb{k}$. Focalizamos especialmente o caso de codimensão 3, estudando propriedades envolvendo o sistema inverso de Macaulay, o grau do socle, o número de redução, e a Cohen-Macaulicidade da álgebra de Rees associada. Uma atenção especial é dedicada ao problema clássico de formas gerais, no espírito da conjectura de Fröberg. Nosso interesse é entender a rarefação de ideais de Gorenstein gerados por formas gerais. Nessa direção conjecturamos que se $I \subset R$ é um ideal gerado por $r \geq n+2$ formas de grau $d \geq 2$, então I é Gorenstein se, e somente se, $d = 2$ e $r = \binom{n+1}{2} - 1$. Provamos esta conjectura para $n = 3$ e uma das implicações para n arbitrário. Outro tema abordado é o aqui denominado *problema do quociente*, relacionado à apresentação de um ideal Gorenstein na forma $I = (x_1^m, \ldots, x_n^m) : \mathfrak{f}$, para certa forma $\mathfrak{f} \in R$. Se I tem co-comprimento finito e resolução linear, estabelecemos sob quais condições a forma $\mathfrak{f}$ é unicamente determinada e qual é seu grau. Mostramos também que esse problema está relacionado com a noção de *dual de Newton*, introduzido por Costa–Simis e posteriormente estudado por vários autores recentes.

Palavras-chave: Ideais de Gorenstein equigerados. Sistema inverso de Macaulay. Problema do quociente. Formas gerais.

Abstract

This thesis deals with equigenerated Gorenstein ideals of finite colength in a standard graded ring $R = \Bbbk[x_1, \ldots, x_n]$ over an infinite field $\Bbbk$. We focus especially on such ideals of codimension 3, by looking at properties involving the Macaulay inverse system, the degree of socle, the reduction number, and the Cohen-Macaulayness of the associated Rees algebra. A special attention is devoted to the classical problem of general forms, as in the well-known conjecture of Fröberg. Our interest is to understand the sparsity of Gorenstein ideals generated by general forms. We conjecture that if $I \subset R$ is an ideal generated by a general set of $r \geq n+2$ forms of degree $d \geq 2$, then I is Gorenstein if and only if $d = 2$ and $r = \binom{n+1}{2} - 1$. We prove this conjecture for $n = 3$ and one of its implications for arbitrary n. Another theme considered in this thesis is what we called the *colon problem*, a subject related to the presentation of a Gorenstein ideal as a link $I = (x_1^m, \ldots, x_n^m) : \mathfrak{f}$, for a form $\mathfrak{f} \in R$. If I has finite colength and linear resolution, we establish under what conditions the form $\mathfrak{f}$ is uniquely determined, in addition to determining its degree. As we show, this problem is related to the so-called *Newton dual* introduced by Costa–Simis and further studied by various recent authors.

Keywords: Equigenerated Gorenstein ideals. Macaulay inverse system. Colon Problem. General foms.

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Contents

Introduction

The terminology and the definition of a Gorenstein ring were introduced by Grothendieck in 1961, as inspired by the work of Gorenstein in [17]. Subsequently, the importance of these rings grew very fast, perhaps culminating at some stage with the celebrated paper of Bass ([3]), who showed that they could be defined in many different equivalent ways. As times went by, additional equivalent forms were established.

Gorenstein rings became indeed ubiquitous, having been studied by many important authors, such as [5], [48], [9], [19], [25], [26], [32], [41], [27], [29], [30], [12], [31], [28], [38]. Most of these refer to a ground Noetherian local ring. Yet, not so much to the case where the ground ring is a standard graded ring over a field and the Gorenstein ideal is generated by forms of the same degree (equigenerated). It would even seem that the case of codimension three itself has not been intensely explored by emphasizing equigeneration.

Following common acceptance, a Noetherian local ring is a Gorenstein ring if it is a Cohen-Macaulay ring having socle dimension one. We are interested in the case where the ring is of the form R/I, where R is regular. In this case, it is customary to call the ideal I itself Gorenstein. By a well-known quirk, if $R = \Bbbk[x_1, \ldots, x_n]$ is a standard polynomial ring over a field $\Bbbk$ and $I \subset R$ is a homogeneous ideal, one says that R/I is Gorenstein if its localization at the maximal homogeneous ideal is Gorenstein. In this case, R/I admits a self-dual graded free resolution over R. Our typical Gorenstein ring will have this form, with the additional hypothesis that I is equigenerated. A good chunk of the results contained in this thesis appear in [33].

In the following, we described in a summarized form the content of each chapter of this thesis.

Chapter 1 is devoted to the notation and concepts adopted throughout of this work. In Section 1.1 we recall the definition of graded rings and Hilbert function. In Section 1.2 we approach the construction of a graded free resolution of a standard graded algebra. In Section 1.3, we present the associated algebras of an ideal, such as the Rees algebra, the associated graded ring, and the fiber cone. In Section 1.4 we present the definition of a graded Artinian Gorenstein algebra and Gorenstein ideal adopted throughout our work. In addition, we recall the definition of the Macaulay inverse system of an ideal. Finally in Section 1.5, we present the definition of the Newton dual of a form f in $\Bbbk[x_1, \ldots, x_n]$.

Chapter 2, is devoted to examine the Gorenstein ideals of codimension three. In Section 2.1 we present a pair of integers (d, r) called the virtual datum, where r and d denote the minimal number of generators and the degree of these generators of the ideal respectively. Keeping in mind this concept we were able to determine in Lemma 2.1.5 the value of the socle degree of codimension three equigenerated Gorenstein ideals in $\Bbbk[x, y, z]$. In Section 2.2, adopting the concept of "proper virtual datum", we prove as a main result the Theorem 2.2.2 which guarantees that for every virtual datum in dimension $n \geq 3$ there is a codimension three Gorenstein ideal considering a method based on a pretty simple reparametrization device and in arbitrary characteristic. In Section 2.3 is presented some basic results involving ideals satisfying the virtual datum $(d, d+1)$.

Chapter 3, is dedicate to study the ideals generated by a general set of r forms of the same degree d. This is divided in four sections. In Section 3.1 we present a brief approach of the parameter spaces used in our context, besides we introduce the definition of the forms (d, r, e)-extremal. In Section 3.2 we work some results involving the definition (d, r, e)-extremal, in particular we present a connection between Fröberg's conjecture and this concept. In Section 3.3, we prove Theorem 3.3.2 which guarantees that the ideals generated by a general set of r forms of degree d in $\Bbbk[x, y, z]$ are Gorenstein if and only if $d = 2$ and $r = 5$. As a consequence of this we conjecture that if $I \subset \Bbbk[x_1, \ldots, x_n]$ is an ideal generated by a general set of $r \geq n+2$ forms of degree $d \geq 2$, then I is Gorenstein if and only if $d = 2$ and $r = \binom{n+1}{2} - 1$. In order to elaborate this conjecture we show in Proposition 3.3.5 that if I is an ideal generated by a general set of $r \geq n$ quadrics in $\Bbbk[x_1, \ldots, x_n]$ then there is no linear forms in the socle of R/I. Furthermore if the minimal number of generators of I is in the interval $\frac{(n+2)(n+1)}{6} \leq r \leq \binom{n+1}{2} - 1$, then by Theorem 3.3.6 we proved that I is Gorenstein if, and only if $r = \binom{n+1}{2} - 1$. We end this chapter with the Section 3.4 by exposing some simple results and questions, involving the socle degree for an arbitrary degree, that arose from computational evidence and from of the literature seen so far.

Chapter 4, we work the so-called "colon problem", exposing conditions between the terms m and $\mathfrak{f}$ of the ideal $I = (x_1^m, \ldots, x_n^m) : \mathfrak{f}$. In Section 4.1, keeping in mind the concepts of Macaulay inverse system and Newton's duality, and taking a suitable directrix vector, we proved in Proposition 4.1.1 that $\mathfrak{f}$ can be retrieved from I by taking the so-called (socle-like) Newton dual of a minimal generator of the Macaulay inverse of I, still in this result we expose also the degree of $\mathfrak{f}$ in terms of m and the socle degree of I and establish under what conditions the form $\mathfrak{f}$ is uniquely determined. In Section 4.2, considering the case where the ideal has linear resolution we solve the colon problem in Theorem 4.2.2, in terms of a degree constraint. As a consequence we show in Proposition 4.2.4 that when the directrix $\mathfrak{f}$ is a power of $x_1 + \cdots + x_n$, the solutions lie on a dense open set of the space of parameters. In addition, we introduce the concept of *pure power gap* of an ideal I, which consists of an integer measuring how far of from the socle degree is an exponent of the powers of independent linear forms lying in the ideal. We give the basic role of this invariant in Proposition 4.2.7. The Section 4.3 is dedicated to expose two models of telescopic alternating matrices and to present a conjecture involving the generator of Macaulay inverse and the colon representation of the ideals generated by maximal Pfaffians from one of these matrices.

In Chapter 5 we work some properties satisfied by Gorenstein ideals involving the blowup algebras of an ideal, the reduction number and the rational map. In Section 5.1, considering I an $\mathfrak{m}$-primary ideal, such that, $I^{m_0} = \mathfrak{m}^{dm_0}$ for some $m_0 \geq 1$ we prove in Proposition 5.1.1 some conditions involving the Cohen-Macaulayness of the blowup algebras, the reduction number and the rational map defined by a set of forms spanning $[I]_d$, where $[I]_d$ is the set of homogeneous forms of degree d in I. In Section 5.2 considering the role played by the syzygetic condition, we prove in Theorem 5.2.1 that if I is a Gorenstein ideal minimally generated by $2d+1$ forms of degree $d \geq 2$ then the reduction number of I is 2 and the algebras of blowup of I are not Cohen-Macaulay.

Furthermore, the rational map defined by a set of forms spanning $[I]_d$ is birational. As a consequence of this theory we prove in Proposition 5.2.2 that if I is an ideal satisfying the virtual datum (d, r), then under some conditions, involving the fiber cone and the reduction number, the rational map is not birational. We end this chapter by exposing some results (in Section 5.3) involving the almost complete intersection and a suitable dense Zariski open set.

Finally, in Appendix A we expose another way to prove Theorem 3.3.2.

Notation

- $\Bbbk[x_1, \ldots, x_n]$ is the standard graded polynomial ring in the variables $x_1, \ldots, x_n$ over a field $\Bbbk$;
- $\mathrm{char}(\Bbbk)$ is the characteristic of a field $\Bbbk$;
- $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$;
- grade I is the grade of an ideal I;
- $\deg(f)$ is the degree of a form $f \in \Bbbk[x_1, \ldots, x_n]$;
- $\mathrm{Ann}(M)$ is the annihilator of a module M;
- $\mathrm{ht}\, I$ is the codimension of an ideal I;
- depth M is the depth of a module or a ring M;
- $\ker(\varphi)$ is the kernel of a homomorphism φ;
- $\mu(I)$ is the minimal number of generators of a homogeneous ideal I;
- M_i is the ith graded piece of a graded module M.

Chapter 1

Ideal theoretic notions

The main goal of this chapter is to recall some fundamental concepts, which will be helpful for a better understanding of this work. For much of the material in this chapter, we refer to [44].

1.1 Graded rings and the Hilbert function

Let A be a ring and let R be an A-algebra. Pretty generally, given a commutative monoid $\mathbb{M}$, R is said to be an $\mathbb{M}$-*graded algebra* if it admits a decomposition into a direct sum of additive subgroups

$$R = \bigoplus_{i \in \mathbb{M}} R_i$$

such that $R_0 = A$ and $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{M}$. R_i turns out to have a natural structure of an A-module, called the *degree i component* of R.

An $\mathbb{M}$-graded R-module M is defined similarly. The sort of monoid in this work is the set $\mathbb{N}$ of positive integers. In particular, the focus is on *standard* $\mathbb{N}$-*graded* algebras $R = A[R_1]$, where R is generated over $A = R_0$ by the component of degree one. In particular, we focus on the standard polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ over a field $\Bbbk$ and its residue rings R/I, where $I \subset R$ is a (standard) homogeneous ideal. The latter inherit same sort of grading as R, where $[R/I]_i = R_i/I_i$. Thus, as $\Bbbk$-vector spaces we have that

$$\dim_{\Bbbk}[R/I]_i = \dim_{\Bbbk} R_i - \dim_{\Bbbk} I_i.$$

This brings over the following notable function:

Definition 1.1.1. The *Hilbert function* of R/I is the numerical function

$$h_{R/I}(-) : \mathbb{N} \longrightarrow \mathbb{N}, \quad i \mapsto \dim_{\mathbb{k}}[R/I]_i.$$

In particular,

$$h_R(i) = \binom{n+i-1}{i}$$

for every $i \geq 0$.

The generating function of $h_{R/I}(-)$ is called the *Hilbert series* of R/I, and is denoted by $H_{R/I}(t)$. Thus,

$$H_{R/I}(t) = \sum_{i \geq 0} h_{R/I}(i) t^i \in \mathbb{Z}[\![t]\!]$$

where $\mathbb{Z}[\![t]\!]$ is the power series ring. A remarkable feature of $H_{R/I}(t)$ is that it is a rational function of the form

$$H_{R/I}(t) = \frac{h_0 + h_1 t + \dots}{(1-t)^n}$$

where $n = \dim_{\mathbb{k}} R/I$ and $h_0 + h_1 t + \ldots \in \mathbb{Z}[t]$. The vector $(h_0, h_1, \ldots)$ is called the *h-vector* of R/I. When R/I is Artinian, $h_i = h_{R/I}(i)$ for every i.

Most important is the fact that the Hilbert function of R/I is polynomial for all $t \gg 0$; the corresponding polynomial is the *Hilbert polynomial* of R/I. It can be expressed in the following form:

$$PH(R/I, t) = e_0(I)\binom{t}{r} + e_1(I)\binom{t}{r-1} + \cdots + e_r(I),$$

where $r := \dim_{\mathbb{k}} R/I - 1$, the e_i's are integers and $e_0 \geq 0$.

The elements $e_i(I)$ are uniquely determined integers, called the *Hilbert coefficients* of R/I (or of I by abuse). The leading coefficient $e_0(I)$ is called the *multiplicity* of R/I (or of I). Note that for the case where R/I is an Artinian algebra, have finitely many nonzero graded components in R/I, thus $h_{R/I}(t) = 0$ for a large t and the Hilbert polynomial is zero.

1.2 Graded free resolutions

Let R be a standard graded algebra such that $R_0 = \Bbbk$ is a field and R_1 is generated as a $\Bbbk$-vector space by a finite set $x_1, \dots, x_n$ of homogeneous elements. The unique maximal homogeneous ideal $\mathfrak{m} = (R_1) = (x_1, \dots, x_n) = \bigoplus_{i>0} R_i$ is often the *graded irrelevant ideal* of R.

For a graded R-module M and an integer i we denote by $M(i)$ the graded R-module with grading given by $M(i)_j = M_{i+j}$. Let M be a graded R-module generated by homogeneous elements $g_1, \dots, g_n$. Consider the surjective R-map

$$R^n \xrightarrow{\phi_0} M \to 0, \quad e_i \mapsto g_i$$

where e_i denotes the ith vector of the canonical base of R^n. To make this map homogeneous as well, one considers the graded module $\bigoplus_{i=1}^n R(-\deg g_i)$ instead R^n. Thus, we work with the homogeneous R-map

$$\phi_0 : \bigoplus_{i=1}^{n} R(-\deg g_i) \to M \to 0, \quad e_i \mapsto g_i.$$

The kernel of this map is the graded submodule of *syzygies* of M (more precisely, of the chosen set of generators):

$$Syz(M) = \{(s_1, \dots, s_n) \in R^n \quad ; \quad s_1 g_1 + \ldots + s_n g_n = 0\},$$

each element $(s_1, \dots, s_n) \in Syz(M)$ is called *syzygy* of module M. Since $Syz(M)$ is finitely generated, we can repeat this process landing in a short exact sequence

$$\bigoplus_{i=1}^{r} R(-a_i) \xrightarrow{\phi_1} \bigoplus_{i=1}^{n} R(-\deg g_i) \xrightarrow{\phi_0} M \to 0,$$

called a *graded presentation* of M. Iterating gives a *graded free resolution* of the module M

$$\cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

where $F_i = \bigoplus_j R(-a_{i,j})$, for certain $a_{i,j}$, with ϕ_i a homogeneous R-map. We say that this resolution is minimal if $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m} F_{i-1}$ for all $i \geq 1$.

When a minimal graded free resolution is written as

$$\cdots \to \bigoplus_j R(-a_{p,j})^{\beta_{p,j}} \to \bigoplus_j R(-a_{p-1,j})^{\beta_{p-1,j}} \to \cdots \to \bigoplus_j R(-a_{0,j})^{\beta_{0,j}} \to M \to 0,$$

the exponents $\beta_{i,j}$ are called the *graded Betti numbers* of M, while the integers $a_{i,j}$ are known as the *shifts* (or degrees) of the resolution. The *projective dimension* (also, *homological dimension*) of the graded R-module M is the minimal possible length of a free resolution of finite length and is denoted by pd M. If no such resolution of finite length exists, then one say M has infinite projective dimension. Note that pd $M = \max\{i : \beta_{i,j} \neq 0\}$. Moreover, we say that the resolution is *pure* if $a_{i,j} = a_i$ for all i, j. If the resolution is pure and $a_i = a_{i-1} + 1$ for all i, then it is called *linear resolution.*

1.3 Rees (blowup) algebras

Given an ideal I in a ring R, the *Rees algebra* of I is the following subalgebra of $R[t]$ (where t is an indeterminate):

$$\mathcal{R}(I) = R + It + I^2t^2 + \cdots$$

Note that the inclusion $\mathcal{R}(I) \subset R[t]$ implies that the Rees algebra is a domain if the ring R is domain. In this case, provided that R is Noetherian, one has $\dim \mathcal{R}(I) = \dim R + 1$ (see, e.g., [44, Proposition 7.3.1]).

Closely related to $\mathcal{R}(I)$ is the *associated graded ring* of I, defined as

$$\mathrm{gr}_I(R) := \mathcal{R}(I)/I\mathcal{R}(I).$$

If $(R, \mathfrak{m})$ is a Noetherian local ring and $I \subset \mathfrak{m}$, then $\dim \mathrm{gr}_I(R) = \dim R$ (see [44, Theorem 7.3.6]).

If R is Noetherian local with maximal ideal $\mathfrak{m}$ or standard graded with homogeneous maximal ideal $\mathfrak{m}$, the *fiber cone* (also, *fiber algebra*) of I is the algebra

$$\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I).$$

The *analytic spread* of I, denoted $\ell(I)$, is the dimension of the fiber cone $\mathcal{F}(I)$. The analytic spread of a proper ideal satisfies the inequality $\mathrm{ht}\, I \leq \ell(I) \leq \dim R$ (see e.g. [44, Proposition 7.3.16 (ii)] and [44, Proposition 7.3.12] for the first and second inequality respectively). In particular, $\ell(I) = \dim R$ if I is $\mathfrak{m}$-primary.

Let J be an ideal contained in I. One says that J is a *reduction* of I if the inclusion $\mathcal{R}(J) \subset \mathcal{R}(I)$ is an integral extension of rings, equivalently, if $I^{n+1} = JI^n$ for $n \gg 0$. If J is a reduction of I, the reduction number of I with respect to a reduction J, denoted $r_J(I)$, is the least integer n such that $I^{n+1} = JI^n$. A reduction of I which is minimal with respect to inclusion is called a *minimal reduction* of I. The (absolute) *reduction number* of I is

$$r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

If $(R, \mathfrak{m})$ is a local ring with infinite residue field $\Bbbk = R/\mathfrak{m}$ then all minimal reductions of I have the same minimal number of generators, namely $\ell(I)$; this follows from the fact that a sequence of elements in I minimally generates a minimal reduction of I if and only if its image in $I/\mathfrak{m}I$ forms a system of parameters of $\mathcal{F}(I)$, a ring of dimension $\ell(I)$.

1.4 Gorenstein ideals

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. Denote by $\mathfrak{m}$ the maximal homogeneous ideal of R. Given a homogeneous $\mathfrak{m}$-primary ideal $I \subset R$, consider the least integer s such that $\mathfrak{m}^{s+1} \subset I$. Then, the graded $\Bbbk$-algebra R/I can be written as

$$R/I = \Bbbk \oplus (R/I)_1 \oplus (R/I)_2 \oplus \cdots \oplus (R/I)_s$$

with $(R/I)_s \neq 0$. The *socle* of R/I, denoted $\operatorname{Soc}(R/I)$, is the ideal $I : \mathfrak{m}/I \subset R/I$. Since I and $\mathfrak{m}$ are homogeneous ideals, then $\operatorname{Soc}(R/I)$ is a homogeneous ideal of R/I. In particular,

$$\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_1 \oplus \cdots \oplus \operatorname{Soc}(R/I)_s.$$

The integer s is the *socle degree* of R/I. The algebra R/I is *level*, if $\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_s$, i.e., if the socle is concentrated at a single degree.

The socle of a graded Artinian algebra can also be obtained from a free resolution:

Lemma 1.4.1. ([32, Lemma 1.3]) *Let $R = \Bbbk[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field $\Bbbk$, and let R/I be a graded Artinian quotient with minimal graded resolution $\mathbb{F}$ over R. If $F_n = \oplus_{i=1}^r R(-d_i)$, then there is a homogeneous isomorphism*

$$\operatorname{Soc}(R/I) \simeq \oplus_{i=1}^r \Bbbk\left(-\left(d_i - \sum_{i=1}^n \deg x_i\right)\right)$$

of graded $\Bbbk$-vector spaces.

A graded Artinian quotient R/I is *Gorenstein* if and only if $\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_s$ and $\dim_{\Bbbk} \operatorname{Soc}(R/I)_s = 1$. More generally, let $J \subset R$ be a homogeneous ideal such that R/J is Cohen-Macaulay. We say that R/J is a graded *Gorenstein algebra* if for a maximal homogeneous R/J-regular sequence $\mathbf{x} \subset R$ the graded algebra $R/(J, \mathbf{x})$ is a graded Artinian Gorenstein algebra. In this case, by abuse, J is also called a *Gorenstein ideal.*

We briefly recall the notion of Macaulay inverse. Let V be a vector space of dimension n over a field $\Bbbk$, with a basis $\{x_1, \ldots, x_n\}$. Let $R = \operatorname{Sym}_{\Bbbk}(V) = \Bbbk[x_1, \ldots, x_n]$ be the standard graded polynomial ring in n variables over $\Bbbk$. Letting $\{y_1, \ldots, y_n\}$ stand for the dual basis on

$V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, consider the divided power ring

$$D_{\Bbbk}(V^*) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\Bbbk}(R_i, \Bbbk) = \Bbbk_{\mathrm{DV}}[y_1, \ldots, y_n]. \tag{1.1}$$

In particular, $\{\mathbf{y}^{[\alpha]} \mid \alpha \in \mathbb{N}^n \text{ and } |\alpha| = j\}$ is the dual basis of $\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } |\alpha| = j\}$ on $D_{\Bbbk}(V^*)_j = \mathrm{Hom}_{\Bbbk}(R_j, \Bbbk)$. If $\alpha \in \mathbb{Z}^n$ then one sets $\mathbf{y}^{[\alpha]} = 0$ if some component of α is negative. Make $D_{\Bbbk}(V^*)$ into a module over R through the following action

$$R \times D_{\Bbbk}(V^*) \to D_{\Bbbk}(V^*), \quad \left(f = \sum_\alpha a_\alpha \mathbf{x}^\alpha, F = \sum_\beta b_\beta \mathbf{y}^\beta \right) \mapsto fF = \sum_{\alpha,\beta} a_\alpha b_\beta \mathbf{y}^{[\beta-\alpha]}.$$

For a homogeneous ideal $I \subset R$ and an R-submodule $M \subset D_{\Bbbk}(V^*)$ one defines:

$$\mathrm{Ann}(I) := \{g \in D_{\Bbbk}(V^*) \mid Ig = 0\} \quad \text{and} \quad \mathrm{Ann}(M) := \{f \in R \mid fM = 0\}.$$

Then $\mathrm{Ann}(I)$ is an R-submodule of $D_{\Bbbk}(V^*)$, while $\mathrm{Ann}(M)$ is an ideal of R. The R-submodule $\mathrm{Ann}(I)$ is called the *Macaulay inverse* (system) of I.

The main basic result regarding this construction is due to Macaulay ([34]). In the present language it can be stated in the following form:

Theorem 1.4.2. (Macaulay Duality, [30, Theorem 1.4]) *There exists a one-to-one correspondence between the set of nonzero homogeneous codimension n Gorenstein ideals of R and the set of nonzero homogeneous cyclic submodules of $D_{\Bbbk}(V^*)$ given by $I \mapsto \mathrm{Ann}(I)$, with inverse $M \mapsto \mathrm{Ann}(M)$. Moreover, the socle degree of R/I is equal to the degree of a homogeneous generator of* $\mathrm{Ann}(I)$.

1.5 Newton duality

The notion of the Newton (complementary) dual of a form $f \in \Bbbk[x_1, \ldots, x_n]$ in a polynomial ring over a field $\Bbbk$ was introduced in [8] and revisited in [10]. Namely, start out with the log matrix A of the constituent monomials of f (i.e, the nonzero terms of f). This is the matrix whose columns are the exponents vectors of the nonzero terms of f in, say, the lexicographic ordering. It is denoted by $\mathcal{N}(f)$. Then, the Newton dual log matrix (or simply the *Newton dual matrix*) of the Newton log matrix $\mathcal{N}(f) = (a_{i,j})$ is the matrix $\widehat{\mathcal{N}(f)} = (\alpha_i - a_{i,j})$, where $\alpha_i = \max_j\{a_{i,j}\}$, with $1 \leq i \leq n$ and j indexes the set of all nonzero terms of f.

In other words, denoting $\boldsymbol{\alpha} := (\alpha_1 \cdots \alpha_n)^t$, one has

$$\widehat{\mathcal{N}(f)} = [\,\boldsymbol{\alpha} \mid \cdots \mid \boldsymbol{\alpha}\,]_{n \times r} - \mathcal{N}(f),$$

where r denotes the number of nonzero terms of f. The vector $\boldsymbol{\alpha}$ is called the *directrix vector* of $\mathcal{N}(f)$ (or of f by abuse).

We note that taking the Newton dual is a true duality upon forms not admitting monomial factor, in the sense that, for such a form f, $\widehat{\widehat{\mathcal{N}(f)}} = \mathcal{N}(f)$ holds.

We define the *Newton dual* of f to be the form $\hat{f}$ whose terms are the ordered monomials obtained form $\widehat{\mathcal{N}(f)}$ affected by the same coefficients as in f.

Example 1.5.1. Let $R = \Bbbk[x, y, z]$ be a polynomial ring and $f = xy^4z - x^2y^2z^2 + z^6$ be a form in R. Then, the Newton matrix of f considering the order lexicographic, is given by

$$N(f) = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 2 & 0 \\ 1 & 2 & 6 \end{bmatrix}.$$

Now, note that

$$\begin{aligned} \alpha_1 &= \max_j\{a_{11}, a_{12}, a_{13}\} = \max\{1, 2, 0\} = 2 \\ \alpha_2 &= \max_j\{a_{21}, a_{22}, a_{23}\} = \max\{4, 2, 0\} = 4 \\ \alpha_3 &= \max_j\{a_{31}, a_{32}, a_{33}\} = \max\{1, 2, 6\} = 6. \end{aligned}$$

Thus, follows that

$$\widehat{\mathcal{N}(f)} = \begin{bmatrix} 2-1 & 2-2 & 2-0 \\ 4-4 & 4-2 & 4-0 \\ 6-1 & 6-2 & 6-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 5 & 4 & 0 \end{bmatrix}.$$

Besides, we have that $\boldsymbol{\alpha} = (2, 4, 6)^t$ and

$$[\,\boldsymbol{\alpha} \,|\cdots|\, \boldsymbol{\alpha}\,]_{n\times r} = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \end{bmatrix}.$$

Therefore, the *Newton dual* of f is given by $\hat{f} = xz^5 - y^2z^4 + x^2y^4$.

Chapter 2

Codimension 3 equigenerated Gorenstein ideal

The literature on Gorenstein ideals of codimension 3 is too extensive to be collected here, but see [5], [23], [7], [9], [19], [41], [48] for some outstanding work. And yet, the equigenerated case has not been so much under focus. In this chapter, we will highlight some properties satisfied by the codimension 3 equigenerated Gorenstein ideals in a standard graded polynomial ring $\Bbbk[x_1, \ldots, x_n]$ over an infinite field $\Bbbk$.

Unless otherwise stated, in this chapter, R will be the standard graded polynomial ring $\Bbbk[x_1, \ldots, x_n]$ over an infinite field $\Bbbk$. Here, by definition, an *alternating matrix* is a square skew-symmetric matrix with null main diagonal entries.

Let Φ denote an $r \times r$ ($r \geq 1$) alternating matrix with entries in R. It is a classical result that $\det \Phi = 0$ when r is odd and it is the square of a polynomial in the entries of Φ if r is even. This polynomial is called the *Pfaffian* of Φ and it is denoted by $Pf(\Phi)$. In the case where $r \geq 3$ is odd it is customary to refer to the Pfaffians arising from the principal $(r-1) \times (r-1)$-minors of Φ as the (maximal) Pfaffians of Φ.

2.1 Numerical data

The main structural result about codimension 3 Gorenstein ideals can be stated as follows:

Theorem 2.1.1. ([5, Theorem 2.1]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(i) *Let $r \geqslant 3$ be an odd integer, and let Φ be an $r \times r$ alternating matrix with entries in $\mathfrak{m}$. Suppose $Pf_{r-1}(\Phi)$ has grade 3. Then $Pf_{r-1}(\Phi)$ is a Gorenstein ideal, minimally generated by r elements.*

(ii) *Every Gorenstein ideal of grade 3 arises as in* (i).

The above result can be stated also when R is a standard graded ring over a field by the usual way of localizing at the irrelevant maximal ideal of R (see the consideration of the Hilbert function after the proof of [5, Theorem 2.1]). As part of the proof, it is shown that a Gorenstein ideal $I \subset R$ of grade 3 has a canonical free resolution

$$0 \to R \to R^r \stackrel{\Phi}{\to} R^r \to R \to R/I \to 0,$$

where Φ is an alternating matrix and I is generated by the maximal Pfaffians of Φ.

In addition, when I is equigenerated, say, in degree $d \geq 1$, then the columns of Φ must be homogeneous of some standard degree d_i, $1 \leq i \leq r$. By the nature of each generator of I as a maximal Pfaffian of Φ, it immediately follows that

$$2d = d_1 + d_2 + \cdots + d_{r-2} + d_{r-1} = d_1 + d_2 + \cdots + d_{r-2} + d_r = \cdots = d_2 + d_3 + \cdots + d_{r-1} + d_r.$$

By an elementary argument, $d_1 = d_2 = \cdots = d_r$. It follows that $2d = (r-1)d'$, where d' is the common values of the d_i's, i.e.,

$$d = \frac{r-1}{2}\, d'. \tag{2.1}$$

Thus, an equigenerated Gorenstein ideal I of codimension 3 complies with this relation, where r is the number of generators, d their common degree and d' is the degree of any entry of the alternating matrix defining I. Thus, the earlier free resolution becomes a graded free resolution of the form

$$0 \to R(-(s+n)) \to \bigoplus_{i=1}^{r} R(-(d+d')) \to \bigoplus_{i=1}^{r} R(-d) \to R \to R/I \to 0,$$

where s is the socle degree of R/I.

Remark 2.1.2. Note that (2.1) implies that the number of generators of I above is at most $2d+1$. This bound has been obtained earlier in [37, Theorem 2.2] for any codimension 3 Gorenstein ideal with initial degree d.

For reference convenience, we introduce the following terminology:

Definition 2.1.3. A *codimension* 3 *equigenerated Gorenstein virtual datum* in dimension $n \geq 3$ is a pair (d, r) of integers such that:
(i) $d \geq 2$ and $r \geq 3$.
(ii) r is odd and $(r-1)/2$ is a factor of d.

If no misunderstanding arises, we will mostly omit 'codimension 3 equigenerated Gorenstein' in the above terminology in what follows. Given such a datum, the integer $d' := 2d/(r-1)$ will be called the *skew-degree* of the datum. The expression of r in terms of the 'generating degree' and the skew-degree is $r = (2d + d')/d'$.

Examples 2.1.4. (i) The case where $d' = 1$, i.e., $r = 2d + 1$ will be referred to as the *linear case.*

(ii) If d is a prime number then either Φ is linear (that is, has linear entries), or else I is a complete intersection.

(iii) In particular, for $d = 3$ (in any number of variables), necessarily $\mu(I) = 7$ and the matrix Φ is linear; for $d = 4$, either $\mu(I) = 9$ and Φ is linear or else $\mu(I) = 5$ and the entries of Φ are quadrics.

When $n = 3$, R/I is a graded Artinian Gorenstein algebra. In this case, the relation between the numerical datas d, r, d' and the socle degree of R/I is given by the following lemma

Lemma 2.1.5. *Let $I \subset R = \Bbbk[x, y, z]$ denote a codimension 3 equigenerated Gorenstein ideal with virtual datum $(d, (2d + d')/d')$, where $d' \geq 1$ is the skew-degree of I as introduced earlier. Then the socle degree of R/I is $2d + d' - 3$.*

Proof. It is well-known that, pretty generally, the socle degree of a graded Artinian Gorenstein quotient R/I of a standard graded polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ is given by $D - n$, where D is the last shift in the minimal graded R-resolution of R/I (see, e.g., Lemma 1.4.1). In the present case, the resolution has length 3 and the first syzygies have shifted degree $d + d'$. Since the graded free resolution of a graded Gorenstein algebra is self-dual we have that $D = d + d + d' = 2d + d'$, hence the socle degree is as stated. □

We will say that the datum (d, r) is *proper* (in dimension n) if there exists a codimension 3 Gorenstein ideal I in R with this datum, satisfying $\operatorname{ht} I_1(\Phi) = n$, where Φ is the skew-symmetric matrix whose maximal Pfaffians generate I. In this case the skew-degree is in fact the degree of any entry in the skew-symmetric matrix Φ, perhaps justifying its designation.

Example 2.1.6. For every $d \geq 1$ consider the virtual datum (d, r) where $r = 2d + 1$. Let $H_d = (L_{i,j})$ be an $r \times r$ skew-symmetric matrix whose above diagonal entries are defined by:

$$L_{(i,j)} = \begin{cases} x & \text{if } j = i + 1 \text{ and } i \text{ is odd} \\ y & \text{if } j = i + 1 \text{ and } i \text{ is even} \\ z & \text{if } j = r - i + 1 \\ 0 & \text{elsewhere.} \end{cases}$$

The ideal $I = \operatorname{Pf}_{2d}(H_d)$ is a Gorenstein ideal of codimension 3 in $R = \Bbbk[x, y, z]$ (see [5, Proposition 6.2]). In addition, $\operatorname{ht} I_1(H_d) = 3$. This takes care of the linear case, i.e., every virtual datum $(d, 2d + 1)$ is proper in dimension 3.

For $n = 3$, this example actually yields the case of any datum as follows:

Proposition 2.1.7. *Every virtual datum is proper in dimension 3.*

Proof. Let $d' := 2d/(r - 1)$ denote the skew-degree of the virtual datum. Consider the following homomophism of $\Bbbk$-algebras

$$\zeta : \Bbbk[x, y, z] \to \Bbbk[x, y, z], \quad F(x, y, z) \mapsto F(x^{d'}, y^{d'}, z^{d'}).$$

For the matrix $H_{(r-1)/2}$ of the Example 2.1.6 we have

$$\mathrm{Pf}_{(r-1)}(\zeta(L_{i,j})) = \zeta(\mathrm{Pf}_{(r-1)}(L_{i,j}))\Bbbk[x,y,z].$$

From this equality and the fact that $\mathrm{Pf}_{(r-1)}(L_{i,j})$ contains powers of the variables x,y,z it follows that $\mathrm{Pf}_{(r-1)}(\zeta(L_{i,j}))$ contains powers of $x^{d'}, y^{d'}$ and $z^{d'}$. In particular, $\mathrm{Pf}_{(r-1)}(\zeta(L_{i,j}))$ is a codimension 3 Gorenstein ideal with datum (d,r). □

In the next section we extend this proposition for arbitrary dimension $n \geq 3$.

2.2 Properness in arbitrary dimension

The argument works in arbitrary characteristic. Our approach for the equigenerated case and $n \geq 3$ is based on a simple reparametrization-like device as follows.

Quite generally, for a standard graded polynomial ring $\Bbbk[z_1,\ldots,z_N]$ over an infinite field $\Bbbk$, and an integer $p \geq 1$, consider the following injective $\Bbbk$-algebra map

$$\zeta_p : \Bbbk[z_1,\ldots,z_N] \to \Bbbk[z_1,\ldots,z_N], \quad z_i \mapsto z_i^p \ (1 \leq i \leq N).$$

If X is any matrix with entries in $\Bbbk[z_1,\ldots,z_N]$, we denote by $\zeta_p(X)$ the matrix obtained by evaluating ζ_p at every entry of X. We focus on the case where r is a given odd integer and Φ denotes the $r \times r$ generic skew-symmetric matrix. Let B stand for the polynomial ring over $\Bbbk$ in the nonzero entries of Φ.

Lemma 2.2.1. *For any integer $p \geq 1$ the $(r-1)$-Pfaffians of the skew-symmetric matrix $\zeta_p(\Phi)$ generate an ideal of B of codimension 3.*

Proof. It is well-known that an $(r-1)$-Pfaffian is a polynomial in the entries of the source skew-symmetric matrix (see, e.g., [39, Proposition 159]). Let $I \subset B$ denote the ideal of $(r-1)$-Pfaffians of Φ. Since ζ_p is a homomorphism of $\Bbbk$-algebras, the ideal of B generated by the $(r-1)$-Pfaffians of $\zeta_p(\Phi)$ is the extended ideal $\zeta_p(I)B$, where $\zeta_p(I)$ is the image of I by ζ_p. Now, clearly, $\zeta_p(I) \subset \zeta_p(B)$ and $I \subset B$ are ideals of the same height. On the other hand, the extension $\zeta_p(B) \subset B$ is integral and $\zeta_p(B)$ is integrally closed in its field of fractions because it is isomorphic to a polynomial ring over $\Bbbk$. Therefore, the going-down property of the extension $\zeta_p(B) \subset B$ implies that $\mathrm{ht}\,\zeta_p(I)B = \mathrm{ht}\,\zeta_p(I) \geq 3$. □

Theorem 2.2.2. *Let $n \geq 3$ be an integer. Then every virtual datum (d,r) in dimension n is proper.*

Proof. Denote by d' the skew-degree of the virtual datum (d,r). Define $u = \binom{r}{2} - n \leq \binom{r}{2} - 3$. We induct on u. If $u = 0$, by Lemma 2.2.1 the ideal generated by the $(r-1)$-Pfaffians of the matrix $\zeta_{d'}(\Phi)$ is a Gorenstein ideal with datum (d,r).

Now, suppose that the result is true for a certain $1 \leq u < \binom{r}{2} - 3$. Then there is an $r \times r$ skew-symmetric matrix $\Psi = (g_{i,j})$ whose entries $g_{i,j}$ are forms of degree d' in $\Bbbk[x_1,\ldots,x_n]$, such that $\mathrm{ht}\, I_1(\Psi) = n$ and $I = \mathrm{Pf}_{r-1}(\Psi)$ is a codimension 3 Gorenstein ideal. Since one can assume that r is fixed, we may argue by descending induction on n instead. Thus, we are assuming that $n > 3$. Since I is an unmixed homogeneous ideal of codimension 3, then by a homogeneous version of the prime avoidance lemma, there is a linear form $\ell \in R = \Bbbk[x_1, \ldots, x_n]$ that is R/I-regular. Without loss of generality, we can suppose that $\ell = x_n - \sum_{i=1}^{n-1} \alpha_i x_i$. Now consider the following surjective $\Bbbk$-algebra homomorphism:

$$\pi : \Bbbk[x_1, \ldots, x_n] \twoheadrightarrow \Bbbk[x_1, \ldots, x_{n-1}], \quad x_i \mapsto x_i \, (1 \leq i \leq n-1), \, x_n \mapsto \sum_{i=1}^{n-1} \alpha_i x_i.$$

Using again the fact that Pfaffians are polynomials in the entries of the source matrix, π induces a $\Bbbk$-algebra isomorphism

$$\Bbbk[x_1, \ldots, x_n]/(\mathrm{Pf}_{r-1}(\Psi), \ell) \simeq \Bbbk[x_1, \ldots, x_{n-1}]/\mathrm{Pf}_{r-1}(\widetilde{\Psi})$$

and

$$\Bbbk[x_1, \ldots, x_n]/(I_1(\Psi), \ell) \simeq \Bbbk[x_1, \ldots, x_{n-1}]/I_1(\widetilde{\Psi})$$

where $\widetilde{\Psi} = (\pi(g_{ij}))$. By the second isomorphism above, $I_1(\widetilde{\Psi})$ has codimension $n-1$. Since ℓ is regular on $\Bbbk[x_1, \ldots, x_n]/I$ then $\Bbbk[x_1, \ldots, x_{n-1}]/\mathrm{Pf}_{r-1}(\widetilde{\Psi})$ is a Gorenstein ring of codimension

$$(n-1) - \dim \Bbbk[x_1, \ldots, x_{n-1}]/\mathrm{Pf}_{r-1}(\widetilde{\Psi}) = (n-1) - (n-4) = 3.$$

Thus, $\mathrm{Pf}_{r-1}(\widetilde{\Psi})$ is a codimension 3 Gorenstein ideal in $\Bbbk[x_1, \ldots, x_{n-1}]$ with datum (d, r), satisfying the condition $\mathrm{ht}\, I_1(\widetilde{\Psi}) = n-1$. $\square$

Remark 2.2.3. We comment on previous results related to the above. In [7] the authors considered the case of homogeneous (but not necessarily, equigenerated) Gorenstein ideals for $n = 3$. They introduced a certain *admissible sequence* of integers involving the degrees of the generators and the minimal number of generators of the ideal, satisfying similar properties as those of a virtual datum. They proved that for every such sequence of integers, there is a codimension 3 Gorenstein ideal in the polynomial ring $\Bbbk[x, y, z]$. In [9] the author focused on the degrees of relations of the syzygy matrix and the number of generators of the ideal, also exhibiting models of matrices in 3 variables ensuring the existence of Gorenstein ideals of codimension 3. Our approach, though restricted to the equigenerated case, gives the same sort of result in arbitrary dimension $n \geq 3$.

2.3 Special cases of the virtual datum

As by Lemma 2.1.5, if I is a Gorenstein ideal in $R = \Bbbk[x, y, z]$ satisfying the virtual datum (d, r), then the initial degree of I is $\geq d$ and the socle degree of R/I is $2d + d' - 3$. To the converse of this statement, has been proven in [30, Proposition 1.8] that if I has initial degree $\geq d$ and the socle degree $2d - 2$ then I is a Gorenstein ideal satisfying the virtual datum $(d, 2d+1)$. One of the simplest structures is that having virtual datum $(d, d+1)$.

Proposition 2.3.1. *Let $I \subset R = \Bbbk[x, y, z]$ be a homogeneous ideal of codimension 3 and initial degree $\geq d$. If I is Gorenstein with socle degree $2d-1$ then it has minimal graded free resolution*

$$0 \to R(-(2d+2)) \to R^{d+1}(-(d+2)) \oplus R^{\beta}(-(d+1)) \to R^{d+1}(-d) \oplus R^{\beta}(-(d+1)) \to R,$$

for suitable β. If, in addition, I is equigenerated then it has virtual datum $(d, d+1)$, hence d is even.

Proof. Since the initial degree of the Gorenstein ideal I is d then its minimal graded free resolution is

$$0 \to R(-c) \to \bigoplus_{i=1}^{r} R(-D_i) \xrightarrow{\rho} \bigoplus_{i=1}^{r} R(-d_i) \to I \to 0 \tag{2.2}$$

where $r = \mu(I)$ and $d = d_1 \leq \ldots \leq d_r$. By Lemma 1.4.1, $c = 2d - 1 + 3 = 2d + 2$. Since this resolution is self-dual, $D_i = c - d_i = 2d + 2 - d_i$ for every $1 \leq i \leq r$. If $d_u > d + 1$ for some $1 \leq u \leq r$, then the degree of the ith entry of the uth column of ρ is $D_u - d_i = 2d + 2 - (d_i + d_u) \leq 0$, contradicting the minimality of the resolution. Therefore, one has $d_i \leq d + 1$ for every $1 \leq i \leq r$. Thus, the resolution becomes

$$0 \to R(-(2d+2)) \to R^{\alpha}(-(d+2)) \oplus R^{\beta}(-(d+1)) \to R^{\alpha}(-d) \oplus R^{\beta}(-(d+1)) \to R$$

for certain integers $\alpha, \beta \geq 0$ such that $\alpha + \beta = r$. Reading the Hilbert series of R/I off its resolution yields

$$\frac{1 - \alpha t^d - \beta t^{d+1} + \beta t^{d+1} + \alpha t^{d+2} - t^{2(d+1)}}{(1-t)^3} = \frac{1 - \alpha t^d + \alpha t^{d+2} - t^{2(d+1)}}{(1-t)^3}.$$

Since the first derivative of the numerator vanishes at $t = 1$, we get $\alpha = d + 1$. Therefore, the resolution becomes

$$0 \to R(-(2d+2)) \to R^{d+1}(-(d+2)) \oplus R^{\beta}(-(d+1)) \to R^{d+1}(-d) \oplus R^{\beta}(-(d+1)) \to R.$$

This gives the first statement. For the second statement, if I is generated in degree $d+1$ then $D_i = d+2$, thus $d' = 1$. On the other hand, since $c = 2d+2$, it follows that $d' = 0$ by Lemma 2.1.5, which is an absurd. Finally, if I is generated in degree d then we have that $d' = 2$ and I has virtual datum $(d, d+1)$. $\square$

Note that the conditions involving the initial degree of the ideal and the value of the socle degree are not sufficient to guarantee that the ideal is Gorenstein with virtual datum $(d, d+1)$. In fact the ideal $I = (x^5, y^5, z^5) : (x + y + z)^3$ has initial degree 5 and socle degree 9, but the minimal free resolution of R/I is given by

$$0 \to R(-12) \to R^6(-7) \oplus R(-6) \to R^6(-5) \oplus R(-6) \to R \to 0.$$

Another information carried by Proposition 2.3.1 is that $\beta = 0$ for the equigenerated case and $\beta = \mu(I) - d - 1$ otherwise. Still by Proposition 2.3.1 we have the following

Corollary 2.3.2. *Let $a \geq d$ and $b = 3a - 2d - 2$. Then, $I = (x^a, y^a, z^a) : (x+y+z)^b \subset R = \Bbbk[x,y,z]$ is a codimension 3 Gorenstein ideal and*

$$0 \to R(-(2d+2)) \to R^{d+1}(-(d+2)) \oplus R^{\beta}(-(d+1)) \to R^{d+1}(-d) \oplus R^{\beta}(-(d+1)) \to R$$

is a minimal free resolution of R/I, for suitable β. If I is equigenerated then it has virtual datum $(d, d+1)$ and d is an even number.

Proof. Firstly note that $R_{2d} \subset I$. In fact, since $J := (x^a, y^a, z^a)$ is a complete intersection, then $\operatorname{Soc}(R/J) = 3a - 3$. Thus, $R_{3a-2} \subset (x^a, y^a, z^a)$. Hence, $(x+y+z)^b R_{2d} \subset R_{3a-2} \subset (x^a, y^a, z^a)$ and therefore $R_{2d} \subset I$. By the [43, Theorem 5] we have that the initial degree of I is d. Besides, note that $(R/I)_{2d-1} \neq 0$. In fact, up to change of variables the element $x^{a-1}y^{a-1}z^{a-1} = x^{b-1}yx^{2d+2-2a}y^{a-2}z^{a-1}$ is a term which is part of generators of the ideal $(x+y+z)^b R_{2d-1}$, but this term is not in (x^a, y^a, z^a). Thus, the statements follow by Proposition 2.3.1. $\square$

Chapter 3

Gorenstein ideals and general forms

Ideals generated by general forms have been vastly discussed in the literature (see e.g., [1], [2], [13], [14], [21], [35], [36]). In this chapter we focus on such ideals which are, moreover, equigenerated Gorenstein ideals in the standard graded polynomial ring over an infinite field.

3.1 Parametrization tools

Fix a ground field $\Bbbk$ and integers $r \geq 1,\ d \geq 1$. Let $g_t = \sum_{|\alpha|=d} Y_\alpha^{(t)} \mathbf{x}^\alpha,\ 1 \leq t \leq r$ denote forms of degree d, whose coefficients are mutually independent indeterminates over $\Bbbk$, while $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a set of independent variables over the polynomial ring $A := \Bbbk[Y_\alpha^{(t)} \,|\, 1 \leq t \leq r,\ |\alpha| = d]$. Note that $A = A_1 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_r$, where each A_t is up to variable names identified with the homogeneous coordinate ring of $\mathbb{P}_{\Bbbk}^N \left(N = \binom{d+n-1}{d} - 1\right)$.

Consider the polynomial ring $S := F[x_1, \ldots, x_n]$, where F denote the fraction field of the domain A.

Fix yet another integer $e \geq 0$ and let $D := \dim_F S_{d+e}$. The F-vector subspace $S_e g_1 + \cdots + S_e g_r$ of S_{d+e} is spanned by the set $\{\mathbf{x}^\beta g_1 \,|\, |\beta| = e\} \cup \cdots \cup \{\mathbf{x}^\beta g_r \,|\, |\beta| = e\}$. The coefficient matrix of this set with respect to the canonical monomial basis of S_{d+e} is a $D \times (r \dim_F S_e)$ matrix

$$M_{d,r,e} := \left(\begin{array}{c|c|c} M_1 & \ldots & M_r \end{array} \right), \tag{3.1}$$

where M_t is the coefficient matrix of the set $\{\mathbf{x}^\beta g_t \,|\, |\beta| = e\}$ with respect to the canonical basis of S_{d+e}. In particular, the entries of the block M_t involve only the coefficients $Y_{d,0,\ldots,0}^{(t)}, \ldots, Y_{0,\ldots,0,d}^{(t)}$ of the form g_t.

Definition 3.1.1. $M_{d,r,e}$ could be called the *parameter matrix of the e-span of r forms of degree d.* A couple of variations of this construct will be discussed in the sequel (Section 3.3, Section 4.2 and Section 5.3).

Of special interest is the specialization to 'rational' forms $\mathbf{f} = \{f_1, \ldots, f_r\}$, i.e., those with coefficients in $\Bbbk$, which is carried via the canonical surjection $A \twoheadrightarrow \Bbbk$. For these, let

$R = \Bbbk[x_1, \ldots, x_n]$ and consider the parameter map $R_d \times \cdots \times R_d \to (\mathbb{P}^N_{\Bbbk})^r$ that associates to the vector $[f_1 \cdots f_r]$ the point

$$P_{\mathbf{f}} := (\lambda^{(1)}_{d,0,\ldots,0} : \cdots : \lambda^{(1)}_{0,\ldots,0,d}; \ldots ; \lambda^{(r)}_{d,0,\ldots,0} : \cdots : \lambda^{(r)}_{0,\ldots,0,d}).$$

Here $Y^{(t)}_\alpha \mapsto \lambda^{(t)}_\alpha$ and $N = \dim_{\Bbbk} R_d - 1 = \binom{d+n-1}{d} - 1$.

We will freely denote $M_{d,r,e}(P_{\mathbf{f}})$ the result of evaluating via $A \twoheadrightarrow \Bbbk$.

The following simple example may help visualizing the shape of $M_{d,r,e}$.

Example 3.1.2. Let $r = d = n = 2$ and $e = 1$. Then $D = 4$ and one gets:

$$M_{2,2,1} = \left(\begin{array}{cc|cc} Y^{(1)}_{2,0} & 0 & Y^{(2)}_{2,0} & 0 \\ Y^{(1)}_{1,1} & Y^{(1)}_{2,0} & Y^{(2)}_{1,1} & Y^{(2)}_{2,0} \\ Y^{(1)}_{0,2} & Y^{(1)}_{1,1} & Y^{(2)}_{0,2} & Y^{(2)}_{1,1} \\ 0 & Y^{(1)}_{0,2} & 0 & Y^{(2)}_{0,2} \end{array}\right).$$

In this example the matrix has maximal rank. The next example shows that the rank can drop in general. Let $n = 3$, $d = 1$, $e = r = 2$. In this case, we have $D = 10$ and $\operatorname{rank} M_{1,2,1} = 9$, where $M_{1,2,1}$ is the following 10×12 matrix:

$$\left(\begin{array}{cccccc|cccccc}
Y^{(1)}_{1,0,0} & 0 & 0 & 0 & 0 & 0 & Y^{(2)}_{1,0,0} & 0 & 0 & 0 & 0 & 0 \\
Y^{(1)}_{0,1,0} & Y^{(1)}_{1,0,0} & 0 & 0 & 0 & 0 & Y^{(2)}_{0,1,0} & Y^{(1)}_{1,0,0} & 0 & 0 & 0 & 0 \\
Y^{(1)}_{0,0,1} & 0 & Y^{(1)}_{1,0,0} & 0 & 0 & 0 & Y^{(2)}_{0,0,1} & 0 & Y^{(2)}_{1,0,0} & 0 & 0 & 0 \\
0 & Y^{(1)}_{0,1,0} & 0 & Y^{(1)}_{1,0,0} & 0 & 0 & 0 & Y^{(2)}_{0,1,0} & 0 & Y^{(2)}_{1,0,0} & 0 & 0 \\
0 & Y^{(1)}_{0,0,1} & Y^{(1)}_{0,1,0} & 0 & Y^{(1)}_{1,0,0} & 0 & 0 & Y^{(2)}_{0,0,1} & Y^{(2)}_{0,1,0} & 0 & Y^{(2)}_{1,0,0} & 0 \\
0 & 0 & Y^{(1)}_{0,0,1} & 0 & 0 & Y^{(1)}_{1,0,0} & 0 & 0 & Y^{(2)}_{0,0,1} & 0 & 0 & Y^{(2)}_{1,0,0} \\
0 & 0 & 0 & Y^{(1)}_{0,1,0} & 0 & 0 & 0 & 0 & 0 & Y^{(2)}_{0,1,0} & 0 & 0 \\
0 & 0 & 0 & Y^{(1)}_{0,0,1} & Y^{(1)}_{0,1,0} & 0 & 0 & 0 & 0 & Y^{(2)}_{0,0,1} & Y^{(2)}_{0,1,0} & 0 \\
0 & 0 & 0 & 0 & Y^{(1)}_{0,0,1} & Y^{(1)}_{0,1,0} & 0 & 0 & 0 & 0 & Y^{(2)}_{0,0,1} & Y^{(2)}_{0,1,0} \\
0 & 0 & 0 & 0 & 0 & Y^{(1)}_{0,0,1} & 0 & 0 & 0 & 0 & 0 & Y^{(2)}_{0,0,1}
\end{array}\right).$$

Let now $\mathbf{f} = \{f_1, \ldots, f_r\}$ denote forms of degree d with coefficients in $\Bbbk$. Set $R := \Bbbk[x_1, \ldots, x_n]$. If $R_e f_1 + \cdots + R_e f_r = R_{d+e}$ we loosely say that the d-forms $f_1, \ldots, f_r$ *span in degree* $d+e$. A central question here asks when the matrix $M_{d,r,e}$ has maximal rank. This is tantamount to having that, for given r, d, e, 'most' tuples $(f_1, \ldots, f_r)$ of rational forms span.

As a matter of further notation, for given data $\{d, r, e\}$ as above denote by $D_{d,r,e}$ the rank of the matrix $M_{d,r,e}$. Let $I_{D_{d,r,e}}(M_{d,r,e}) \subset A$ denote the ideal of minors of $M_{d,r,e}$ of order $D_{d,r,e}$, and $V(I_{D_{d,r,e}}(M_{d,r,e})) \subset (\mathbb{P}^N)^r$ the corresponding projective subvariety. The following properties are immediately verified:

(A) $D_{d,r,e} \leq \min\{\dim_F S_{d+e}, r \dim_F S_e\} = \min\{\dim_{\Bbbk} R_{d+e}, r \dim_{\Bbbk} R_e\}$.

Equivalently,

$$\dim_{\Bbbk} R_{d+e} - D_{d,r,e} \geq \max\{\dim_{\Bbbk} R_{d+e} - r\dim_{\Bbbk} R_e,\, 0\}. \tag{3.2}$$

(B) For every set $\mathbf{f} \in (R_d)^r$, $\dim_{\Bbbk}[(\mathbf{f})]_{d+e} = \operatorname{rank} M_{d,r,e}(P_{\mathbf{f}}) \leq D_{d,r,e}$.

Equivalently,

$$\dim_{\Bbbk}[R/(\mathbf{f})]_{d+e} \geq \dim_{\Bbbk} R_{d+e} - D_{d,r,e}. \tag{3.3}$$

(C) For every set $\mathbf{f} \in (R_d)^r$ such that $P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,e}}(M_{d,r,e}))$,

$$\dim_{\Bbbk}[R/(\mathbf{f})]_{d+e} = \dim_{\Bbbk} R_{d+e} - D_{d,r,e}. \tag{3.4}$$

Definition 3.1.3. Let $R = \Bbbk[x_1, \ldots, x_n]$. Given d, r, e, a tuple of forms $\mathbf{f} \in (R_d)^r$ is (d,r,e)-*extremal* if

$$\dim_{\Bbbk}[R/I]_{d+e} = \max\{\dim_{\Bbbk} R_{d+e} - r\dim_{\Bbbk} R_e,\, 0\}$$

where $I = (\mathbf{f}) \subset R$.

For convenience, we isolate some basic facts in the following lemma.

Lemma 3.1.4. *Let $R = \Bbbk[x_1, \ldots, x_n]$. Given integer data d, r, e as above, one has:*

(i) *If there is a (d,r,e)-extremal set in $(R_d)^r$ then every tuple $\mathbf{f} \in (R_d)^r$ such that $P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,e}}(M_{d,r,e}))$ is (d,r,e)-extremal. In particular, for any such $\mathbf{f}$ one has*

$$\dim_{\Bbbk} R_{d+e} - D_{d,r,e} = \max\{\dim_{\Bbbk} R_{d+e} - r\dim_{\Bbbk} R_e,\, 0\}.$$

(ii) *For $e = 0$, every tuple $\mathbf{f} \in (R_d)^r$ such that $P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,0}}(M_{d,r,0}))$ is $(d,r,0)$-extremal.*

(iii) *For $e = 1$, every tuple $\mathbf{f} \in (R_d)^r$ such that $P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,1}}(M_{d,r,1}))$ is $(d,r,1)$-extremal.*

(iv) *If $e > n(d-1)$ and $r \geq n$ then every $\mathbf{f} \in (R_d)^r$ such that $P_{\mathbf{f}} \in (\mathbb{P}^N)^r \setminus V(I_{D_{d,r,e}}(M_{d,r,e}))$ is (d,r,e)-extremal.*

Proof. (i) This follows from properties (A), (B) and (C) above.

(ii) This is just linear independence over $\Bbbk$.

(iii) This is the content of [21, Theorem 1].

(iv) In fact, for every $e > n(d-1)$, $\max\{\dim_{\Bbbk} R_{d+e} - r\dim_{\Bbbk} R_e,\, 0\} = 0$. For a sequence $\mathbf{f}_0 \in (R_d)^r$ such that $\{x_1^d, \ldots, x_n^d\} \subset \mathbf{f}_0$ we have $\dim_{\Bbbk}[R/(\mathbf{f}_0)]_{d+e} = 0$, that is, $\mathbf{f}_0$ is (d,r,e)-extremal. Thus, the claim follows by (i). $\square$

3.2 Around Fröberg's conjecture

Recall that a set $\mathbf{f} = \{f_1, \dots, f_r\} \subset R$ of forms of the same degree is *general* when the corresponding parameter point $P_{\mathbf{f}}$ is generic in the sense of its coordinates; that is, the corresponding vector of coefficients belongs to a suitable dense Zariski open set of the parameter space.

Though the definition sounds a bit fluid, when working with such general set of forms, in practice, one deals with some property $\mathcal{P}$ that holds for them, meaning that there is a dense open set $U_{\mathcal{P}}$ (depending on $\mathcal{P}$) of the parameter space such that $\mathcal{P}$ holds for every set of forms whose corresponding point lies in $U_{\mathcal{P}}$. The stronger the given property, the smaller, and possibly the harder to describe, the corresponding dense open set in the parameter space, and the objective is often attained by indirect argument.

Let $\mathbf{f} = \{f_1, \dots, f_r\}$ be a general set in a polynomial ring $R = \Bbbk[x, y, z]$ over an infinite field $\Bbbk$. In [1] the Hilbert series of $R/(f_1, \dots, f_r)$ is completely determined. In the next result we give an alternative argument in the case of an almost complete intersection $R/(f_1, f_2, f_3, f_4)$. Although a consequence of [1], the actual argument reduces to computing the socle degree.

Proposition 3.2.1. (char($\Bbbk$) = 0) *The tuple of a general set of four forms of degree $d \geq 1$ in $R = \Bbbk[x, y, z]$ is $(d, 4, e)$-extremal, for arbitrary $e \geq 1$ and generates an ideal of R with socle degree $2d - 2$.*

Proof. By Lemma 3.1.4, it suffices to show the existence of one set of four forms of degree d satisfying the statement.

Let then $J = (x^d, y^d, z^d) \subset R = \Bbbk[x, y, z]$. The minimal graded free resolution of R/J is given by the Koszul complex

$$0 \to R(-3d) \to R(-2d)^3 \to R(-d)^3 \to R \to R/J \to 0,$$

from which one readily gets the Hilbert series of R/J:

$$H_{R/J}(t) = \frac{1 - 3t^d + 3t^{2d} - t^{3d}}{(1-t)^3}.$$

Therefore, the coefficients a_i of $H_{R/J}(t)$ are:

$$a_i = \begin{cases} \dim_{\Bbbk} R_i & \text{if } 0 \leq i \leq d-1 \\ \dim_{\Bbbk} R_i - 3\dim_{\Bbbk} R_{i-d}, & \text{if } d \leq i \leq 2d-1 \\ \dim_{\Bbbk} R_i - 3\dim_{\Bbbk} R_{i-d} + 3\dim_{\Bbbk} R_{i-2d}, & \text{if } 2d \leq i \leq 3d-3. \end{cases} \tag{3.5}$$

Now, by [46] (also [49, Corollary 3.5]), R/J has the Strong Lefschetz property, that is, there is a linear form $L \in R$ such that for $f := L^d$ the multiplication map $[R/J]_{i-d} \to [R/J]_i$ by f has maximal rank for all $i \geq d$. Consequently, setting $I := (x^d, y^d, z^d, f)$, since the image of this map is the vector space $(fR_{i-d}, J_i)/J_i = I_i/J_i$, then

$$\dim_{\Bbbk} I_i/J_i = \min\{\dim_{\Bbbk}[R/J]_{i-d}, \dim_{\Bbbk}[R/J]_i\},$$

hence

$$\begin{aligned}\dim_{\Bbbk}[R/I]_i &= \dim_{\Bbbk}[R/J]_i - \min\{\dim_{\Bbbk}[R/J]_{i-d}, \dim_{\Bbbk}[R/J]_i\}\\ &= \max\{\dim_{\Bbbk}[R/J]_i - \dim_{\Bbbk}[R/J]_{i-d},\, 0\}\end{aligned}$$

for all $i \geq d$.

Thus, by (3.5) and an obvious calculation, we have:

$$\dim_{\Bbbk}[R/I]_i = \begin{cases} \dim_{\Bbbk} R_i & \text{if } 0 \leq i \leq d-1\\ \max\{\dim_{\Bbbk} R_i - 4\dim_{\Bbbk} R_{i-d},\, 0\}, & \text{if } d \leq i \leq 2d-1\\ \max\{\dim_{\Bbbk} R_i - 4\dim_{\Bbbk} R_{i-d} + 6\dim_{\Bbbk} R_{i-2d},\, 0\}, & \text{if } 2d \leq i \leq 3d-3.\end{cases}$$

Therefore, it follows that

$$\dim_{\Bbbk}[R/I]_i = \dim_{\Bbbk} R_i - 4\dim_{\Bbbk} R_{i-d} > 0, \quad \text{for } d \leq i \leq 2d-2$$

and $\dim_{\Bbbk}[R/I]_{2d-1} = 0$, thus showing that the socle degree is $2d-2$. □

Remark 3.2.2. As a consequence of the above proposition, the socle degree of an almost complete intersection I of a general set of forms in $\Bbbk[x, y, z]$ is $2d-2$. This value, has been previously established in [35, Lemma 2.5] for arbitrary codimension and not necessarily equigenerated ideal in a different way.

As a last issue in this section we highlight a connection, in the d-equigenerated case, between Fröberg's conjecture ([13]) in arbitrary dimension and the concept of a (d, r, e)-extremal set of forms. Using Fröberg's notation, in the d-equigenerated case $((1-t^d)^r/(1-t)^n)_+$ denotes the initial positive segment of the power series $(1-t^d)^r/(1-t)^n$.

Fröberg's conjecture [13] for equigenerated ideals reads as follows:

Conjecture 3.2.3. *For a general set of forms $\mathbf{f} = \{f_1, \ldots, f_r\}$ in $R = \Bbbk[x_1, \ldots, x_n]$ of degree $d \geq 1$, the Hilbert series of $R/(f_1, \ldots, f_r)$ is $((1-t^d)^r/(1-t)^n)_+$.*

In [15, Question 2.5] it is asked what are the coefficients of the Hilbert series. In the equigenerated case, one has the following expression:

Proposition 3.2.4. *Let $R = \Bbbk[x_1, \ldots, x_n]$ and $r \geq n$ and $d \geq 1$ be integers. Then*

$$\left(\frac{(1-t^d)^r}{(1-t)^n}\right)_+ = \sum_{j=0}^{j_0-1}\left(\sum_{i=0}^{r}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_{j-id}\right)t^j,$$

with j_0 the least integer in the interval $d \leq j \leq dr-n$ such that $\sum_{i=0}^{r}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_{j-id} \leq 0$.

Proof. The arbitrary term of the series $(1-t^d)^r/(1-t)^n$, is certainly well-known. We recall how to get its expression. Clearly, $1/(1-t)^n = \sum_{u=0}^{\infty}\dim_{\Bbbk} R_u t^u$. Thus,

$$\frac{(1-t^d)^r}{(1-t)^n} = \left(\sum_{i=0}^{r}(-1)^i\binom{r}{i}t^{di}\right)\left(\sum_{u=0}^{\infty}\dim_{\Bbbk} R_u t^u\right)$$

$$= \sum_{i=0}^{r}\sum_{u=0}^{\infty}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_u t^{u+id}$$
$$= \sum_{j=0}^{\infty}\left(\sum_{\substack{u\geq 0,\, 0\leq i\leq r\\ u+id=j}}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_u\right)t^j$$
$$= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{r}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_{j-id}\right)t^j.$$

From this and the definition of $((1-t^d)^r/(1-t)^n)_+$ the stated expression follows suit. $\square$

Remark 3.2.5. The above clearly translates [15, Question 2.5] into the quest for an explicit expression of the integer j_0 in terms of n, d, r. Also, we see that an affirmative answer to Fröberg's conjecture implies that a general set of $r \geq n$ forms $\mathbf{f} = \{f_1, \ldots, f_r\}$ of degree d is (d, r, e)-extremal for every $0 \leq e \leq \min\{d-1, j_0 - d\}$.

The following question is fundamental as regards the present method:

Question 3.2.6. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of $r \geq n$ forms of degree $d \geq 2$. Does an equality*

$$\dim_{\Bbbk} I_j = \min\{r \cdot \dim_{\Bbbk} R_{j-d}, \dim_{\Bbbk} R_j\}$$

hold in the interval $d \leq j \leq 2d-1$?

Note that the above question is tantamount to saying that the set of forms that generate I is (d, r, e)-extremal for $0 \leq e \leq d-1$. Letting j_0 be as in Proposition 3.2.4, assuming Fröberg's conjecture, by Remark 3.2.5, we have that $0 \leq e \leq d-1$ whenever $j_0 > 2d-1$. Letting a_j be the coefficient $\sum_{i=0}^{r}(-1)^i\binom{r}{i}\dim_{\Bbbk} R_{j-id}$, since j_0 is some j, we have that $a_j > 0$ for $d \leq j \leq 2d-1$. Thus, assuming Fröberg's conjecture and $j_0 > 2d-1$, Question 3.2.6 is affirmative.

3.3 Gorensteiness under general forms

We now draw some consequences of the previous section in dimension three. First, one has the following constraint for Gorenstein ideals in this dimension, provided the structural skew-symmetric matrix has linear entries, i.e., the skew-degree $d' = 1$. By the relation of the virtual datum, $d = (r-1)/2$ in this situation.

Proposition 3.3.1. *Let $r \geq 5$ be an odd integer and $d = (r-1)/2$. If there is a Gorenstein ideal $I \subset R = \Bbbk[x, y, z]$ generated by a general set of r forms of degree d, then $d = 2$, i.e., $r = 5$.*

Proof. Let I denote such an ideal. By the symmetry of the Hilbert function of R/I we have:

$$h_{R/I}(d+1) = h_{R/I}(d-3) = \binom{d-1}{2}.$$

Then, since I is generated by a general set of forms, drawing upon Lemma 3.1.4 (iii), we get

$$\binom{d-1}{2} = h_{R/I}(d+1) = \max\left\{\binom{d+3}{2} - 3r,\, 0\right\} = \max\left\{\binom{d+3}{2} - 3(2d+1),\, 0\right\}.$$

A direct calculation shows that the only possibility for d a positive integer is that $\binom{d-1}{2} = 0$, hence $d = 2$. $\square$

The main theorem is now a simple consequence:

Theorem 3.3.2. (char($\Bbbk$) = 0) *Let $I \subset R = \Bbbk[x, y, z]$ be a Gorenstein ideal generated by a general set of $r \geq 5$ forms of degree $d \geq 2$. Then $r = 5$ and $d = 2$.*

Proof. Let $J \subset I$ be generated by $n + 1 = 4$ of the forms. By Proposition 3.2.1, the socle degree of R/J is $2d - 2$. Then the socle degree of R/I is at most $2d - 2$. But, since I is Gorenstein, the socle degree of R/I is $2d + d' - 3$ (Lemma 2.1.5). Therefore, $d' = 1$. Now apply Proposition 3.3.1. $\square$

Remark 3.3.3. An alternative proof of the above result is available by cooking up a couple of explicit calculations. Since these are rather lengthy and distracting, we have decided to defer them to the Appendix (Theorem A.4).

For arbitrary n, we state the following conjecture.

Conjecture 3.3.4. *Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of $r \geq n + 2$ forms of degree $d \geq 2$. Then the following are equivalent:*

(i) *I is Gorenstein.*

(ii) *$d = 2$ and $r = \binom{n+1}{2} - 1$.*

To elaborate on this conjecture we introduce the following notation, in the spirit of Definition 3.1.1. Given integers $n \geq 3$, $d \geq 2$ and $r \geq 5$, consider the following $\binom{d+n-1}{d} \times r$ generic matrix

$$\mathbf{Y} := \mathbf{Y}_{n,d,r} = \begin{pmatrix} Y^{(1)}_{d,\ldots,0} & \cdots & Y^{(r)}_{d,\ldots,0} \\ \vdots & \ddots & \vdots \\ Y^{(1)}_{0,\ldots,d} & \cdots & Y^{(r)}_{0,\ldots,d} \end{pmatrix}, \tag{3.6}$$

whose entries along each column can be thought of as the coordinates of a copy of $\mathbb{P}_{\Bbbk}^{\binom{d+n-1}{d}-1}$ or the indeterminate coefficients of a form of degree d.

Given r rational forms $\mathbf{f} = \{f_1, \ldots, f_r\} \subset R = \Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ of degree $d \geq 1$, they form a vector equal to the matrix product $[\mathbf{x}_d] \cdot \mathbf{Y}(P_\mathbf{f})$, where $\mathbf{x}_d$ is the list of monomials of degree d in $\mathbf{x}$ and $\mathbf{Y}(P_\mathbf{f})$ is the matrix resulting from evaluating the entries of $\mathbf{Y}$ at the point $P_\mathbf{f}$ (i.e., at the corresponding coefficients of $\mathbf{f}$).

Throughout, set $\mathfrak{m} = (\mathbf{x})$. Our main technical result concerns the case where $d = 2$.

Proposition 3.3.5. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of r quadrics, where $n \leq r \leq N - 1$ and $N = \binom{n+1}{2}$. Then the socle of R/I has no nonzero linear forms.*

Proof. A quadric in $\Bbbk[x_1,\ldots,x_n]$ depends on $N=\binom{n+1}{2}$ coefficients. We will accordingly denote the corresponding indices by $\{1,1\},\{1,2\},\ldots,\{n-1,n\},\{n,n\}$. Thus, the matrix $\mathbf{Y}_{n,2,r}$ has the shape

$$\begin{pmatrix} Y_{1,1}^{(1)} & \cdots & Y_{1,1}^{(r)} \\ \vdots & \ddots & \vdots \\ Y_{n,n}^{(1)} & \cdots & Y_{n,n}^{(r)} \end{pmatrix},$$

where delimiters have been omitted. Let $\mathbf{f}=\{f_1,\ldots,f_r\}\subset R$ be a general set of quadrics generating I. For simplicity, set $P:=P_{\mathbf{f}}$. As explained above, one has the following matrix equality

$$[\mathbf{f}]=[\mathbf{x}_2]\cdot\mathbf{Y}_{n,2,r}(P).$$

The $N\times r$ matrix $\mathbf{Y}_{n,2,r}$ can be decomposed in two vertical blocks as

$$\mathbf{Y}_{n,2,r}=\begin{pmatrix} B \\ L \end{pmatrix}$$

where

$$B=\begin{pmatrix} Y_{1,1}^{(1)} & \cdots & Y_{1,1}^{(r)} \\ \vdots & \ddots & \vdots \\ Y_{u,v}^{(1)} & \cdots & Y_{u,v}^{(r)} \end{pmatrix}_{r\times r} \quad \text{and} \quad L=\begin{pmatrix} Y_{u',v'}^{(1)} & \cdots & Y_{u',v'}^{(r)} \\ \vdots & \ddots & \vdots \\ Y_{n,n}^{(1)} & \cdots & Y_{n,n}^{(r)} \end{pmatrix}_{(N-r)\times r},$$

for suitable indices $1\le u\le v\le n$ and $1\le u'\le v'\le n$. Thus,

$$[\mathbf{f}]\cdot\mathrm{cof}(B)(P)=[\mathbf{x}_2]\cdot\begin{pmatrix} \Delta(P)\mathbb{I}_r \\ L(P)\cdot\mathrm{cof}(B)(P) \end{pmatrix} \tag{3.7}$$

where $\mathrm{cof}(B)$ is the matrix of cofactors of B, $\mathbb{I}_r$ is the identity matrix of order r and $\Delta=\det B$. Write, say,

$$L\cdot\mathrm{cof}(B)=\begin{pmatrix} \mathfrak{g}_{1,1}^{u',v'} & \cdots & \mathfrak{g}_{u,v}^{u',v'} \\ \vdots & \ddots & \vdots \\ \mathfrak{g}_{1,1}^{n,n} & \cdots & \mathfrak{g}_{u,v}^{n,n} \end{pmatrix}. \tag{3.8}$$

Since $\mathbf{f}$ is a general set, then $\Delta(P)\neq 0$. Thus, by (3.7):

$$I=\left(\Delta(P)x_ix_j+\sum_{(u',v')\le(s,t)\le(n,n)}\mathfrak{g}_{i,j}^{s,t}(P)x_sx_t \,\middle|\, (1,1)\le(i,j)\le(u,v)\right).$$

Let $\ell=a_1x_1+\cdots+a_nx_n$ be a linear form in the socle of R/I. For any given $1\le k\le n$, one has

$$a_1x_1x_k+\cdots+a_nx_nx_k=\sum_{(1,1)\le(i,j)\le(u,v)}\alpha_{i,j}\left(\Delta(P)x_ix_j+\sum_{(u',v')\le(s,t)\le(n,n)}\mathfrak{g}_{i,j}^{s,t}(P)x_sx_t\right). \tag{3.9}$$

Comparing coefficients we get a linear system

$$\mathcal{A}_r(P)\cdot\mathbf{a}=0$$

where $\mathbf{a}$ is the transpose of the matrix $(a_1\cdots a_n)$ and $\mathcal{A}_r(P)$ is an $(N-r)n\times n$ matrix whose entries belong to the set

$$\{0\}\cup\{\mathfrak{g}^{s,t}_{i,j}(P)\,:\,(1,1)\leq(i,j)\leq(u,v),\,(u',v')\leq(s,t)\leq(n,n)\}\cup\{\Delta(P)\}.$$

We claim that rank $(\mathcal{A}(P)_r)=n$, which says that the assumed linear form is zero. We divide the proof in two cases:

Case 1: $r=N-1$.

In this case, the relation(3.9) has the following format for $1\leq t\leq n$:

$$a_1x_1x_t+\cdots+a_nx_nx_t=\sum_{\substack{1\leq i\leq j\leq n\\(i,j)\neq(n,n)}}\alpha_{i,j}(\Delta(P)x_ix_j-\mathfrak{g}^{n,n}_{i,j}(P)x_n^2). \tag{3.10}$$

Comparing coefficients yields

$$\mathcal{A}_{N-1}(P)=\begin{pmatrix}\mathfrak{g}^{n,n}_{1,1}(P)&\cdots&\mathfrak{g}^{n,n}_{1,n-1}(P)&\mathfrak{g}^{n,n}_{1,n}(P)\\\vdots&\ddots&\vdots&\vdots\\\mathfrak{g}^{n,n}_{1,n-1}(P)&\cdots&\mathfrak{g}^{n,n}_{n-1,n-1}(P)&\mathfrak{g}^{n,n}_{n-1,n}(P)\\\mathfrak{g}^{n,n}_{1,n}(P)&\cdots&\mathfrak{g}^{n,n}_{n-1,n}(P)&\Delta(P)\end{pmatrix}. \tag{3.11}$$

Since the entries of $\mathbf{Y}_{n,2,N-1}$ are mutually independent indeterminates, there is a $\Bbbk$-homomorphism $\phi:\Bbbk[\mathbf{Y}_{n,2,N-1}]\to\Bbbk[L]$ mapping B to the identity matrix and fixing the entries of L. Thus, by (3.8)

$$L=[\phi(\mathfrak{g}^{n,n}_{1,1})\cdots\phi(\mathfrak{g}^{n,n}_{1,n})\;\phi(\mathfrak{g}^{n,n}_{2,2})\cdots\phi(\mathfrak{g}^{n,n}_{2,n})\cdots\phi(\mathfrak{g}^{n,n}_{n-1,n-1})\,\phi(\mathfrak{g}^{n,n}_{n-1,n})].$$

In particular, the determinant

$$\mathbb{D}:=\det\begin{pmatrix}\mathfrak{g}^{n,n}_{1,1}&\cdots&\mathfrak{g}^{n,n}_{1,n-1}&\mathfrak{g}^{n,n}_{1,n}\\\vdots&\ddots&\vdots&\vdots\\\mathfrak{g}^{n,n}_{1,n-1}&\cdots&\mathfrak{g}^{n,n}_{n-1,n-1}&\mathfrak{g}^{n,n}_{n-1,n}\\\mathfrak{g}^{n,n}_{1,n}&\cdots&\mathfrak{g}^{n,n}_{n-1,n}&\Delta\end{pmatrix}$$

specializes to

$$\phi(\mathbb{D})=\det\begin{pmatrix}Y^{(1)}_{n,n}&\cdots&Y^{(n-1)}_{n,n}&Y^{(n)}_{n,n}\\\vdots&\ddots&\vdots&\vdots\\Y^{(n-1)}_{n,n}&\cdots&Y^{(N-1)}_{n,n}&Y^{(N)}_{n,n}\\Y^{(n)}_{n,n}&\cdots&Y^{(N)}_{n,n}&1\end{pmatrix}.$$

The latter does not vanish as it is the sum of two forms in two different degrees, none of which vanishes. Hence, $\mathbb{D} \neq 0$ as well.

Case 2: $r < N-1$.

It is enough to find a point P' such that $\operatorname{rank}(\mathcal{A}_r(P')) = n$. For this, consider a general set of quadrics $\mathbf{f}' = \{f'_1, \ldots, f'_{N-1}\}$. By the previous case, as applied to $J := (f'_1, \ldots, f'_{N-1})$, the socle of R/J has no linear forms. Since $r \geq n$ and $\{f'_1, \ldots, f'_{N-1}\}$ is general, we can assume that the r first elements $f'_1, \ldots, f'_r$ of $\mathbf{f}'$ are such that $J' = (f'_1, \ldots, f'_r)$ is $\mathfrak{m}$-primary and $\Delta(P') \neq 0$, where $P' \in (\mathbb{P}^{N-1})^r$ is the point corresponding to the set $\{f'_1, \ldots, f'_r\}$.

If $\operatorname{rank}(\mathcal{A}_r(P')) < n$ then $\mathcal{A}_r(P') \cdot \mathbf{a} = 0$ has a nonzero solution $\mathbf{a}$, so $\operatorname{Soc}(R/J')_1 \neq 0$. But, this is nonsense since $\operatorname{Soc}(R/J')_1 \subset \operatorname{Soc}(R/J)_1$ because $J' \subset J$ lives in degree 2. $\square$

The next result follows suit.

Theorem 3.3.6. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of r quadrics, where $\frac{(n+2)(n+1)}{6} \leq r \leq N-1$ and $N = \binom{n+1}{2}$. Then $\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_2$ and $\dim_{\Bbbk} \operatorname{Soc}(R/I)_2 = N - r$. In particular, R/I is Gorenstein if and only if $r = N-1$.*

Proof. We have

$$\begin{aligned} \dim_{\Bbbk}[R/I]_3 &= \max\{\dim_{\Bbbk} R_3 - rn,\, 0\} \quad \text{(by Lemma 3.1.4 (iii))} \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{(because } (n+2)(n+1)/6 \leq r \leq N-1). \end{aligned}$$

Thus, $R/I = \Bbbk \oplus (R/I)_1 \oplus (R/I)_2$ as a $\Bbbk$-vector space, where $\dim_{\Bbbk}(R/I)_1 = n$ and $\dim_{\Bbbk}(R/I)_2 = N - r$. Hence, by Proposition 3.3.5,

$$\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_2 \quad \text{and} \quad \dim_{\Bbbk} \operatorname{Soc}(R/I)_2 = N - r.$$

In particular, $\dim_{\Bbbk} \operatorname{Soc}(R/I) = 1$ if and only if $r = N-1$, that is, R/I is an Artinian Gorenstein algebra if and only if $r = N-1$. $\square$

Corollary 3.3.7. Conjecture 3.3.4 *holds true for $n = 3$.*

Proof. One implication has already been proved in Theorem 3.3.2. For the converse, apply the previous theorem, by which the socle is $\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_2$ and $\dim_{\Bbbk} \operatorname{Soc}(R/I)_2 = 1$. Since R/I is Cohen-Macaulay, it is Gorenstein. $\square$

Remark 3.3.8. An argument is available when $n = 3$ for the last assertion in Theorem 3.3.6. Namely, by the argument in the proof of [23, Proposition 2.3], if the socle of I contains a linear form then $\mu(I^2) < 15$. However, if I is generated by a general set of five forms, one has $\mu(I^2) = 15$, for which it suffices to check for the quadrics $\{x^2, y^2, z^2, xy + yz, xz - yz\}$. This gives a contradiction. This argument breaks down for $n \geq 4$, whereas the theorem fixes it for $n \leq 5$.

Another consequence of Theorem 3.3.6 is as follows.

Proposition 3.3.9. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be a Gorenstein ideal generated by a general set of $r \geq n \geq 4$ forms of degree $d \geq 2$. Then the socle degree of R/I is $\neq 3$.*

Proof. Set $N = \binom{n+1}{2}$. Suppose that the socle degree of R/I is 3. Then the h-vector of R/I is given by $h = (1, n, h_2, 1)$. On the other hand, as I is generated by a general set of $r \geq n$ forms, then R/I is Artinian, thus its h-vector is symmetric, hence $h_2 = n$. Since I is equigenerated and the socle degree of R/I is 3 then d is necessarily 2, thus it suffices to consider the case where $d = 2$. Note that $r = N - h_2 = N - n$, hence $\frac{(n+2)(n+1)}{6} \leq r$, otherwise we would have $n \leq 3$. By Theorem 3.3.6, the socle degree of R/I is 2, a contradiction. □

3.4 The socle in arbitrary degree

The essential content of [21, Theorem 1] can be restated as follows:

Proposition 3.4.1. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of r forms of degree d and set $N = \binom{n+d-1}{d}$ for the maximal number of such forms in R. If $r \leq N-1$ and $\dim_{\Bbbk} R_{d+1} \leq rn$ then $\mathrm{Soc}(R/I) = \mathrm{Soc}(R/I)_{d-1} \oplus \mathrm{Soc}(R/I)_d$ with $\dim_{\Bbbk} \mathrm{Soc}(R/I)_d = N - r$ and $\dim_{\Bbbk} \mathrm{Soc}(R/I)_{d-1} = \alpha$, for some $\alpha \geq 0$. In particular, if R/I is level then R/I is Gorenstein if and only if $r = N-1$.*

Proof. Denoting by $\mathbf{f}$ the general set of forms generating I, we have, by Lemma 3.1.4 (iii), that $\mathbf{f}$ is $(d, r, 1)$-extremal, thus follows that $\dim_{\Bbbk}[R/I]_{d+1} = 0$. On the other hand being $r < N$ then $\dim_{\Bbbk}[R/I]_d \neq 0$. Therefore $\mathrm{Soc}(R/I) = \mathrm{Soc}(R/I)_{d-1} \oplus \mathrm{Soc}(R/I)_d$ with $\dim_{\Bbbk} \mathrm{Soc}(R/I)_d = N - r$ and $\dim_{\Bbbk} \mathrm{Soc}(R/I)_{d-1} = \alpha$. For the supplementary statement, note that the socle has elements in degree d since by assumption $\mathfrak{m}^d \subset I : \mathfrak{m}$. Thus, if R/I is level it must be the case that $\mathrm{Soc}(R/I)_d$ and $\dim_{\Bbbk} \mathrm{Soc}(R/I)_d = N - r$, hence R/I is Gorenstein if and only if $r = N - 1$. □

A challenging problem is to determine the value of α. For $d = 2$, we had Theorem 3.3.6, independently of r. For $d \geq 3$ the algebra R/I is not always level for $r \leq N-1$. The following result gives a more precise statement.

Proposition 3.4.2. *Let $R = \Bbbk[x_1, \ldots, x_n]$ and $I \subset R$ be an ideal generated by a general set of r forms of degree $d \geq 3$ and let N be as in* Proposition 3.4.1. *If $\dim_{\Bbbk} R_d - \frac{1}{n} \dim_{\Bbbk} R_{d-1} < r \leq N - 1$, then R/I is not level.*

Proof. Let $\mathbf{f} = \{f_1, \ldots, f_r\}$ be the general set of forms of degree d generating I. Consider the matrix equality $[\mathbf{f}] = [\mathbf{x}_d] \cdot \mathbf{Y}_{n,d,r}(P)$ where $\mathbf{x}_d$ is the list of monomials of degree d in R and $\mathbf{Y}_{n,d,r}(P)$ is the matrix defined in (3.6) evaluated at the point $P_{\mathbf{f}}$. In a similar way as in Proposition 3.3.5 supposing $g = \sum_{|\sigma|=d-1} b_\sigma \mathbf{x}^\sigma$ a form of degree $d-1$ in $\mathrm{Soc}(R/I)$ we get a linear system $\mathcal{B}_r(P) \cdot \mathbf{b} = 0$ where $\mathbf{b}$ is the transpose of the matrix (b_σ) and $\mathcal{B}_r(P)$ is an $(N-r)n \times \dim_{\Bbbk} R_{d-1}$ matrix. That is, the homogeneous linear system that characterizes when forms of degree $d-1$ are in $\mathrm{Soc}(R/I)_{d-1}$ has $(\dim_{\Bbbk} R_d - r)n$ equations and $\dim_{\Bbbk} R_{d-1}$ variables. Since $\dim_{\Bbbk} R_d - \frac{1}{n} \dim_{\Bbbk} R_{d-1} < r$ then such system has solution and hence $\mathrm{Soc}(R/I)_{d-1} \neq 0$. Applying Proposition 3.4.1 yields $\mathrm{Soc}(R/I)_d \neq 0$. Therefore, our statement is proved. □

Note that the main assumption of Proposition 3.4.1 can be rephrased as $r \geq \frac{1}{n} \dim_{\Bbbk} R_{d+1}$. If I is an ideal generated by a general set of r forms of degree d in $\Bbbk[x_1, \ldots, x_n]$ such that the socle degree is $d + e$ with $e \geq 1$, then $\frac{(n+d+e)!(e+1)!}{(d+e+1)!(n+e)!} \leq r < \frac{(n+d)!}{(d+1)!n!}$. Thus, if I is an ideal generated by a general set of r quadrics and such that the socle degree s of R/I is 3 then $\frac{(n+3)(n+2)}{12} \leq r < \frac{(n+2)(n+1)}{6}$. The converse of this statement is not true in general, e.g., with $n = 8$ and $r = 10$, r is in the above range, but $s = 4$. Furthermore, note that if I is an ideal generated by r forms of degree d with $r < \frac{(n+d)!}{(d+1)!n!}$ then $s \geq d+1$. In particular, if $d = 2$ and $r < \frac{(n+2)(n+1)}{6}$ then $s \geq 3$. In fact, suppose $s < d+1$. Note that, since I is equigenerated in degree d then it suffices to consider $s = d$. Thus, we have that $nr \geq \dim_{\Bbbk} R_{d+1}$ hence $r \geq \frac{(n+d)!}{(d+1)!n!}$.

For illustration, some not so inspiring tables were obtained by computation using Macaulay 2 ([18]), as follows.

Example 3.4.3. Let $5 \leq n \leq 10$. Then:

n	**range of r with $s = 3$**	**Conclusion**
5	$r = 6$	For $n = 5$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 1 \leq r < (n+2)(n+1)/6$
6	$7 \leq r \leq 9$	For $n = 6$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 1 \leq r < (n+2)(n+1)/6$
7	$9 \leq r \leq 11$	For $n = 7$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 1 \leq r < (n+2)(n+1)/6$
8	$11 \leq r \leq 14$	For $n = 8$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 1 \leq r < (n+2)(n+1)/6$
9	$13 \leq r \leq 18$	For $n = 9$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 2 \leq r < (n+2)(n+1)/6$
10	$15 \leq r \leq 21$	For $n = 10$, r quadrics have socle degree 3 in the interval $(n+3)(n+2)/12 + 2 \leq r < (n+2)(n+1)/6$

The following conjecture-like questions are either computationally inspired or else stem from a careful look at the preceding material so far.

Question 3.4.4. *Let I be an ideal generated by a general set of r quadrics in $R = \Bbbk[x_1, \ldots, x_n]$. Is R/I level?*

By Theorem 3.3.6 the answer is affirmative for $r \geq n$ and $n \leq 5$.

Question 3.4.5. *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of r forms of degree d. Then R/I is either level, or else $\mathrm{Soc}(R/I)$ lives in two consecutive degrees $\geq d$.*

Question 3.4.6. *Let $J \subset R = \Bbbk[x, y, z]$ be an almost complete intersection of finite colength generated by r forms of degree d, and $I = (J, f)$, where f is a form of degree d such that $f \notin J$. Under what condition can we ensure the inequality*

$$\textit{socle degree of } R/I < \textit{socle degree of } R/J?$$

Of a more particular nature, we pose the following question inspired from the linear case with $n = 3$:

Question 3.4.7. *Let I be an ideal generated by a general set of r forms of degree d in the polynomial ring $R = \Bbbk[x_1, ..., x_n]$. If $\binom{r+1}{2} \geq \binom{2d+n-1}{n-1}$, then:*

(a) *Is $2d - 2$ an upper bound for the maximum degree of the generators of* $\mathrm{Soc}(R/I)$?

(b) *With $\mathfrak{m} = (x_1, \ldots, x_n)$, is $I^2 = \mathfrak{m}^{2d}$?*

Chapter 4

On the $(x_1^m, \ldots, x_n^m)$-colon problem

It is known (see [4, Proposition 1.3]) that any homogeneous Gorenstein ideal of codimension n in $\Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ can be obtained as a quotient ideal $(x_1^m, \ldots, x_n^m) : \mathfrak{f}$, for some integer $m \geq 1$ and some form $\mathfrak{f}$. In this chapter we deal with some of the main questions regarding this representation.

It is first established under which condition the form $\mathfrak{f}$ is uniquely determined and what is its degree in terms of m and the socle degree of I. Then we prove that $\mathfrak{f}$ can be retrieved from I by taking the so-called (socle-like) Newton dual of a minimal generator of the Macaulay inverse of I.

Then we give conditions under which the Gorenstein ideal I is equigenerated in terms of the exponent m and the form $\mathfrak{f}$. We solve this problem in the case where I has linear resolution

$$0 \to R(-2d-n+2) \to R(-d-n+2)^{b_{n-1}} \to \cdots \to R(-d-1)^{b_2} \to R(-d)^{b_1} \to R$$

where $b_1 = \mu(I)$.

These questions will be subsumed under the designation *the colon problem*, to avoid 'link' which has already many uses. For convenience, call $\mathfrak{f}$ a *directrix form* (of I) associated to the regular sequence $\{x_1^m, \ldots, x_n^m\}$.

4.1 Macaulay inverse system versus Newton duality

The Macaulay–Matlis duality meets yet another version in terms of the Newton polyhedron nature of the homogeneous forms involved so far.

Our next result asserts that directrix forms and Macaulay inverse generators obey a duality in terms of the Newton dual. Given a directrix form $\mathfrak{f}$ associated to the regular sequence $\{x_1^m, \ldots, x_n^m\}$ – i.e., $(x_1^m, \ldots, x_n^m) : \mathfrak{f} = I$ – it will typically admit monomial terms belonging to the ideal $(x_1^m, \ldots, x_n^m)$. In order to fix this inconvenient, we redefine the *socle-like*

Newton dual of such directrix form by taking as directrix vector $\nu := (m-1 \cdots m-1)^t$.

Proposition 4.1.1. *Let $I \subset R = \Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$ be a homogeneous codimension n Gorenstein ideal with socle degree s. Given an integer $m \geq 1$, suppose that I admits a directrix form $\mathfrak{f}$ associated to the regular sequence $\{x_1^m, \ldots, x_n^m\}$. Then:*

(i) $\deg \mathfrak{f} = n(m-1) - s$ *and $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(x_1^m, \ldots, x_n^m)$.*

(ii) *The socle-like Newton dual of $\mathfrak{f}$ is a minimal generator of the Macaulay inverse to I (having dual degree s), and its socle-like Newton dual retrieves $\mathfrak{f}$.*

Proof. Suppose $\mathfrak{f} = \sum_{|\alpha| = \deg \mathfrak{f}} a_\alpha \mathbf{x}^\alpha$. Then, the socle-like Newton dual of $\mathfrak{f}$ is $\hat{\mathfrak{f}} = \sum_\alpha a_\alpha \mathbf{y}^{\hat{\alpha}}$, where $\hat{\alpha} := \nu - \alpha$ (in particular, $\deg \hat{\mathfrak{f}} = n(m-1) - \deg \mathfrak{f}$). Given a homogeneous polynomial $h = \sum_{|\beta| = \deg h} b_\beta \mathbf{x}^\beta \in R$ one has:

$$h\mathfrak{f} = \sum_{|\gamma| = \deg \mathfrak{f} + \deg h} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) \mathbf{x}^\gamma \quad \text{and} \quad h\hat{\mathfrak{f}} = \sum_{|\gamma| = \deg \mathfrak{f} + \deg h} \left(\sum_{\hat{\alpha} - \beta = \hat{\gamma}} a_\alpha b_\beta \right) \mathbf{y}^{\hat{\gamma}}$$

with $\hat{\gamma} = \nu - \gamma$. In particular, for every γ, the coefficient of $\mathbf{x}^\gamma$ as a term in $h\mathfrak{f}$ is equal to the coefficient of $\mathbf{y}^{\hat{\gamma}}$ as a term of $h\hat{\mathfrak{f}}$. Moreover, the ith coordinate of γ is larger than m if and only if the ith coordinate of $\hat{\gamma}$ is negative. Thus,

$$\begin{aligned} h \in (x_1^m, \ldots, x_n^m) : \mathfrak{f} \quad &\Leftrightarrow \quad \text{for every} \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \neq 0, \gamma \ \text{ has a coordinate larger than}\, m \\ &\Leftrightarrow \quad \text{for every} \sum_{\hat{\alpha}-\beta=\hat{\gamma}} a_\alpha b_\beta \neq 0, \hat{\gamma} \ \text{ has a negative coordinate} \qquad (4.1) \\ &\Leftrightarrow \quad h\hat{\mathfrak{f}} = 0 \Leftrightarrow h \in \operatorname{Ann}(\hat{\mathfrak{f}}). \end{aligned}$$

Therefore, $I = \operatorname{Ann}(\hat{\mathfrak{f}})$, that is, $\hat{\mathfrak{f}}$ is a minimal generator of the Macaulay inverse to I. By construction, one has $\deg \hat{\mathfrak{f}} = n(m-1) - \deg \mathfrak{f}$. On the other hand, it is well known that the degree of a minimal generator of the Macaulay inverse to I is the socle degree of I, i.e., $\deg \hat{\mathfrak{f}} = s$. Therefore, $\deg \mathfrak{f} = n(m-1) - s$. Since $\hat{\mathfrak{f}}$ is uniquely determined, up to a scalar coefficient, the form $\mathfrak{f}$ is uniquely determined as well, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(x_1^m, \ldots, x_n^m)$. Thus, assertion (i) follows.

Assertion (ii) follows from the above discussion. □

Remark 4.1.2. Item (i) of Proposition 4.1.1 is stable under a change of coordinates. In other words, it holds true replacing the sequence $\{x_1^m, \ldots, x_n^m\}$ by a sequence $\{\ell_1^m, \ldots, \ell_n^m\}$, where $\{\ell_1, \ldots, \ell_n\}$ are independent linear forms. Thus, if I is a homogeneous codimension n Gorenstein ideal such that $(\ell_1^m, \ldots, \ell_n^m) : \mathfrak{f} = I$, for some form $\mathfrak{f} \in R$, then $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$, written as a polynomial in $\ell_1, \ldots, \ell_n$, belongs to the ideal $(\ell_1^m, \ldots, \ell_n^m)$.

4.2 The linear case and the pure power gap

In this section we characterize when I is a codimension n equigenerated Gorenstein ideal with linear resolution in terms of the exponent m and the form $\mathfrak{f} \in R = \Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$. The preliminaries remain valid in arbitrary characteristic, but characteristic zero is called upon in item (ii) of Proposition 4.2.4 below.

Let e, e' be positive integers and let $\mathfrak{f} = \sum_{|\alpha|=e} a_\alpha \mathbf{x}^\alpha \in R_e$ and $g = \sum_{|\beta|=e'} b_\beta \mathbf{x}^\beta \in R_{e'}$ be forms. Given an integer $m \geq 1$, write

$$g\mathfrak{f} = \sum_{\mathbf{x}^\gamma \notin (x_1^m,\ldots,x_n^m)} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathbf{x}^\gamma + \sum_{\mathbf{x}^\gamma \in (x_1^m,\ldots,x_n^m)} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathbf{x}^\gamma, \tag{4.2}$$

where $\gamma \in \mathbb{N}^n$ is a running n-tuple. To this writing associate a matrix $\mathcal{M}_{e,e',m}$ whose rows are indexed by the n-tuples γ such that $|\gamma| = e + e'$ and whose columns are indexed by the n-tuples β such that $|\beta| = e'$. The entries of the matrix are specified as follows:

$$\text{the } (\gamma, \beta)\text{-entry of } \mathcal{M}_{e,e',m} = \begin{cases} 0, & \text{if some coordinate of } \alpha = \gamma - \beta \text{ is } < 0 \\ a_\alpha, & \text{if each coordinate of } \alpha = \gamma - \beta \text{ is } \geq 0. \end{cases}$$

In addition, let χ denote the row matrix $[\mathbf{x}^\gamma]$ with the monomial entries $\mathbf{x}^\gamma \notin (x_1^m, \ldots, x_n^m)$, and let $\mathbf{b}$ stand for the column matrix whose entries are the coefficients b_β of g.
Then equality(4.2) can be rewritten as

$$g\mathfrak{f} = \chi \cdot \mathcal{M}_{e,e',m} \cdot \mathbf{b} + \sum_{\mathbf{x}^\gamma \in (x_1^m,\ldots,x_n^m)} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathbf{x}^\gamma. \tag{4.3}$$

It is important to observe that the matrix $\mathcal{M}_{e,e',m}$ depends only on the integers e, e' and m, and not on the details of g.

From this, it follows immediately:

Lemma 4.2.1. *Let $I = (x_1^m, \ldots, x_n^m) : \mathfrak{f}$. Then $g \in I$ if and only if $\mathcal{M}_{e,e',m} \cdot \mathbf{b} = 0$. In particular, $I_{e'} = \{0\}$ if and only if* $\operatorname{rank} \mathcal{M}_{e,e',m} = \binom{e'+n-1}{n-1}$.

To tie up the ends, consider the parameter map

$$R_e \to \mathbb{P}^{\binom{e+n-1}{n-1}-1}, \quad \mathfrak{f} = \sum_{|\alpha|=e} a_\alpha \mathbf{x}^\alpha \mapsto P_\mathfrak{f} = (a_{(e,\ldots,0)} : \cdots : a_{(0,\ldots,e)})$$

in the notation of Section 3.1. Let $\{Y_{e,\ldots,0}, \ldots, Y_{0,\ldots,e}\}$ denote the coordinates of $\mathbb{P}^{\binom{e+n-1}{n-1}-1}$ and let $\mathcal{MG}_{e,e',m}$ stand for the matrix whose entries are obtained by replacing each a_α in $\mathcal{M}_{e,e',m}$ by the corresponding Y_α.

Next, we will present the main result of this section.

Theorem 4.2.2. *Let $m \geq 1$ be an integer and let $\mathfrak{f} \in R = \Bbbk[x_1, \ldots, x_n]$ be a form. The following are equivalent:*

(i) $I = (x_1^m, \dots, x_n^m) : \mathfrak{f}$ *is a codimension* n *equigenerated Gorenstein ideal with linear resolution.*

(ii) *The integer* $s := n(m-1) - \deg \mathfrak{f}$ *is even and* $\operatorname{rank} \mathcal{M}_{\deg \mathfrak{f}, s/2, m} = \binom{s/2+n-1}{n-1}$.

(iii) *The integer* $s := n(m-1) - \deg \mathfrak{f}$ *is even and* $P_{\mathfrak{f}}$ *is a point in the Zariski open set* $\mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m}))$, *with* $k := \binom{s/2+n-1}{n-1}$ *and* $e = \deg \mathfrak{f}$.

Proof. (i)$\Rightarrow$(ii) Suppose that I is equigenerated in degree d. Since I has linear resolution then the socle degre of I is $2d-2$. Thus, by the Proposition 4.1.1, $s = 2d-2$. In particular, s is an even integer. On the other hand, since I is generated in degree d then $I_{s/2} = I_{d-1} = \{0\}$. Hence, by the Lemma 4.2.1, $\operatorname{rank} \mathcal{M}_{\deg \mathfrak{f}, s/2, m} = \binom{s/2+n-1}{n-1}$.

(ii)$\Rightarrow$(i) We claim that I is codimension n Gorenstein ideal generated in degree $t = s/2+1$. The ideal I is Gorenstein of codimension n because it is the link of the homogeneous almost complete intersection $J = (x_1^m, \dots, x_n^m, \mathfrak{f})$ with respect to the complete intersection of pure powers $(x_1^m, \dots, x_n^m)$. By Proposition 4.1.1, the socle degree of I is s. Thus, $(R/I)_{2t-1} = (R/I)_{s+1} = \{0\}$. On the other hand, since $\operatorname{rank} \mathcal{M}_{\deg \mathfrak{f}, s/2, m} = \binom{s/2+n-1}{n-1}$, then $I_{t-1} = I_{s/2} = \{0\}$ by Lemma 4.2.1. Since I is a codimension n Gorenstein ideal and $(R/I)_{2t-1} = \{0\}$ and $I_{t-1} = 0$ it follows from [30, Proposition 1.8] that I is generated in degree t and has linear resolution.

(ii) $\Leftrightarrow$ (iii) This is a mere language transcription. $\square$

Remark 4.2.3. The key point for proving the implication (i) $\Rightarrow$ (ii) is the use of [30, Proposition 1.8], which characterizes the $\mathfrak{m}$-primary Gorenstein ideals with linear resolution through estimates for the initial degree and the socle degree. For other classes of equigenerated Gorenstein ideals the examples show that a similar characterization must take into account not only the initial degree and the socle degree. For example, the ideals $(x^5, y^5, z^5) : (x+y+z)^5$ and $(x^5, y^5, z^5) : x^3y^2 + y^3z^2 + x^2z^3$ have the same initial degree and the same socle degree. However, the first ideal is equigenerated in degree 4 while the second is minimally generated in degree 4 and 5. Extending Theorem 4.2.2 to other ideals should include additional conditions.

For a full treatment of the case where $n = 3$ and $I = (x^m, y^m, z^m) : (x+y+z)^m$ see [42, Subsection 8.4.1.3], along with the conjecture that I is equigenerated (necessarily in degree $d-1$) if and only if m is odd.

The question remains as to when the Zariski open set $\mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m}))$ is nonempty, where $k := \binom{s/2+n-1}{n-1}$ and $e = \deg \mathfrak{f}$. The next result determines all pair of integers $m, e \geq 1$, with even $s = n(m-1) - e$, for this to be the case when $\mathfrak{f} = (x_1 + \cdots + x_n)^e$.

Proposition 4.2.4. ($\operatorname{char}(\Bbbk) = 0$) *Let* $m, e \geq 1$ *integers such that* $s = n(m-1) - e$ *is even. Set* $d := s/2 + 1$.

(i) *If* $m < d$ *then* $\mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m})) = \emptyset$.

(ii) *If* $m \geq d$ *then* $I = (x_1^m, \dots, x_n^m) : (x_1 + \cdots + x_n)^e$ *is a codimension* n *Gorenstein ideal generated by forms of degree* d *with linear resolution. In particular,* $\mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m}))$ *is a dense open set.*

Proof. (i) We claim that there is no form $\mathfrak{f}$ of degree e such that $I = (x_1^m, \dots, x_n^m) : \mathfrak{f}$ is a codimension n equigenerated Gorenstein ideal with linear resolution. In fact, otherwise I would be an ideal generated in degree d with $(x_1^m, \dots, x_n^m) \subset I$ – an absurd. Hence, by Theorem 4.2.2, $\mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m})) = \emptyset$.

(ii) We mimic the argument of [30, Proposition 7.24]. Namely, by applying [30, Proposition 1.8], it is sufficient to show that $R_{2d-1} \subset I$ and $I_{d-1} = \{0\}$. Clearly,

$$R_{n(m-1)+1} \subset (x_1^m, \dots, x_n^m).$$

Moreover,

$$(x_1 + \cdots + x_n)^e R_{2d-1} \subset R_{2d-1+e} = R_{n(m-1)+1}.$$

Hence, $R_{2d-1} \subset I$. On the other hand, the initial degree of $I/(x_1^m, \dots, x_n^m)$ is at least d as a consequence of the Lefschetz like result of R. Stanley, as proved in [43, Theorem 5]. Since $m \geq d$ by assumption then the initial degree of I is at least d. Therefore, $I_{d-1} = \{0\}$, as was to be shown. In particular, for $\mathfrak{f} = (x_1 + \cdots + x_n)^e$ Theorem 4.2.2 gives $P_{\mathfrak{f}} \in \mathbb{P}^{\binom{e+n-1}{n-1}-1} \setminus V(I_k(\mathcal{MG}_{e,s/2,m}))$. □

Remark 4.2.5. The only place where one needs characteristic zero above is in the use of [43, Theorem 5] – for a different proof of this typical charming result of characteristic zero, see [6, Théorème, Appendix]. It is reasonable to expect that the above proposition be valid in arbitrary characteristic.

Yet another numerical invariant seems to play some role in regard to the colon representation.

Let $\boldsymbol{\ell} = \{\ell_1, \dots, \ell_n\} \in R = \Bbbk[x_1, \dots, x_n]$ be a regular sequence of linear forms and let $I \subset R$ be a homogeneous codimension n Gorenstein ideal with socle degree s. Denote by $m(I, \boldsymbol{\ell})$ the least index m such that $\{\ell_1^m, \dots, \ell_n^m\} \subset I$. Since $R_{s+1} = I_{s+1}$, then $m(I, \boldsymbol{\ell}) \leq s+1$. The *pure power gap* of I with respect to the regular sequence $\boldsymbol{\ell}$ is $\mathfrak{g}(I, \boldsymbol{\ell}) := s+1-m(I, \boldsymbol{\ell})$. The *absolute pure power gap* of I (or simply, the pure power gap of I) is $\mathfrak{g}(I) := s+1-\min_{\boldsymbol{\ell}}\{m(I, \boldsymbol{\ell})\}$.

To start we have the following basic ring-theoretic result:

Lemma 4.2.6. *Let $m_1, \dots, m_n \geq 1$ be integers and $\mathfrak{f}$ a form in $R = \Bbbk[x_1, \dots, x_n]$. Then*

$$(\ell_1^{m_1}, \dots, \ell_n^{m_n}) : \mathfrak{f} = (\ell_1^{m_1}, \dots, \ell_i^{m_i+1}, \dots, \ell_n^{m_n}) : \ell_i \mathfrak{f}$$

for every $1 \leq i \leq n$. In particular,

$$(\ell_1^{m_1}, \dots, \ell_n^{m_n}) : \mathfrak{f} = (\ell_1^{m_1+k}, \dots, \ell_i^{m_i+k}, \dots, \ell_n^{m_n+k}) : (\ell_1 \cdots \ell_n)^k \mathfrak{f}$$

for each $k \geq 0$.

Proof. One can assume that $i = 1$. The inclusion $(\ell_1^{m_1}, \dots, \ell_n^{m_n}) : \mathfrak{f} \subset (\ell_1^{m_1+1}, \dots, \ell_n^{m_n}) : \ell_1 \mathfrak{f}$ is immediate. Thus, consider $h \in (\ell_1^{m_1+1}, \dots, \ell_n^{m_n}) : \ell_1 \mathfrak{f}$. Then,

$$\ell_1 \mathfrak{f} h = p_1 \ell_1^{m_1+1} + \cdots + p_n \ell_n^{m_n}$$

for certain $p_1,\dots,p_n \in R$. In particular, ℓ_1 divide $p_2\ell_2^{m_2}+\cdots+p_n\ell_n^{m_n}$. We can write

$$p_i = \ell_1 q_i + r_i, \quad \text{for each } 2 \le i \le n,$$

where $r_2,\dots,r_n$ are polynomials in $\Bbbk[\ell_2,\dots,\ell_n]$. Thus,

$$p_2\ell_2^{m_2}+\cdots+p_n\ell_n^{m_n} = q_2\ell_1\ell_2^{m_2}+\cdots+q_n\ell_1\ell_n^{m_n}+r_2\ell_2^{m_2}+\cdots+r_n\ell_n^{m_n}.$$

Since ℓ_1 divides $p_2\ell_2^{m_2}+\cdots+p_n\ell_n^{m_n}$ and $r_2\ell_2^{m_2}+\cdots+r_n\ell_n^{m_n} \in \Bbbk[\ell_2,\dots,\ell_n]$ then

$$p_2\ell_2^{m_2}+\cdots+p_n\ell_n^{m_n} = q_2\ell_1\ell_2^{m_2}+\cdots+q_n\ell_1\ell_n^{m_n}.$$

Thus,

$$\mathfrak{f}h = p_1\ell_1^{m_1}+q_2\ell_2^{m_2}+\cdots+q_n\ell_n^{m_n},$$

that is, $h \in (\ell_1^{m_1},\dots,\ell_n^{m_n}) : \mathfrak{f}$. Therefore, $(\ell_1^{m_1},\dots,\ell_n^{m_n}) : \mathfrak{f} = (\ell_1^{m_1+1},\dots,\ell_n^{m_n}) : \ell_1\mathfrak{f}$ as stated.
□

To see an application, recall from Remark 4.1.2 that if I is a homogeneous codimension n Gorenstein ideal such that $(\ell_1^m,\dots,\ell_n^m) : \mathfrak{f} = I$, where $\ell_1,\dots,\ell_n$ are linear forms, then $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$, written as a polynomial in $\ell_1,\dots,\ell_n$, belongs to the ideal $(\ell_1^m,\dots,\ell_n^m)$.

Proposition 4.2.7. *Let $I \subset R$ be a homogeneous codimension n Gorenstein ideal with socle degree s. Suppose that as above, $\ell_1,\dots,\ell_n$ are linear forms such that $I = (\ell_1^{s+1},\dots,\ell_n^{s+1}) : \mathfrak{f}$ with $\mathfrak{f}$ uniquely determined, up to a scalar coefficient, by the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(\ell_1^{s+1},\dots,\ell_n^{s+1})$. Then, $\mathfrak{g}(I,\boldsymbol{\ell})$ is the largest index such that $(\ell_1\cdots\ell_n)^{\mathfrak{g}(I,\boldsymbol{\ell})}$ divides $\mathfrak{f}$.*

Proof. Denote $m_0 := m(I,\boldsymbol{\ell})$ and $\mathfrak{g} := \mathfrak{g}(I,\boldsymbol{\ell})$. Then $(\ell_1^{m_0},\dots,\ell_n^{m_0}) : I$ is an almost complete intersection $J = (\ell_1^{m_0},\dots,\ell_n^{m_0},\mathfrak{f}_0)$, for some form $\mathfrak{f}_0 \in R$. Since $R = \Bbbk[\ell_1,\dots,\ell_n]$, we can write $\mathfrak{f}_0$ as a polynomial in these linear forms and get rid of the terms belonging to the ideal $(\ell_1^{m_0},\dots,\ell_n^{m_0})$. This way, the latter is part of a minimal set of generators of J. Therefore, $(\ell_1^{m_0},\dots,\ell_n^{m_0}) : J = (\ell_1^{m_0},\dots,\ell_n^{m_0}) : \mathfrak{f}_0$ is Gorenstein and $I = (\ell_1^{m_0},\dots,\ell_n^{m_0}) : \mathfrak{f}_0$.

By Lemma 4.2.6 one has

$$\begin{aligned} I = (\ell_1^{m_0},\dots,\ell_n^{m_0}) : \mathfrak{f}_0 &= (\ell_1^{m_0+\mathfrak{g}},\dots,\ell_n^{m_0+\mathfrak{g}}) : (\ell_1\cdots\ell_n)^{\mathfrak{g}}\mathfrak{f}_0 \\ &= (\ell_1^{s+1},\dots,\ell_n^{s+1}) : (\ell_1\cdots\ell_n)^{\mathfrak{g}}\mathfrak{f}_0. \end{aligned} \tag{4.4}$$

Consider

$$\mathfrak{f}_0 = \sum_{|\alpha|=\deg \mathfrak{f}_0} a_\alpha \ell_1^{\alpha_1}\cdots\ell_n^{\alpha_n}.$$

Then,

$$(\ell_1\cdots\ell_n)^{\mathfrak{g}}\mathfrak{f}_0 = \sum_{|\alpha|=\deg \mathfrak{f}_0} a_\alpha \ell_1^{\alpha_1+\mathfrak{g}}\cdots\ell_n^{\alpha_n+\mathfrak{g}}.$$

For each nonzero a_α, one has $\alpha_i \le m_0 - 1$ for each $1 \le i \le n$. Hence, $\alpha_i + \mathfrak{g} \le$

$m_0 + \mathfrak{g} - 1 = s$ for each $1 \leq i \leq n$. Thus, no nonzero term of $(\ell_1 \cdots \ell_n)^{\mathfrak{g}}\mathfrak{f}_0$ belongs to the ideal $(\ell_1^{s+1}, \ldots, \ell_n^{s+1})$. Then, since $\mathfrak{f}$ is uniquely determined, up to a scalar coefficient, by $I = (\ell_1^{s+1}, \ldots, \ell_n^{s+1}) : \mathfrak{f}$ and the condition that no nonzero term of $\mathfrak{f}$ belongs to the ideal $(\ell_1^{s+1}, \ldots, \ell_n^{s+1})$, one has $\mathfrak{f} = \lambda(\ell_1 \cdots \ell_n)^{\mathfrak{g}}\mathfrak{f}_0$ for some nonzero $\lambda \in \Bbbk$. Hence, $(\ell_1 \cdots \ell_n)^{\mathfrak{g}}$ divides $\mathfrak{f}$.

Finally, we assert that $\mathfrak{g}$ is the largest index with this property, a claim that is obvious if $m_0 = 1$, because in this case $\mathfrak{f}_0$ is a nonzero scalar. Thus, suppose $m_0 \geq 2$. If $\mathfrak{g}$ is not the largest index such that $(\ell_1 \cdots \ell_n)^{\mathfrak{g}}$ divides $\mathfrak{f}$ then $\ell_1 \cdots \ell_n$ divides $\mathfrak{f}_0$. Hence, by Lemma 4.2.6,

$$I = (\ell_1^{m_0}, \ldots, \ell_n^{m_0}) : \mathfrak{f}_0 = (\ell_1^{m_0-1}, \ldots, \ell_n^{m_0-1}) : \frac{\mathfrak{f}_0}{\ell_1 \cdots \ell_n},$$

so, $\{\ell_1^{m_0-1}, \ldots, \ell_n^{m_0-1}\} \subset I$, contradicting that m_0 is least such that $\{\ell_1^{m_0}, \ldots, \ell_n^{m_0}\} \subset I$. $\square$

4.3 Example: telescopic alternating matrices

We exhibit two models of alternating matrices with linear entries in $\Bbbk[x, y, z]$ such that, up to $d \leq 11$, the maximal Pfaffians generate an ideal of codimension 3. These are defined for any odd valued sizes. All computations were done with Macaulay 2 [18].

• Matrix $M_r = (m_{i,j})$.

Here, with $m_{i,j}$ above the diagonal:

$$m_{i,j} = \begin{cases} x & \text{if } j = i+1 \text{ and } i \text{ is odd} \\ y & \text{if } j = i+1 \text{ and } i \text{ is even} \\ z & \text{if } j = i+s \ (s = 2, 3, 5, 7, \ldots, r-2) \text{ and } i \text{ is odd} \\ z & \text{if } j = i+s \ (s = 2, 3, 5, 7, \ldots, r-2) \text{ and } i \text{ is even} \\ 0 & \text{elsewhere.} \end{cases}$$

We illustrate for the first two relevant values of r:

$$M_5 = \begin{bmatrix} 0 & x & z & z & 0 \\ -x & 0 & y & z & z \\ -z & -y & 0 & x & z \\ -z & -z & -x & 0 & y \\ 0 & -z & -z & -y & 0 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 0 & x & z & z & 0 & z & 0 \\ -x & 0 & y & z & z & 0 & z \\ -z & -y & 0 & x & z & z & 0 \\ -z & -z & -x & 0 & y & z & z \\ 0 & -z & -z & -y & 0 & x & z \\ -z & 0 & -z & -z & -x & 0 & y \\ 0 & -z & 0 & -z & -z & -y & 0 \end{bmatrix}.$$

For M_5, one finds that the ideal generated by the maximal Pfaffians is given by

$$\begin{aligned} I &= (x^2 + yz - z^2, xz - z^2, xy - z^2, yz - z^2, y^2 + xz - z^2) \\ &= (x^2, y^2, xy - z^2, xz - z^2, yz - z^2), \end{aligned}$$

a well-known example from [23].

- Matrix $N_r = (n_{i,j})$.

$$n_{i,j} = \begin{cases} x & \text{if } j = i+1 \text{ and } i \text{ is odd} \\ y & \text{if } j = i+1 \text{ and } i \text{ is even} \\ z & \text{if } j = i+2 \text{ and } i \text{ is odd} \\ z & \text{if } j = i+2 \text{ and } i \text{ is even} \\ 0 & \text{elsewhere.} \end{cases}$$

The first two with $r \geq 5$ are:

$$N_5 = \begin{bmatrix} 0 & x & z & 0 & 0 \\ -x & 0 & y & z & 0 \\ -z & -y & 0 & x & z \\ 0 & -z & -x & 0 & y \\ 0 & 0 & -z & -y & 0 \end{bmatrix}, \quad N_7 = \begin{bmatrix} 0 & x & z & 0 & 0 & 0 & 0 \\ -x & 0 & y & z & 0 & 0 & 0 \\ -z & -y & 0 & x & z & 0 & 0 \\ 0 & -z & -x & 0 & y & z & 0 \\ 0 & 0 & -z & -y & 0 & x & z \\ 0 & 0 & 0 & -z & -x & 0 & y \\ 0 & 0 & 0 & 0 & -z & -y & 0 \end{bmatrix},$$

with corresponding ideals of maximal Pfaffians:

$$(x^2 - z^2, x\,z, x\,y, y\,z, y^2 - z^2),$$

also discussed in [23], and

$$(x^3 - 2\,x\,z^2, x^2 z - z^3, x^2 y - y\,z^2, x\,y\,z, x\,y^2 - x\,z^2, y^2 z - z^3, y^3 - 2\,y\,z^2).$$

Note that both versions are 'telescopic' in the sense that, for any given odd r, the principal submatrix of M_r (respectively, N_r) order $r-2$ is M_{r-2} (respectively, N_{r-2}).

Another matrix which also satisfies this configuration is the matrix H_d presented in Example 2.1.6, where was mentioned that $I := Pf_{2d}(H_d)$ is a Gorenstein ideal of codimension 3 in $\Bbbk[x,y,z]$. In addition, it was proved in [30, Proposition 7.10] that the Macaulay inverse system for I is generated by

$$\phi_d = \sum_{i=0}^{d-1} (-1)^i c_i y_1^{(d-1-i)} y_2^{(d-1-i)} y_3^{(2i)}$$

where c_i is the ith Catalan number $c_i = \frac{1}{i+1}\binom{2i}{i}$. From this, note that by Proposition 4.1.1 the ideal I can be written as $(x^{2d-1}, y^{2d-1}, z^{2d-1}) : \sum_{i=0}^{d-1} c_i (-1)^{d-1+i} x^{d-1+i} y^{d-1+i} z^{2(d-1)-2i}$. The following related conjecture has been verified up to $r \leq 23$:

Conjecture 4.3.1. (char($\Bbbk$)=0) *Set $r := 2d+1$ and let $I(d) \subset \Bbbk[x,y,z]$ denote the ideal generated by the $(2d)$-Pfaffians of N_{2d+1}. Then $I(d)$ is Gorenstein and $I(d) = (x^{2d-1}, y^{2d-1}, z^{2d-1}) : \mathfrak{f}(d)$, with directrix*

$$\mathfrak{f}(d) = x^2 y^2 \mathfrak{f}(d-1) + \sum_{i=0}^{d-1} c_i c_{d-1-i} x^{2(d-1-i)} y^{2i} z^{2d-2},$$

where $\mathfrak{f}(d-1)$ *is the directrix of the telescoped down ideal* $I(d-1)$. *Equivalently, letting* $\hat{\mathfrak{f}}(d)$ *denote a generator of the Macaulay inverse system for* $I(d)$, *then*

$$\hat{\mathfrak{f}}(2) = x^{*2} + y^{*2} + z^{*2} \quad \textit{and} \quad \hat{\mathfrak{f}}(d) = z^{*2}\hat{\mathfrak{f}}(d-1) + \sum_{i=0}^{d-1} c_i c_{d-1-i} x^{*(2i)} y^{*2(d-1-i)} \quad (d \geq 3).$$

Remark 4.3.2. Note that at least the degree of $\mathfrak{f}$ is right $(= 4(d-1))$ according to Proposition 4.1.1.

Chapter 5

Properties at large

In this chapter we look at further aspects of codimension 3 equigenerated Gorenstein ideals, as related to reduction number, rational maps and the associated Rees algebra and fiber cone.

5.1 Basic general properties

We first establish a general landscape regarding homogeneous equigenerated ideals of finite colength in an arbitrary number of variables.

Proposition 5.1.1. *Let $R = \Bbbk[x_1, \ldots, x_n]$ be a standard graded polynomial ring over an infinite field and let $\mathfrak{m}$ be its maximal homogeneous ideal. Let I be a homogeneous d-equigenerated $\mathfrak{m}$-primary ideal.*

(a) *If $I^{m_0} = \mathfrak{m}^{dm_0}$ for some $m_0 \geq 1$, then $I^m = \mathfrak{m}^{md}$ for $m \geq m_0$.*

(b) *Let $m = m_0$ be minimal possible in* (a)*. Then the reduction number of I is at most $\max\{m_0, r(\mathfrak{m}^d)\}$, where $r(\mathfrak{m}^d)$ denotes the reduction number of $\mathfrak{m}^d$.*

(c) *The* (*regular*) *rational map $\mathfrak{F} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{\mu(I)-1}$ defined by a set of forms spanning $[I]_d$ is birational onto the image.*

(d) *The Rees algebra $\mathcal{R}(I)$ satisfies the condition R_1 of Serre.*

(e) depth $\mathrm{gr}_I(R) = 0$.

Proof. (a) Now, $I^{m_0} \subset \mathfrak{m}^d I^{m_0-1} \subset \mathfrak{m}^d \mathfrak{m}^{(m_0-1)d} = I^{m_0}$, hence, $I^{m_0} = \mathfrak{m}^d I^{m_0-1}$. Then, for $m \geq m_0$,

$$I^{m+1} = I^{m+1-m_0} I^{m_0} = I^{m+1-m_0} \mathfrak{m}^d I^{m_0-1} = I^m \mathfrak{m}^d = \mathfrak{m}^{(m+1)d},$$

and so on.

(b) Let $J \subset I$ be a homogeneous minimal reduction. Since $\mathfrak{m}^d$ is the integral closure of I, then J is also a minimal reduction of $\mathfrak{m}^d$. But, as is well-known, the latter has reduction number at most $n-1$ for any minimal reduction (see, e.g., [11, Corollary 7.12]). Setting $N = \max\{m_0, r(\mathfrak{m}^d)\}$, one has:

$$\begin{aligned} I^{N+1} &= (\mathfrak{m}^d)^{N+1} && \text{by (a)} \\ &= J(\mathfrak{m}^d)^N && \text{because } J \text{ is a minimal reduction of } \mathfrak{m}^d \\ &= JI^N && \text{by (a).} \end{aligned}$$

(c) By (a), the Hilbert polynomial $HP(\mathcal{F}(I), m)$ of the fiber cone $\mathcal{F}(I)$ is

$$HP(\mathcal{F}(I), m) = \binom{md+n-1}{n-1} = \frac{d^{n-1}}{(n-1)!}m^{n-1} + \text{lower degree terms of } m.$$

Hence, the multiplicity $e(\mathcal{F}(I))$ of $\mathcal{F}(I)$ is d^{n-1}. On the other hand, by [45, Theorem 6.6 (a)] the degree $\deg(\mathfrak{F})$ of the rational map $\mathfrak{F}$ is

$$\deg(\mathfrak{F}) \quad = \quad \frac{d^{n-1}}{e(\mathcal{F}(I))}.$$

Thus, $\deg(\mathfrak{F}) = 1$, as asserted.

(d) Consider the Hilbert-Samuel polynomial ($m \gg 0$)

$$\lambda(R/I^{m+1}) = e_0(I)\binom{n+m}{n} - e_1(I)\binom{n+m-1}{n-1} + \text{lower degree terms of } m$$

and the Hilbert polynomial

$$\lambda(R/\overline{I^{m+1}}) = \overline{e}_0(I)\binom{n+m}{n} - \overline{e}_1(I)\binom{n+m-1}{n-1} + \text{lower degree terms of } m$$

where $\overline{I^{m+1}}$ denotes the integral closure of I^{m+1}. By (a), $I^m = \overline{I^m}$ for every $m \geq m_0$. Thus, in particular, $e_1(I) = \overline{e}_1(I)$. Hence, by [22, Proposition 3.2], $\mathcal{R}(I)$ satisfies the condition R_1 of Serre.

(e) By (a), one has an exact sequence

$$0 \to \mathcal{R}(I) \to \mathcal{R}(\mathfrak{m}^d) \to C \to 0,$$

with C a module of finite length. In particular, depth $C = 0$. Since $\mathcal{R}(\mathfrak{m}^d)$ is Cohen-Macaulay, then depth $\mathcal{R}(I) = $ depth $C + 1 = 1$.

Now, clearly depth $\mathrm{gr}_I(R) \leq$ depth $\mathcal{R}(I) = 1$. Supposing that depth $\mathrm{gr}_I(R) > 0$, let $a \in I \setminus I^2$ be such that its image in $I/I^2 \subset \mathrm{gr}_I(R)$ is a regular element. Then one has an exact sequence

$$0 \to \mathrm{gr}_I(R)(-1) \to \mathcal{R}(I)/a\mathcal{R}(I) \to \mathcal{R}_{R/(a)}(I/(a)) \to 0$$

(see [47, Proposition 5.1.11]). Since a is regular on $\mathcal{R}(I)$ then the middle term has depth zero, while the rightmost term – being a Rees algebra over a Cohen-Macaulay ring of dimension

≥ 1 – has depth ≥ 1. It follows that $\mathrm{gr}_I(R) \simeq \mathrm{gr}_I(R)(-1)$ has depth zero; a contradiction. $\square$

5.2 The role of the syzygetic condition

For the main result in this part recall that a *syzygetic* ideal I in a Noetherian ring R is one such that the natural surjection $\mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_R(I)$ is an isomorphism in degree ≤ 2, where $\mathcal{S}_R(I)$ is the symmetric algebra of I. In particular, for such an ideal, the minimal number of generators of I^2 is given by $\binom{\mu(I)+1}{2}$, which is the minimal number of generators of the second symmetric power of I.

Some of the numerical invariants are more easily accessed in the linear case with $n = 3$.

Theorem 5.2.1. *Let $I \subset R = \Bbbk[x, y, z]$ denote a codimension 3 homogeneous d-equigenerated Gorenstein ideal minimally generated by $2d + 1$ forms. Assuming $d \geq 2$, the following hold.*

(a) $I^m = \mathfrak{m}^{md}$ *for every* $m \geq 2$.

(b) *The reduction number of I is* 2.

(c) *The (regular) rational map $\mathfrak{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{2d}$ defined by a set of forms spanning $[I]_d$ is birational onto the image.*

(d) *The Rees algebra $\mathcal{R}(I)$ satisfies the condition R_1 of Serre.*

(e) depth $\mathrm{gr}_I(R) = 0$.

(f) *The fiber cone $\mathcal{F}(I)$ is not Cohen-Macaulay.*

Recall that I is syzygetic ([20, Proposition 2.10] for the case where $\mathrm{char}(\Bbbk) \neq 2$ and [40, Theorem 1] for any $\Bbbk$).

Proof. (a) Since I is syzygetic then

$$\mu(I^2) = \binom{\mu(I) + 1}{2} = \binom{2d + 2}{2} = \mu(\mathfrak{m}^{2d}).$$

Thus, $I^2 \subset \mathfrak{m}^{2d}$ is a inclusion of homogeneous ideals in the same degree $2d$, having the same minimal number of homogeneous generators. Hence, $I^2 = \mathfrak{m}^{2d}$ and, by Proposition 5.1.1, $I^m = \mathfrak{m}^{md}$ for every $m \geq 2$.

Items (c) through (e) follow directly from the analogous statements in Proposition 5.1.1. It remains to deal with (b) and (f).

(b) By (a) and Proposition 5.1.1, one has $r(I) \leq 2$. On the other hand, since I is syzygetic one has $2 \leq r(I)$. Hence, $r(I) = 2$.

(f) Suppose that the fiber cone is Cohen-Macaulay. Then the reduction number $r(I)$ is the Castelnuovo-Mumford regularity $\mathrm{reg}(\mathcal{F}(I))$ of $\mathcal{F}(I)$ (see, e.g., [16, Proposition 1.2]). By (b), the latter is 2. But since I is syzygetic, the defining ideal of $\mathcal{F}(I)$ over $S := \Bbbk[T_1, \ldots, T_{2d+1}]$ admits no forms of degree 2, hence is generated in the single degree 3. Together these imply that the minimal graded free resolution of $\mathcal{F}(I)$ over S is linear. Moreover, the length of the resolution is $2d+1-\ell(I)=2d-2$. Again, since $\mathcal{F}(I)$ is Cohen-Macaulay, by [24, Theorem 1.2] the multiplicity of the fiber cone $\mathcal{F}(I)$ is

$$e(\mathcal{F}(I))=\binom{3+2d-2-1}{2d-2}=\binom{2d}{2}=d(2d-1).$$

Now consider the rational map $\mathfrak{F}:\mathbb{P}^2 \dashrightarrow \mathbb{P}^{2d}$ defined by the given generators of I in degree d, and let $\deg(\mathfrak{F})$ denote the degree of $\mathfrak{F}$. Since I is equigenerated then $\mathcal{F}(I)$ is isomorphic to the $\Bbbk$-subalgebra $\Bbbk[[I]_d]\subset R$, while the latter is up to degree normalization the homogeneous defining ideal of the image of $\mathfrak{F}$. Then, since $\mathfrak{F}$ is birational, by [45, Theorem 6.6 (a)] one has $e(\mathcal{F}(I))=d^2$, i.e., $2d-1=d$, which is an absurd for $d\geq 2$. □

Away from the linear case, one can still obtain some information.

Proposition 5.2.2. *Let $I\subset R=\Bbbk[x,y,z]$ be a codimension 3 Gorenstein ideal with datum (d,r) and skew-degree d'. Let $\mathfrak{F}:\mathbb{P}^2\dashrightarrow\mathbb{P}^{r-1}$ be the rational map defined by a set of forms spanning $[I]_d$. If the reduction number of I is at most 2 and $\mathcal{F}(I)$ is Cohen-Macaulay then:*

(a) *$(r-2)$ divides d'^2.*

(b) *If $r\geq 5$ then $\mathfrak{F}$ is not birational onto the image.*

Proof. (a) As in the proof of Theorem 5.2.1 (b), which did not depend on the linear assumption, the reduction number of I is 2. Since $\mathcal{F}(I)$ is Cohen-Macaulay then the same argument as in the proof of Theorem 5.2.1 (f) yields $e(\mathcal{F}(I))=\binom{r-1}{2}$. Again, by [45, Theorem 6.6 (a)], $\binom{r-1}{2}\deg(\mathfrak{F})=d^2$. By definition, $d=(r-1)d'/2$, hence

$$2(r-2)\deg(\mathfrak{F})=(r-1)d'^2. \tag{5.1}$$

Since $\gcd\{(r-2),(r-1)\}=1$ then $(r-2)$ divides d'^2, as desired.

(b) Since $(r-1)/2>1$ then (5.1) forces $\deg(\mathfrak{F})>1$. Hence, $\mathfrak{F}$ is not birational. □

5.3 Almost complete intersection ideals of codimension three

Almost complete intersections speak for themselves, and yet there seems to be yet quite a bit to say about them even in codimension three. There is some knowledge coming from [23, Proposition 2.7] and [42, Proposition 8.4.24] in the case where the generators are quadrics. Here we focus on the case where the quadrics are moreover general, and also extend consideration to the case of general cubic generators.

First, a straightforward consequence of previous chapters.

Proposition 5.3.1. *Let $I\subset R=\Bbbk[x,y,z]$ be a codimension 3 ideal generated by a general set of 4 forms of degree 2. Then*

(i) *The type of R/I is 2.*

(ii) *The socle degree of R/I is 2.*

(iii) *The minimal free resolution of R/I is given by*

$$0 \to R^2(-5) \to R^2(-3) \oplus R^3(-4) \to R^4(-2) \to R \to R/I.$$

Proof. (i) and (ii) follow immediately from Theorem 3.3.6.

As for (iii), one knows the general shape of minimal graded free resolution of an almost complete intersection R/I, where I is 2-equigenerated of codimension 3 ([50, Theorem 4.1]):

$$0 \to \bigoplus_{i=1}^{t} R(-(8-s_i)) \to R^3(-4) \oplus \bigoplus_{i=1}^{t} R(-s_i) \to R^4(-2) \to R \to R/I.$$

Then (iii) follows from this and (i), (ii). $\square$

Remark 5.3.2. A similar result was obtained in [35].

As a matter of discussion, properties of $r \leq n$ general forms of degree d can be dealt with by trading out by the dth powers of r among the variables $x_1, \ldots, x_n$. Not so much for $r \geq n+1$. However, for $r = n+1$ one can walk a good mile by trading in the forms $x_1^d, \ldots, x_n^d, f$, where f is a sufficiently general form no term of which is divisible by any x_i^d. The meaning of 'sufficiently general' will depend on the property in question.

Proposition 5.3.3. (char($\Bbbk$) $\neq 2$) *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of $r = n+1$ forms of degree $d \geq 2$. Then $\mu(I^2) = \binom{n+2}{2}$.*

Proof. Consider the set $\mathbf{h} = \{h_1, \ldots, h_{n+1}\}$ of $n+1$ forms of degree d given by:

$$h_1 := x_1^d,\ h_2 := x_2^d, \ldots,\ h_n := x_n^d,\ h_{n+1} := \begin{cases} x_1x_2 + x_2x_3, & \text{if } d = 2 \\ x_1^{d-1}x_2, & \text{if } d \geq 3. \end{cases} \tag{5.2}$$

The defining equation of the fiber cone $\Bbbk[h_1t, \ldots, h_{n+1}t] \subset \Bbbk[x_1, \ldots, x_n, t]$ is

$$F(t_1, \ldots, t_{n+1}) = \begin{cases} (t_{n+1}^2 - t_1t_2 - t_2t_3)^2 - 4t_1t_2^2t_3, & \text{if } d = 2 \\ t_{n+1}^d - t_1^{d-1}t_2, & \text{if } d \geq 3. \end{cases}$$

as is straightforward to verify.

In any case, $\deg F(t_1, \ldots, t_{n+1}) > 2$ which shows that the set $\{h_ih_j \mid 1 \leq i \leq j \leq r\}$ has no $\Bbbk$-linear dependencies.

Next, we draw on a construct similar to the one in Definition 3.1.1. However, since the landscape is slightly different, we prefer to argue ab initio. Borrowing from the notation in Section 3.1, letting $\mathbf{g} = \{g_1, \ldots, g_{n+1}\}$ be forms of degree d with indeterminate coefficients, we consider the $\binom{2d+n-1}{2d} \times \binom{n+2}{2}$ matrix $M_{\mathbf{g}}$ such that, for given $1 \leq i < j \leq n+1$, the entries of the (i,j)th column are the coefficients of the product g_ig_j – note that every such entry is a quadric form in the original indeterminate coefficients of g_i and g_j. In other words, $M_{\mathbf{g}}$ is the so-called content matrix of $[g_ig_j \mid 1 \leq i < j \leq n+1]$ with respect to the set of orderly monomials of R of degree $2d$.

Now, set $\mathfrak{n} = \binom{n+2}{2}$. Since the set of products $\{h_ih_j \,|\, 1 \leq i \leq j \leq r\}$ is $\Bbbk$-linearly independent then the rank of the evaluated matrix $M_{\mathbf{g}}(P_{\mathbf{h}})$ is the maximum $\mathfrak{n}$. Equivalently, this means that the Zariski open set $(\mathbb{P}^N)^{n+1} \setminus V(I_{\mathfrak{n}}(M_{\mathbf{g}}))$ is nonempty. Therefore, for a general set of forms $\mathbf{f} = \{f_1, \ldots, f_{n+1}\} \subset R_d$ the set of products $\{f_if_j \,|\, 1 \leq i \leq j \leq r\}$ is $\Bbbk$-linearly independent. In other words, $\mu(I^2) = \dim_{\Bbbk}[I^2]_{2d} = \mathfrak{n}$. $\square$

Proposition 5.3.4. (char($\Bbbk$) $\neq 2, 3$) *Let $I \subset R = \Bbbk[x_1, \ldots, x_n]$ be an ideal generated by a general set of $n+1$ forms of degree d with $2 \leq d \leq 3$. Then $\mu(I^t) = \binom{n+t}{t}$ for $t \leq d^2 - 1$.*

Proof. In fact, for $d = 2$ follows by Proposition 5.3.3. For $d = 3$ in a similar way as in Proposition 5.3.3 consider the forms $h_1 := x_1^3,\ h_2 := x_2^3, \ldots,\ h_n := x_n^3,\ h_{n+1} := x_1^2x_2 + x_2^2x_3$. In this case, we have that the defining equation of the fiber cone $\Bbbk[h_1t, \ldots, h_{n+1}t]$ is given by

$$F(t_1, \ldots, t_{n+1}) = (t_{n+1}^3 - t_1^2t_2 - t_2^2t_3)^3 - 27t_1^2t_2^3t_3t_{n+1}^3.$$

Note that, $\deg F(t_1, \ldots, t_{n+1}) > 8$. Hence, by a similar argument as in Proposition 5.3.3, with $t \leq 3^2 - 1 = 8$ we have

$$\mu(I^t) = \binom{n+t}{t}.$$

$\square$

Proposition 5.3.5. *Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an almost complete intersection generated by a general set of $n+1$ forms of degree $d \geq 2$. Then $\deg(\mathcal{F}(I)) \geq 3$.*

Proof. Since $\mathcal{F}(I)$ is a hypersurface ring, $r(I) = \deg(\mathcal{F}(I)) - 1$. Thus, it is enough to show that $r(I) \geq 2$. Letting $I = (\mathbf{f}) = (f_1, \ldots, f_{n+1})$ be a minimal generating set, we may assume that $\{f_1, \ldots, f_n\}$ is a regular sequence and that $J = (f_1, \ldots, f_n)$ is a minimal reduction of I. Then, it suffices to prove that $I^2 \neq JI$.

We argue in terms of the respective minimal numbers of generators. Quite generally, with the above numbers, $\mu(JI) \leq n(n+1) - \binom{n}{2} = n(n+3)/2$. On the other hand, by Proposition 5.3.3 we have that $\mu(I^2) = \binom{n+2}{2}$. Therefore, $r(I) \neq 1$ and hence $\deg(\mathcal{F}(I)) \geq 3$. $\square$

Remark 5.3.6. Note that the expected value $\mu(I^2) = \binom{n+2}{2}$ for itself does not imply the syzygetic property. In fact, almost complete intersections of general forms will admit quadratic relations with polynomial coefficients of degree ≥ 1.

It has been proved in [23, Proposition 2.7] that if I is a codimension 3 ideal in $R = \Bbbk[x, y, z]$ generated by 4 quadrics, then $r(I) = 1$ or $r(I) = 3$, and that when $r(I) = 1$ then $\mathcal{R}(I)$ is Cohen-Macaulay and does not satisfy the condition R_1 of Serre. Based on our previous findings plus computer experiment using the Macaulay2 [18], we pose:

Conjecture 5.3.7. *Let d be a prime number and let $I \subset R = \Bbbk[x, y, z]$ be a strict almost complete intersection of forms of degree d. Then $r(I) = d - 1$ or else $r(I) = d^2 - 1$. In addition, if the forms are general then:*

(i) $r(I) = d^2 - 1$.

(ii) *The rational map defined by any linear system spanning* $[I]_d$ *is birational.*

(iii) $\mathcal{R}(I)$ *satisfies the condition* R_1.

(iv) $\mathcal{R}(I)$ *is not Cohen-Macaulay.*

Note that for the first statement is enough to apply [45, Theorem 6.6 (a)]. For lower cases, such that $d = 2, 3$ we have the following results essentially traced back to [23, Proposition 2.7] and to [22, Proposition 3.3] in their main facets.

Proposition 5.3.8. *Let* $I \subset R = \Bbbk[x, y, z]$ *be a codimension* 3 *ideal generated by a general set of four quadrics. Then the following statements hold:*

(i) $r(I) = 3$.

(ii) *The rational map defined any linear system spanning* $[I]_2$ *is birational.*

(iii) $\mathcal{R}(I)$ *is strictly almost Cohen-Macaulay.*

Proof. Since R/I has finite colength, applying [45, Theorem 6.6 (a)] we have that $\deg(\mathcal{F}(I)) = \frac{4}{\deg(\mathfrak{F})}$ where $\deg(\mathfrak{F})$ is the degree of rational map defined by the a set of forms spanning $[I]_2$.

Therefore, $\deg(\mathcal{F}(I))$ divides 4, so the previous proposition implies that $\deg(\mathcal{F}(I)) = 4$. This proves both (i) and (ii).

(iii) This is proved in [23, Proposition 2.7]. $\square$

Proposition 5.3.9. *Let* $I \subset R = \Bbbk[x, y, z]$ *be a codimension* 3 *ideal generated by* 4 *cubics. Then* $r(I) = 2$ *or* $r(I) = 8$. *In addition, if the forms are general, then:*

(i) $r(I) = 8$.

(ii) *The rational map defined by any linear system spanning* $[I]_3$ *is birational.*

(iii) $\mathcal{R}(I)$ *satisfies the condition* R_1.

(iv) $\mathcal{R}(I)$ *is not Cohen-Macaulay.*

Proof. By [45, Theorem 6.6 (a)] we have that $\deg(\mathcal{F}(I)) = 3$ or $\deg(\mathcal{F}(I)) = 9$, and consequently $r(I) = 2$ or $r(I) = 8$.

Assume now that the forms are general. (i) Similarly as in Proposition 5.3.5 consider $I = (\mathbf{f}) = (f_1, f_2, f_3, f_4)$, $J = (f_1, f_2, f_3)$ a minimal reduction of I. By a simple calculation, we have that $\mu(JI^2) = 19$, and by Proposition 5.3.4 follows that $\mu(I^3) = 20$. Therefore $r(I) = 8$. (ii) Is enough to note that $\deg(\mathcal{F}(I)) = \frac{9}{\deg(\mathfrak{F})}$, since $r(I) = 8$ then $\deg(\mathcal{F}(I)) = 9$ and therefore $\deg(\mathfrak{F}) = 1$. (iii) Follows by the [22, Proposition 3.3]. (iv) Suposse $\mathcal{R}(I)$ Cohen–Macaulay then $\mathcal{R}(I)$ is normal and consequently I would be normal, but I is not even integrally closed, because $\mathfrak{m}^3$ is integral over I. $\square$

Appendix

Appendix A

An alternative proof of Theorem 3.3.2

In this part we give some explicit results, which can be used in particular to give an alternative proof of Theorem 3.3.2 without using either [21] or [46]. Throughout this Appendix, $\Bbbk$ is a field of characteristic zero.

Assume the following setup:

Setup A.1. *Set* $R = \Bbbk[x, y, z]$.

- $r \geq 5$ *is a odd integer and* $d := (r-1)d'/2$ *for some integer* $d' \geq 1$.
- $e := d + d' - 3$.

Proposition A.2. *With the data of* Setup A.1*, let* $I_{d',r} \subset R = \Bbbk[x, y, z]$ *be the ideal*

$$I_{d',r} = (z^{d'}(x^{d'}, y^{d'})^{(r-3)/2}, x^{2d'}(x^{d'}, y^{d'})^{(r-5)/2}, y^d, z^d).$$

Then,

(a) *The minimal graded free resolution of* $R/I_{d',r}$ *has the form*

$$0 \to \begin{array}{c} R(-(d+2d'))^{(r-5)/2} \\ \oplus \\ R(-(d+3d')) \\ \oplus \\ R(-(2d))^{(r-3)/2} \end{array} \to \begin{array}{c} R(-(d+d'))^{\frac{3(r-1)-6}{2}} \\ \oplus \\ R(-(d+2d')) \\ \oplus \\ R(-(2d-d'))^{(r-1)/2} \end{array} \to R(-d)^r \to R \to R/I_{d',r} \to 0$$

(b) *The decomposition structure of the socle of* $I_{d',r}$ *is*

$$\mathrm{Soc}(R/I_{d',r}) = \begin{cases} \mathrm{Soc}(R/I)_{4d'-3} \oplus \mathrm{Soc}(R/I)_{5d'-3}, & \text{if } r = 5 \\ \mathrm{Soc}(R/I)_{d+2d'-3} \oplus \mathrm{Soc}(R/I)_{d+3d'-3} \oplus \mathrm{Soc}(R/I)_{2d-3}, & \text{if } r \geq 7 \end{cases}$$

where

$$\begin{cases} \dim_{\Bbbk} \operatorname{Soc}(R/I)_{4d'-3} = 1, \\ \dim_{\Bbbk} \operatorname{Soc}(R/I)_{5d'-3} = 1, \\ \dim_{\Bbbk} \operatorname{Soc}(R/I)_{d+2d'-3} = (r-5)/2, \\ \dim_{\Bbbk} \operatorname{Soc}(R/I)_{d+3d'-3} = 1, \\ \dim_{\Bbbk} \operatorname{Soc}(R/I)_{2d-3} = (r-3)/2. \end{cases}$$

(c) *If, moreover, $r \geq 7$ then $V(I_D(M_{d,r,d+d'-3}))$ is a proper subset of $(\mathbb{P}^N)^r$, where $D := \dim_{\Bbbk}(R_{2d+d'-3})$.*

Proof. (a) We will first prove the case $d' = 1$ (in particular, $r = 2d+1$). For convenience, we write the generators of I in the following block shaped matrix:

$$\phi_0 := \begin{bmatrix} z\mathbf{f} & x^2\mathbf{g} & y^d & z^d \end{bmatrix}, \tag{A.1}$$

where $\mathbf{f} := [x^{d-1}\ x^{d-2}y\ \cdots\ y^{d-1}]$ and $\mathbf{g} := [x^{d-2}\ x^{d-3}y\ \cdots\ y^{d-2}]$. Then ϕ_0 is the matrix of the map $R(-d)^{2d+1} \to R$.

Throughout $\mathbf{e}_j$ and $\mathbf{0}_{i\times j}$ will denote the identity matrix of order j and the $i \times j$ null matrix, respectively. Consider further the following matrices, which will be candidates to first and second syzygies.

$$\phi_1 = \left[\begin{array}{ccccc|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0}_{d\times(d-2)} & \mathbf{0}_{d\times 1} & \mathbf{a}_7 \\ \mathbf{b}_1 & \mathbf{0}_{(d-1)\times 1} & \mathbf{b}_3 & \mathbf{0}_{(d-1)\times 1} & \mathbf{b}_5 & \mathbf{b}_6 & \mathbf{0}_{(d-1)\times d} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(d-1)} & \mathbf{c}_4 & \mathbf{0}_{1\times(d-2)} & \mathbf{c}_6 & \mathbf{0}_{1\times d} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(d-1)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(d-2)} & 0_{1\times 1} & \mathbf{d}_7 \end{array}\right], \tag{A.2}$$

with three blocks in standard degrees $1, 2$ and $d-1$ from left to right, respectively – if needed we will refer to these three blocks as *degree blocks*.

Here

$$\mathbf{a}_1 = \begin{bmatrix} y\mathbf{e}_{d-2} \\ \mathbf{0}_{2\times(d-2)} \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} \mathbf{0}_{(d-2)\times 1} \\ -y \\ x \end{bmatrix},\ \mathbf{a}_3 = \begin{bmatrix} x\mathbf{e}_{d-1} \\ \mathbf{0}_{1\times(d-1)} \end{bmatrix},\ \mathbf{a}_4 = \begin{bmatrix} \mathbf{0}_{(d-1)\times 1} \\ y \end{bmatrix},\ \mathbf{a}_7 = -z^{d-1}\mathbf{e}_d,$$

$$\mathbf{b}_1 = \begin{bmatrix} \mathbf{0}_{1\times(d-2)} \\ -z\mathbf{e}_{d-2} \end{bmatrix}, \quad \mathbf{b}_3 = -z\mathbf{e}_{d-1}, \quad \mathbf{b}_5 = \begin{bmatrix} -y & 0 & \cdots & 0 \\ x & -y & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & x & -y \\ 0 & \cdots & 0 & x \end{bmatrix}, \quad \mathbf{b}_6 = \left[\begin{array}{c} \mathbf{0}_{(d-2)\times 1} \\ \hline -y^2 \end{array}\right],$$

$$\mathbf{c}_4 = \begin{bmatrix} -z \end{bmatrix}, \quad \mathbf{c}_6 = \begin{bmatrix} x^2 \end{bmatrix}, \quad \mathbf{d}_7 = \begin{bmatrix} x^{d-1} & x^{d-2}y & \cdots & y^{d-1} \end{bmatrix}.$$

The second matrix is

$$\phi_2 = \begin{bmatrix} \mathbf{s}_1 & \mathbf{0}_{(d-2)\times 1} & \mathbf{u}_1 & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{t}_2 & \mathbf{0}_{1\times(d-2)} & \mathbf{v}_2 \\ \mathbf{s}_3 & \mathbf{t}_3 & \mathbf{u}_3 & \mathbf{0}_{(d-1)\times 1} \\ \mathbf{0}_{1\times d-2} & \mathbf{t}_4 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} \\ \mathbf{s}_5 & \mathbf{0}_{(d-2)\times 1} & \mathbf{0}_{(d-2)\times(d-2)} & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{t}_6 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} \\ \mathbf{0}_{d\times(d-2)} & \mathbf{0}_{d\times 1} & \mathbf{u}_7 & \mathbf{v}_7 \end{bmatrix}, \tag{A.3}$$

where

$$\mathbf{s}_1 = x\mathbf{e}_{d-2}, \quad \mathbf{s}_3 = \begin{bmatrix} -y\mathbf{e}_{d-2} \\ \mathbf{0}_{1\times(d-2)} \end{bmatrix}, \quad \mathbf{s}_5 = z\mathbf{e}_{d-2},$$

$$\mathbf{t}_2 = \begin{bmatrix} -xy \end{bmatrix}, \quad \mathbf{t}_3 = \begin{bmatrix} \mathbf{0}_{(d-2)\times 1} \\ -y^2 \end{bmatrix}, \quad \mathbf{t}_4 = \begin{bmatrix} x^2 \end{bmatrix}, \quad \mathbf{t}_6 = \begin{bmatrix} z \end{bmatrix},$$

$$\mathbf{u}_1 = -z^{d-1}\mathbf{e}_{d-2}, \quad \mathbf{u}_3 = \begin{bmatrix} \mathbf{0}_{1\times(d-2)} \\ z^{d-1}\mathbf{e}_{d-2} \end{bmatrix}, \quad \mathbf{u}_7 = \begin{bmatrix} -y & 0 & \cdots & 0 \\ x & -y & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & x & -y \\ 0 & \cdots & 0 & x \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\mathbf{v}_2 = \begin{bmatrix} z^{d-1} \end{bmatrix}, \quad \mathbf{v}_7 = \begin{bmatrix} \mathbf{0}_{(d-2)\times 1} \\ -y \\ x \end{bmatrix}.$$

The goal of this seemingly bizarre block wise way of writing matrices is to adjust page fitting and reading easiness.

Claim 1. The sequence of R-maps

$$0 \to R^{2d-2} \xrightarrow{\phi_2} R^{4d-2} \xrightarrow{\phi_1} R^{2d+1} \xrightarrow{\phi_0} R \tag{A.4}$$

is a complex of R-modules.

The fact that the composite

$$\phi_0{\cdot}\phi_1 = \begin{bmatrix} z\mathbf{fa}_1 + x^2\mathbf{gb}_1 & z\mathbf{fa}_2 & z\mathbf{fa}_3 + x^2\mathbf{gb}_3 & z\mathbf{fa}_4 + y^d\mathbf{c}_4 & x^2\mathbf{gb}_5 & x^2\mathbf{gb}_6 + y^d\mathbf{c}_6 & z\mathbf{fa}_7 + z^d\mathbf{d}_7 \end{bmatrix}$$

is a null matrix is a routine exercise in syzygies of monomials as reduced Koszul relations. Yet, the shape of ϕ_1 will be of relevance later for rank and minors computation.

The other composite

$$\phi_1\cdot\phi_2=\left[\begin{array}{cccc}\mathbf{a}_1\mathbf{s}_1+\mathbf{a}_3\mathbf{s}_3 & \mathbf{a}_2\mathbf{t}_2+\mathbf{a}_3\mathbf{t}_3+\mathbf{a}_4\mathbf{t}_4 & \mathbf{a}_1\mathbf{u}_1+\mathbf{a}_3\mathbf{u}_3+\mathbf{a}_7\mathbf{u}_7 & \mathbf{a}_2\mathbf{v}_2+\mathbf{a}_7\mathbf{v}_7\\ \mathbf{b}_1\mathbf{s}_1+\mathbf{b}_3\mathbf{s}_3+\mathbf{b}_5\mathbf{s}_5 & \mathbf{b}_3\mathbf{t}_3+\mathbf{b}_6\mathbf{t}_6 & \mathbf{b}_1\mathbf{u}_1+\mathbf{b}_3\mathbf{u}_3 & \mathbf{0}_{(d-1)\times 1}\\ \mathbf{0}_{1\times(d-2)} & \mathbf{c}_4\mathbf{t}_4+\mathbf{c}_6\mathbf{t}_6 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1}\\ \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} & \mathbf{d}_7\mathbf{u}_7 & \mathbf{d}_7\mathbf{v}_7\end{array}\right]$$

is a bit more delicate, but all calculations are straightforward.

Claim 2. The complex (A.4) is acyclic.

For the argument we use the Buchsbaum-Eisenbud acyclicity criterion. Obviously, $\operatorname{rank}\phi_0=1$ and $\operatorname{ht} I_1(\phi_0)\geq 1$.

We next focus on ϕ_1, aiming at showing that $\operatorname{rank}\phi_1\geq 2d$ (hence, $\operatorname{rank}\phi_1=2d$) and $\operatorname{ht} I_{2d}(\phi_1)\geq 2$.

For this, we single out the following $2d\times 2d$ submatrices.

$$A:=\left[\begin{array}{cccc|c}\mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0}_{d\times(d-2)} & \mathbf{0}_{d\times 1} & \\ \mathbf{b}_3 & \mathbf{0}_{(d-1)\times 1} & \mathbf{b}_5 & \mathbf{b}_6 & \boldsymbol{\rho}\\ \mathbf{0}_{1\times(d-1)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} & \end{array}\right],$$

formed with rows $1,2,\ldots,2d-1,2d+1$ and columns $d,d+1,\ldots,3d-1$, where

$$\boldsymbol{\rho}=(-z^{d-1}\ 0\ \ldots\ 0\ x^{d-1})^t$$

is the first column of the rightmost degree block of ϕ_1, and

$$B=\left[\begin{array}{ccc}\mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_7\\ \mathbf{b}_3 & \mathbf{0}_{(d-1)\times 1} & \mathbf{0}_{(d-1)\times d}\\ \mathbf{0}_{1\times(d-1)} & \mathbf{c}_4 & \mathbf{0}_{1\times d}\end{array}\right],$$

formed with rows $1,2,\ldots,2d$ and columns $d,d+1,\ldots,2d,3d,\ldots,4d-2$.

Expanding conveniently, we have

$$\begin{aligned}\det A &= x^{d-1}\det\left[\begin{array}{cccc}\mathbf{a}_3 & \mathbf{a}_4 & \mathbf{0}_{d\times(d-2)} & \mathbf{0}_{d\times 1}\\ \mathbf{b}_3 & \mathbf{0}_{(d-1)\times 1} & \mathbf{b}_5 & \mathbf{b}_6\end{array}\right]\\ &= x^{d-1}\det\left[\begin{array}{cc}\mathbf{a}_3 & \mathbf{a}_4\end{array}\right]\det\left[\begin{array}{cc}\mathbf{b}_5 & \mathbf{b}_6\end{array}\right]\\ &= x^{d-1}\det\left[\begin{array}{cc}x\mathbf{e}_{d-1} & \mathbf{0}_{(d-1)\times 1}\\ \mathbf{0}_{1\times(d-1)} & y\end{array}\right]\det\left[\begin{array}{ccccc}-y & 0 & \cdots & 0 & 0\\ x & -y & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & x & -y & 0\\ 0 & \cdots & 0 & x & -y^2\end{array}\right]\\ &= x^{d-1}(yx^{d-1})((-1)^{d-1}y^d)=(-1)^{d-1}x^{2d-2}y^{d+1}\in I_{2d}(\phi_1)\end{aligned}\tag{A.5}$$

and

$$\begin{aligned}\det B &= (-1)^d \mathbf{c}_4 \det \mathbf{a}_7 \det \mathbf{b}_3 = (-1)^d(-z)(-1)^d z^{(d(d-1))}(-1)^{d-1}(z^{d-1}) \\ &= (-1)^{3d} z^{d^2} \in I_{2d}(\phi_1).\end{aligned} \tag{A.6}$$

Therefore, we are through.

Next get to ϕ_2, for which we want to prove that $\operatorname{rank} \phi_2 \geq 2d-2$ and $\operatorname{ht} I_{2d-2}(\phi_2) \geq 3$. Note that, since $\operatorname{rank} \phi_1 = 2d$, then $\operatorname{rank} \phi_2 \leq 2d-2$, so we have derived the sought equality.

The determinants of the following three $(2d-2) \times (2d-2)$ submatrices of ϕ_2 will be shown to form a regular sequence:

$$S = \begin{bmatrix} \mathbf{s}_1 & \mathbf{0}_{(d-2)\times 1} & \mathbf{u}_1 & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{t}_4 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} \\ \mathbf{0}_{(d-1)\times(d-2)} & \mathbf{0}_{(d-1)\times 1} & \tilde{\mathbf{u}}_7 & \tilde{\mathbf{v}}_7 \end{bmatrix},$$

$$T = \begin{bmatrix} \mathbf{s}_3 & \mathbf{t}_3 & \mathbf{u}_3 & \mathbf{0}_{(d-1)\times 1} \\ \mathbf{0}_{(d-1)\times(d-2)} & \mathbf{0}_{(d-1)\times 1} & \bar{\mathbf{u}}_7 & \bar{\mathbf{v}}_7 \end{bmatrix}$$

and

$$U = \begin{bmatrix} \mathbf{s}_1 & \mathbf{0}_{(d-2)\times 1} & \mathbf{u}_1 & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{t}_2 & \mathbf{0}_{1\times(d-2)} & \mathbf{v}_2 \\ \mathbf{s}_5 & \mathbf{0}_{(d-2)\times 1} & \mathbf{0}_{(d-2)\times(d-2)} & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & \mathbf{t}_6 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} \end{bmatrix}.$$

Here

- $\tilde{\mathbf{u}}_7$ and $\tilde{\mathbf{v}}_7$ are the submatrices obtained from $\mathbf{u}_7$ and $\mathbf{v}_7$, respectively, by omitting the first row.
- $\bar{\mathbf{u}}_7$ and $\bar{\mathbf{v}}_7$ are the submatrices obtained from $\mathbf{u}_7$ and $\mathbf{v}_7$, respectively, by omitting the last row.

The calculation is straightforward:

$$\begin{aligned}\det S &= \mathbf{t}_4 \det \mathbf{s}_1 \det \begin{bmatrix} \tilde{\mathbf{u}}_7 & \tilde{\mathbf{v}}_7 \end{bmatrix} \\ &= \mathbf{t}_4 \det \mathbf{s}_1 \det \begin{bmatrix} x & -y & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & -y & 0 \\ 0 & \cdots & 0 & x & -y \\ 0 & \dots & 0 & 0 & x \end{bmatrix} \\ &= x^2 \cdot x^{d-2} \cdot x^{d-1} = x^{2d-1},\end{aligned} \tag{A.7}$$

$$\begin{aligned}
\det T &= \det\begin{bmatrix} \mathbf{s}_3 & \mathbf{t}_3 \end{bmatrix} \det\begin{bmatrix} \bar{\mathbf{u}}_7 & \bar{\mathbf{v}}_7 \end{bmatrix} \\
&= \det\begin{bmatrix} -y\mathbf{e}_{d-2} & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{0}_{1\times(d-2)} & -y^2 \end{bmatrix} \det\begin{bmatrix} -y & 0 & \cdots & 0 & 0 \\ x & -y & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & x & -y & 0 \\ 0 & \cdots & 0 & x & -y \end{bmatrix} \\
&= y^{2d-1},
\end{aligned}$$

$$\begin{aligned}
\det U &= \pm \det \mathbf{s}_5 \cdot \det\begin{bmatrix} \mathbf{0}_{(d-2)\times 1} & \mathbf{u}_1 & \mathbf{0}_{(d-2)\times 1} \\ \mathbf{t}_2 & \mathbf{0}_{1\times(d-2)} & \mathbf{v}_2 \\ \mathbf{t}_6 & \mathbf{0}_{1\times(d-2)} & \mathbf{0}_{1\times 1} \end{bmatrix} \\
&= \pm \det \mathbf{s}_5 \det \mathbf{u}_1 \det \mathbf{v}_2 \det \mathbf{t}_6 = \pm z^{d-2}(-1)^d z^{(d-1)(d-2)} z^{d-1} z = \pm(-1)^d z^{d(d-1)}.
\end{aligned}$$

This completes the argument on the acyclicity criterion, hence also the proof for the case $d' = 1$.

Now, deal with the case of arbitrary $d' > 1$. Consider the endomorphism of $\Bbbk$-algebras

$$\zeta : \Bbbk[x, y, z] \to \Bbbk[x, y, z], \quad x \mapsto x^{d'},\ y \mapsto y^{d'},\ z \mapsto z^{d'}.$$

Note that $I_{d',r}$ is the extension of $I_{1,r}$ by the endomorphim ζ. Consider the matrices ϕ_0, ϕ_1 and ϕ_2 as in (A.1), (A.2) and (A.3). Let $\zeta(\phi_i)$ $(i = 0, 1, 2)$ denote the matrix obtained from ϕ_i by evaluating ζ in each of its entries. Obviously,

$$\zeta(\phi_0) \cdot \zeta(\phi_1) = 0 \quad \text{and} \quad \zeta(\phi_1) \cdot \zeta(\phi_2) = 0.$$

Hence, the sequence

$$0 \to R^{r-3} \xrightarrow{\zeta(\phi_2)} R^{2r-4} \xrightarrow{\zeta(\phi_1)} R^r \xrightarrow{\zeta(\phi_0)} R \tag{A.8}$$

is a complex.

As shown above, there are $(r-1) \times (r-1)$ submatrices A and B of ϕ_1 such that

$$\det A = (-1)^{d-1} x^{r-3} y^{(r+1)/2} \in I_{2d}(\phi_1) \quad \text{and} \quad \det B = (-1)^{3d} z^{((r-1)/2)^2} \in I_{2d}(\phi_1).$$

Thus,

$$\det \zeta(A) = (-1)^{d-1} x^{(r-3)d'} y^{((r+1)/2)d'} \in I_{r-1}(\zeta(\phi_1)),\ \det \zeta(B) = (-1)^{3d} z^{((r-1)/2)^2 d'} \in I_{r-1}(\zeta(\phi_1)).$$

Hence, $\operatorname{rank} \zeta(\phi_1) = 2d$ and $\operatorname{ht} I_{r-1}(\zeta(\phi_1)) \geq 2$.

The proof for $d' = 1$ also guarantees the existence of $(r-3) \times (r-3)$ submatrices S, T, U

of ϕ_2 such that

$$\det S = x^{r-2},\ \det T = y^{r-2},\ \det U = \pm(-1)^d z^{(r-1)(r-3)/4}.$$

Thus, $\det \zeta(S) = x^{(r-2)d'}$, $\det \zeta(T) = y^{(r-2)d'}$, $\det \zeta(U) = \pm(-1)^d z^{((r-1)(r-3)/4)d'}$, hence

$$\operatorname{rank} \zeta(\phi_2) = r-3 \quad \text{and} \quad \operatorname{ht} I_{r-3}(\zeta(\phi_2)) \geq 3. \tag{A.9}$$

Therefore, the complex (A.8) is acyclic.

By construction, the cokernel of $\zeta(\phi_0)$ is $R/I_{d',r}$. So,

$$0 \to R^{r-3} \xrightarrow{\zeta(\phi_2)} R^{2r-4} \xrightarrow{\zeta(\phi_1)} R^r \xrightarrow{\zeta(\phi_0)} R \to R/I_{d',r} \to 0 \tag{A.10}$$

is a free resolution of $R/I_{d',r}$. Finally, by observing the degrees of the entries of the matrices $\zeta(\phi_0)$, $\zeta(\phi_1)$ and $\zeta(\phi_2)$, the resolution (A.10) we see that it is a minimal graded free resolution for $R/I_{d',r}$ as stated in the proposition.

(b) This is a consequence of (a) via a well-known argument (see, e.g., Lemma 1.4.1).

(c) Let $f_1, \ldots, f_r$ denote the given set of generators of $I_{d',r}$. Then:

$$\begin{aligned} R_{2d+d'-3} &= I_{2d+d'-3} \quad \text{(by item (b))} \\ &= R_{d+d'-3} I_d \\ &= R_e f_1 + \cdots + R_e f_r. \end{aligned} \tag{A.11}$$

Hence, by property (B) (in Section 3.1), $P_{\mathbf{f}} \notin V(I_D(M_{d,r,e}))$. In particular, $V(I_D(M_{d,r,e}))$ is a proper subset of $(\mathbb{P}^N)^r$. □

In order to overcome the obstruction $r \geq 7$ in Proposition A.2 (c), we introduce the following particular construct in five generators.

Proposition A.3. *Given an integer $d' \geq 2$, consider the following ideal of $R = \Bbbk[x, y, z]$*

$$I = (x^{2d'}, y^{2d'}, z^{2d'}, x^{d'}y^{d'}, xz^{2d'-1}).$$

Then:

(a) *The minimal graded free resolution of R/I has the form:*

$$0 \to \begin{array}{c} R(-(4d'+1)) \\ \oplus \\ R(-(5d'-1))^2 \end{array} \to \begin{array}{c} R(-(2d'+1)) \\ \oplus \\ R(-3d')^2 \\ \oplus \\ R(-(4d'-1))^2 \\ \oplus \\ R(-4d')^2 \end{array} \to R(-2d')^5 \to R \to R/I \to 0 \tag{A.12}$$

(b) *The structure decomposition of the socle of R/I is*

$$\operatorname{Soc}(R/I) = \operatorname{Soc}(R/I)_{4d'-2} \oplus \operatorname{Soc}(R/I)_{5d'-4} \tag{A.13}$$

where $\dim \operatorname{Soc}(R/I)_{4d'-2} = 1$ *and* $\dim \operatorname{Soc}(R/I)_{5d'-4} = 2$.

(c) $V(I_D(M_{2d',5,3d'-3}))$ *is a proper subset of* $(\mathbb{P}^N)^5$*, where* $D := \dim_{\Bbbk}(R_{d+3d'-3})$.

Proof. Once more, write $\phi_0 = [x^{2d'}\ y^{2d'}\ z^{2d'}\ x^{d'}y^{d'}\ xz^{2d'-1}]$ and introduce first and second syzygies candidates:

$$\phi_1 = \begin{bmatrix} 0 & 0 & -y^{d'} & 0 & -z^{2d'-1} & 0 & 0 \\ 0 & x^{d'} & 0 & 0 & 0 & -z^{2d'} & -xz^{2d'-1} \\ x & 0 & 0 & 0 & 0 & y^{2d'} & 0 \\ 0 & -y^{d'} & x^{d'} & -z^{2d'-1} & 0 & 0 & 0 \\ -z & 0 & 0 & x^{d'-1}y^{d'} & x^{2d'-1} & 0 & y^{2d'} \end{bmatrix}$$

and

$$\phi_2 = \begin{bmatrix} y^{2d'} & 0 & 0 \\ 0 & 0 & z^{2d'-1} \\ 0 & z^{2d'-1} & 0 \\ 0 & x^{d'} & -y^{d'} \\ 0 & -y^{d'} & 0 \\ -x & 0 & 0 \\ z & 0 & x^{d'-1} \end{bmatrix}.$$

A straightforward calculation yields

$$\phi_0 \cdot \phi_1 = 0 \quad \text{and} \quad \phi_1 \cdot \phi_2 = 0.$$

Thus, the sequence

$$0 \to R^3 \xrightarrow{\phi_2} R^7 \xrightarrow{\phi_1} R^5 \xrightarrow{\phi_0} R$$

is a complex.

Now consider the following 4×4 submatrices of ϕ_1

$$A = \begin{bmatrix} 0 & 0 & -z^{2d'-1} & 0 \\ 0 & 0 & 0 & -z^{2d'} \\ 0 & -z^{2d'-1} & 0 & 0 \\ -z & x^{d'-1}y^{d'} & x^{2d'-1} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & x^{d'} & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & -y^{d'} & x^{d'} & 0 \\ -z & 0 & 0 & x^{2d'-1} \end{bmatrix}.$$

As easily seen, $\det A = -z^{6d'-1}$ and $\det B = -x^{4d'}$. Therefore, $\operatorname{rank} \phi_1 = 4$ and $\operatorname{ht} I_4(\phi_1) \geq 2$.

On the other hand, for the following 3×3 submatrices of ϕ_2

$$S = \begin{bmatrix} y^{2d'} & 0 & 0 \\ 0 & x^{d'} & -y^{d'} \\ 0 & -y^{d'} & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & z^{2d'-1} \\ 0 & z^{2d'-1} & 0 \\ z & 0 & x^{d'-1} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & x^{d'} & -y^{d'} \\ -x & 0 & 0 \\ z & 0 & x^{d'-1} \end{bmatrix}$$

one has $\det S = -y^{4d'}$, $\det T = -z^{4d'-1}$ and $\det U = x^{2d'}$, forming a regular sequence. Thus, (a) holds.

(b) and (c) follow easily as before. $\square$

We now give an alternative proof of Theorem 3.3.2.

Theorem A.4. (Theorem 3.3.2 (bis)) *Let $I \subset R = \Bbbk[x, y, z]$ be a Gorenstein ideal generated by a general set of $r \geq 5$ forms of degree $d \geq 2$. Then $r = 5$ and $d = 2$.*

Proof. Since I is generated by a general set of sufficiently many forms, it has finite colength (see Lemma 3.1.4 (iv)). Let $\{f_1, \ldots, f_r\}$ be a set of such forms of degree d. Since the socle degree of I is $2d + d' - 3$ (see Lemma 2.1.5), then $R_{d+d'-3}f_1 + \cdots + R_{d+d'-3}f_r \subsetneq R_{2d+d'-3}$. Therefore, (B) implies that $P_{\mathbf{f}} \in V(I_D(M_{d,r,d+d'-3})) = V(I_D(M_{((r-1)/2)d',r,((r-1)/2)d'+d'-3}))$, where $D = \dim_{\Bbbk}(R_{2d+d'-3})$.

Suppose that $d \geq 3$. Then $((r-1)/2)d' = d \geq 3$, i.e., $(r-1)d' \geq 6$. Now, either $d' = 1$ and $r \geq 7$, or else $d' \geq 2$. Thus, in any case, either Proposition A.2 or Proposition A.3 implies that $V(I_D(M_{d,r,d+d'-3}))$ is a proper Zariski closed subset of $(\mathbb{P}^N)^r$. This shows that all Gorenstein ideals of codimension 3, of given degree $d \geq 3$ and $r \geq 5$ number of generators, are parameterized by a proper Zariski closed subset of the space of parameters. $\square$

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