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Some aspects of local cohomology theory

por

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sob orientação do

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e sob coorientação do

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Este trabalho trata de algumas características da teoria da cohomologia local. Desenvolvemos uma nova ferramenta chamada sequência espectral de Mayer-Vietoris que nos permite estudar vários módulos de cohomologia local suportados em diferentes ideais, o que nos levou a generalizar ou recuperar resultados anteriores de vários autores e também a produzir novos, especialmente no que diz respeito a anéis polinomiais multigraduados. Também lidamos com módulos Cohen-Macaulay generalizados e módulos de deficiência, fornecendo relações entre números de Bass e Betti destes de modo a tanto generalizar resultados clássicos quanto a provar novos como um caso da a conjectura de Auslander e Reiten. Finalmente, cohomologia local é vista como uma importante ferramenta para o estudo da interação entre a finitude de dimensões homológicas e de anulamento de módulos Ext.

Palavras-chave: Cohomologia local; sequência espectral de Mayer-Vietoris; dimensão cohomológica; regularidade de Castelnuovo-Mumford; característica de Euler; módulos de deficiência; módulo Cohen-Macaulay generalizado; conjectura de Auslander-Reiten; cohomologia local generalizada; dimensão homológica.

Abstract

This work is about some features of local cohomology theory. We develop a new tool called Mayer-Vietoris spectral sequence that allows us to study several local cohomology modules supported in different ideals, which led us to generalize or retrieve previous results of several authors and also produce new ones, especially in what concerns multigraded polynomial rings. We also deal with generalized Cohen-Macaulay modules and deficiency modules, providing relations between their Bass and Betti numbers in order to both generalize classical results and produce new ones as a case of the conjecture of Auslander and Reiten in a particular case. Finally, local cohomology is viewed as an important tool for the studying of the interplay between finiteness of homological dimensions and the vanishing of Ext modules.

Keywords: Local cohomology; Mayer-Vietoris spectral sequence; cohomological dimension; Castelnuovo-Mumford regularity; Euler characteristics; deficiency modules; generalized Cohen-Macaulay module; Auslander-Reiten conjecture; generalized local cohomology; homological dimension.

"Hay que trabajar, pero sin perder la ternura jamás."

Roberto Callejas Bedregal

Dedicatória

Aos meus pais, Hilma e Paulo Cezar

Contents

Introduction

Serre's fundamental work (1955) [79] set up cohomology theory as an essential tool in modern algebraic geometry. Serre observes that many statements about projective varieties can be understood as statements about graded or complete local rings so that, after proved, they may be of use in obtaining global outcomes. For instance, the duality theorem for projective varieties becomes a duality theorem for local cohomology modules. Thus, it suggested many themes which are largely studied in local cohomology theory until nowadays, but only in Hartshorne's book (1967) [45] (on a Grothendieck's 1961 seminar) the effectiveness of local cohomology is recognised in local algebra. Despite Grothendieck and Hartshorne's viewpoints were still geometric, their notes have shown the indispensable benefit of local cohomology theory for the study of commutative Noetherian rings.

In light of this, Broadmann-Sharp [17] provides an excellent algebraic introduction to Grothendieck's local cohomology theory with some geometric flavour (as the title of the book suggests), which is the approach that we intend to follow up here. To be more exact, we still rely on the geometric viewpoint since in this thesis local cohomology modules — over a non-necessarily Noetherian ring — are defined via cohomology of Čech complexes. Such a generality provides an algebro-geometric link that allows the reader to approach the theory here developed through the better way it fits for their purpose.

As the title of this thesis says, we work on different aspects of local cohomology theory. We will explain them in general lines and next describe each one highlighting those different features.

In this thesis, we make some progress regarding local cohomology theory, especially in a multigraded setting, on deficiency modules and the use of local cohomology to approach problems on homological dimensions. Important tools of the general theory are extended providing through the chapters new achievements — such as a duality, Artinianess and more information about (numerical) homological invariants — and improvements of known results. On what concerns deficiency modules, which are, in a sense (due to local duality), the Noetherian versions of local cohomology modules, we develop some theory on generalized Cohen-Macaulay modules and then provide comparisons between Bass and Betti numbers of modules and their deficiencies. Finally, the theory of generalized local cohomology turns out to be quite useful in the study of the finiteness of homological dimensions.

Namely, we investigate a quasi-isomorphism between complexes so that the construction of a spectral sequence, which we call it by Mayer-Vietoris spectral sequence, since it generalizes the well-known Mayer-Vietoris long exact sequence, see [17], is made possible. Although there are other generalizations of the Mayer-Vietoris long exact sequence (see [1, 66, 78, 81]), ours still allow us to work on problems involving more than two ideals. Note that the spectral sequence in [66] has the same spirit but Noetherianess is required and its construction is different (though similar somehow) so that it is useful in this work but not sufficient for our goals. Having such tools in hands, in a multigraded setting, we provide bounds for cohomological dimension (sometimes determining them), determine depth of some local cohomology modules, study the support (in the sense of Definition 1.2.2) of multigraded local cohomology modules and multigraded regularity. Yet in a multigraded setting, we study the interplay between supports (the non-vanishing region) of Tor modules and local cohomology modules. After that, we work with deficiency modules. We see that the relation between projective and injective dimensions of a given module and its deficiency modules provides several consequences such as a generalization of a Foxby result [38], sufficient homological conditions on deficiency modules for a local ring be Cohen-Macaulay, a characterization of the complete intersection property in terms of Bass numbers of the residual field and a case of the long-standing Auslander-Reiten conjecture [7]. In the last part, local cohomology allows an investigation of the interplay between homological dimensions and Ext vanishing in such a way that we are able to give positive answers under suitable depth hypothesis to some questions that have appeared in the literature, as the one raised by Jorgensen in [60] about fourteen years ago, and investigate the finiteness of other homological dimensions such as Gorenstein (injective) dimension. We now will give a more detailed description of these outcomes.

Chapter 1 describes a few general and known facts on local cohomology, sheaf cohomology, Castelnuovo-Mumford regularity, deficiency modules, generalized local cohomology and homological dimensions. It sets the way we will work on local cohomology modules throughout the thesis. The first section 1.1 deals with the definition of local cohomology modules as the cohomologies of Čech complexes, see Subsection 1.1.1. Although it is not the usual definition in the Noetherian case (see [18, 17, 58] for example) since they are defined as derived functors of torsion functors, it allows us to work on a more general setting as [22] did and to establish the link between local cohomology and sheaf cohomology, described in Subsection 1.1.2. The second section 1.2 motivates the Castelnuovo-Mumford regularity in Subsection 1.2.1 towards its multigraded version (in the sense of [14]), presented on the last subsection 1.2.2. Next, in a Noetherian setting, we introduce deficiency modules in Section 1.3 and generalized local cohomology modules in Section 1.4. We finish this preliminary chapter presenting some homological dimensions in Section 1.5.

Mayer-Vietoris spectral sequence

Chapter 2 is dedicated to the construction of one of the main tools in this work, the so-called Mayer-Vietoris spectral sequence. The main reason for building up this cohomological tool is that it allows us to work on a wider setting in the sense that we can work with any finite number of ideals, as the Mayer-Vietoris long exact sequence just works with two ideals. To be clearer, we show that given ideals $I_1, I_2, ..., I_n$ in a (commutative with non-zero unity) ring R and an R-module M, there exists a spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0<...<i_p} H^q(U_{I_{i_0}I_{i_1}...I_{i_p}}, \widetilde{M}) \Rightarrow_p H^{p+q}(U_{I_1+I_2+...+I_n}, \widetilde{M}). \qquad (0.0.1)$$

Throughout the chapter, we provide two different constructions of such spectral sequence and two ways — a geometric and a topological one — to get its second page. The first construction is made in Section 2.1, see Theorem 2.1.8. This construction is treated as a consequence of a quasi-isomorphism between the tensor product of Čech complexes and the Čech complex of the product of the sequences involved, see propositions 2.1.4 and 2.1.7. The two first natural questions about this spectral sequence are "*does it degenerate at the second page?*" and "*what its second page looks like?*". The second question is what we answer in Section 2.2. Actually, we provide two answers for that. Theorem 2.2.1 is our second construction of the Mayer-Vietoris spectral sequence which also provides the second page, as we proved that such spectral sequence is a particular case of a Čech spectral sequence and thus it has a geometric view. At the end of Section 2.2 we display the second page in terms of general sheaf theory and inverse limits under a certain topological space, but we postpone the arguments to the Appendix A. The last section 2.3 deals with the first applications of the Mayer-Vietoris spectral sequence 0.0.1. The first one is naturally the Mayer-Vietoris long exact sequence, given in Theorem 2.3.1, and the others are about a version of the Mayer-Vietoris long exact sequence for three ideals (see Theorem 2.3.6) and some general results on cohomological dimension, especially Corollary 2.3.3 which regards relations between the cohomological dimension with respect to the product of ideals and cohomological dimensions with respect to each ideal. Further applications of the Mayer-Vietoris spectral sequence (and its first applications) are provided in the next chapter.

Multigraded local cohomology

In Chapter 3 we study local cohomology modules over a multigraded polynomial ring. It was motivated by the research around Castelnuovo-Mumford regularity in its multigraded version, firstly defined through geometrical terms by Hoffman-Wang and Maclagan-Smith [53, 69], and then Botbol-Chardin [14] defined such regularity in an algebraic sense and even in a way more general setting, which is the context we are working with here. But before dealing with multigraded regularity we will study some relations of multigraded cohomology modules in particular cases, cohomological dimensions and vanishing of multigraded pieces of Tor modules (Betti numbers) and local cohomology modules, as we are going to describe now.

The first section 3.1 is divided into two parts. The first one, Subsection 3.1.1, begins with a simple generalization of a result of Chardin and Nemati [23] for the non-Noetherian case, see Proposition 3.1.1. It establishes the cohomological dimension of a multigraded free module with respect to the irrelevant ideal (and therefore, an upper bound for the cohomological dimension of any multigraded module with respect to the irrelevant ideal), see Corollary 3.1.2. As consequence, for the Noetherian case, in Corollary 3.1.4 we conclude that cohomology modules of a multigraded free module supported in ideals generated by variables of the same degree cannot be Artinian. We finish the section by determining the cohomological dimension of such cohomology modules supported in the priorly mentioned ideals, see Corollary 3.1.9. As for Subsection 3.1.2, we furnish a duality between the local cohomology modules of multigraded modules supported in the ideals generated by variables of the same degree. Namely, we prove the following.

Theorem (Theorem 3.1.12)**.** *Let* $\mathbf{k}$ *be a field and denote* $R = \mathbf{k}[X_0, ..., X_m, Y_0, ..., Y_n]$ *the standard bigraded polynomial ring. Write* $B_1 = (X_0, ..., X_m)$ *and* $B_2 = (Y_0, ..., Y_n)$*. If* M *is a finitely generated bigraded* R*-module then one has functorial graded isomorphism*

$$H^{n+1-i}_{B_2}(M) \simeq {}^*\operatorname{Ext}^i_R(M, H^{m+1}_{B_1}(\omega_R))^\vee$$

for all $i \geq 0$*, where* $\_^\vee ={}^* \operatorname{Hom}_{\mathbf{k}}(\_, \mathbf{k})$ *and* $\omega_R = R(-(m+1), -(n+1))$ *is the canonical module of* R*.*

We also notice that the roles of B_1 and B_2 in the theorem above are interchangeably (see Theorem 3.1.14) and as first corollaries we conclude that

$$\operatorname{depth}_R H^{m+1}_{B_1}(F) = \operatorname{cd}_{B_2} H^{m+1}_{B_1}(F) = n+1$$

and

$$\operatorname{depth}_R H^{n+1}_{B_2}(F) = \operatorname{cd}_{B_1} H^{n+1}_{B_2}(F) = m+1,$$

where F is a finitely generated bigraded free R-module, see Corollary 3.1.16. Further, it should be noted that by taking $B_1 = 0$ in the duality above we recover graded local duality and that together with a duality of Herzog and Rahimi [51] one has

$$H^{n+1}_{B_2}(R) \simeq H^{m+1}_{B_1}(\omega_R)^\vee \text{ and } H^{n+1}_{B_2}(R)^\vee \simeq H^{m+1}_{B_1}(\omega_R).$$

Section 3.2 begins with an application of Corollary 2.3.3 for the multigraded case in Proposition 3.2.1. There we bound the cohomological dimension of a module with respect to the irrelevant ideal. This result allows us to provide a version of

Proposition 3.1.1 for three variables; it is shown in Proposition 3.2.2. As a consequence, Corollary 3.2.3 generalizes Corollary 3.1.2. The last result of this section is Proposition 3.2.5 which characterizes local cohomology modules of a free module supported in the irrelevant ideal in terms of local cohomology modules supported in ideals generated by variables (that are much more treatable). It should be noticed that it is the first result that we are using the Mayer-Vietoris spectral sequence B.2.2 defined by Lyubeznik [66].

Section 3.3 is about support (as in Definition 1.2.2), Castelnuovo-Mumford regularity and Euler characteristics. The subsection 3.3.1 contains results on the support of local cohomology modules, some of them are applications of the duality above and others are useful in Subsection 3.3.2. We also investigate the relation between the support of Tor modules and of local cohomology modules and find another spectral sequence, see Corollary 3.3.11, quite similar to that of Lyubeznik B.2.2. In the next subsection, we work on multigraded regularity. The Mayer-Vietoris spectral sequence 0.0.1 plays a crucial role. It is indeed the key in the proof of Proposition 3.3.29, Theorem 3.3.30 and Proposition 3.3.31. All these results provide relations between multigraded regularity with respect to arbitrary products and sums of a finite number of given ideals.

Section 3.4 is the last section of Chapter 3. It deals with the Euler characteristic of a multigraded module with respect to an ideal and an element of the grading group, see Definition 3.4.1. The main result of this section is Theorem 3.4.7; it shows that the Euler characteristic of a module with respect to the product of ideals generated by variables is the alternating sum of the Euler characteristics of the module with respect to the arbitrary sum of such ideals. We then show that over a standard multigraded polynomial ring with coefficients in a field, the Euler characteristic with respect to the product of ideals generated by pure powers of the variables is written in terms of the Euler characteristics with respect to the alternating sum of the ideals generated by the arbitrary sum of such variables, see Corollary 3.4.8.

In the next two chapters, we explore the interplay between local cohomology and homological dimensions. We seek first the classic ones, projective and injective dimensions. Next, we also involve the complete intersection and Gorenstein (injective) dimensions.

Deficiency modules

The content in Chapter 4 is joint work with Thiago Fiel which contains three sections and it is independent of chapters 2 and 3. It has already been submitted for publication, see [36]. The ring is supposed to be a factor of a Gorenstein local ring.

In Section 4.1 we find relations between a given generalized Cohen-Macaulay module and its deficiency modules. With this in hands, we show that the canonical module of a module, the most important deficiency module, is generalized Cohen-Macaulay provided the given module is also generalized Cohen-Macaulay, see Corollary 4.1.5. Further, when in addition the module M has depth at least two, then $M \simeq K(K(M))$, generalizing thus a Schenzel's result [76], see Corollary 4.1.10. Another interesting consequence regards characterization of the Cohen-Macaulay property in equidimensional terms, see Corollary 4.1.9.

The next section 4.2 provides a bounding for the Bass number of a module in terms of the Betti numbers of its deficiency modules, see Theorem 4.2.1. As main consequences, Corollary 4.2.3 ensures that for a ring in our context to be Cohen-Macaulay it is sufficient to admit a finitely generated module such that all its deficiency modules have finite projective dimension, and Theorem 4.2.4 relates Bass and Betti numbers of modules and its canonical modules in a more general setting that of Foxby [38, Corollary 3.6], also improving [39, Corollary 3.3]. In the rest of the section, we try to weaken the hypothesis in Theorem 4.2.4 in order to get relations between Bass and Betti numbers in a general setting.

We end the chapter with Section 4.3. Its main result, Theorem 4.3.1, contains a corollary of a result of Schenzel [76, Theorem 3.2] and provides several relations between the numbers involved when varying the dimension of the module, see for example Corollary 4.3.4 which gives the equality

$$\beta_2(M) - \beta_1(M) = \mu^2(K(M)) - \mu^1(K(M))$$

for an Artinian module. An application of this formula is a characterization of the complete intersection property in terms of the first and second Bass numbers of the residue field, see Corollary 4.3.6. Another consequence is a case of the Auslander-Reiten conjecture. Namely, we prove that the Auslander-Reiten conjecture holds true for finitely generated modules having deficiency modules of finite injective dimension over Noetherian rings which are factors of Gorenstein local rings, see Corollary 4.3.10. This generalizes the recent achievement in [39, Theorem 4.8].

Generalized local cohomology

The content in the next chapter of this thesis is joint work with Cleto B. Miranda-Neto and it is in preparation for submission for publication [54]. We see local cohomology as a fundamental tool in our study. Another aspect of local cohomology is its usefulness to solve other problems in commutative algebra and algebraic geometry. In Chapter 5, generalized local cohomology turns out to be one of the main tools required to investigate the finiteness of some homological dimensions and the interplay with Ext vanishing.

Our first goal is to address some problems from the existing literature on finite projective dimension via Ext vanishing; this is done in Section 5.1. We begin by considering a couple of problems from [61], one of them regarding freeness of modules over one-dimensional Gorenstein local rings (see Question 5.1.1), and then special attention is paid to a question raised by D. Jorgensen in [59] on prescribed projective dimension over complete intersection rings by means of the vanishing of a certain Ext module (see Question 5.1.10), which in fact motivated the first part of this chapter. Our results, the main one being Theorem 5.1.8, are in the direction of detecting some additional condition under which such questions turn out to admit an affirmative answer, whereas on the other hand, we are able to relax some of the hypotheses. In the case of our approach to Question 5.1.1, our result is Theorem 5.1.4 (which in turn is related to the notion of cohomological dimension), and for Jorgensen's problem, we require that the depth of certain Ext modules be sufficiently high in a sense that will be made precise, while the ring is only assumed to be Gorenstein (see Corollary 5.1.11). It is worth observing that such a depth hypothesis plays a key role in this chapter, being present in most of the results.

Still in Section 5.1, we use the class formed by the so-called "rigid" modules (see Definition 5.1.14) to establish sufficient conditions for a given module to have finite projective dimension (see Theorem 5.1.16 and its corollaries). We combine this approach with some results from the previous part in order to estimate this invariant, which in particular allows for characterizations of freeness.

In Section 5.2, we study connections between the vanishing of Ext modules and the finiteness of the injective dimension, the Gorenstein injective dimension, and the Gorenstein dimension of a module over a given Cohen-Macaulay local ring. The main result is Theorem 5.2.2, and some of its byproducts, which put us again in the scenario of finite projective dimension, are Corollary 5.2.4 and Corollary 5.2.7 (the latter, in particular, is crucially used in Section 5.3). Freeness criteria for modules over Gorenstein local rings are also provided.

Our interest in Section 5.3 meets some of the preceding ones, but via different methods. First, we employ the Burch property of ideals to study finite projective dimension over a Gorenstein local ring by means, in particular, of the vanishing of suitable Ext and Tor modules (see Corollary 5.3.3). Then we turn to another auxiliary tool, namely, the notion of strongly rigid module, the main results, in this case, being Corollary 5.3.8 and Corollary 5.3.10. We also consider a special instance that arises from ideals that are weakly full with respect to a suitable power of the maximal ideal, e.g., integrally closed ideals (provided that the ring has positive depth). The main result in this regard is Theorem 5.3.13. More freeness criteria follow readily in this last section as well.

Chapter 6 gathers questions about the other chapters of this thesis.

There are two appendices. As already mentioned, Appendix A treats the development of the second page of the Mayer-Vietoris spectral sequence 2.1.8 in topological terms. It is based on Jensen's work [59] and either sheaf theory and inverse limits play a fundamental role through this construction.

The Appendix B is devoted to spectral sequences. It contains a little information on the general theory and several examples that are quite useful in this work. The major reason for this appendix is the difficulty of this machinery at first view — at second and third views too — that a student can easily find, just as this author did. The more examples of spectral sequences are given the more students feel confident and comfortable using this marvellous homological tool.

Chapter 1

Preliminaries

This chapter concerns fundamental known concepts and techniques that shall be used throughout this work. Most of the content here can be found in standard textbooks such as [18], [17] and [30].

1.1 Local cohomology and sheaf cohomology

In this section, we comment on the general setting of both local cohomology and sheaf cohomology theories and their deep relation. The interested reader can find more about these topics and their relations in [17] and [58] which work only on Noetherian rings and see [45] for a way more general setting.

1.1.1 Local cohomology and Čech complex

The Čech complex is a basic and important tool in commutative algebra and algebraic geometry. Our goal here is to recall its definition and comment on its close relation with local cohomology modules and sheaf cohomology groups. Although the environment here is not necessarily Noetherian, the definition and some properties of Čech complexes follow the same lines as those of some classic textbooks; see for example [17], [18] and [58].

Let R be a ring. Let also $\mathfrak{a} = a_1, ..., a_n$ be a sequence of elements in R, I the ideal in R generated by $\mathfrak{a}$ and let M be an R-module.

Definition 1.1.1 (Čech complex)**.** *The* Čech complex of M with respect to the se- quence $\mathfrak{a}$ *is defined as the sequence of R-modules and R-homomorphisms*

$$C^{\bullet}_{\mathfrak{a}}(M): \quad 0 \to C^0_{\mathfrak{a}}(M) \to C^1_{\mathfrak{a}}(M) \to \ldots \to C^n_{\mathfrak{a}}(M) \to 0$$

where

(i) $C^0_{\mathfrak{a}}(M) = M$*;*

(ii) $C^p_{\mathfrak{a}}(M) = \bigoplus_{i_1<\ldots<i_p} M_{a_{i_1}\cdot\ldots\cdot a_{i_p}}$ *for* $p \geq 1$*;*

(iii) The homomorphism $C^p_{\mathfrak{a}}(M) \to C^{p+1}_{\mathfrak{a}}(M)$ *is induced by the homomorphisms*

$$\begin{array}{rcl} f^p : M_{a_{i_1}\cdot\ldots\cdot a_{i_p}} & \to & M_{a_{j_1}\cdot\ldots\cdot a_{j_{p+1}}} \\ x & \mapsto & f^p(x) \end{array}$$

with

$$f^p(x) = \begin{cases} \frac{(-1)^k x}{1}, & \text{if } \{i_1, ..., i_p\} = \{j_1, ..., \hat{j_k}, ..., j_{p+1}\}, \\ 0, & \text{else.} \end{cases}$$

It can easily be verified that $C^{\bullet}_{\mathfrak{a}}(M)$ is indeed a chain complex. It can also be proved that if another sequence $\mathfrak{b}$ generates the same radical ideal that of I then $H^i(C^{\bullet}_{\mathfrak{a}}(M)) \simeq H^i(C^{\bullet}_{\mathfrak{b}}(M))$ for all i. It brings us to the central object of this work.

Definition 1.1.2 (Local cohomology modules)**.** *Let M be an R-module. Given a finitely generated ideal I of R and an integer $i \geq 0$, we define the i-*th local cohomology module of M with support in I *as the cohomology module $H^i(C^{\bullet}_{\mathfrak{a}}(M))$ where $\mathfrak{a}$ is a finite sequence of elements generating I and we will denote it by $H^i_I(M)$.*

The interested reader may see [17] and [22] (in the non-Noetherian case) for an in-depth study of such modules.

Notice that the isomorphism

$$H^0_I(M) \simeq \bigcup_{n\geq 0} 0 :_M I^m \simeq \varinjlim_n \operatorname{Hom}_R(R/I^n, M)$$

is functorial in M so that the i-th right derived functors of H^0_I coincide with the functors $\varinjlim_n \operatorname{Ext}^i_R(R/I^n, \_)$. It is isomorphic to H^i_I whenever R is Noetherian or I is generated by a regular sequence. Furthermore, Botbol and Chardin have used the Mayer-Vietoris sequence in [14, Theorem 2.3] to prove that H^i_I is also the i-th right derived functor of H^0_I in the case of R being a polynomial ring over an arbitrary ring and I being a finitely generated monomial ideal.

The next example is of great importance in the next chapters. It follows the same lines as [17, Example 13.5.3]. Although [17] requires Noetherianity, this hypothesis is not needed in this construction.

Example 1.1.3. *Let G be an abelian group, S be a G-graded commutative unitary ring and $R = S[X_1, ..., X_n]$, the ring of polynomials over S, graded by $G \oplus \mathbb{Z}^n$. (The degree of the i-th variable X_i is the i-th element e_i of the canonical basis of $\mathbb{Z}^n$.) By writing d^{n-1} for the differential $C^{n-1}_{(X_1,...,X_n)}(R) \to C^n_{(X_1,...,X_n)}(R)$, one has*

$$H^n_{(X_1,...,X_n)}(R) \simeq R_{X_1 \cdot ... \cdot X_n} / \operatorname{coker} d^{n-1}.$$

Both $R_{X_1 \cdot ... \cdot X_n}$ and $\operatorname{coker} d^{n-1}$ are free S-modules in a such way that $H^n_{(X_1 \cdot ... \cdot X_n)}(R)$ is also a free S-module with basis $\{X_1^{i_1} \cdot ... \cdot X_n^{i_n} : i_1, ..., i_n < 0\}$. Meanwhile it has a $G \oplus \mathbb{Z}^n$-graded R-module structure such that

$$X_t(X_1^{i_1} \cdot ... \cdot X_n^{i_n}) = \begin{cases} X_1^{i_1} \cdot ... \cdot X_{t-1}^{i_{t-1}} X_t^{i_t+1} X_{t+1}^{i_{t+1}} \cdot ... \cdot X_n^{i_n}, \ \textit{if } i_t < -1, \\ 0, \ \textit{if } i_t = -1 \end{cases}$$

for $i_1, ..., i_n < 0$ and $1 \leq t \leq n$, and

$$\deg(s_g X_1^{i_1} \cdot ... \cdot X_n^{i_n}) = (g, (i_1, ..., i_n))$$

for $s_g \in S_g \backslash 0$. This $G \oplus \mathbb{Z}^n$-graded R-module is called the module of inverse polynomials in $X_1, ..., X_n$ over S, *and we will denote it by $S[X_1^-, ..., X_n^-]$.*

A similar argument allows us to extend this example for more variables of same degree. Namely, by writing $R = S[X_{1,1}, ..., X_{1,n_1}, ..., X_{k,1}, ..., X_{k,n_k}]$ with $\deg(X_{i,j}) = e_i \in \mathbb{Z}^k$ for all i, j, $\mathfrak{m}$ the ideal generated by the variables and $d = n_1 + ... + n_k$, then

$$H^d_{\mathfrak{m}}(R) \simeq S[X^-_{1,1}, ..., X^-_{1,n_1}, ..., X^-_{k,1}, ..., X^-_{k,n_k}]$$

and

$$\deg(s_g X_{1,1}^{i_{1,1}} \cdot ... \cdot X_{1,n_1}^{i_{1,n_1}} \cdot ... \cdot X_{k,1}^{i_{k,1}} \cdot ... \cdot X_{k,n_k}^{i_{k,n_k}}) = (g, (i_{1,1} + ... + i_{1,n_1}, ..., i_{k,1} + ... + i_{k,n_k})).$$

We finish this subsection by recording a useful lemma by Botbol.

Lemma 1.1.4. ([13, Lemma 6.4.7]) *Let S be a ring and consider the standard $\mathbb{Z}^k$-graded polynomial ring $R = S[X_{1,1}, ..., X_{1,n_1}, ..., X_{k,1}, ..., X_{k,n_k}]$. Set $B_i = (X_{i,1}, ..., X_{i,n_i})$ and $B = B_1 \cdot ... \cdot B_k$. Then, for any $l \geq 0$,*

$$H^l_B(R) \simeq \bigoplus_{\substack{1 \leq i_1 < ... < i_p \leq k \\ n_{i_1} + ... + n_{i_p} - (p-1) = l}} H^{n_{i_1} + ... + n_{i_p}}_{B_{i_1} + ... + B_{i_p}}(R).$$

1.1.2 Local cohomology and sheaf cohomology

Now we introduce some concepts on sheaf cohomology in a general setting and then explain a fundamental relation between local cohomology and sheaf cohomology. See [45] for more information on this topic and [40] and [46] for general and basic algebraic geometry notions.

Definition 1.1.5. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ be an $\mathcal{O}_X$-module, $Z \subseteq X$ be a closed subset and write $U = X \setminus Z$. We define the* sections of $\mathcal{F}$ supported in Z *as*

$$H^0_Z(X, \mathcal{F}) := \ker(H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})).$$

Once $H^0_Z(X, \_)$ is a left exact functor we shall write $H^i_Z(X, \_)$ for its i-th right derived functor and it will be called the *i-th local cohomology of $\mathcal{F}$ with support in Z*. We are going to see that on an affine scheme X, for a quasi-coherent sheaf $\widetilde{M}$ and a complement of a closed variety Z, the groups $H^i_Z(X, \widetilde{M})$ coincide with the i-th local cohomology modules $H^i_I(M)$. It is therefore another way to define local cohomology modules.

Consider $(X, \mathcal{O}_X)$ as a ringed space, $\mathcal{F}$ an $\mathcal{O}_X$-module, $Z \subseteq X$ a closed subset and write $U = X \setminus Z$. Since the morphism $H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$ is surjective whenever $\mathcal{F}$ is a flasque sheaf one has long exact sequence

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}) \to \cdots$$

See [46] for details. Now we make use of the well-known fact that quasi-coherent sheaves on affine schemes have no higher cohomology, see for example [44, Théorème 1.3.1]. In other words, when $X = \operatorname{Spec}(R)$ is an affine scheme and $\mathcal{F} = \widetilde{M}$ is a quasi-coherent sheaf on X, we must have $H^i(X, \widetilde{M}) = 0$ for all $i > 0$ so that the sequence

$$0 \to H^0_Z(X, \widetilde{M}) \to H^0(X, \widetilde{M}) \to H^0(U, \widetilde{M}) \to H^1_Z(X, \widetilde{M}) \to 0$$

is exact and, for all $i > 0$,

$$H^{i+1}_Z(X, \widetilde{M}) \simeq H^i(U, \widetilde{M}),$$

where $Z = V(I)$ and $U = X \setminus Z$.

We also have a *geometric* notion of the Čech complex. Namely, let X be a topological space and consider U as an open subset of X, $\mathcal{U}$ an open cover of U and $\mathcal{F}$ a sheaf on X. Thus the complex $C^\bullet(\mathcal{U},\mathcal{F})$, where

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_p})$$

and which differentials being induced by the restriction morphisms, is called the *Čech complex of the open covering* $\mathcal{U}$ *with coefficients in the sheaf* $\mathcal{F}$. Its cohomology groups are denoted by $\check{H}^i(\mathcal{U},\mathcal{F})$. Coming back to the case where $X = \operatorname{Spec}(R), \mathcal{F} = \widetilde{M}, Z = \mathrm{V}(I)$ and $U_I = X \setminus Z$, with I being generated by a sequence $\mathfrak{a} = a_1, \ldots, a_n$ of elements in R, the basic open subsets defined by the $a_i's$ clearly cover the open subset U_I; denote such open covering by $\mathcal{U}_\mathfrak{a}$. By [45, Theorem D] one has

$$\check{H}^i(\mathcal{U}_\mathfrak{a},\widetilde{M}) \simeq H^i(U_I,\widetilde{M})$$

for all i, that is, the Čech cohomology of quasi-coherent sheaves computes the sheaf cohomology of the open subset corresponding to the open cover. Moreover, it may be showed that $C^{p+1}_\mathfrak{a}(M) = C^p(\mathcal{U}_\mathfrak{a},\widetilde{M})$ for all p and that there exists exact sequence

$$0 \to C^\bullet(\mathcal{U}_\mathfrak{a},\widetilde{M})[-1] \to C^\bullet_\mathfrak{a}(M) \to M \to 0$$

where M denotes the complex centered in the R-module M at degree 0. Therefore there exists exact sequence

$$0 \to H^0_I(M) \to M \to H^0(U_I,\widetilde{M}) \to H^1_I(M) \to 0$$

and, for all $i > 0$,

$$H^{i+1}_I(M) \simeq H^i(U_I,\widetilde{M}).$$

All that discussion allows us to conclude that, for a finitely generated ideal I in R and an R-module M,

$$H^i_I(M) \simeq H^i_Z(X,\widetilde{M})$$

for all $i \geq 0$, where $X = \operatorname{Spec}(R)$ and $Z = \mathrm{V}(I)$.

We also need the "sheafified" version of the Čech complex, see [19, Section 4.3] or [46, Chapter III]. Let X be a topological space and consider $\mathcal{U}$ an open cover of X.

Let $j_{i_0 \dots i_p} : U_{i_0} \cap \dots \cap U_{i_p} \to X$ be the canonical inclusion of a non-void intersection $U_{i_0} \cap \dots \cap U_{i_p}$ of elements of $\mathcal{U}$. Given an sheaf $\mathcal{F}$ on X, for every $p \geq 0$ define the sheaf

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} (j_{i_0 \dots i_p})_* \mathcal{F}|_{U_{i_0} \cap \dots \cap U_{i_p}}$$

where $(j_{i_0 \dots i_p})_*$ is the direct image functor. More explicitly, given an open subset $U \subseteq X$,

$$\check{C}^p(\mathcal{U}, \mathcal{F})(U) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U \cap U_{i_0} \cap \dots \cap U_{i_p}).$$

The usual Čech differentials induces sheaf morphisms $\check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$. The next proposition follows from definition and [46, Chapter III, Proposition 4.3].

Proposition 1.1.6. *The following statements hold true.*

(i) There are isomorphisms

$$\Gamma(X, \check{C}^p(\mathcal{U}, \mathcal{F})) \simeq C^p(\mathcal{U}, \mathcal{F});$$

(ii) If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

1.2 Castelnuovo-Mumford regularity

The *Castelnuovo-Mumford regularity* is an invariant of fundamental importance in both commutative algebra and algebraic geometry. It measures the complexity of a module or a sheaf, so to speak, for the regularity of a module bounds the largest degree of its minimal generators, and the regularity of a sheaf estimates the smallest twist for which it is generated by the global sections. The two textbooks [17] and [30], and [21] are excellent references for this topic. Besides that, Chardin, Jouanolu and Rahimi [22] work on the regularity without assuming Noetherianity.

We start this section by recalling Castelnuovo-Mumford regularity in its classical case to motivate its multigraded notion, which was first studied in [69] and [53], and then generalized in [14].

1.2.1 The classical case

First we will see a motivation to define regularity. Suppose $R = \mathbf{k}[X_1, \dots, X_n]$ with $\mathbf{k}$ being a field and write $\mathfrak{m} = (X_1, \dots, X_n)$. Let M be a finitely generated R-module.

Grothendieck's vanishing theorem asserts that $H^i_{\mathfrak{m}}(M) = 0$ for all $i > \dim(M)$ or $i < \operatorname{depth}_R(M)$, as well as the non-vanishing of these modules for $i = \dim(M)$ and $i = \operatorname{depth}_R(M)$, see [17] or [58]. Also, Serre's vanishing theorem implies the vanishing of the sheaf cohomology groups $H^i(\operatorname{Proj}(R), \widetilde{M}(\mu))$ for all $i > 0$ and μ big enough, see [46]. A similar argument as that of Subsection 1.1.2 shows that there exists a graded isomorphism

$$H^i_{\mathfrak{m}}(M) \simeq \bigoplus_{\mu} H^{i+1}(\operatorname{Proj}(R), \widetilde{M}(\mu))$$

for all $i > 0$ so that Serre's vanishing theorem can be stated in terms of the graded pieces of local cohomology modules. The Castelnuovo-Mumford regularity is a measure of this vanishing degree.

If $H^i_{\mathfrak{m}}(M) \neq 0$, we set

$$a_i(M) := \sup\{\mu : H^i_{\mathfrak{m}}(M)_\mu \neq 0\},$$

and if $H^i_{\mathfrak{m}}(M) = 0$, set $a_i(M) := -\infty$. The *Castelnuovo-Mumford regularity* is defined as

$$\operatorname{reg}(M) := \sup_i \{a_i(M) + i\}.$$

On the other hand, Eisenbud and Goto in [31] have proved that the Castelnuovo-Mumford regularity can also be obtained from minimal free resolutions. In other words, this invariant can be defined in terms of graded Betti numbers. Let $F_\bullet$ be a minimal free resolution of M with $F_i = \oplus_j R(-j)^{\beta_{ij}}$. Thus $\beta_{ij} = \dim_{\mathbf{k}}(\operatorname{Tor}_i^R(M, \mathbf{k})_j)$ for all i, j. If $\operatorname{Tor}_i^R(M, \mathbf{k}) \neq 0$, set

$$b_i(M) := \sup\{\mu : \operatorname{Tor}_i^R(M, \mathbf{k})_\mu \neq 0\},$$

else, $b_i(M) := -\infty$. Therefore $b_i(M)$ is the maximal degree of a minimal generator of F_i, and so of the module of the i-th syzygies of M. As an immediate consequence of [21, Corollary 1.2.2] one has

$$\operatorname{reg}(M) = \max_i \{b_i(M) - i\}.$$

The next lemma shall be useful in this work.

Lemma 1.2.1. ([22, Lemma 2.1]) *Let S be a ring and consider the standard $\mathbb{Z}$-graded polynomial ring $R = S[X_1, \ldots, X_n]$. Let also M be a graded R-module. Consider the following properties:*

(i) $M_\nu = 0$ *for all* $\nu \gg 0$*;*

(ii) $M = H^0_{(X_1,...,X_n)}(M)$*;*

(iii) $H^i_{(X_1,...,X_n)}(M) = 0$ *for all* $i > 0$.

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii), (ii)* $\Rightarrow$ *(i) if* M *is finitely generated or* $\operatorname{reg}(M) < \infty$*, and (iii)* $\Rightarrow$ *(ii) if* $M_\nu = 0$ *for* $\nu \ll 0$.

1.2.2 Multigraded regularity

One of the motivations for a multigraded version of the Castelnuovo-Mumford regularity comes from toric geometry. Cox in [25] defines the coordinate ring of a simplicial toric variety X as being a polynomial ring graded by the divisor class group G of X. The dictionary linking the geometry of X with the theory of G-graded modules leads to geometric interpretations and applications for multigraded regularity. Basic notions of multigraded commutative algebra can be found in [15, 17, 41, 42].

The *multigraded Castelnuovo-Mumford regularity* concept has first appeared in the work of Hoffman and Wang [53], where they work on a bigraded setting motivated by the geometry of $\mathbb{P}^1 \times \mathbb{P}^1$. Then, also motivated by toric geometry, Maclagan and Smith in [69] work on a more general setting of multigraded regularity. Later, Botbol and Chardin in [14] introduce a further generalization of regularity, by working over any commutative ring and by considering local cohomology modules supported in any finitely generated graded ideal. That is the way we shall work with multigraded regularity in this thesis.

Here notations and concepts follow [14] and [23].

Let S be a commutative ring, G be an abelian group and write $R := S[X_1, ..., X_n]$, with $\deg(X_i) = \gamma_i \in G$ and $\deg(s) = 0$ for $s \in S$. Let $B \subseteq (X_1, ..., X_n)$ be a finitely generated G-graded ideal of R and denote by $\mathcal{C}$ the monoid generated by $\{\gamma_1, ..., \gamma_n\}$.

Definition 1.2.2. *The* support *of a* G*-graded* R*-module* M *is*

$$\operatorname{Supp}_G(M) := \{\gamma \in G : M_\gamma \neq 0\}.$$

Given an R-module M, for $\mu \in G$ we set the R-module $M(-\mu)$ with grading defined by $M(-\mu)_\nu := M_{\mu-\nu}$.

Lemma 1.2.3. *Let* M *be a graded* R*-module, then* $\operatorname{Supp}_G(M(-\mu)) = \operatorname{Supp}_G(M) + \mu$.

Proof. Indeed $\nu \in \mathrm{Supp}_G(M(-\mu))$ if and only if $\nu - \mu \in \mathrm{Supp}_G(M)$, equivalently $\nu \in \mathrm{Supp}_G(M) + \mu$. ■

Notation 1.2.4. *Let M be a graded R-module. For a graded ideal I we set*

$$C_I^i(M) := \mathrm{Supp}_G(H_I^i(M)) \text{ and } C_I(M) := \cup_{i \geq 0} C_I^i(M).$$

Example 1.2.5. *By considering the $\mathbb{Z}^k$-graded ring*

$$R = S[X_{1,1}, ..., X_{1,n_1}, ..., X_{k,1}, ..., X_{k,n_k}]$$

as in Example 1.1.3 one has

$$C_{\mathfrak{m}}(R) = C_{\mathfrak{m}}^d(R) = \mathrm{Supp}_{\mathbb{Z}^k}(H_{\mathfrak{m}}^d(R)) = \prod_{i=1}^{k} \mathbb{Z}_{\leq -n_i}.$$

Furthermore, writing $B_j = (X_{j,1}, ..., X_{j,n_j})$ and $d_{i_1 \dots i_p} = n_{i_1} + ... + n_{i_p}$ one has

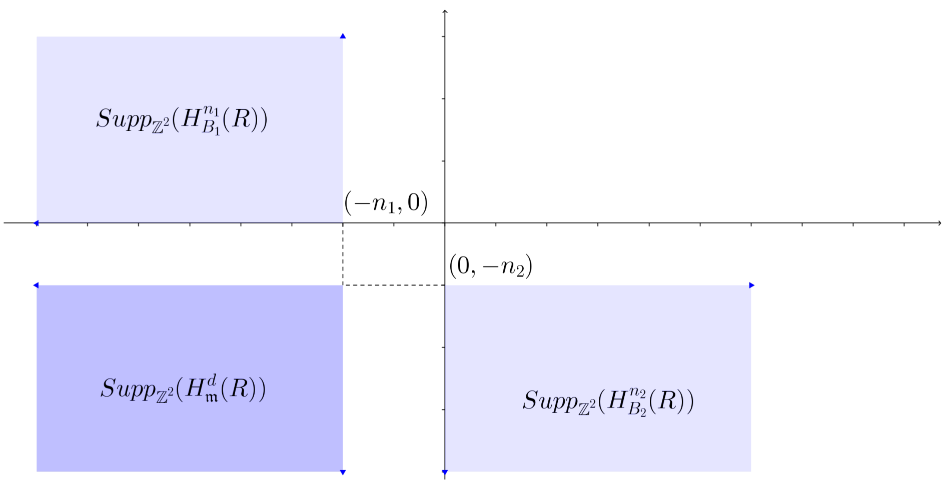
$$H_{B_{i_1}+...+B_{i_p}}^{d_{i_1 \dots i_p}}(R) = \\ S[X_{i_j,l}^{-} : \ j = 1, ..., p \text{ and } l = 1, ..., n_{i_j}][X_{j,l_j} : \ j \neq i_1, ..., i_p \text{ and } l_j = 1, ..., n_j].$$

Thus

$$C_{B_{i_1}+...+B_{i_p}}(R) = \mathrm{Supp}_{\mathbb{Z}^k}(H_{B_{i_1}+...+B_{i_p}}^{d_{i_1 \dots i_p}}(F)) = \\ \mathbb{Z}_{\geq 0}^{i_1 - 1} \times \mathbb{Z}_{\leq -n_{i_1}} \times \mathbb{Z}_{\geq 0}^{i_2 - i_1 - 1} \times \mathbb{Z}_{\leq -n_{i_2}} \times ... \times \mathbb{Z}_{\leq -n_{i_p}} \times \mathbb{Z}_{\geq 0}^{k - i_p}.$$

The following example illustrates some supports.

Example 1.2.6. *By taking $k = 2$ in Example 1.2.5, we have the following regions $C_{\mathfrak{m}}(R) = \mathbb{Z}_{\leq -n_1} \times \mathbb{Z}_{\leq -n_2} = (-n_1, -n_2) + \mathbb{Z}_{\leq 0}, C_{B_1}(R) = \mathbb{Z}_{\leq -n_1} \times \mathbb{Z}_{\geq 0} = (-n_1, 0) + \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}$ and $C_{B_2}(R) = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq -n_2} = (0, -n_2) + \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0}$.*



The following lemma shows the close relation between some Tor modules and degree shifts in free resolutions of a module.

Lemma 1.2.7. ([14, Lemma 3.12]) *Let M be a graded module with* $\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu + \mathbb{Z}^k_{\geq 0}$ *for some $\mu \in \mathbb{Z}^k$. Then,*

(i) there exists a graded free R-resolution $F_\bullet$ of M such that

$$F_i = \bigoplus_{j \in E_i} R(-\gamma_{ij}) \quad \text{with} \quad \gamma_{ij} \in \bigcup_{0 \leq l \leq i} \mathrm{Supp}_{\mathbb{Z}^k}(\mathrm{Tor}^R_l(M, S)), \forall j;$$

(ii) if further M is finitely generated and S is Noetherian, each F_j could be chosen finitely generated;

(iii) if $S = \mathbf{k}$ is a field, M is finitely generated and $F_\bullet$ is minimal, then

$$F_j = \sum_{\mu \in \mathbb{Z}^k} R(-\mu)^{\dim_{\mathbf{k}} \mathrm{Tor}^R_j(M, \mathbf{k})_\mu}.$$

We now introduce two families of sets that play important roles in regularity.

Definition 1.2.8. *Set $\mathcal{E}_0 := \{0\}, \mathcal{E}_l := \{\gamma_{i_1} + ... + \gamma_{i_l} : i_1 < ... < i_l\}$ for all $l > 0$, $\mathcal{E}_{-1} := -\mathcal{E}_1$ and $\mathcal{E}_l = \emptyset$ for all $l < -1$.*

For all $i > 0$, $\mathcal{F}_i := \{\gamma_{j_1} + ... + \gamma_{j_i} : j_1 \leq ... \leq j_i\}$ and $\mathcal{F}_i = \mathcal{E}_i$ for all $i \leq 0$.

Observe that $\mathcal{E}_i \subset \mathcal{F}_i$ for all i and if $\gamma_i = \gamma$ for all i, then $\mathcal{E}_l = \{l\gamma\}$ when $\mathcal{E}_l \neq \emptyset$ and $\mathcal{F}_l = \{l\gamma\}$ when $\mathcal{F}_l \neq \emptyset$.

Definition 1.2.9 (Multigraded regularity)**.** *Given $\gamma \in G$ and $l \in \mathbb{Z}_{\geq 0}$, a G-graded R-module M is* weakly (B, γ)-regular at level l *if*

$$\gamma \notin \bigcup_{i \geq l} C^i_B(M) + \mathcal{F}_{i-1}.$$

M is weakly (B, γ)-regular if it is weakly (B, γ)-regular at level 0.

If further M is weakly (B, γ')-regular (respectively, weakly (B, γ')-regular at level l) for any $\gamma' \in \gamma + \mathcal{C}$, then M is said to be (B, γ)-regular *(respectively,* (B, γ)-regular at level l*). One writes* $\mathrm{reg}_B(M) := \mathrm{reg}^0_B(M)$ *with*

$$\mathrm{reg}^l_B(M) := \{\gamma \in G : M \textit{ is } (B, \gamma)\textit{-regular at level } l\}.$$

It immediately follows from the definition that $\mathrm{reg}^l_B(M)$ is the maximal set $\mathcal{S}$ of elements in G such that $\mathcal{S} + \mathcal{C} = \mathcal{S}$ — that is, $\mathcal{S}$ is $\mathcal{C}$-stable — and M is weakly (B, γ)-regular at level l for any $\gamma \in \mathcal{S}$.

Example 1.2.10. *Consider $k = 2$ and $n_1 = n_2 = 2$ in Example 1.2.5. Thus*

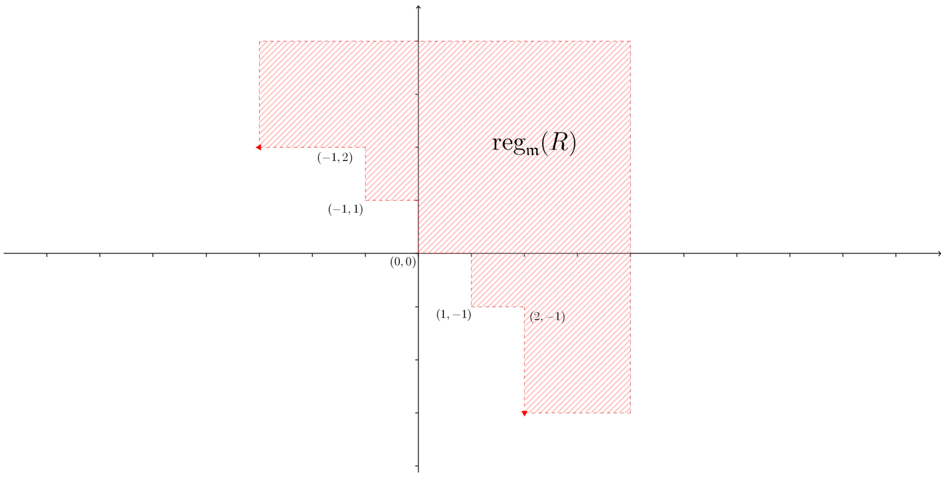
$$\bigcup_{i \geq 0} C^i_{\mathfrak{m}}(R) + \mathcal{F}_{i-1} = C_{\mathfrak{m}}(R) + \mathcal{F}_3.$$

Once $C_{\mathfrak{m}}(R) = \mathbb{Z}^2_{\leq -2}$ and $\mathcal{F}_3 = \{(3,0),(2,1),(1,2),(0,3)\}$, we must have

$$\bigcup_{i\geq 0} C^i_{\mathfrak{m}}(R) + \mathcal{F}_{i-1} = \mathbb{Z}_{\leq 1}\times\mathbb{Z}_{\leq -2}\cup\mathbb{Z}_{\leq 0}\times\mathbb{Z}_{\leq -1}\cup\mathbb{Z}_{\leq -1}\times\mathbb{Z}_{\leq 0}\cup\mathbb{Z}_{\leq -2}\times\mathbb{Z}_{\leq 1}$$

and since the complement of the set above is $\mathbb{Z}^2_{\geq 0}$-stable we conclude that

$$\operatorname{reg}_{\mathfrak{m}}(R) = \mathbb{Z}_{\leq -1}\times\mathbb{Z}_{\geq 2}\cup\mathbb{Z}_{\geq -1}\times\mathbb{Z}_{\geq 1}\cup\mathbb{Z}^2_{\geq 0}\cup\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq -1}\cup\mathbb{Z}_{\geq 2}\times\mathbb{Z}_{\leq -1}.$$



1.3 Deficiency modules

In this section, we introduce deficiency modules and the notion of canonical module of a module.

Let $(R,\mathfrak{m})$ be a Noetherian local ring which is factor of a Gorenstein local ring $(S,\mathfrak{n})$ of dimension s, that is, there exists a surjective local homomorphism $S\to R$. Denote by $\_^{\vee}$ the Matlis dual. The local duality theorem (see for example [18, Theorem 11.2.6]) assures that for all finitely generated R-module M there exists isomorphism

$$H^j_{\mathfrak{m}}(M)\simeq \operatorname{Ext}^{s-j}_S(M,S)^{\vee}$$

for all $j\geq 0$. See [17, 18, 58] for all these concepts not defined here.

Schenzel [76] generalized the notion of canonical module in the following sense.

Definition 1.3.1. *Given a finitely generated R-module M, the j-th* deficiency module of M *is defined as*

$$K^j(M) := \operatorname{Ext}^{s-j}_S(M,S)$$

for all $j=0,...,\dim_R M$. Particularly, $K(M) := K^{\dim_R M}(M)$ is called the canonical module of M*.*

Local duality assures that these modules are well-defined (i.e., they do not depend on the Gorenstein ring S). In a certain sense, since

$$H^j_{\mathfrak{m}}(M) \simeq K^j(M)^\vee$$

for all $j = 0, ..., \dim_R M$, the deficiency modules of M measure the extent of the failure of M to be Cohen-Macaulay.

We say that a finitely generated R-module M satisfies *Serre's condition S_k*, for k being a non-negative integer, provided

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geq \min\{k, \dim_{R_\mathfrak{p}} M_\mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Supp} M$.

Lemma 1.3.2. ([76, Lemma 1.9]) *Let M be a finitely generated R-module of dimension t. The modules $K^j(M)$ satisfy the following properties.*

(i) $\dim_R K^j(M) \leq j$ *for all integer j and* $\dim_R K(M) = t$*;*

(ii) Suppose that M is equidimensional. Then M satisfies Serre's condition S_k if and only if $\dim_R K^j(M) \leq j - k$ *for all* $0 \leq j < t$.

1.4 Generalized local cohomology

In this section R denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. Also, denote by $\_^\vee$ the Matlis dual and by $\widehat{\_}$ the completion with respect to $\mathfrak{m}$. Further, ω_R denotes the canonical module of R whenever it exists. See [17, 18, 58] for all those basic notions.

The theory of generalized local cohomology, initiated by Herzog [49] in his *habilitationss*, and further developed by Suzuki [80], Bijan-Zadeh [12], Yassemi [84] and Herzog and Zamani [52]. It also has attracted the attention of many other authors, see for example [2, 29, 48, 67].

Definition 1.4.1. ([49]) Let R be a ring and M, N be finitely generated R-modules. Given an ideal I of R and an integer $i \geq 0$, the *ith generalized local cohomology module of M and N with respect to I* is defined as

$$H^i_I(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/I^n M, N).$$

Notice that by taking $M = R$ we retrieve the ordinary local cohomology module $H_I^i(N)$ of N. A systematic approach of this notion has been done first by Suzuki [80], whose next lemmas and spectral sequences in Appendix B.2.5 are fundamental in some of the main results in Chapter 5.

Lemma 1.4.2. ([80, Theorem 2.3]) *Let $(R, \mathfrak{m})$ be a local ring and M, N be finitely generated R-modules. Set $t = \operatorname{depth}_R N$. Then, $H_{\mathfrak{m}}^t(M, N) \neq 0$ and $H_{\mathfrak{m}}^j(M, N) = 0$ for all $j < t$.*

Suzuki also provides a local duality theorem for generalized local cohomology modules.

Lemma 1.4.3. ([80, Theorem 3.5]) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, and let M, N be a pair of finitely generated R-modules. If $\operatorname{pd}_R M < \infty$ then, for each $j \geq 0$, there is an isomorphism*

$$H_{\mathfrak{m}}^j(M, N)^\vee \cong \operatorname{Ext}_{\widehat{R}}^{d-j}(\widehat{N}, \widehat{M} \otimes_{\widehat{R}} \omega_{\widehat{R}}).$$

Now we recall the local duality version for finite injective dimension.

Lemma 1.4.4. ([52, Theorem 2.1(b)]) *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R, and let M, N be finitely generated R-modules. If $\operatorname{id}_R N < \infty$ then, for each $j \geq 0$, there is an isomorphism*

$$H_{\mathfrak{m}}^j(M, N)^\vee \cong \operatorname{Ext}_{\widehat{R}}^{d-j}(\operatorname{Hom}_{\widehat{R}}(\omega_{\widehat{R}}, \widehat{N}), \widehat{M}).$$

1.5 Homological dimensions

In this section, we assume all rings are Noetherian.

1.5.1 Complete intersection dimension

Needless to say, the projective dimension $\operatorname{pd}_R M$ of a finitely generated module M over a local ring $(R, \mathfrak{m})$ is a fundamental classical invariant. One of its refinements is the so-called *complete intersection dimension* of M. The theory about this homological dimension was introduced in [9] by Avramov, Gasharov and Peeva, and it features many interesting properties as well.

Definition 1.5.1. [9] *First, a* quasi-deformation *(of codimension c) of a local ring R is a diagram of local homomorphisms $R \to R' \leftarrow S$, the first being flat and the second* *surjective with kernel generated by an S-regular sequence (of length c). Now, if M is a finitely generated R-module, then the* complete intersection dimension of M over R *is defined as*

$$\text{CI-dim}_R M := \inf\{\operatorname{pd}_S M \otimes_R R' - \operatorname{pd}_S R' : R \to R' \leftarrow S \textit{ is a quasi-deformation}\}.$$

Recall that a Noetherian local ring $(R, \mathfrak{m})$ is a complete intersection if its completion in the $\mathfrak{m}$-adic topology is isomorphic to the quotient of a regular local ring by an ideal generated by a regular sequence. We note some interesting properties of $\text{CI-dim}_R M$. For instance, $\text{CI-dim}_R M < \infty$ for every finitely generated R-module M if (and only if) R is a complete intersection ring. Another remarkable property is that $\text{CI-dim}_R M \leq \operatorname{pd}_R M$ for every finitely generated R-module M, with equality whenever $\operatorname{pd}_R M < \infty$. More precisely, if $\text{CI-dim}_R M$ is finite then it satisfies the Auslander-Buchsbaum type formula $\text{CI-dim}_R M = \operatorname{depth} R - \operatorname{depth} M$. Therefore, the class consisting of the modules of finite projective dimension is (strictly) contained in the class of modules having finite complete intersection dimension. See details in [9].

The next lemma turns out to be useful in our approach in Chapter 5.

Lemma 1.5.2. ([8, Theorem 4.2]) *Let R be a local ring and M be a finitely generated R-module such that* $\text{CI-dim}_R M < \infty$ (*e.g., R is a complete intersection ring*). *Then,* $\operatorname{pd}_R M < \infty$ *if and only if* $\operatorname{Ext}^e_R(M, M) = 0$ *for some even integer $e \geq 2$.*

1.5.2 Gorenstein dimension

We invoke yet another homological dimension. Let R be a ring. An R-module M is said to be *totally reflexive* if M is reflexive and $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for all $i > 0$, with $M^* = \operatorname{Hom}_R(M, R)$ being the algebraic dual of M.

Definition 1.5.3. [6] *A non-zero R-module M is said to have* Gorenstein dimension of M at most t *if there exists an exact sequence*

$$0 \to X_t \to X_{t-1} \to \cdots \to X_0 \to M \to 0$$

where X_j is a finitely generated totally reflexive R-module for each $j = 0, ..., t$. We denote this by $\text{G-dim}_R M \leq t$. *In this case, we in particular write* $\text{G-dim}_R M < \infty$; *otherwise,* $\text{G-dim}_R M = \infty$. *If* $\text{G-dim}_R M < \infty$ *we define*

$$\text{G-dim}_R M := \inf\{t : \text{G-dim}_R M \leq t\}.$$

Clearly, $\text{G-dim}_R M = 0$ *if and only if M is totally reflexive.*

We briefly note some properties of G-$\dim_M$ and report a lemma. If R is Gorenstein then G-$\dim_R N < \infty$ for any R-module N. As is also well-known, the basic relation between Gorenstein, complete intersection and projective dimensions can be expressed as

$$\text{G-}\dim_R N \leq \text{CI-}\dim_R N \leq \operatorname{pd}_R N$$

, which are all equalities whenever $\operatorname{pd}_R N < \infty$. See [9] for details.

Lemma 1.5.4. [55, Theorem 2.2] *If M is an R-module with $\operatorname{id}_R M < \infty$, then* G-$\dim_R M = \operatorname{pd}_R M$.

1.5.3 Gorenstein injective dimension

Definition 1.5.5. [33] *A* complete injective resolution *is an exact sequence of injective R-modules*

$$\mathbf{I}: \ \cdots \to I_1 \to I^0 \to I^1 \to I^2 \to \cdots$$

such that $\operatorname{Hom}_R(E, \mathbf{I})$ *is exact for every injective R-module E. An R-module M is said to be* Gorenstein injective *if there exists a complete injective resolution* $\mathbf{I}$ *with $M = \ker(I^0 \to I^1)$. Now, given a non-negative integer r, we say that a non-zero R-module M is of* Gorenstein injective dimension at most r, *which is denoted by* $\operatorname{Gid}_R M \leq r$, *if there exists an exact sequence*

$$0 \to M \to G^0 \to G^1 \to \cdots \to G^r \to 0$$

such that each G^j is Gorenstein injective. We then define the Gorenstein injective dimension of M *as*

$$\operatorname{Gid}_R M := \inf\{r : \operatorname{Gid}_R M \leq r\}.$$

For the trivial module $M = 0$, we set $\operatorname{Gid}_R 0 = \infty$.

The Gorenstein injective dimension of an R-module M generalizes the usual injective dimension in the sense that $\operatorname{Gid}_R M \leq \operatorname{id}_R M$, with equality if $\operatorname{id}_R M < \infty$, according to [24, Proposition 3.10]. Another useful property is that, if R is Gorenstein, then $\operatorname{Gid}_R M < \infty$ for every R-module M; this follows from [34, Theorem 3.2]. The next lemma is a version of the well-known Ischebeck's formula (see, e.g., [18, Exercise 3.1.24]) in the context of the Gorenstein injective dimension.

Lemma 1.5.6. ([75, Theorem 2.10]) *Let M be a finitely generated R-module with $\operatorname{id}_R M < \infty$ and let N be a finitely generated R-module with* $\operatorname{Gid}_R N < \infty$. *Then,* $e_R(M, N) = \operatorname{depth} R - \operatorname{depth}_R M$.

Chapter 2

Mayer-Vietoris spectral sequence

In this chapter, we provide a generalization for the Mayer-Vietoris long exact sequence, see [17] or [58]. We construct a spectral sequence, which will be called *Mayer-Vietoris spectral sequence*, that depends on the number of ideals one is working with in such a way that it degenerates to the well-known Mayer-Vietoris long exact sequence when one takes just two ideals. Such a spectral sequence is a distinct generalization of the Mayer-Vietoris long exact sequence, but somehow similar to [1] and [66], which has been quite useful in this work. Besides the possibility of working with more than two ideals, our construction does not require Noetherianness, and from these generalizations, a quite amount of results will follow. We should notice that the non-Noetherianness in the Mayer-Vietoris long exact sequence is a well-known result, which may be seen, built through different paths, for instance, in [78] and [81].

We provide two ways to construct the Mayer-Vietoris spectral sequence. One of them is an application of the Čech spectral sequence which is studied in [19, 40, 43] and the other one is a direct construction of filtrations of complexes that builds such a spectral sequence up. Each one has its own advantage. Although the second construction is direct, the Čech spectral sequence alternative gives immediately the second page of the spectral sequence (in geometrical terms). We also calculate the second page in terms of sheaf cohomology groups and inverse limits.

The textbooks [70] and [74] are nice introductory references for the reader not acquainted with spectral sequences. Besides we devote Appendix B for spectral sequences and Section B.2.1 especially treats the Čech spectral sequence. As already mentioned, [19, 40, 43] discuss extensively the Čech spectral sequence.

The construction of the Mayer-Vietoris spectral sequence by filtrations may be seen as an application of the studying of relations between Čech complexes of finite sequences 1.1.1 and the Čech complex defined by the product of these sequences. To make it easier to read, we shall fix and modify some notations.

Let R be a ring and M be an R-module. Consider a sequence of elements $\mathfrak{a} = a_1, ..., a_n$ of R and let I be the ideal of R generated by $\mathfrak{a}$. One may consider the Čech complex $C^\bullet(\mathcal{U}_\mathfrak{a}, \widetilde{M})$ where $\mathcal{U}_\mathfrak{a}$ is the open covering of $U_\mathfrak{a} = \operatorname{Spec}(R) \setminus \operatorname{V}(\mathfrak{a})$ given by the basic open subsets defined by the $a_i's$, see Section 1.1.2. Write

$$\mathscr{C}^\bullet_\mathfrak{a}(M) := C^\bullet(\mathcal{U}_\mathfrak{a}, \widetilde{M}) \simeq \Gamma(X, \check{C}^\bullet(\mathcal{U}_\mathfrak{a}, \widetilde{M})).$$

With this notation, what we have from Section 1.1.2 is an exact sequence

$$0 \to H^0_I(M) \to M \to H^0(\mathscr{C}^\bullet_\mathfrak{a}(M)) \to H^1_I(M) \to 0$$

and isomorphisms

$$H^{i+1}_I(M) \simeq H^i(\mathscr{C}^\bullet_\mathfrak{a}(M))$$

for all $i > 0$.

We should also notice that, as already seen in Section 1.1.1, in the case of R being Noetherian, $H^i_I(\_)$ coincides with the i-th right derived functor of the I-torsion functor $\varinjlim_n \operatorname{Hom}_R(R/I^n, \_)$. But even more, $H^i(\mathscr{C}^\bullet_\mathfrak{a}(\_))$ coincides with the i-th right derived functor of the ideal transform functor $D_I(\_)$. [17] shows many properties of ideal transform functors as well as their geometrical significance.

2.1 Construction

Given an R-module M and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ sequences of elements of R, it is well-known that the tensor product of Čech complexes $C^\bullet_{\mathfrak{a}_1}(R) \otimes_R \ldots \otimes_R C^\bullet_{\mathfrak{a}_n}(R) \otimes_R M$ is quasi-isomorphic (i.e., they have isomorphic cohomology modules) to the Čech complex $C^\bullet_{\mathfrak{a}_1 \cup \ldots \cup \mathfrak{a}_n}(M)$. We wonder if there is a similar relation considering the complexes $\mathscr{C}^\bullet_\mathfrak{a}(M)$ instead. That is what we investigate in the first subsection.

2.1.1 A quasi-isomorphism

The following preliminary results were obtained in this thesis.

Lemma 2.1.1. *If* $\mathfrak{a} = \{a_1, \ldots, a_r\}$ *and* $\mathfrak{b} = \{b_1, \ldots, b_s\}$ *are two finite sequences of elements in* R*,* $\mathfrak{ab} = \{a_i b_j : i = 1, \ldots, r \text{ and } j = 1, \ldots, s\}$ *is their product and* M *is an* R*-module then there exists isomorphism*

$$H^0(\mathscr{C}^\bullet_{\mathfrak{ab}}(M)) \simeq H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M)$$

and this isomorphism is functorial in M*.*

In case of R *is Noetherian and of both* I *and* J *are the ideals generated by* $\mathfrak{a}$ *and* $\mathfrak{b}$ *respectively, we have a functorial isomorphism*

$$D_{IJ}(\_) \simeq D_I(D_J(\_)).$$

Proof. Note that $\left(\frac{x_{ij}}{(a_i b_j)^{s_{ij}}}\right)_{i,j} \in H^0(\mathscr{C}^\bullet_{\mathfrak{ab}}(M))$ if and only if there is $r = r(i,j,k,l)$ such that

$$x_{kl}(a_i b_j)^{s_{kl}}(a_i b_j a_k b_l)^{s_{ij}+r} = x_{ij}(a_k b_l)^{s_{ij}}(a_i b_j a_k b_l)^{s_{kl}+r}.$$

After some identifications we can give $H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M)$ a similar characterization. Indeed, $\left(\frac{x_{ij}}{(a_i b_j)^{s_{ij}}}\right)_{i,j} \in H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M)$ if and only if there are $u = u(i,j,l), v = v(i,k,l), w = w(i,j,k)$ and $z = z(j,k,l)$ such that

$$x_{ij} b_l^{s_{ij}} (a_i b_j b_l)^{s_{il}+u} = x_{il} b_j^{s_{il}} (a_i b_j b_l)^{s_{ij}+u},$$

$$x_{kl} a_i^{s_{kl}} (a_i a_k b_l)^{s_{il}+v} = x_{il} a_k^{s_{il}} (a_i a_k b_l)^{s_{kl}+v},$$

$$x_{kj} a_i^{s_{kj}} (a_i a_k b_j)^{s_{ij}+w} = x_{ij} a_k^{s_{ij}} (a_i a_k b_j)^{s_{ij}+w}$$

and

$$x_{kj} b_l^{s_{kj}} (a_k b_j b_l)^{s_{kl}+z} = x_{kl} b_j^{s_{kl}} (a_k b_j b_l)^{s_{kj}+z}.$$

Fix (i,j) and (k,l) with $(i,j) \leq (k,l)$. The image of $\left(\frac{x_{ij}}{(a_i b_j)^{s_{ij}}}\right)_{i,j}$ through the differential composed with the projection onto $M_{a_i a_k b_j}$ is

$$\frac{x_{kj} a_i^{s_{kj}}}{(a_i a_k b_j)^{s_{kj}}} - \frac{x_{ij} a_k^{s_{ij}}}{(a_i a_k b_j)^{s_{ij}}}.$$

We may see the three other equations by looking at the image of $\left(\frac{x_{ij}}{(a_i b_j)^{s_{ij}}}\right)_{i,j}$ through the differential composed with the projections onto $M_{a_i a_k b_l}, M_{a_i b_j b_l}$ and $M_{a_k b_j b_l}$.

By taking $i = k$ in the first equation of the proof we have

$$x_{il} b_j^{s_{il}} (a_i b_j b_l)^{s_{ij}+r} a_i^{s_{il}+s_{ij}+r} = x_{ij} b_l^{s_{ij}} (a_i b_j b_l)^{s_{il}+r} a_i^{s_{ij}+s_{ij}+r}.$$

Since a_i is invertible in $M_{a_ib_jb_l}$ we have the first of the last four equations above. Similarly one can find the three others and this proves that

$$H^0(\mathscr{C}^\bullet_{\mathfrak{ab}}(M)) \subseteq H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M).$$

Now if the four equations that characterize $H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M)$ hold true then by multiplying the first one of them by $a_k^{s_{ij}+s_{il}+u}(a_ib_ja_kb_l)^{s_{kl}+v}$ we have

$$x_{ij}(a_kb_j)^{s_{ij}}(a_ib_ja_kb_l)^{s_{kl}+(s_{il}+u+v)} = x_{ij}a_k^{s_{il}}(a_ia_kb_l)^{s_{kl}+v}(a_ib_ja_kb_l)^{s_{ij}+u}b_j^{s_{il}+s_{kl}+v}.$$

From the second equation we conclude that

$$x_{ij}(a_kb_j)^{s_{ij}}(a_ib_ja_kb_l)^{s_{kl}+(s_{il}+u+v)} = x_{kl}(a_ib_j)^{s_{kl}}(a_ib_ja_kb_l)^{s_{ij}+(s_{il}+u+v)}.$$

This proves the other inclusion. The functoriality stems from the functoriality of the identifications involved.

Finally, the tensor product of the complexes $\mathscr{C}^\bullet_{\mathfrak{a}}(R)$ and $\mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M$ yields a spectral sequence converging to $H^\bullet(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M)$ and in particular

$$H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{b}}(R) \otimes_R M) \simeq H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(H^0(\mathscr{C}^\bullet_{\mathfrak{b}}(M)))).$$

The lemma now follows from [17, Proposition 5.1.23]. ■

We want to extend Lemma 2.1.1 in the sense that the complexes involved are quasi-isomorphic. For that, we need to use some geometric tools.

Consider finite sequences $\mathfrak{a}$ and $\mathfrak{b}$ of elements of R and write $X = \operatorname{Spec}(R)$. If $j_{\mathfrak{a}} : U_{\mathfrak{a}} \to X$ and $j_{\mathfrak{b}} : U_{\mathfrak{b}} \to X$ are the canonical inclusions, then we denote by $D^\bullet(\mathcal{F}) = \check{C}^\bullet(\mathcal{U}_{\mathfrak{a}}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \check{C}^\bullet(\mathcal{U}_{\mathfrak{b}}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}$ the total complex

$$(j_{\mathfrak{a}})_*\check{C}^\bullet(\mathcal{U}_{\mathfrak{a}}, \mathcal{O}_X|_{U_{\mathfrak{a}}}) \otimes_{\mathcal{O}_X} (j_{\mathfrak{b}})_*\check{C}^\bullet(\mathcal{U}_{\mathfrak{b}}, \mathcal{O}_X|_{U_{\mathfrak{b}}}) \otimes_{\mathcal{O}_X} \mathcal{F}$$

where $\mathcal{F}$ is an $\mathcal{O}_X$-module.

Lemma 2.1.2. *If $\mathfrak{a}$ and $\mathfrak{b}$ are finite sequences of elements of R, then the sequence of functors $\{H^p(\Gamma(X, D^\bullet(\_)))\}$ is a universal δ-functor in the category of quasi-coherent sheaves.*

Proof. First, once $C^\bullet(\mathcal{U}_{\mathfrak{a}}, \mathcal{O}_X) \otimes_R C^\bullet(\mathcal{U}_{\mathfrak{b}}, \mathcal{O}_X)$ is composed by flat R-modules, $D^\bullet(\_)$ preserves exact sequences of sheaves so that $\{H^p(\Gamma(X, D^\bullet(\_)))\}$ defines a δ-functor.

Now, a result of Gabber [32, Corollary 3.5] assures that the category of quasi-coherent sheaves on X has enough injective objects, and thus flasque sheaves, see [46, Lemma 2.4]. Then let $\mathcal{M}$ be a quasi-coherent flasque sheaf on X and M an R-module such that $\mathcal{M} \simeq \widetilde{M}$. By applying the global sections functor to $D^\bullet(\mathcal{M})$ we obtain the total complex given by the double complex

$$\Gamma(X, D^\bullet(\mathcal{M}))^{p,q} = C^p(\mathcal{U}_\mathfrak{a}, \mathcal{O}_X) \otimes_R C^q(\mathcal{U}_\mathfrak{b}, \mathcal{O}_X) \otimes_R M = \mathscr{C}^p_\mathfrak{a}(R) \otimes_R \mathscr{C}^q_\mathfrak{b}(R) \otimes_R M.$$

Such a double complex gives rise to a spectral sequence E converging to $H^\bullet(\Gamma(X, D^\bullet(\mathcal{M})))$ whose first terms are

$$E_1^{p,q} = \mathscr{C}^p_\mathfrak{a}(R) \otimes_R H^q(\mathscr{C}^\bullet_\mathfrak{b}(M)) \simeq \mathscr{C}^\bullet_\mathfrak{a}(H^q(\mathscr{C}^\bullet_\mathfrak{b}(M))).$$

(See Appendix B for spectral sequences.) Once $\mathcal{M}$ is flasque, Proposition 1.1.6 assures that $\check{C}^\bullet(\mathcal{U}_\mathfrak{b}, \mathcal{M})$ is a flasque resolution of $\mathcal{M}$ and thus $E_1^{p,q} = 0$ for all $q > 0$. It follows by convergence of E that

$$H^p(\Gamma(X, D^\bullet(\mathcal{M}))) \simeq E_2^{p,0} = H^p(\mathscr{C}^\bullet_\mathfrak{a}(H^0(\mathscr{C}^\bullet_\mathfrak{b}(M)))).$$

Finally, since again $\mathcal{M}$ is flasque, it should be noticed that $H^0(\widetilde{\mathscr{C}^\bullet_\mathfrak{b}(M)}) = H^0(\widetilde{U_\mathfrak{b}, \mathcal{M}})$ is flasque as well. Therefore

$$H^p(\Gamma(X, D^\bullet(\mathcal{M}))) = 0$$

for all $p > 0$. ■

Proposition 2.1.3. *If $\mathfrak{a}$ and $\mathfrak{b}$ are two finite sequences of elements of R, then given an R-module M, for all $p \geq 0$ there exists functorial isomorphism*

$$H^p(\mathscr{C}^\bullet_{\mathfrak{ab}}(M)) \simeq H^p(\mathscr{C}^\bullet_\mathfrak{a}(R) \otimes_R \mathscr{C}^\bullet_\mathfrak{b}(R) \otimes_R M).$$

Proof. By Lemma 2.1.2 we have that

$$\{H^p(\mathscr{C}^\bullet_\mathfrak{a}(R) \otimes_R \mathscr{C}^\bullet_\mathfrak{b}(R) \otimes_R \_)\} \simeq \{H^p(\Gamma(X, D^\bullet(\_)))\}$$

is a universal δ-functor in the category of quasi-coherent sheaves. On the other hand, $\{H^p(\mathscr{C}^\bullet_{\mathfrak{ab}}(\_)\}$ is also a universal δ-functor in the same category. The result follows by Lemma 2.1.1. ■

Proposition 2.1.3 allows us to construct a filtration to the complex $\mathscr{C}^\bullet_{\mathfrak{a},\mathfrak{b}}(M)$ with the cohomology of the complexes involved in the filtration being known. Indeed, consider the tensor product of the Čech complexes $D^\bullet := C^\bullet_\mathfrak{a}(R) \otimes C^\bullet_\mathfrak{b}(R) \otimes M$, see Definition 1.1.1. Switching D^0 by 0 and applying a shift in such a way that D^1 is centered at level 0 one gets a complex isomorphic to $\mathscr{C}^\bullet_{\mathfrak{a},\mathfrak{b}}(M)$. Moreover, by considering the complex $F^\bullet$ such that

$$F^p = \begin{cases} 0, \ p = 0 \\ \bigoplus_{i+j=p+1} C^i_\mathfrak{a}(R) \otimes C^j_\mathfrak{b}(R) \otimes M, \ p \neq 0 \end{cases}$$

with differentials induced by that ones of $C^\bullet_{\mathfrak{a}}(R)\otimes C^\bullet_{\mathfrak{b}}(R)\otimes M$, it may be seen that $F^\bullet \simeq \mathscr{C}^\bullet_{\mathfrak{a}}(R)\otimes\mathscr{C}^\bullet_{\mathfrak{b}}(R)\otimes M[-1]$ that, by Proposition 2.1.3, has as cohomology modules those of $\mathscr{C}^\bullet_{\mathfrak{a}\mathfrak{b}}(M)[-1]$.

We would like to extend this construction to more than two ideals. What we get from such a construction is a filtered complex which cohomology can be approximated by sheaf cohomology groups of the sort $H^\bullet(U_I,\widetilde{M})$, see Section 1.1.2. This can be done because from a filtered complex a spectral sequence arises. (The reader may see this construction, for example, in [70] or [83]). It may be noticed, for instance, that from the filtered complex given above, a spectral sequence with only two columns arise and which abutment is $H^\bullet(U_{I+J},\widetilde{M})$, where I and J are, respectively, the ideals of R generated by $\mathfrak{a}$ and $\mathfrak{b}$. From this case, in particular, the spectral sequence degenerates in a long exact sequence, which is the well-known Mayer-Vietoris long exact sequence. We shall prove this in the next section. Therefore, summing up, we are seeking a construction that generalizes that of the Mayer-Vietoris long exact sequence.

Induction on the number of sequences provides a generalization to Proposition 2.1.3.

Proposition 2.1.4. *If M is an R-module, $\mathfrak{a}_1,\mathfrak{a}_2,...,\mathfrak{a}_n$ are finite sequences of elements in R and $\mathfrak{a}$ is the sequence defined by all distinct products of n elements, where only one element belongs to a sequence $\mathfrak{a}_i$ for $i=1,...,n$, then*

$$H^p(\mathscr{C}^\bullet_{\mathfrak{a}}(M))\simeq H^p(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R)\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_2}(R)\otimes_R...\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_n}(R)\otimes_R M),\ \textit{for all } p\geq 0.$$

Proof. The tensor product of the complex $\mathscr{C}^\bullet_{\mathfrak{a}_1}(R)\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_2}(R)\otimes_R...\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_{n-1}}(R)$ by the complex $\mathscr{C}^\bullet_{\mathfrak{a}_n}(M)$ gives rise to a spectral sequence E which abutment assures functorial isomorphism

$$H^0(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R)\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_2}(R)\otimes_R...\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_n}(R)\otimes_R M)\simeq H^0(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R)\otimes_R...\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_{n-1}}(R)(H^0(\mathscr{C}^\bullet_{\mathfrak{a}_n}(M))))$$

so that, by induction hypothesis,

$$\begin{aligned}H^0(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R)\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_2}(R)\otimes_R...\otimes_R\mathscr{C}^\bullet_{\mathfrak{a}_n}(R)\otimes_R M)&\simeq H^0(\mathscr{C}^\bullet_{\mathfrak{a}_1...\mathfrak{a}_{n-1}}(R)(H^0(\mathscr{C}^\bullet_{\mathfrak{a}_n}(M))))\\&\simeq H^0(\mathscr{C}^\bullet_{\mathfrak{a}}(M)).\end{aligned}$$

Further, given an R-module N such that $\widetilde{N}$ is flasque (such N exists by a Gabber's result [32, Corollary 3.5]), by induction hypothesis, the second page of this spectral sequence is such that

$$E_2^{p,q}\simeq H^p(\mathscr{C}^\bullet_{\mathfrak{a}_1\mathfrak{a}_2...\mathfrak{a}_{n-1}}(H^q(\mathscr{C}^\bullet_{\mathfrak{a}_n})(N)))=0$$

for all $p > 0$ or $q > 0$ so that $\{H^p(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_2}(R) \otimes_R ... \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R \_)\}_p$ is a universal δ-functor, whence the result. ■

Now we shall provide a more direct proof for Proposition 2.1.4.

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be sequences of elements in R and M be an R-module. We denote the total complex $(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_2}(R) \otimes_R \dots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M)[-1]$ by $\mathscr{C}^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$ and the (augmented) complex

$$0 \to M \to \mathscr{C}^1_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \to \mathscr{C}^2_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \to \cdots$$

by $C^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$. We set $H^i_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) := H^i(C^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M))$ for all $i \geq 0$. Further, by defining

$$D_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) := H^1(\mathscr{C}^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)) = H^0(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_2}(R) \otimes_R \dots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M),$$

the exact sequence of complexes

$$0 \longrightarrow \mathscr{C}^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow C^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow M \longrightarrow 0$$

where M also denotes the complex centered in the R-module M at degree 0, as we did in Section 1.1.2. Thus there is an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow M \longrightarrow D_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow H^1_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow 0 \tag{2.1.1}$$

as the first one at the beginning of this chapter, and isomorphisms

$$H^i(\mathscr{C}^\bullet_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)) \simeq H^i_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$$

for all $i > 1$.

From now on, let $\mathfrak{a}$ be the sequence defined by all distinct products of n elements, where only one element belongs to a sequence $\mathfrak{a}_i$ for $i = 1, \dots, n$. Let I be the ideal generated by $\mathfrak{a}$. It should be noticed that the sequence $0 \longrightarrow C^0_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) \longrightarrow C^1_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$ is identical to $0 \longrightarrow C^0_{\mathfrak{a}}(M) \longrightarrow C^1_{\mathfrak{a}}(M)$; in particular, $H^0_I(M) = H^0_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$.

Lemma 2.1.5. *Assume that* $M = H^0_I(M)$*, then* $C^i_{\mathfrak{a}}(M) = C^i_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M) = 0$ *for any* $i > 0$.

Proof. For $i > 0$, the summands of $C^i_{\mathfrak{a}}(M)$ or $C^i_{\mathfrak{a}_1,\dots,\mathfrak{a}_n}(M)$ are localizations M_w, where w is a multiple of an element of the form $a_1 \cdot \ldots \cdot a_n$, for some product with $a_i \in \mathfrak{a}_i$. ■

For an ideal J generated by a sequence of elements $\mathfrak{b}$ in R, we set $D_J(M) := H^0(\mathscr{C}^\bullet_{\mathfrak{b}}(M))$ (even though the ring is not Noetherian). Moreover, as Chardin, Jouanolou and Rahimi have shown in [22], any element of $H^q_J(M)$ is annihilated by a power of the ideal J. This extends to sums of localizations $C^p_{\mathfrak{a}} H^q_J(M)$ and hence to the quotients $H^p_I(H^q_J(M))$ and submodules $D_I(H^q_J(M))$.

Lemma 2.1.6. *For any $i \geq 0$, $H^i_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M) = H^0_I(H^i_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M))$.*

Proof. First, notice that if $E_2 \Rightarrow_p H$ is a spectral sequence such that $H^0_I(E_2^{p,q}) = E_2^{p,q}$ for all p, q then $H^0_I(H^i) = H^i$ for all $i \geq 0$. Thus, the result will follow by considering the spectral sequence arising from the double complex $\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \ldots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M = (\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \ldots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_{n-1}}(R)) \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M$:

$$E_2^{i,j} = H^i(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \ldots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_{n-1}}(H^j(\mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M)))$$

that abuts to $H^{i+j}(\mathscr{C}^\bullet_{\mathfrak{a}_1}(R) \otimes_R \ldots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_n}(R) \otimes_R M) = H^{i+j+1}(\mathscr{C}^\bullet_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M))$, and showing that $E_2^{i,j} = H^0_I(E_2^{i,j})$. We shall proceed by induction in n. We commented the case $n = 1$ before this lemma. Suppose $n > 1$ and that the result holds true for any module and $n - 1$ sequences.

If $j > 0$, then $H^j(\mathscr{C}^\bullet_{\mathfrak{a}_n} \otimes_R M) = H^{j+1}_{(\mathfrak{a}_n)}(M)$ so that

$$H^j(\mathscr{C}^\bullet_{\mathfrak{a}_n} \otimes_R M) = H^0_I(H^j(\mathscr{C}^\bullet_{\mathfrak{a}_n} \otimes_R M))$$

and thus $E_2^{i,j} = H^0_I(E_2^{i,j})$. Suppose $j = 0$. If $i > 0$, then

$$E_2^{i,0} = H^{i+1}(\mathscr{C}^\bullet_{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n-1}}(D_{(\mathfrak{a}_n)}(M)) = H^{i+1}_{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n-1}}(D_{(\mathfrak{a}_n)}(M))$$

so $E_2^{i,0} = H^0_I(E_2^{i,0})$. Once $H^i(\mathscr{C}^\bullet_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M)) = H^i_{\mathfrak{a}_1,\ldots\mathfrak{a}_n}(M)$ for all $i > 1$, we conclude that $H^i_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M) = H^0_I(H^i_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M))$ for all $i > 1$.

Now, suppose $i = j = 0$ in the spectral sequence and consider the exact sequence 2.1.1. Given $a_1 \in \mathfrak{a}_1, \ldots, a_n \in \mathfrak{a}_n$ and $x \in D_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M) = D_{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n-1}}(D_{(\mathfrak{a}_n)}(M))$, by induction hypothesis there exists N such that $(a_1 \cdot \ldots \cdot a_{n-1})^N x \in D_{(\mathfrak{a}_n)}(M)$ and then $(a_1 \cdot \ldots \cdot a_n)^N x \in M$ for $N \gg 0$. In other words, $I^n x = 0$ in $H^1_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M)$. ■

The next proposition is another proof for Proposition 2.1.4.

Proposition 2.1.7. *If M is an R-module, $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ are finite sequences of elements in R, $\mathfrak{a}$ is the sequence defined by all distinct products of n elements, where only one element belongs to a sequence $\mathfrak{a}_i$ for $i = 1, \ldots, n$, and I is the ideal generated by $\mathfrak{a}$, then*

$$D_I(M) \simeq D_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M) \quad \textit{and} \quad H^i_I(M) \simeq H^i_{\mathfrak{a}_1,\ldots,\mathfrak{a}_n}(M)$$

for all $i \geq 0$.

Proof. First, Lemma 2.1.5 assures that it suffices to prove the proposition when $H^0_I(M) = 0$. Thus, from the exact sequences

$$0 \to M \to D_I(M) \to H^1_I(M) \to 0 \quad \text{and} \quad 0 \to M \to D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M) \to H^1_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M) \to 0$$

and from lemmas 2.1.5 and 2.1.6 we obtain isomorphism of complexes $\mathscr{C}^\bullet_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M) \simeq \mathscr{C}^\bullet_{\mathfrak{a}_1,...,\mathfrak{a}_n}(D_I(M))$ and $\mathscr{C}^\bullet_{\mathfrak{a}}(M) \simeq \mathscr{C}^\bullet_{\mathfrak{a}}(D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M))$. In particular,

(i) $D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M) \simeq D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(D_I(M))$ and $D_I(M) \simeq D_I(D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M))$;

(ii) $H^i_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M) \simeq H^i_{\mathfrak{a}_1,...,\mathfrak{a}_n}(D_I(M))$ and $H^i_I(M) \simeq H^i_I(D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M))$ for all $i \geq 2$.

Now, the double complex

$$\begin{tikzcd}
 & \vdots & & \vdots & \\
0 \arrow[r] & \mathscr{C}^0_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^1_{\mathfrak{a}_1,...,\mathfrak{a}_n}(R) \otimes_R M \arrow[r] \arrow[u] & & \mathscr{C}^1_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^1_{\mathfrak{a}_1,...,\mathfrak{a}_n}(R) \otimes_R M \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathscr{C}^0_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^0_{\mathfrak{a}_1,...,\mathfrak{a}_n}(R) \otimes_R M \arrow[r] \arrow[u] & & \mathscr{C}^1_{\mathfrak{a}}(R) \otimes_R \mathscr{C}^0_{\mathfrak{a}_1,...,\mathfrak{a}_n}(R) \otimes_R M \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

gives rise to two spectral sequences that collapses at second page in such a way that

$$D_I(D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M)) \simeq D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(D_I(M)) \quad \text{and} \quad H^i_I(D_{\mathfrak{a}_1,...,\mathfrak{a}_n}(M)) \simeq H^i_{\mathfrak{a}_1,...,\mathfrak{a}_n}(D_I(M))$$

for all $i \geq 2$. The result follows from (i) and (ii). ■

2.1.2 The spectral sequence

We are now ready to construct the promised Mayer-Vietoris spectral sequence. The reader not familiarized with the construction of a spectral sequence has [70, 74] and [83] as pretty good references.

Theorem 2.1.8 (Mayer-Vietoris Spectral Sequence)**.** *Let M be an R-module. Given $I, I_1, ..., I_n$ finitely generated ideals of R such that $I = I_1 + ... + I_n$, then there exists spectral sequence*

$$E_1^{p,q} = \bigoplus_{i_0 < ... < i_p} H^q(U_{I_{i_0} I_{i_1} ... I_{i_p}}, \widetilde{M}) \Rightarrow_p H^{p+q}(U_I, \widetilde{M}).$$

Proof. Given $p \geq 1$ and m integers define

$$X_p^m := \{(i_1, ..., i_n) \in \mathbb{N}^n : i_1 + ... + i_n = m \text{ and at most } n - p - 1 \text{ of the } i_j's \text{ are zero}\}.$$

Let $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_n$ be finite sequences of elements of R such that I_i is generated by $\mathfrak{a}_i$ for all $i = 1, ..., n$. Thus I is generated by $\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2 \cup ... \cup \mathfrak{a}_n$ and consider

$$F_p^m := \bigoplus_{(i_1,i_2,\dots,i_n)\in X_p^m} C_{\mathfrak{a}_1}^{i_1}(R)\otimes_R C_{\mathfrak{a}_2}^{i_2}(R)\otimes_R \dots \otimes_R C_{\mathfrak{a}_n}^{i_n}(R)\otimes_R M$$

where the $C^\bullet_{\mathfrak{a}_i}(R)$ are the Čech complexes 1.1.1. Consider also morphisms

$$F_p^m \to F_p^{m+1}$$

as the restriction of the differential $C^m_{\mathfrak{a}}(R)\otimes_R M \to C^{m+1}_{\mathfrak{a}}(R)\otimes_R M$ to F_p^m.

Now, by considering the complex $\mathscr{C}^\bullet_{\mathfrak{a}}(M)$ from the beginning of the section, notice that it can be obtained from $C^\bullet_{\mathfrak{a}}(M)[1]$ by replacing $C^0_{\mathfrak{a}}(M)$ to 0, as we have already seen in Section 1.1.2. By writing in the same way the complexes F_p after the same shift, the family $\{F_p\}_p$ turns out to be a limited (descending) filtration of $\mathscr{C}^\bullet_{\mathfrak{a}}(M)$ and therefore it yields a spectral sequence E converging to

$$H^\bullet(\mathscr{C}^\bullet_{\mathfrak{a}}(M)) \simeq H^\bullet(U_I, \widetilde{M}).$$

Moreover, it may be seen that

$$F_p/F_{p+1} \simeq \left(\bigoplus_{i_0<\dots<i_p} \mathscr{C}^\bullet_{\mathfrak{a}_{i_0}}(R)\otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_{i_1}}(R)\otimes_R \dots \otimes_R \mathscr{C}^\bullet_{\mathfrak{a}_{i_p}}(R)\otimes_R M\right)[-p]$$

for all $p>0$ so that, by Proposition 2.1.4 (or 2.1.7),

$$E_1^{p,q} = H^{p+q}(F_p/F_{p+1}) \simeq \bigoplus_{i_0<\dots<i_p} H^q(\mathscr{C}^\bullet_{\mathfrak{a}_{i_0}\mathfrak{a}_{i_1}\dots\mathfrak{a}_{i_p}}(M)) \simeq \bigoplus_{i_0<\dots<i_p} H^q(U_{I_{i_0}I_{i_1}\dots I_{i_p}}, \widetilde{M})$$

for all $p>0$ and $q\geq 0$, and

$$H^q(\mathscr{C}^\bullet_{\mathfrak{a}}(M)/F_1) = \bigoplus_j H^q(U_{I_j}, \widetilde{M})$$

for all $q \geq 0$. ∎

2.2 Second page

A natural question that has arisen is what the second page of the Mayer-Vietoris spectral sequence looks like. We suggest two alternatives to respond to that. The first one was already mentioned, it is another way to construct a Mayer-Vietoris spectral sequence. Indeed our spectral sequence may be seen as a particular case of the Čech spectral sequence (see Appendix B.2.1) so that its second page is given in terms of objects having a geometrical meaning. Another great advantage of this construction is that the ring does not need to be Noetherian. The second alternative has ended up having a more topological aspect as the finite sequences $\mathfrak{a}_1, ..., \mathfrak{a}_n$ form a basis of a certain topological space. This way gives the second page in terms of right derived functors of inverse limit functors.

Theorem 2.2.1 (Mayer-Vietoris Spectral Sequence)**.** *Let $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_n$ be finite sequences of elements of R and M an R-module. If $\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2 \cup \ldots \cup \mathfrak{a}_n$ and I is the ideal of R generated by $\mathfrak{a}$ then there exists a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{U}_{\mathfrak{a}}, \mathcal{H}^q(\widetilde{M})) \Rightarrow_p H^{p+q}(U_I, \widetilde{M})$$

where $\mathcal{U}_{\mathfrak{a}}$ is the family (U_{I_i}), $i = 1, ..., n$, with U_{I_i} being the complement in $\mathrm{Spec}(R)$ *of the variety defined by the ideal generated by the sequence $\mathfrak{a}_i$. Moreover,*

$$E_1^{p,q} = \bigoplus_{i_0 < \ldots < i_p} H^q(U_{I_{i_0} \cdot I_{i_1} \cdot \ldots \cdot I_{i_p}}, \widetilde{M})$$

and the first page coincides with the Mayer-Vietoris spectral sequence's 2.1.8.

Proof. The existence of the desired spectral sequence follows directly from [40, Théorème 5.4.1]. The first page is obtained by noticing that, for every p, $U_{I_{i_0}} \cap U_{I_{i_1}} \cap ... \cap U_{I_{i_p}} = U_{I_{i_0} \cdot I_{i_1} \cdot \ldots \cdot I_{i_p}}$. ■

The second alternative demands some work on general sheaf theory and inverse limits. Its too long and technical arguments on basic issues of these two subjects do not seem to show us much relevance in what concerns the Mayer-Vietoris spectral sequence performance. Hence we attach this discussion in Appendix A and here we only display what the second page of the Mayer-Vietoris spectral sequence looks like.

Let $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_n$ be finite sequences of elements of R and $I_1, I_2, ..., I_n$ be the ideal generated by them respectively. The set

$$\Sigma = \{I_{i_0} \cdot I_{i_1} \cdot ... \cdot I_{i_p} : p = 0, ..., n-1 \text{ and } i_0 < i_1 < ... < i_p\}$$

endowed with the inclusion order turns out to be a topological space with basis being composed by the subsets $(-\infty, I] := \{J \in \Sigma : J \subseteq I\}$. The second page of the Mayer-Vietoris spectral sequence is given by

$$E_2^{p,q} = \varprojlim_{I \in \Sigma}{}^{(p)} H^q(U_I, \widetilde{M}).$$

2.3 The long exact sequence and further applications

In this section, we provide some general applications to the Mayer-Vietoris spectral sequence. The first one is naturally the Mayer-Vietoris long exact sequence. The major advantage of this construction is the no need for Noetherianity. It allows us to generalize many results proved in the Noetherian case that we naturally shall prove some soon. We should remember that the Mayer-Vietoris long exact sequence in the non-Noetherian case is a well-known tool. For instance, it is constructed (by different ways) in [78, Theorem 9.4.3] and in [81].

Theorem 2.3.1 (Mayer-Vietoris Long Exact Sequence)**.** *If I and J are finitely generated ideals of R and M is an R-module then there exists long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & H^0_{I+J}(M) \arrow[r] & H^0_I(M) \oplus H^0_J(M) \arrow[r] & H^0_{IJ}(M) \arrow[dll] \\
 & H^1_{I+J}(M) \arrow[r] & H^1_I(M) \oplus H^1_J(M) \arrow[r] & H^1_{IJ}(M) \arrow[dll] \\
 & H^2_{I+J}(M) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. By the relation between local and sheaf cohomology that we have already seen in Section 1.1.2, the Mayer-Vietoris Spectral Sequence with respect to I and J 2.2.1 has as first page

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 \arrow[r] & H^3_I(M) \oplus H^3_J(M) \arrow[r] & H^3_{IJ}(M) \arrow[r] & 0 \\
0 \arrow[r] & H^2_I(M) \oplus H^2_J(M) \arrow[r] & H^2_{IJ}(M) \arrow[r] & 0 \\
0 \arrow[r] & H^0(U_I, \widetilde{M}) \oplus H^0(U_J, \widetilde{M}) \arrow[r] & H^0(U_{IJ}, \widetilde{M}) \arrow[r] & 0
\end{tikzcd}$$

Since this spectral sequence converges to $H^\bullet(U_{I+J}, \widetilde{M})$ it degenerates in the following long exact sequence.

$$0 \to H^0(U_{I+J}, \widetilde{M}) \to H^0(U_I, \widetilde{M}) \oplus H^0(U_J, \widetilde{M}) \to H^0(U_{IJ}, \widetilde{M}) \to H^2_{I+J}(M) \to \cdots$$

The exactness of the sequence

$$0 \to H^0_K(M) \to M \to H^0(U_K, \widetilde{M}) \to H^1_K(M) \to 0$$

for any ideal K (see again Section 1.1.2) assures us the exactness of the rows in the following double complex

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0_{I+J}(M) \arrow[r] \arrow[d, "\alpha"'] & M \arrow[r] \arrow[d] & H^0(U_{I+J}, \widetilde{M}) \arrow[r] \arrow[d] & H^1_{I+J}(M) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0_I(M) \oplus H^0_J(M) \arrow[r] \arrow[d, "\beta"'] & M \oplus M \arrow[r] \arrow[d] & H^0(U_I, \widetilde{M}) \oplus H^0(U_J, \widetilde{M}) \arrow[r] \arrow[d, "\varphi"] & H^1_I(M) \oplus H^1_J(M) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & H^0_{IJ}(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & H^0(U_{IJ}, \widetilde{M}) \arrow[r] \arrow[d] & H^1_{IJ}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

(The first and fourth columns are induced by the second and third ones. The second column is the canonical exact sequence $x \mapsto (x, x)$ and $(x, y) \mapsto x - y$.)

This double complex provides two spectral sequences. One of them (taking horizontal homology first) is composed of zeros so that the other spectral sequence E (obtained by taking vertical homology) converges to zero. Since the second column is an exact complex and the third one has homology only in $H^0(U_{IJ}, \widetilde{M})$, the first page of E is given by

$$
\begin{tikzcd}
\ker\alpha & 0 & 0 & \ker\gamma \\
\ker\alpha/\operatorname{im}\beta & 0 & 0 & \ker\psi/\operatorname{im}\gamma \\
\operatorname{coker}\beta \arrow[uurrr, dashed, "\simeq"] & 0 & \operatorname{coker}\varphi \arrow[r, "\simeq"] & \operatorname{coker}\psi
\end{tikzcd}
$$

where the dotted homomorphism is the only one on the third page. The convergence gives us the exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^0_{I+J}(M) \arrow[r] & H^0_I(M) \oplus H^0_J(M) \arrow[r] & H^0_{IJ}(M) \arrow[r] & \operatorname{coker}\beta \arrow[dllll, "\simeq"] \\
\ker\gamma \arrow[r] & H^1_{I+J}(M) \arrow[r] & H^1_I(M) \oplus H^1_J(M) \arrow[r] & H^1_{IJ}(M) &
\end{tikzcd}
$$

and isomorphism $\operatorname{coker}\varphi \simeq \operatorname{coker}\psi$. This isomorphism, in turn, induces another double complex

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\varphi \arrow[r] \arrow[d] & H^0(U_{IJ}, \widetilde{M}) \arrow[r] \arrow[d] & H^2_{I+J}(M) \arrow[r] \arrow[d, equal] & H^2_I(M) \oplus H^2_J(M) \arrow[r] \arrow[d, equal] & \dots \\
0 \arrow[r] & \operatorname{im}\psi \arrow[r] & H^1_{IJ}(M) \arrow[r] & H^2_{I+J}(M) \arrow[r] & H^2_I(M) \oplus H^2_J(M) \arrow[r] & \cdots
\end{tikzcd}
$$

and a similar discussion as above completes the proof. ■

The first applications of the Mayer-Vietoris long exact sequence is a generalization of a result of Dibaei and Vahidi on cohomological dimension [28, Corollary 2.2]. It is a homological invariant that plays an important role in both commutative algebra and algebraic geometry, see [17, 22] for interesting properties and applications.

Definition 2.3.2. *If I is a finitely generated ideal of R and M is an R-module, the* cohomological dimension of M with respect to the ideal I *is defined as being*

$$\mathrm{cd}_I(M) := \sup\{p \in \mathbb{N} \cup \{-\infty\} : H_I^p(M) \neq 0\}.$$

Corollary 2.3.3. *If I and J are two finitely generated ideals of R and M is an R-module then*

$$\mathrm{cd}_{IJ}(M) \leq \mathrm{cd}_I(R/ann_R(M)) + \mathrm{cd}_J(M).$$

In case of M being finitely generated we have

$$cd_{IJ}(M) \leq \mathrm{cd}_I(M) + \mathrm{cd}_J(M).$$

Proof. The Mayer-Vietoris long exact sequence 2.3.1 gives us exact sequence

$$H_I^i(M) \oplus H_J^i(M) \longrightarrow H_{IJ}^i(M) \longrightarrow H_{I+J}^{i+1}(M)$$

for all i. The result follows from [22, Proposition 4.1 b), Proposition 4.1 c), Corollary 4.2]. ∎

Corollary 2.3.4. *Let M be a finitely generated R-module and let I be a finitely generated ideal of R. Given finitely generated ideals $\mathfrak{q}_1, ..., \mathfrak{q}_n$ of R such that $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap ... \cap \mathfrak{q}_n$, then*

$$\mathrm{cd}_I(M) \leq \sum_{i=1}^{n} \mathrm{cd}_{\mathfrak{q}_i}(M).$$

Definition 2.3.5. *If I is a finitely generated R-ideal and M an R-module, we set*

$$\mathrm{depth}_I(M) := max\{p \in \mathbb{N} \cup \{+\infty\} : H_I^i(M) = 0, \text{ for all } i < p\}.$$

When R is local and I is its maximal ideal, $\mathrm{depth}_R(M)$ *stands for* $\mathrm{depth}_I(M)$.

The next result is a version of the Mayer-Vietoris long exact sequence 2.3.1 for three ideals.

Theorem 2.3.6. *Let I, J and K be three ideals of R generated, respectively, by the finite sequences $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$. If M is an R-module such that $g = \mathrm{depth}_{I+J+K}(M) > 2$ then there exists long exact sequence*

$$
\begin{tikzcd}
0 \arrow[r] & H^0(U_{I+J+K}, \widetilde{M}) \arrow[r] & H^0(U_I, \widetilde{M}) \oplus H^0(U_J, \widetilde{M}) \oplus H^0(U_K, \widetilde{M}) \arrow[dl] \\
& H^0(U_{IJ}, \widetilde{M}) \oplus H^0(U_{IK}, \widetilde{M}) \oplus H^0(U_{JK}, \widetilde{M}) \arrow[r] & H^0(U_{IJK}, \widetilde{M}) \arrow[dl] \\
& H^2_I(M) \oplus H^2_J(M) \oplus H^2_K(M) \arrow[r] & H^2_{IJ}(M) \oplus H^2_{IK}(M) \oplus H^2_{JK}(M) \arrow[dl] \\
& H^2_{IJK}(M) \arrow[r] & \cdots \\
& \cdots \arrow[r] & H^{g-1}_{IJ}(M) \oplus H^{g-1}_{IK}(M) \oplus H^{g-1}_{JK}(M) \arrow[dl] \\
& H^1(\mathcal{U}_{\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}, \mathcal{H}^{g-2}(\widetilde{M})) \arrow[r] & 0.
\end{tikzcd}
$$

Proof. From the construction of the Mayer-Vietoris Spectral Sequence E 2.2.1 one sees that it has three columns and the only possibly nonzero differentials at the second page are $E_2^{0,q} \to E_2^{2,q-1}$. Since $E_3^{0,q} \simeq E_\infty^{0,q} = 0$ for all $q < g-1$, $E_2^{1,q} \simeq E_\infty^{1,q} = 0$ for all $q < g-2$ and $E_3^{2,q} \simeq E_\infty^{2,q} = 0$ for all $q < g-3$. Therefore all nonzero differentials $E_2^{0,q} \to E_2^{2,q-1}$ are isomorphisms for $q < g-2$ and all sequences

$$E_1^{0,q} \to E_1^{1,q} \to E_1^{2,q}$$

are exact for $q < g-2$ so that one obtains a long exact sequence from the sequence

$$
\begin{tikzcd}
&&&&& 0 \arrow[d] && \\
0 \arrow[r] & E_2^{0,0} \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] \arrow[dr] & E_2^{2,0} \arrow[r] \arrow[d] & 0 & \\
&&&&& E_1^{0,1} \arrow[d] && \\
&&&&& E_1^{1,1} \arrow[d] && \\
&&&&& E_1^{2,1} \arrow[d] \arrow[dr] && \\
&&&& 0 \arrow[r] & E_2^{2,1} \arrow[r] \arrow[d] & E_1^{0,2} \arrow[r] & \cdots \\
&&&&& 0 &&
\end{tikzcd}
$$

The result follows from definition of depth and from the second page characterization 2.2.1. ■

The remaining results of this section are attempts to get general information about sheaf cohomology modules from the Mayer-Vietoris spectral sequence.

Proposition 2.3.7. *Let $I_1, I_2, ..., I_n$ be finite generated ideals of R, $I = I_1+I_2+...+I_n$, M an R-module and*

$$m = \sup\{\mathrm{cd}_{I_{i_0}\cdot I_{i_1}\cdot\ldots\cdot I_{i_p}}(M) \mid i_0 < i_1 < ... < i_p,\ p = 0, \ldots, n-1\}.$$

Consider the Mayer-Vietoris spectral sequence E 2.2.1 defined by finite generators sets of the I_i's. If $m \le 1$ then

$$H^p(\mathcal{U}_I, \widetilde{M}) \simeq H^p(U_I, \widetilde{M})$$

for all $p \ge 0$. In particular $\mathrm{cd}_I(M) \le n$. If $m > 1$ then

$$\mathrm{cd}_I(M) \le n+m-1,$$

$$H^{n+m-2}(U_I, \widetilde{M}) \simeq \mathrm{coker}(E_1^{n-2,m-1} \to E_1^{n-1,m-1})$$

and there exists exact sequence

$$0 \to H^{n-1}(\mathcal{U}_I, \mathcal{H}^{m-2}(\widetilde{M})) \to H^{m+n-3}(U,\widetilde{M}) \to H^{n-2}(\mathcal{U}_I, \mathcal{H}^{m-1}(\widetilde{M})) \to 0.$$

Proof. Since the Mayer-Vietoris spectral sequence 2.2.1 is such that

$$E_1^{p,q} = \bigoplus_{i_0<\ldots<i_p} H^q(U_{I_{i_0}\cdot I_{i_1}\cdot\ldots\cdot I_{i_p}}, \widetilde{M}) \simeq \bigoplus_{i_0<\ldots<i_p} H^{q+1}_{I_{i_0}\cdot I_{i_1}\cdot\ldots\cdot I_{i_n}}(M)$$

for $q > 0$, we have $E_1^{p,q} = 0$ for all $q > 0$ provided $m \le 1$.

In case of $m > 1$, it is enough to prove when m is finite. From the corner

$$\begin{array}{ccccccc}
\cdots & & 0 & & 0 & & 0 \\
\cdots & \longrightarrow & E_1^{n-2,m-1} & \longrightarrow & E_1^{n-1,m-1} & \longrightarrow & 0 \\
\cdots & \longrightarrow & E_1^{n-2,m-2} & \longrightarrow & E_1^{n-1,m-2} & \longrightarrow & 0
\end{array}$$

and by convergence,

$$H_I^q(M) = H^{q-1}(U_I, \widetilde{M}) = 0$$

whenever $q > n+m-1$,

$$H^{n+m-2}(U_I, \widetilde{M}) \simeq E_2^{n-1,m-1} \simeq \mathrm{coker}(E_1^{n-2,m-1} \to E_1^{n-1,m-1})$$

and there is exact sequence

$$0 \to E_2^{n-1,m-2} \to H^{m+n-3}(U_I, \widetilde{M}) \to E_2^{n-1,m-1} \to 0.$$

The result follows by the geometric characterization 2.2.1. ∎

The next definition is a generalization of the invariant $a_i(M)$ defined in the Section 1.2.1.

Definition 2.3.8. *If $R = \bigoplus_{n\in\mathbb{N}} R_n$ is a positively graded ring, I is a finitely generated graded ideal of R and M is a graded R-module then we set, for each $i \in \mathbb{N}$,*

$$a_I^i(M) := \sup\{\mu : H_I^i(M)_\mu \neq 0\}$$

if $H_I^i(M) \neq 0$ and $a_I^i(M) := -\infty$ else.

Proposition 2.3.9. *Suppose $R = \bigoplus_{n\in\mathbb{N}} R_n$ is a positively graded ring, M is a graded R-module, let $I_1, I_2, ..., I_n$ be finitely generated homogeneous ideals of R and $I = I_1 + I_2 + ... + I_n$. If $j > 1$ then*

$$a_I^j(M) \leq \max\{\mu \mid \exists p \leq j,\ H^{j-p}(U_{I_{i_0}I_{i_1}...I_{i_p}}, \widetilde{M})_\mu \neq 0\}.$$

Proof. Let E be the Mayer-Vietoris spectral sequence 2.2.1 associated to the n sequences defined by the finite generators of the I_i's. Given μ such that $H_I^j(M)_\mu \neq 0$ we cannot have $(E_1^{p,j-p})_\mu = 0$ for all p because of convergence of E. Therefore $H^{j-p}(U_{I_{i_0}I_{i_1}...I_{i_p}}, \widetilde{M})_\mu \neq 0$ for some $p \leq j$. It proves that $\{\mu \mid H_I^j(M)_\mu \neq 0\} \subseteq \{\mu \mid \exists p \leq j,\ H^{j-p}(U_{I_{i_0}I_{i_1}...I_{i_p}}, \widetilde{M})_\mu \neq 0\}$ whence the result. ∎

Chapter 3

Local cohomology over multigraded polynomial rings

Most of the results in this chapter concern relations, in a polynomial ring, between local cohomology modules supported in the irrelevant ideal and on ideals generated by variables. Our main is to bring information about local cohomology modules supported on these ideals. For instance, a duality-like theorem and Artinianness are proved in the bigraded case and information about invariants such as cohomological dimension are settled in general.

We need to set some notation to use throughout this section.

Notation 3.0.1. *Let S be a commutative unitary ring. Let $k \geq 1$ be an integer and consider the $\mathbb{Z}^k$-graded polynomial ring $R = S[X_{1,0}, ..., X_{1,n_1}, ..., X_{k,0}, ..., X_{k,n_k}]$ with* $\deg(X_{i,j}) = e_i$ *for all $j = 0, ..., n_i$, where e_i denotes the i-th element of the canonical basis of $\mathbb{Z}^k$. By a graded R-module we just mean a $\mathbb{Z}^k$-graded R-module. Write $B_i = (X_{i,0}, ..., X_{i,n_i})$ for $i = 1, ..., k$, $B = B_1 \cap ... \cap B_k$, $B_{i_0 ... i_p} = B_{i_0} \cap ... \cap B_{i_p}$ for $p = 0, ..., k-2$ and $\mathfrak{m} = B_1 + ... + B_k$.*

3.1 The bigraded case

In this section, we consider $k = 2$ in Notation 3.0.1. To simplify even more the notation, we write $n_1 = m$, $n_2 = n$ and $R = S[X_0, ..., X_m, Y_0, ..., Y_n]$. Since here we work on the bigraded case, it is natural to ask whether the results in this section hold true for $k \geq 3$. We, indeed, generalize some of these results later.

3.1.1 Cohomological dimension and artinianness

This section presents some information about the cohomological dimension of local cohomology modules and its Artinianness. We first show a proposition that generalizes Chardin and Nemati [23, Proposition 3.4] though its proof essentially follows the same lines.

Proposition 3.1.1. *Let F be a graded free R-module. One has*

$$H_B^{m+n+1}(F) \simeq H_{\mathfrak{m}}^{m+n+2}(F).$$

Furthermore, if $m = n$ then

$$H_B^i(F) \simeq \begin{cases} H_{B_1}^{m+1}(F) \oplus H_{B_2}^{m+1}(F), & i = m+1 \\ 0, & \textit{else} \end{cases}$$

and if $m \neq n$ then

$$H_B^i(F) \simeq \begin{cases} H_{B_1}^{m+1}(F), & i = m+1 \\ H_{B_2}^{n+1}(F), & i = n+1 \\ 0, & \textit{else.} \end{cases}$$

Proof. Once $H_{B_1}^i(F) = 0$ if $i \neq m+1$, $H_{B_2}^j(F) = 0$ if $j \neq n+1$ and $H_{\mathfrak{m}}^l(F) = 0$ if $l \neq m+n+2$, the result follows from the Mayer-Vietoris long exact sequence 2.3.1

$$\cdots \to H_{\mathfrak{m}}^i(F) \to H_{B_1}^i(F) \oplus H_{B_2}^i(F) \to H_B^i(F) \to H_{\mathfrak{m}}^{i+1}(F) \to \cdots$$

■

The first consequence of the Proposition 3.1.1 has to do with cohomological dimension, see Definition 2.3.2.

Corollary 3.1.2. *If F is a graded free R-module, then*

$$\mathrm{cd}_B(F) = m+n+1.$$

In particular, $\mathrm{cd}_B(M) \leq m+n+1$ for any graded R-module M.

Proof. The first part follows immediately from Proposition 3.1.1. The inequality follows from the spectral sequence

$$\mathrm{Tor}_p^R(M, H_B^q(R)) \Rightarrow_p H_B^{q-p}(M).$$

(The not acquainted reader might see Appendix B.2.6 for the construction of this spectral sequence.) ■

The next result is similar to [28, Corollary 2.6] but here we work with intersection instead of sum.

Proposition 3.1.3. *If F is a graded finitely generated free R-module then*

$$H_{B_1}^{m+1}(H_{B_2}^{n+1}(F)) \simeq H_B^{m+n+1}(F).$$

In particular, the following statements hold true.

(i) $\mathrm{cd}_{B_1}(H_{B_2}^{n+1}(F)) = m+1$,

(ii) $\mathrm{cd}_{B_2}(H_{B_1}^{m+1}(F)) = n+1$.

Proof. Proposition 2.1.3 induces spectral sequence

$$E_2^{p,q} = H^p(\mathscr{C}_{B_1}^\bullet(H^q(\mathscr{C}_{B_2}^\bullet(F)))) \Rightarrow_p H^{p+q}(\mathscr{C}_B^\bullet(F)).$$

By [22, Proposition 4.7] one has $E_2^{p,q} = 0$ for $p > m+1$ or $q > n+1$ so that the isomorphism desired follows directly. Corollary 3.1.2 assures the others statements. ∎

In the Noetherian case, Proposition 3.1.3 helps us to set the non-Artinianness of the local cohomology modules we are working on. Note that Dibaei and Vahidi have also proved Corollary 3.1.4 in [28, Proposition 4.1] following a different path but here we suggest a more general method for proving this result.

Corollary 3.1.4. *If S is Noetherian and F is a graded finitely generated free R-module then*

$$\dim(H_{B_1}^{m+1}(F)) \geq n+1$$

and

$$\dim(H_{B_2}^{n+1}(F)) \geq m+1.$$

In particular, both local cohomology modules are not Artinian.

Proof. By the Grothendieck's Vanishing Theorem (see [18, 17] or [58]) one has

$$\dim(H_{B_1}^{n_1+1}(F)) \geq \mathrm{cd}_{B_2}(H_{B_1}^{n_1+1}(F)) = n_2+1.$$

The proof of the other inequality is alike. ∎

Next lemma removes the Noetherianity in [10, Lemma 2.8].

Lemma 3.1.5. *Let I be a proper ideal of R and M a finitely presented R-module. If $\mathrm{cd}_I(M) = t \geq 1$ then $H_I^t(M) = IH_I^t(M)$. In particular, if R is local then $H_I^t(M)$ cannot be finitely generated.*

Proof. Note that M is also finitely generated as $R/ann(M)$-module so that there is exact sequence of $R/ann(M)$-modules

$$0 \to K \to (R/ann(M))^n \to M \to 0$$

for some positive integer n. It yields an exact sequence

$$H_I^t(R/ann(M))^n \xrightarrow{\varphi} H_I^t(M) \to H_I^{t+1}(K).$$

Since $\mathrm{Supp}(K) \subseteq \mathrm{Supp}(R/ann(M)) = \mathrm{Supp}(M)$ by [22, Proposition 4.7] one has $\mathrm{cd}_I(K) \leq \mathrm{cd}_I(M) = t$ so that φ is a surjection. We have thus that in order to check the equality $H_I^t(M) = IH_I^t(M)$ it suffices to prove that $H_I^t(R/ann(M)) = IH_I^t(R/ann(M))$. Moreover, once $\mathrm{Supp}(R/ann(M)) = \mathrm{Supp}(M)$ we have

$$\begin{aligned} t = \mathrm{cd}_I^R(M) &= \mathrm{cd}_{I+ann(M)/ann(M)}^{R/ann(M)}(M) \\ &= \mathrm{cd}_{I+ann(M)/ann(M)}^{R/ann(M)}(R/ann(M)) = \mathrm{cd}_I(R/ann(M)), \end{aligned}$$

hence we may suppose $M = R$ and $\mathrm{cd}_I(R) = t \geq 1$.

For an arbitrary R-module N the spectral sequence

$$\mathrm{Tor}_p^R(N, H_I^q(R)) \Rightarrow_p H_I^{q-p}(N)$$

(see Appendix B.2.6) assures the existence of a functorial isomorphism

$$H_I^t(N) \simeq N \otimes_R H_I^t(R).$$

From this,

$$H_I^t(R)/IH_I^t(R) \simeq R/I \otimes_A H_I^t(R) \simeq H_I^t(R/I) = 0$$

since $t \geq 1$.

The last part follows directly from Nakayama's lemma. ■

As immediate consequence of Corollary 3.1.2 and Lemma 3.1.5 we have the next relation.

Corollary 3.1.6. *If F is a finitely generated graded free R-module, then*

$$H_B^{m+n+1}(F) = BH_B^{m+n+1}(F).$$

The two following results have also been proved in the Noetherian case in [28].

Lemma 3.1.7. *If I and J are two ideals of R generated by the finite sequences $\mathfrak{a}$ and $\mathfrak{b}$ respectively, M is an R-module and s, t are two non-negative integers such that*

(i) $H_I^{s+t-i}(H_J^i(M)) = 0$ *for all* $i \in \{0, ..., s+t\} \setminus \{t\}$,

(ii) $H_I^{s+t-i+1}(H_J^i(M)) = 0$ *for all* $i \in \{0, ..., t-1\}$, *and*

(iii) $H_I^{s+t-i-1}(H_J^i(M)) = 0$ *for all* $i \in \{t+1, ..., s+t\}$

then we have isomorphism $H_I^s(H_J^t(M)) \simeq H_{I+J}^{s+t}(M)$.

Proof. Consider the Čech complexes $C_{\mathfrak{a}}(R)$ and $C_{\mathfrak{b}}(M)$ (see Definition 1.1.1). The double complex $C_{\mathfrak{a}}(R) \otimes_R C_{\mathfrak{b}}(M)$ induces a spectral sequence

$$E_2^{p,q} = H_I^p(H_J^q(M)) \Rightarrow_p H_{I+J}^{p+q}(M).$$

(The interested reader may see Appendix B.2.7 for the construction of this spectral sequence.) The hypothesis (i) says that $E_2^{p,q} = 0$ whenever $p+q = s+t$ and $q \neq t$. Meanwhile (ii) and (iii) implies $E_r^{s-r,t+r-1} = 0$ and $E_r^{s+r,t-r+1} = 0$ for $r \geq 2$. By convergence we have $E_\infty^{s,t} \simeq H_{I+J}^{s+t}(M)$ and the differentials

$$E_r^{s-r,t+r-1} \longrightarrow E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$$

are all zero for $r \geq 2$ so that $E_2^{s,t} \simeq E_\infty^{s,t}$. ■

Corollary 3.1.8. *If I and J are two finitely generated ideals of R and M is a finitely presented R-module, then*

$$H_{I+J}^{\mathrm{cd}_I(M)+\mathrm{cd}_J(M)}(M) \simeq H_I^{\mathrm{cd}_I(M)}(H_J^{\mathrm{cd}_J(M)}(M)).$$

Moreover, the following statements are equivalent.

(i) $\mathrm{cd}_{I+J}(M) = \mathrm{cd}_I(M) + \mathrm{cd}_J(M)$.

(ii) $\mathrm{cd}_I(M) = \mathrm{cd}_I(H_J^{\mathrm{cd}_J(M)}(M))$

(iii) $\mathrm{cd}_J(M) = \mathrm{cd}_J(H_I^{\mathrm{cd}_I(M)}(M))$.

Proof. The isomorphism follows directly from [22, Proposition 4.7] and Lemma 3.1.7. If (i) holds then $H_I^{\mathrm{cd}_I(M)}(H_J^{\mathrm{cd}_J(M)}(M)) \neq 0$ so that

$$\mathrm{cd}_I(M) \leq \mathrm{cd}_I(H_J^{\mathrm{cd}_J(M)}(M)).$$

[22, Proposition 4.7] assures equality. Conversely, if (ii) holds true then by the isomorphism we have $H_{I+J}^{\mathrm{cd}_I(M)+\mathrm{cd}_J(M)}(M) \neq 0$ which implies

$$\mathrm{cd}_I(M) + \mathrm{cd}_J(M) \leq \mathrm{cd}_{I+J}(M).$$

We have equality by [22, Proposition 4.2]. Similarly, one proves the equivalence between (i) and (iii). ■

The next result is an immediate consequence of Proposition 3.1.3 and Corollary 3.1.8. It provides another proof (independent of the Mayer-Vietoris long exact sequence 2.3.1) for the first isomorphism in Proposition 3.1.1. Notice that Corollary 3.1.8 also proves again what concerns cohomological dimension in Proposition 3.1.3.

Corollary 3.1.9. *If F is a finitely generated graded free R-module, then*

$$H_B^{m+n+1}(F) \simeq H_{\mathfrak{m}}^{m+n+2}(F),$$

$$\mathrm{cd}_{B_1}(H_{B_2}^{n+1}(F)) = m+1 \text{ and } \mathrm{cd}_{B_2}(H_{B_1}^{m+1}(F)) = n+1.$$

3.1.2 Duality

In this section, we present an interesting duality involving local cohomology modules supported in the ideals B_1 and B_2 which is quite similar to local duality; see, for instance, [17] and [58].

Here, throughout this section, we suppose that $S = \mathbf{k}$ is a field and denote $\_^{\vee} = {}^*\operatorname{Hom}_{\mathbf{k}}(\_, \mathbf{k})$. By [17, Example 14.5.17] and Example 1.2.5 we have that the canonical module ω_R of R is

$$\omega_R = R(-(m+1), -(n+1)).$$

Proposition 3.1.10. *There exists graded isomorphism of R-modules*

$$H_{B_2}^{n+1}(R) \simeq H_{B_1}^{m+1}(\omega_R)^{\vee}.$$

Proof. First we shall use the $\mathbf{k}$-vector space structure of both

$$H_{B_1}^{m+1}(R) = \mathbf{k}[X_0^{-1}, ..., X_m^{-1}][Y_0, ..., Y_n]$$

and

$$H_{B_2}^{n+1}(R) = \mathbf{k}[X_0, ..., X_m][Y_0^{-1}, ..., Y_n^{-1}]$$

to define an isomorphism between $\mathbf{k}$-vector spaces; see Example 1.1.3.

Given $(a,b) \in \mathbb{Z}^2$, once

$${}^*\operatorname{Hom}_{\mathbf{k}}(H_{B_1}^{m+1}(R), \mathbf{k})_{(a,b)} = \operatorname{Hom}_{\mathbf{k}}(H_{B_1}^{m+1}(R)_{(-a,-b)}, \mathbf{k})$$

we define

$$\begin{array}{rcl} \varphi_{(a,b)} : {}^*\operatorname{Hom}_{\mathbf{k}}(H_{B_1}^{m+1}(R), \mathbf{k})_{(a,b)} & \to & H_{B_2}^{n+1}(R)_{(a-(m+1), b-(n+1))} \\ \left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^* & \mapsto & \underline{X}^{\underline{i}-\underline{1}}/\underline{Y}^{\underline{j}+\underline{1}} \end{array}$$

where the underlines denote the sequences $\underline{X}^{\underline{i}} = X_0^{i_0} \cdot \ldots \cdot X_m^{i_m}$, $\underline{X}^{\underline{i}-\underline{1}} = X_0^{i_0-1} \cdot \ldots \cdot X_m^{i_m-1}$ and so on.

Since this mapping sends basis to basis we only need to prove that it is well-defined and surjective. Indeed, an element $\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*$ having degree (a,b) means that $\sum_{l=0}^{m} i_l = a$ and $\sum_{l=0}^{n} j_l = -b$ which implies that $\sum_{l=0}^{m} i_l - 1 = a-(m+1)$ and $\sum_{l=0}^{n} j_l + 1 = -b+(n+1)$ and thus $\underline{X}^{\underline{i}-\underline{1}}/\underline{Y}^{\underline{j}+\underline{1}}$ has degree $(a-(m+1), b-(n+1))$, so $\varphi_{(a,b)}$ is well-defined. Furthermore, given $\underline{X}^{\underline{i}'}/\underline{Y}^{\underline{j}'}$ in $H^{n+1}_{B_2}(R)_{(a-(m+1),b-(n+1))}$, by writing $i_l = i_l'+1$ and $j_l = j_l'-1$ for all l one has $\varphi_{(a,b)}\left(\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\right) = \underline{X}^{\underline{i}'}/\underline{Y}^{\underline{j}'}$.

Therefore we have constructed a $\mathbf{k}$-vector space isomorphism

$$\varphi : {}^*\operatorname{Hom}_{\mathbf{k}}(H^{m+1}_{B_1}(R), \mathbf{k}) \to H^{n+1}_{B_2}(R)(-(m+1), -(n+1)).$$

The result will follow by proving that φ is indeed an R-homomorphism. For this, it suffices to prove its R-linearity on monomials $\underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}$.

First, notice that

$$\begin{aligned}
\left[\left(\underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}\right)\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\right]\left(\underline{Y}^{\underline{p}}/\underline{X}^{\underline{q}}\right) &= \left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\left(\underline{Y}^{\underline{s}+\underline{p}}/\underline{X}^{\underline{q}-\underline{r}}\right) \\
&= \begin{cases} 1, & \text{if } \underline{j} = \underline{s}+\underline{p} \text{ and } \underline{i} = \underline{q}-\underline{r}, \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} 1, & \text{if } \underline{p} = \underline{j}-\underline{s} \text{ and } \underline{q} = \underline{i}+\underline{r}, \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

That is,

$$\left[\left(\underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}\right)\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\right] = \left(\underline{Y}^{\underline{j}-\underline{s}}/\underline{X}^{\underline{i}+\underline{r}}\right)^*$$

and thus

$$\varphi\left(\left(\underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}\right)\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\right) = \underline{X}^{\underline{i}+\underline{r}-\underline{1}}/\underline{Y}^{\underline{j}-\underline{s}+\underline{1}} = \underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}\left(\underline{X}^{\underline{i}-\underline{1}}/\underline{Y}^{\underline{j}+\underline{1}}\right) = \underline{X}^{\underline{r}}\underline{Y}^{\underline{s}}\varphi\left(\left(\underline{Y}^{\underline{j}}/\underline{X}^{\underline{i}}\right)^*\right).$$

■

Remark 3.1.11. *Herzog and Rahimi in [51, Lemma 1.2] proved that there exists isomorphism of bigraded R-modules*

$$H^{n+1}_{B_2}(R)^\vee \simeq H^{m+1}_{B_1}(\omega_R).$$

Hence by Proposition 3.1.10 we conclude that

$$H^{n+1}_{B_2}(R) \simeq H^{m+1}_{B_1}(\omega_R)^\vee \text{ and } H^{n+1}_{B_2}(R)^\vee \simeq H^{m+1}_{B_1}(\omega_R).$$

We are now ready to state the duality-type theorem.

Theorem 3.1.12. *If M is a finitely generated graded R-module then one has functorial graded isomorphism*

$$H_{B_2}^{n+1-i}(M) \simeq {}^*\operatorname{Ext}_R^i(M, H_{B_1}^{m+1}(\omega_R))^\vee$$

for all $i \geq 0$.

Proof. We consider the graded version of Lemma B.3 and Theorem B.4. By taking $N = H_{B_1}^{m+1}(R)$ and $P = \mathbf{k}$ in these two results we have two spectral sequences

$${}^*\operatorname{Ext}_{\mathbf{k}}^p({}^*\operatorname{Ext}_R^q(M, H_{B_1}^{m+1}(R)), \mathbf{k}) \Rightarrow_p H^{q-p}$$

and

$$\operatorname{Tor}_p^R(M, {}^*\operatorname{Ext}_{\mathbf{k}}^q(H_{B_1}^{m+1}(R), \mathbf{k})) \Rightarrow_p H^{p-q}.$$

Both spectral sequences collapses at their second pages so that

$${}^*\operatorname{Hom}_{\mathbf{k}}\left({}^*\operatorname{Ext}_R^i(M, H_{B_1}^{m+1}(R)), \mathbf{k}\right) \simeq H^i \simeq \operatorname{Tor}_i^R\left(M, {}^*\operatorname{Hom}_{\mathbf{k}}(H_{B_1}^{m+1}(R), \mathbf{k})\right)$$

for all $i \geq 0$. Now Proposition 3.1.10 implies that

$$\operatorname{Tor}_i^R\left(M, {}^*\operatorname{Hom}_{\mathbf{k}}(H_{B_1}^{m+1}(R), \mathbf{k})\right) \simeq \operatorname{Tor}_i^R\left(M, H_{B_2}^{n+1}(R)\right)(-(m+1), -(n+1))$$

for all $i \geq 0$. Since B_2 is generated by a R-regular sequence, the Čech complex of R with respect to B_2 is a flat resolution of $H_{B_2}^{n+1}(R)$, thus

$$\operatorname{Tor}_i^R\left(M, H_{B_2}^{n+1}(R)\right) \simeq H_{B_2}^{n+1-i}(M)$$

for all $i \geq 0$, whence the result. ∎

By taking $B_1 = (0)$ in Theorem 3.1.12 we recover the graded local duality in the standard case.

Corollary 3.1.13. *One has*

$${}^*\operatorname{Ext}_R^i(\mathbf{k}, H_{B_1}^{m+1}(R)) = 0$$

for all $i \neq n+1$, and

$${}^*\operatorname{Hom}_{\mathbf{k}}({}^*\operatorname{Ext}_R^{n+1}(\mathbf{k}, H_{B_1}^{m+1}(R)), \mathbf{k}) \simeq \mathbf{k}(-(m+1), -(n+1)).$$

Particularly, $\operatorname{depth}_R H_{B_1}^{m+1}(R) = n+1.$

Proof. It is an immediate consequence of Theorem 3.1.12 and from the fact that $M = \mathbf{k}$ has finite length. ∎

It should be noticed that the roles of $H^{m+1}_{B_1}(R)$ and $H^{n+1}_{B_2}(R)$ are interchangeable in the demonstrations of Proposition 3.1.10 and Theorem 3.1.12. For this reason we just enunciate such similar duality and its corollary.

Theorem 3.1.14. *If M is a finitely generated graded R-module then one has functorial graded isomorphism*

$$H^{m+1-i}_{B_1}(M) \simeq {}^*\operatorname{Ext}^i_R(M, H^{n+1}_{B_2}(\omega_R))^\vee$$

for all $i \geq 0$.

Corollary 3.1.15. *One has*

$${}^*\operatorname{Ext}^i_R(\mathbf{k}, H^{n+1}_{B_2}(R)) = 0$$

for all $i \neq m+1$, and

$${}^*\operatorname{Hom}_{\mathbf{k}}({}^*\operatorname{Ext}^{m+1}_R(\mathbf{k}, H^{n+1}_{B_2}(R)), \mathbf{k}) \simeq \mathbf{k}(-(m+1), -(n+1)).$$

Particularly, $\operatorname{depth}_R H^{n+1}_{B_2}(R) = m+1$.

As immediate consequence of corollaries 3.1.9, 3.1.13 and 3.1.15 we get the following.

Corollary 3.1.16. *If F is a finitely generated graded free R-module, then*

$$\operatorname{depth}_R(H^{m+1}_{B_1}(F)) = \operatorname{cd}_{B_2}(H^{m+1}_{B_1}(F)) = n+1$$

and

$$\operatorname{depth}_R(H^{n+1}_{B_2}(F)) = \operatorname{cd}_{B_1}(H^{n+1}_{B_2}(F)) = m+1.$$

3.2 The general case

Remember that we are considering the notation in 3.0.1. The first Proposition of this section provides a bound for the cohomological dimension of graded R-modules with respect to the ideals $B_{i_0 \dots i_p}$.

Proposition 3.2.1. *Let M be a graded R-module. For all $p \in \{0, ..., k-1\}$ the following inequality holds true*

$$\operatorname{cd}_{B_{i_0 \dots i_p}}(M) \leq \sum_{j=0}^{p} n_{i_j} + (p+1).$$

In particular, $\operatorname{cd}_B(M) \leq \sum_{j=1}^{k} n_j + k$.

Proof. Let F be a graded free R-module. Since every ideal B_i is generated by an R-regular sequence of length n_i+1, from Corollary 2.3.3 we have

$$\mathrm{cd}_{B_{i_0\dots i_p}}(F) \le \sum_{j=0}^{p} \mathrm{cd}_{B_{i_j}}(F) = \sum_{j=0}^{p}(n_{i_j}+1) = \sum_{j=0}^{p} n_{i_j} + (p+1).$$

The result follows from the spectral sequence

$$\mathrm{Tor}_p^R(M, H^q_{B_{i_0\dots i_p}}(F)) \Rightarrow_p H^{q-p}_{B_{i_0\dots i_p}}(M).$$

(See Appendix B.2.6.) ■

The next result is an attempt to extent Proposition 3.1.1 to $k=3$. But first we need a lemma.

Proposition 3.2.2. *Suppose $k=3$. If F is a graded free R-module, then*

$$H_B^{n_1+n_2+n_3+1}(F) \simeq H_{\mathfrak{m}}^{n_1+n_2+n_3+3}(F).$$

Moreover, if $n_1 \neq n_2$ then

$$H^i_{B_{12}}(F) \simeq \begin{cases} H^{n_1+1}_{B_1}(F), & i = n_1+1, \\ H^{n_2+1}_{B_2}(F), & i = n_2+1, \\ 0, & \textit{else}. \end{cases}$$

And if $n_1 = n_2$ then

$$H^i_{B_{12}}(F) \simeq \begin{cases} H^{n_1+1}_{B_1}(F) \bigoplus H^{n_1+1}_{B_2}(F), & i = n_1+1 \\ 0, & \textit{else}. \end{cases}$$

There are also similar isomorphisms by comparing either n_1 and n_3 or n_2 and n_3.

Proof. We shall prove first the isomorphisms concerning $H^i_{B_{12}}(F)$. Consider the ring $T = S[X_{1,0}, \dots, X_{1,n_1}, X_{2,0}, \dots, X_{2,n_2}]$. From the canonical morphisms

$$T \hookrightarrow T[X_{3,0}, \dots, X_{3,n_3}] \xrightarrow{\simeq} R$$

one has isomorphism

$$H^i_{B_{12}}(F) \simeq H^i_{B_{12}\cap T}(T) \otimes_T F$$

which implies the claim about $H^i_{B_{12}}(F)$ because of Proposition 3.1.1.

A completely analogous argument assures isomorphisms involving $H^i_{B_{13}}(F)$ and $H^i_{B_{23}}(F)$.

Now consider the Mayer-Vietoris spectral sequence 2.2.1

$$E_1^{p,q} = \bigoplus_{i_0<\dots<i_p} H^q(U_{i_0\dots i_p}, \widetilde{F}) \Rightarrow_p H^{p+q}(U_{\mathfrak{m}}, \widetilde{F})$$

and write $d = n_1 + n_2 + n_3 + 3$. Since $d - 3 > 0$ and, by Proposition 3.2.1, $d - 2 > \max\{\mathrm{cd}_{B_j}(F), \mathrm{cd}_{B_{il}}(F)\}$, one has $E_1^{0,d-2} = E_1^{1,d-3} = E_1^{1,d-2} = E_1^{0,d-1} = 0$ so that

$$H_B^{d-2}(F) = E_1^{2,d-3} \simeq H^{d-1}(U_{\mathfrak{m}}, \widetilde{F}) \simeq H_{\mathfrak{m}}^d(F).$$

■

Proposition 3.2.2 and induction allow us to generalize Corollary 3.1.2. It improves the bound obtained in Proposition 3.2.1 (by taking $p = k - 1$).

Corollary 3.2.3. *For any graded free R-module F there exists isomorphism*

$$H_B^{n_1+\ldots+n_k+1}(F) \simeq H_{\mathfrak{m}}^{n_1+\ldots+n_k+k}(F)$$

and $\mathrm{cd}_B(F) = n_1 + \ldots + n_k + 1$. *In particular* $\mathrm{cd}_B(M) \leq n_1 + \ldots + n_k + 1$ *for any graded R-module M.*

Proof. By Propositions 3.1.1 and 3.2.2 and induction one can suppose

$$H^i_{B_{i_0 \ldots i_p}}(F) = 0$$

for $p \leq k-2$ and $i > \max\{n_{i_0} + 1, n_{i_1} + 1, \ldots, n_{i_p} + 1\}$. The convergence of the Mayer-Vietoris Spectral Sequence 2.1.8 gives us the desired isomorphism and the vanishing of $H_B^i(F)$ for $i > n_1 + \ldots + n_k + 1$. The last result follows from the convergence of the spectral sequence B.2.6

$$\mathrm{Tor}_p^R(M, H_B^q(R)) \Rightarrow_p H_B^{q-p}(M).$$

■

Corollary 3.2.3 and Lemma 3.1.5 imply immediately a result similar to Corollary 3.1.6.

Corollary 3.2.4. *If F is a graded free R-module, then*

$$H_B^{n_1+\ldots+n_k+1}(F) = BH_B^{n_1+\ldots+n_k+1}(F).$$

The spectral sequence defined in [66] (which he also calls Mayer-Vietoris spectral sequence) allows us to obtain the local cohomology module of a free module supported on the irrelevant ideal in terms of the local cohomologies of such free module supported on ideals generated by the variables. To some extent, these last local cohomology modules are easier to work with. For ease of consultation we put that spectral sequence in Appendix B.2.2.

Proposition 3.2.5. *Assume R Noetherian. Suppose $n = n_1 = n_2 = ... = n_k$ and let F be a graded free R-module. If $l \geq 0$ then there exists a unique pair (p,q) such that $q - p = l$ and*

$$H^l_B(F) \simeq \bigoplus_{i_0<...<i_p} H^q_{B_{i_0}+...+B_{i_p}}(F).$$

Proof. Consider the spectral sequence in Appendix B.2.2

$$E_1^{-p,q} = \bigoplus_{i_0<...<i_p} H^q_{B_{i_0}+...+B_{i_p}}(F) \Rightarrow_p H^{q-p}_B(F).$$

Since $B_{i_0} + ... + B_{i_p}$ is generated by a R-regular sequence of length $(p+1)(n+1)$ one has $E_1^{-p,q} = 0$ for $q \neq (p+1)(n+1)$. Moreover, if $p \geq 1$ and $p' < p$ then the system of equations

$$\begin{aligned} -p' &= -p + r \\ (p'+1)(n+1) &= (p+1)(n+1) + 1 - r \end{aligned}$$

is equivalent to $-rn = 1$ which has no solution whenever $r \geq 1$. It means that there is no homomorphism $E_1^{-p,(p+1)(n+1)} \to E_1^{-p',(p'+1)(n+1)}$ from which follows that $E_1 = E_\infty$ and the result. ■

3.3 Vanishing

The vanishing of local cohomology modules in the multigraded case is determined by the support 1.2.2 of these modules. Indeed, the support is the region of the "lattice" group G (that graduates the ring) on which the module lies. Thus we present interesting relations involving supports of local cohomology modules. Afterwards, we work on multigraded regularity 1.2.9 and finally we provide conditions for when the graded pieces of local cohomology modules have finite length.

3.3.1 Support

The first proposition allows us to determine the support of local cohomology modules of free modules supported in the irrelevant ideal in terms of the support of local cohomology modules of such free module supported in ideals generated by variables, which are well known when, for instance, the graded free module is a direct sum of the same shifts $R(-\gamma)$ where $\gamma \in \mathbb{Z}^k$.

In this section, we consider again Notation 3.0.1.

Proposition 3.3.1. *Assume R Noetherian. If $F = R^{(L)}$ is a graded free R-module then*

$$C_B^l(F) = \bigcup_{q-p=l} \bigcup_{i_0<...<i_p} C^q_{B_{i_0}+...+B_{i_p}}(F)$$

for all $l \geq 0$ and this union is disjoint.

Proof. Note that

$$C^q_{B_{i_0}+...+B_{i_p}}(F) = C^q_{B_{i_0}+...+B_{i_p}}(R) = \emptyset$$

if $q \neq (n_{i_0}+1) + ... + (n_{i_p}+1)$ and writing $d := (n_{i_0}+1) + ... + (n_{i_p}+1)$, by Example 1.2.5 one has

$$\begin{aligned}&H^d_{B_{i_0}+...+B_{i_p}}(R) = \\ &S[X^-_{i_j,l} : \ j = 0, ..., p \text{ and } l = 0, ..., n_{i_j}][X_{j,l_j} : \ j \neq i_0, ..., i_p \text{ and } l_j = 0, ..., n_j]\end{aligned}$$

and

$$\begin{aligned}&C_{B_{i_0}+...+B_{i_p}}(R) = \mathrm{Supp}_{\mathbb{Z}^k}(H^d_{B_{i_0}+...+B_{i_p}}(F)) = \\ &\mathbb{Z}^{i_0-1}_{\geq 0} \times \mathbb{Z}_{\leq -(n_{i_0}+1)} \times \mathbb{Z}^{i_1-i_0-1}_{\geq 0} \times \mathbb{Z}_{\leq (n_{i_1}+1)} \times ... \times \mathbb{Z}_{\geq (n_{i_p}+1)} \times \mathbb{Z}^{k-i_p}_{\geq 0}.\end{aligned}$$

Given $\{i_0, ..., i_p\}, \{j_0, ..., j_q\} \subseteq \{1, ..., k\}$ with $\{i_0, ..., i_p\} \neq \{j_0, ..., j_q\}$ the intersection

$$\mathrm{Supp}_{\mathbb{Z}^k}(H^{(n_{i_0}+1)+...+(n_{i_p}+1)}_{B_{i_0}+...+B_{i_p}}(F)) \cap \mathrm{Supp}_{\mathbb{Z}^k}(H^{(n_{j_0}+1)+...+(n_{j_q}+1)}_{B_{j_0}+...+B_{j_q}}(F))$$

must be empty because if $l \in \{j_0, ..., j_q\} \setminus \{i_0, ..., i_p\}$ then the l-th coordinate of an element in this intersection should has positive and negative sign, an absurd. It proves that the union in the statement is disjoint.

Now, consider the spectral sequence defined in Appendix B.2.2

$$E_1^{-p,q} = \bigoplus_{i_0<...<i_p} H^q_{B_{i_0}+...+B_{i_p}}(F) \Rightarrow_p H_B^{q-p}(F).$$

For all

$$\gamma \in \bigcup_{q-p=l} \bigcup_{i_0<...<i_p} C^q_{B_{i_0}+...+B_{i_p}}(F)$$

there exists a unique $(-p, q)$ such that $\gamma \in \mathrm{Supp}_{\mathbb{Z}^k}(H^q_{B_{i_0}+...+B_{i_p}}(F))$ and thus

$$[E_1^{-p',q'}]_\gamma \simeq \begin{cases} [H^q_{B_{i_0}+...+B_{i_p}}(F)]_\gamma, & (p', q') = (p, q) \\ 0, & \text{else.} \end{cases}$$

By convergence,

$$[H^l_B(F)]_\gamma \simeq [H^q_{B_{i_0}+...+B_{i_p}}(F)]_\gamma \neq 0,$$

that is, $\gamma \in C_B^l(F)$. The result follows from the convergence of the spectral sequence and by noticing that

$$\mathrm{Supp}_{\mathbb{Z}^k}(E_1^{-p,q}) = \bigcup_{q-p=l} \bigcup_{i_0<...<i_p} C^q_{B_{i_0}+...+B_{i_p}}(F)$$

■

It should be noticed that Lemma 1.1.4 has been already characterized the local cohomology modules of the ring supported in B in terms of the local cohomology modules supported in the ideals generated by the variables. Proposition 3.3.1 characterizes completely the cohomology module $H^l_B(F)$ because its γ-th graded piece is isomorphic to exactly one of the pieces $[H^q_{B_{i_0}+\ldots+B_{i_p}}(F)]_\gamma$. Note also that this proposition holds if we consider the same torsion in each component of F, that is, $F = R(-\gamma)^{(L)}$ satisfies Proposition 3.3.1 as well.

Corollary 3.3.2. *Assume R Noetherian and let M be a finitely generated graded R-module. Let I be an ideal in R generated by homogeneous elements of the same degree that form a M-regular sequence. If $K_\bullet$ denotes the Koszul complex of such a sequence then for all $l \geq 0$ one has*

$$C^l_B(M/IM) \subseteq \bigcup_{q-p=l} \bigcup_{u-v=q} \bigcup_{i_0<\ldots<i_v} C^u_{B_{i_0}+\ldots+B_{i_v}}(K_p).$$

Proof. The double complex $K_\bullet \otimes C^\bullet_B(R)$ (Appendix B.2.3) defines a spectral sequence

$$E_1^{p,q} = H^q_B(K_p) \Rightarrow_p H^{q-p}_B(M/IM)$$

so that

$$\operatorname{Supp}_{\mathbb{Z}^k}(H^l_B(M/IM)) \subseteq \bigcup_{q-p=l} \operatorname{Supp}_{\mathbb{Z}^k}(H^q_B(K_p)).$$

The result follows from Proposition 3.3.1. ■

The next two results are consequences of the duality 3.1.12.

Proposition 3.3.3. *If $S = \mathbf{k}$ is a field and $k = 2$, then*

$$\operatorname{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Ext}^{n+1}_R(\mathbf{k}, H^{m+1}_{B_1}(R))) = (m+1, n+1)$$

and

$$\operatorname{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Ext}^{m+1}_R(\mathbf{k}, H^{n+1}_{B_2}(R)) = (m+1, n+1).$$

Proof. It follows immediately from corollaries 3.1.13 and 3.1.15. ■

Definition 3.3.4. *Given a graded R-module M, define*

$T^{i_0,\ldots,i_p}_j(M) := \operatorname{Supp}_{\mathbb{Z}^k}(\operatorname{Tor}^R_j(M, R/(B_{i_0} + \cdots + B_{i_p}))$ and $T^{i_0,\ldots,i_p}(M) := \cup_j T^{i_0,\ldots,i_p}_j(M)$

and set

$$\hat{T}^{i_0,\ldots,i_p}_j(M) := T^{i_0,\ldots,i_p}_j(M) - \sum_{l=0}^{p}(n_{i_l}+1)e_{i_l} \text{ and } \hat{T}^{i_0,\ldots,i_p}(M) := \cup_j \hat{T}^{i_0,\ldots,i_p}_j(M).$$

For simplicity,

$$T_j(M) := T^{1,\ldots,k}_j(M) = \operatorname{Supp}_{\mathbb{Z}^k}(\operatorname{Tor}^R_i(M, S)) \quad \text{and} \quad T(M) := \cup_i T_i(M),$$

and similarly for $\hat{T}_j(M)$ and $\hat{T}(M)$.

Proposition 3.3.5. *If $S = \mathbf{k}$ is a field, $k = 2$ and M is a finitely generated graded R-module, then*

$$C^p_{B_2}(M) \subseteq C_{B_2}(R) + T_{n+1-p}(M)$$

and

$$C^p_{B_1}(M) \subseteq C_{B_1}(R) + T_{m+1-p}(M).$$

for all $p \geq 0$.

Proof. We shall prove the first inclusion. Given $p \geq 0$, by the duality 3.1.12,

$$\begin{aligned} C^p_{B_2}(M) &= \mathrm{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Hom}_{\mathbf{k}}({}^*\operatorname{Ext}^{n+1-p}_R(M, H^{m+1}_{B_1}(R)), \mathbf{k})) - (m+1, n+1) \\ &\subseteq -\mathrm{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Ext}^{n+1-p}_R(M, H^{m+1}_{B_1}(R))) - (m+1, n+1). \end{aligned}$$

Let $F_\bullet$ be the minimal graded free R-resolution of M. By [14, Lemma 3.12 (1)] one has

$$\mathrm{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Hom}_R(F_i, H^{m+1}_{B_1}(R))) = C_{B_1}(R) - T_i(M)$$

for all $i \geq 0$. The result follows the fact that

$$\mathrm{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Ext}^{n+1-p}_R(M, H^{m+1}_{B_1}(R))) \subseteq \mathrm{Supp}_{\mathbb{Z}^2}({}^*\operatorname{Hom}_R(F_{n+1-p}, H^{m+1}_{B_1}(R)))$$

and

$$C_{B_1}(R) + (m+1, n+1) = -C^{n+1}_{B_2}(M)$$

by Proposition 3.1.10.

The second part follows from the same argument applied to the second version of the duality, see Theorem 3.1.14. ∎

The next corollary follows immediately from last proposition and from the Mayer-Vietoris long exact sequence 2.3.1.

Corollary 3.3.6. *If $S = \mathbf{k}$ is a field, $k = 2$ and M is a finitely generated R-module, then for all $i \geq 0$ one has*

$$C^i_B(M) \subseteq (C_{B_1}(R) + T_{m+1-i}(M)) \cup (C_{B_2}(R) + T_{n+1-i}(M)) \cup C^{i+1}_{\mathfrak{m}}(M).$$

In the next proposition we do not require $\mathbb{Z}^k$ as being the group for which the ring R is graded; it does hold for any abelian group G. Hence we consider the same ring $R = S[X_{1,0}, ..., X_{1,n_1}, ..., X_{k,0}, ..., X_{k,n_k}]$ but with $\deg(X_{i,j}) = \gamma_i \in G$ for all i and j, and $\mathfrak{m}$ is the ideal generated by the variables $X_{i,j}$.

Proposition 3.3.7. *Suppose S Noetherian and let M be a graded R-module. If x is a homogeneous M-regular element of degree η in a graded ideal $\mathfrak{n}$ then*

$$C^{i+1}_{\mathfrak{n}}(M) + \eta \subseteq C^i_{\mathfrak{n}}(M/xM)$$

for all $i \geq 0$.

If in addition M *is Cohen-Macaulay of dimension* d *and* $x \in \mathfrak{m}$ *then*

$$C^d_{\mathfrak{m}}(M) + \eta = C^{d-1}_{\mathfrak{m}}(M/xM).$$

More generally, if $\mathbf{x} = x_1, ..., x_d$ *is a maximal* M*-regular sequence in* $\mathfrak{n}$ *where* x_i *is homogeneous of degree* η_i *then*

$$C^{i+l}_{\mathfrak{n}}(M) + \sum_{j=1}^{l} \eta_i \subseteq C^i_{\mathfrak{n}}(M/(x_1, ..., x_l)M)$$

and in case of M *being Cohen-Macaulay and* $\mathbf{x} \subseteq \mathfrak{m}$ *one has*

$$C^d_{\mathfrak{m}}(M) + \sum_{i=1}^{l} \eta_i = C^{d-l}_{\mathfrak{m}}(M/(x_1, ..., x_l)M)$$

for all $l \in \{1, ..., d\}$.

Proof. From the exact sequence $0 \to M(-\eta) \xrightarrow{x} M \to M/xM \to 0$ one has long exact sequence

$$\cdots \to H^i_{\mathfrak{n}}(M/xM) \to H^{i+1}_{\mathfrak{n}}(M(-\eta)) \xrightarrow{x} H^{i+1}_{\mathfrak{n}}(M) \to \cdots$$

Hence if $\theta \notin C^i_{\mathfrak{n}}(M/xM)$ then the multiplication by x

$$[H^{i+1}_{\mathfrak{n}}(M)]_{\theta-\eta} \xrightarrow{x} [H^{i+1}_{\mathfrak{n}}(M)]_{\theta}$$

is injective. Since every element of $H^{i+1}_{\mathfrak{n}}(M)$ is annihilated by a power of $\mathfrak{n}$ and $x \in \mathfrak{n}$ we must have $[H^{i+1}_{\mathfrak{n}}(M)]_{\theta-\eta} = 0$, that is, $\theta \notin C^{i+1}_{\mathfrak{n}}(M) + \eta$.

In case of M being Cohen-Macaulay of dimension d one has exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(M/xM) \to H^d_{\mathfrak{m}}(M(-\eta)) \xrightarrow{x} H^d_{\mathfrak{m}}(M) \to 0$$

so that

$$C^{d-1}_{\mathfrak{m}}(M/xM) \subseteq C^d_{\mathfrak{m}}(M(-\eta)) = C^d_{\mathfrak{m}}(M) + \eta$$

by Lemma 1.2.3.

The rest of the proposition follows by induction. ∎

Supports of Tor and of local cohomology

In this subsection, we assume Notation 3.0.1 unless mentioned otherwise.

Definition 3.3.8. *A graded* R*-module* M *is bounded if there exists* μ *and* ν *in* $\mathbb{Z}^k$ *such that*

$$\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu + \mathbb{Z}^k_{\geq 0} \quad \text{and} \quad C_B(M) \cap (\nu + \mathbb{Z}^k_{\geq 0}) = \emptyset.$$

The following result shows that many interesting modules are bounded.

Theorem 3.3.9. *Let M be a finitely generated graded module over R. If S is Noetherian, then M is bounded.*

Proof. Since M is finitely generated, there must exists $\mu \in \mathbb{Z}^k$ such that $\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu + \mathbb{Z}^k_{\geq 0}$. (Note that S does not need to be Noetherian.) Now, due to [14, Theorem 4.14], there exists $\nu \in \mathrm{reg}_B(M)$. Thus $C_B(M) \cap (\nu + \mathbb{Z}^k_{\geq 0}) = \emptyset$. ■

We first show a close connection between the support of local cohomology and the one of Tor modules. For its proof, we need the following result.

Theorem 3.3.10. *Let G be an abelian group, R be a G-graded polynomial ring in n variables over a ring, I a graded ideal generated by a regular sequence of length r and M a G-graded R-module.*

If $F_\bullet$ is a G-graded free R-resolution of M, then there exists a degree zero graded isomorphism

$$H^p_I(M) \simeq H_{r-p}(H^r_I(F_\bullet))$$

for all $p \geq 0$.

Proof. Let $C^\bullet$ be the Čech complex of R with respect to a regular sequence of length r generating I and consider the first quadrant double complex $C^\bullet \otimes_R F_\bullet$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \cdots & \vdots \arrow[d] & \\
0 \arrow[r] & C^0 \otimes_R F_s \arrow[r] \arrow[d] & C^1 \otimes_R F_s \arrow[r] \arrow[d] & \cdots \arrow[r] & C^r \otimes_R F_s \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \\
0 \arrow[r] & C^0 \otimes_R F_1 \arrow[r] \arrow[d] & C^1 \otimes_R F_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C^r \otimes_R F_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^0 \otimes_R F_0 \arrow[r] \arrow[d] & C^1 \otimes_R F_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & C^r \otimes_R F_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & \cdots & 0 &
\end{tikzcd}$$

This double complex gives rise to two spectral sequences converging to a graded module H. The first one E, by taking first homologies in the vertical, is such that $E^{p,0}_2 = H^p_I(M)$ and $E^{p,q}_2 = 0$ for $q \neq 0$. Thus $H^p_I(M) \simeq H^{-p}$ for all $p \geq 0$. On the other hand, if $'E$ denotes the spectral sequence by taking first cohomologies in the horizontal, is such that $'E^{p,r}_2 = H_p(H^r_I(F_\bullet))$ and $'E^{p,q}_2 = 0$ if $q \neq r$ so that $H_p(H^r_I(F_\bullet)) \simeq H^{p-r}$ for all $p \geq 0$. Therefore

$$H^p_I(M) \simeq H^{-p} = H^{(r-p)-r} \simeq H_{r-p}(H^r_I(F_\bullet)).$$

■

As consequence of Theorem 3.3.10 and Lemma 1.1.4 we obtain the following spectral sequence.

Corollary 3.3.11. *Let M be a graded R-module. There exists a spectral sequence of graded modules,*

$$\bigoplus_{1\le i_0<\cdots<i_p\le k} H^{p-q}_{B_{i_0}+\cdots+B_{i_p}}(M) \Rightarrow H^q_B(M).$$

Proof. Let $F_\bullet$ be a minimal free resolution of M, $C^\bullet$ be the Čech complex of R with respect to a sequence of elements generating B and consider the third quadrant double complex $F_\bullet \otimes_R C^\bullet$:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \cdots & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \to & F_m \otimes_R C^0 & \to & \cdots & \to & F_1 \otimes_R C^0 & \to & F_0 \otimes_R C^0 & \to & 0\\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \to & F_m \otimes_R C^1 & \to & \cdots & \to & F_1 \otimes_R C^1 & \to & F_0 \otimes_R C^1 & \to & 0\\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & \ddots & & \vdots & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \to & F_m \otimes_R C^n & \to & \cdots & \to & F_1 \otimes_R C^n & \to & F_0 \otimes_R C^n & \to & 0\\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & \vdots & & \cdots & & \vdots & & \vdots & &
\end{array}$$

Such a double complex yields two spectral sequences that converges to the same graded module H. Also, by taking homologies in the horizontal first we get a spectral sequence $'E$ such that $'E_2^{0,-j} = H^j_B(M)$ and $'E_2^{-i,-j} = 0$ whenever $i \neq 0$; hence $H^j_B(M) \simeq H^j$ for all $j \geq 0$. On the other hand, by Lemma 1.1.4, the other spectral sequence E is such that

$$E_2^{-i,-j} = H_i\left(\bigoplus_{\substack{1\le i_0<\ldots<i_p\le k\\ n_{i_0}+\ldots+n_{i_p}=j-1}} H^{n_{i_0}+\ldots+n_{i_p}+(p+1)}_{B_{i_0}+\ldots+B_{i_p}}(F_\bullet)\right) \simeq \bigoplus_{1\le i_0<\ldots<i_p\le k} H^{p+(j-i)}_{B_{i_0}+\ldots+B_{i_p}}(M)$$

and $E_2^{-i,-j} \Rightarrow_i H^{i-j} \simeq H^{i-j}_B(M)$, whence the result. ∎

It is worth mentioning that the spectral sequence above is a graded version of Lyubeznik's one B.2.2 although ours does not require Noetherianity. In particular, we can remove such a hypothesis from Proposition 3.3.1 and its corollary once the main tool there is the Lyubeznik spectral sequence.

Lemma 3.3.12. *Let $E \subset \mathbb{Z}^k$. The smallest set containing E such that its complement is stable under the addition of $\mathbb{Z}^k_{\geq 0}$ is $E + \mathbb{Z}^k_{\leq 0}$.*

Proof. Let $E^* := E + \mathbb{Z}^k_{\leq 0}$. If $e \notin E^*$, $e+n$ is not in E^* for all $n \in \mathbb{Z}^k_{\geq 0}$ as otherwise $e+n = e'-n'$ with $e' \in E$ and $n' \in \mathbb{Z}^k_{\geq 0}$, the equality $e = e'-n-n'$ then contradicting the fact that $e \notin E^*$. Now E^* is minimal as if $E \subseteq E' \subseteq E^*$, with the complement of E' stable, if $e \in E^* \setminus E'$, then $e+n \in E \subseteq E'$ for some $n \in \mathbb{Z}^k_{\geq 0}$, but also $e+n \notin E'$ due to the stability of the complement of E', thus a contradiction. ■

From now on, we denote by E^* the set $E + \mathbb{Z}^k_{\leq 0}$ as in the proof above.

Theorem 3.3.13. *Let M be a graded module with $\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu + \mathbb{Z}^k_{\geq 0}$ for some $\mu \in \mathbb{Z}^k$. Then*

$$C_{\mathfrak{m}}(M)^* = T(M) + C_{\mathfrak{m}}(R) = \hat{T}(M)^*.$$

Proof. Recall that $C_{\mathfrak{m}}(R) = a + \mathbb{Z}^k_{\leq 0}$ with $a := -(n_1+1, \ldots, n_k+1)$. Thus $T(M) + C_{\mathfrak{m}}(R) = \hat{T}(M) + \mathbb{Z}^k_{\leq 0} = \hat{T}(M)^*$.

Let $F_\bullet$ be a graded free R-resolution of M as in Lemma 1.2.7 *i*). Then,

$$H^i_{\mathfrak{m}}(M) \simeq H_{d-i}(H^d_{\mathfrak{m}}(F_\bullet)),$$

for all $i \geq 0$ by Theorem 3.3.10, where $d := (n_1+1) + \cdots + (n_k+1)$ is the number of variables of R. This shows that $C^i_{\mathfrak{m}}(M) \subseteq \bigcup_{p \leq d-i} T_p(M) + C_{\mathfrak{m}}(R)$, hence $C_{\mathfrak{m}}(M) \subseteq T(M) + C_{\mathfrak{m}}(R)$ and so $C_{\mathfrak{m}}(M)^* \subseteq T(M) + C_{\mathfrak{m}}(R)$.

To show the inverse inclusion, notice that for all $\mu \notin C_{\mathfrak{m}}(M)^*$, $(\mu + \mathbb{Z}^k_{\geq 0}) \cap C_{\mathfrak{m}}(M) = \emptyset$. Write $\mathbf{X} = \{X_i\}$ and consider the first quadrant double complex $C^\bullet_{\mathbf{X}}(R) \otimes_R K_\bullet(\mathbf{X}; M)$. It gives rise to a spectral sequence with first terms $K_i(\mathbf{X}; H^j_{\mathfrak{m}}(M))$ that abuts to $\mathrm{Tor}^R_{i-j}(M, S)$. Once all shifts in $K_i(\mathbf{X}; H^j_{\mathfrak{m}}(M))$ have all p-th coordinates at most n_p for any $p = 1, ..., k$, in degree $\mu - a$ all terms are zero because $K_i(\mathbf{X}; H^j_{\mathfrak{m}}(M))_{\mu-a}$ is a sum of copies of $H^j_{\mathfrak{m}}(M)$ sitting in degrees $\mu + \delta$ for $\delta \in \mathbb{Z}^k_{\geq 0}$. It follows that $\mu - a \notin T(M)$, as claimed. ■

Lemma 3.3.14. *Let M be a graded R-module and $i_0, \ldots, i_p$ be distinct elements in $\{1, \ldots, k\}$. If $\nu \in \mathbb{Z}^{k-(p+1)} \setminus \sum_{j=0}^p e_{i_j}\mathbb{Z}$, then the following statements hold.*

i) $H^j_{B_{i_0}+\cdots+B_{i_p}}(M)_{*,\nu} \simeq H^j_{B_{i_0}+\cdots+B_{i_p}}(M_{*,\nu})$ *as graded T-modules, for all $j \geq 0$;*

ii) $\mathrm{Tor}^R_j(M/R/B_{i_0}+\cdots+B_{i_p})_{*,\nu} \simeq \mathrm{Tor}^T_j(M_{*,\nu}, S)$ *as graded T-modules, for all $j \geq 0$.*

Proof. It follows by noticing that both Čech and Koszul complexes $C^\bullet_{B_{i_0}+\ldots+B_{i_p}}(M)$ and $K_\bullet(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; M)$, with $\mathbf{X}_{i_j} = \{X_{i_j,0}, ..., X_{i_j,n_{i_j}}\}$, are such that $C^\bullet_{B_{i_0}+\ldots+B_{i_p}}(M)_{*,\nu} \simeq C^\bullet_{B_{i_0}+\ldots+B_{i_p}}(M_{*,\nu})$ and $K_\bullet(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; M)_{*,\nu} \simeq K_\bullet(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; M_{*,\nu})$. ■

Let $\mathcal{E}^{i_0,\ldots,i_p}_j \subset \mathbb{Z}^k_{\geq 0}$ be the shifts in $K_j(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; R)$.

In the next theorem, we provide a precise relation between support of Tor modules (or multigraded Betti numbers, when over a field) and support of local cohomology modules.

Theorem 3.3.15. *Let M be a graded R-module with $\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu + \mathbb{Z}^k_{\geq 0}$ for some $\mu \in \mathbb{Z}^k$, and $i_0, \dots, i_p$ be distinct elements in $\{1, \dots, k\}$. Then*

$$\hat{T}^{i_0,\dots,i_p}(M) \subseteq C_{B_{i_0}+\dots+B_{i_p}}(M) + \sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}$$

and

$$C_{B_{i_0}+\dots+B_{i_p}}(M) \subseteq \hat{T}^{i_0,\dots,i_p}(M) + \sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}.$$

As a consequence,

$$C_{B_{i_0}+\dots+B_{i_p}}(M)^* = \hat{T}^{i_0,\dots,i_p}(M)^*.$$

Proof. Write $\mathbf{X}_{i_j} = \{X_{i_j,0}, ..., X_{i_j,n_{i_j}}\}$. By analysing the spectral sequences arising from the double complex $C^{\bullet}_{\mathbf{X}_{i_0},\dots,\mathbf{X}_{i_p}}(R) \otimes_R K_{\bullet}(\mathbf{X}_{i_0}, ..., \mathbf{X}_{i_p}; M)$ we obtain the a spectral sequence whose terms in the second are $K_i(\mathbf{X}_{i_0}, ..., \mathbf{X}_{i_p}; H^j_{B_{i_0}+\dots+B_{i_p}}(M))$ and converges to $\mathrm{Tor}^R_{i-j}(M, R/B_{i_0} + \dots + B_{i_p})$. It follows that

$$T^{i_0,\dots,i_p}_j(M) \subseteq \bigcup_{s \leq \sum_{l=0}^{p}(n_{i_l}+1)-j} C^s_{B_{i_0}+\dots+B_{i_p}}(M) + \mathcal{E}^{i_0,\dots,i_p}_{s+j}$$

for all $j \geq 0$. Note that any element of $\mathcal{E}^{i_0,\dots,i_p}_t$ can be written as a sum of the form $\sum_{l=0}^{p}(n_{i_l}+1)e_{i_l} + \sum_{l=0}^{p} a_l e_{i_l}$ with $a_l \in \mathbb{Z}_{\leq 0}$ for all l. Hence

$$\hat{T}^{i_0,\dots,i_p}(M) \subseteq C_{B_{i_0}+\dots+B_{i_p}}(M) + \sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}$$

and in particular $\hat{T}^{i_0,\dots,i_p}(M)^* \subseteq C_{B_{i_0}+\dots+B_{i_p}}(M)^*$.

Now, consider $T = S[\mathbf{X}_{i_0}, \dots, \mathbf{X}_{i_p}]$, fix $\nu \in \mathbb{Z}^{k-(p+1)} \setminus \sum_{j=0}^{p} e_{i_j}\mathbb{Z}$ and $n = (n_{i_0} + 1) + \dots + (n_{i_p} + 1)$.

Let $F^{\nu}_{\bullet}$ be a graded free T-resolution of $M_{*,\nu}$ as in Lemma 1.2.7 *i*). By Theorem 3.3.10 there exists isomorphism

$$H^j_{B_{i_0}+\dots+B_{i_p}}(M_{*,\nu}) \simeq H_{n-j}(H^n_{B_{i_0}+\dots+B_{i_p}}(F^{\nu}_{\bullet}))$$

for all $j \geq 0$ so that, by Lemma 3.3.14 *i*),

$$\mathrm{Supp}_{\mathbb{Z}^{p+1}}(H^j_{B_{i_0}+\dots+B_{i_p}}(M)_{*,\nu}) = C^j_{B_{i_0}+\dots+B_{i_p}}(M_{*,\nu}) \subseteq \bigcup_{l \leq n-j} T_l(M_{*,\nu}) + C_{B_{i_0}+\dots+B_{i_p}}(T).$$

Once $T_l(M_{*,\nu}) = \mathrm{Supp}_{\mathbb{Z}^{p+1}}(\mathrm{Tor}^R_l(M, R/B_{i_0} + \dots + B_{i_p})_{*,\nu})$ by Lemma 3.3.14 *ii*), and also $C_{B_{i_0}+\dots+B_{i_p}}(T) = -\sum_{l=0}^{p}(n_{i_l}+1)e_{i_l} + \sum_{l=0}^{p} e_{i_l}\mathbb{Z}_{\leq 0} \subseteq \mathbb{Z}^{p+1}$ we have

$$\begin{aligned} C^j_{B_{i_0}+\dots+B_{i_p}}(M) &\subseteq \bigcup_{l \leq n-j} T^{i_0,\dots,i_p}_l(M) - \sum_{l=0}^{p}(n_{i_l}+1)e_{i_l} + \sum_{l=0}^{p} e_{i_l}\mathbb{Z}_{\leq 0} \\ &= \bigcup_{l \leq n-j} \hat{T}^{i_0,\dots,i_p}_l(M) + \sum_{l=0}^{p} e_{i_l}\mathbb{Z}_{\leq 0} \end{aligned}$$

and so

$$C_{B_{i_0}+\cdots+B_{i_p}}(M) \subseteq \hat{T}^{i_0,\ldots,i_p}(M) + \sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}.$$

Therefore $C_{B_{i_0}+\cdots+B_{i_p}}(M)^* \subseteq \hat{T}^{i_0,\ldots,i_p}(M)^*$ ■

Proposition 3.3.16. *Let M be a graded R-module and $i_0, \ldots, i_p$ be distinct elements in $\{1, \ldots, k\}$. Then, for any finitely generated graded ideal $I \subseteq \sqrt{B_{i_0} + \cdots + B_{i_p} + \operatorname{ann}_R(M)}$,*

$$\hat{T}_j^{i_0,\ldots,i_p}(M) \subseteq \bigcup_{r \leq \sum_{l=0}^{p}(n_{i_l}+1)-j} C_I^r(M) + \mathcal{E}_{j+r}^{i_0,\ldots,i_p} - \sum_{s=0}^{p}(n_{i_s}+1)e_{i_s}$$

for all $j \geq 0$.

Proof. Let $C^\bullet$ be the Čech complex of R with respect to a finite generating set of I and consider the double complex $C^\bullet \otimes_R K_\bullet(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; M)$. Such double complex gives rise to a spectral sequence with first terms $K_i(\mathbf{X}_{i_0}, \ldots, \mathbf{X}_{i_p}; H_I^j(M))$ that abuts to $\operatorname{Tor}_{i-j}^R(M, R/B_{i_0} + \cdots + B_{i_p})$ since $I \subseteq \sqrt{B_{i_1} + \cdots + B_{i_s} + \operatorname{ann}_R(M)}$.

Now,

$$T_j^{i_0,\ldots,i_p}(M) \subseteq \bigcup_{l-r=j} C_I^r(M) + \mathcal{E}_l^{i_0,\ldots,i_p} = \bigcup_{r \leq \sum_{l=0}^{p}(n_{i_l}+1)-j} C_I^r(M) + \mathcal{E}_{j+r}^{i_0,\ldots,i_p}$$

whence the result. ■

Corollary 3.3.17. *Let M be a finitely generated graded module and $i_0, \ldots, i_p$ be distinct elements in $\{1, \ldots, k\}$.*

Then, for any finitely generated graded ideal $I \subseteq \sqrt{B_{i_0} + \cdots + B_{i_p} + \operatorname{ann}_R(M)}$,

$$C_{B_{i_0}+\cdots+B_{i_p}}(M) \subseteq C_I(M) + 2\sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}.$$

In particular, if $p \geq 1$,

$$C_{B_{i_0}+\cdots+B_{i_p}}(M)^* \subseteq C_{B_{i_0}+\cdots+B_{i_{p-1}}}(M)^*.$$

Proof. By taking union over j in Proposition 3.3.16 we obtain

$$\hat{T}^{i_0,\ldots,i_p}(M) \subseteq C_I(M) + \sum_{j=0}^{p} e_{i_j}\mathbb{Z}_{\leq 0}.$$

The result follows from Theorem 3.3.15. ■

The next corollary says that, by taking star, all local cohomology modules with respect to the product of the B_i's vanish if and only if all local cohomology modules with respect to each B_i vanish. Note that it is not true if we do not take stars, see Example 1.2.6.

Corollary 3.3.18. *Let M be a finitely generated graded R-module. Then,*

$$\bigcup_i C_{B_i}(M)^* = \bigcup_{i_0,\dots,i_p} C_{B_{i_0}+\dots+B_{i_p}}(M)^* = C_B(M)^*.$$

Proof. First by Corollary 3.3.17,

$$\bigcup_i C_{B_i}(M)^* = \bigcup_{i_0,\dots,i_p} C_{B_{i_0}+\dots+B_{i_p}}(M)^* \subseteq C_B(M)^*.$$

On the other hand, $C_B(M) \subseteq \bigcup_{i_0,\dots,i_p} C_{B_{i_0}+\dots+B_{i_p}}(M)$ by [14, Lemma 2.1]. ■

Proposition 3.3.19. *Let M be a graded R-module. Then,*

$$T_j^{i_0,\dots,i_p,j_0,\dots,j_q}(M) \subseteq \bigcup_{l\leq j} T_{j-l}^{i_0,\dots,i_p}(M) + \mathcal{E}_l^{j_0,\dots,j_q}$$

for all $j \geq 0$.

Proof. From the first quadrant double complex

$$K_\bullet(\mathbf{X}_{j_0},\dots,\mathbf{X}_{j_q}; K_\bullet(\mathbf{X}_{i_0},\dots,\mathbf{X}_{i_p}; M)) \simeq K_\bullet(\mathbf{X}_{i_0},\dots,\mathbf{X}_{i_p},\mathbf{X}_{j_0},\dots,\mathbf{X}_{j_q}; M)$$

yields a spectral sequence with first terms

$$K_\bullet(\mathbf{X}_{j_0},\dots,\mathbf{X}_{j_q}; \mathrm{Tor}_\bullet^R(M, R/B_{i_0}+\dots+B_{i_p}))$$

that abuts to a filtration of $\mathrm{Tor}_\bullet^R(M, R/B_{i_0}+\dots+B_{i_p}+B_{j_0}+\dots+B_{j_q})$. ■

Corollary 3.3.20. *Let M be a graded R-module Then,*

$$\hat{T}^{i_0,\dots,i_p,j_0,\dots,j_q}(M)^* \subseteq \hat{T}^{i_0,\dots,i_p}(M)^*.$$

Definition 3.3.21. *For a graded R-module M and distinct elements $i_0,\dots,i_p$ in the set $\{1,\dots,k\}$, we define*

$$\mathcal{C}^j_{B_{i_0}+\dots+B_{i_p}}(M) = C^j_{B_{i_0}+\dots+B_{i_p}}(M)^* + \mathcal{E}_j^{i_0,\dots,i_p},$$

$$\mathcal{C}_{B_i}(M) := \bigcup_j \mathcal{C}^j_{B_i}(M) \quad \text{and} \quad \mathcal{C}(\mathrm{M}) := \bigcup_{\mathrm{i}} \mathcal{C}_{\mathrm{B_i}}(\mathrm{M}).$$

Remark 3.3.22. *Notice that, with this definition,*

$$(\bigcup_{\nu\in\mathbb{Z}^{k-1}} (-\infty, \mathrm{reg}_{B_1}(M_{*,\nu})) \times \{\nu\})^* = \mathcal{C}_{B_1}(M).$$

Indeed, if $r < \mathrm{reg}_{B_1}(M_{*,\nu})$ then there are $a > 0$ and $j \geq 0$ such that $r \in C^j_{B_1}(M_{*,\nu}) - a + j$, i.e., $(r,\nu) \in C^j_{B_1}(M) + \mathbb{Z}^k_{<0} + \mathcal{E}^1_j \subseteq \mathcal{C}_{B_1}(M)$. On the other hand, given $j \geq 0$, for all $(r,\nu) \in C^j_{B_1}(M) + \mathcal{E}^1_j$, with $\nu \in \mathbb{Z}^{k-1}$, we have $H^j_{B_1}(M_{*,\nu})_{r-j} \neq 0$ and in particular $r \leq \mathrm{reg}_{B_1}(M_{*,\nu})$. But this inequality must be strict once $r \in C^j_{B_1}(M_{*,\nu}) + j$.

Lemma 3.3.23. *Suppose that $S[X_1, \dots, X_n]$ is a standard $\mathbb{Z}$-graded polynomial ring. Let N be a finitely generated graded $S[X_1, \dots, X_n]$ module and t an integer. If $N_{t+\mathbb{Z}_{\geq 0}}$ and $\mathfrak{m}$ denote, respectively, the truncation of N in t and the irrelevant ideal, then*

(i) $C^0_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) = C^0_{\mathfrak{m}}(N) \cap (t + \mathbb{Z}_{\geq 0})$;

(ii) $C^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) = C^1_{\mathfrak{m}}(N) \cup \{\mathrm{Supp}_{\mathbb{Z}}(N/H^0_{\mathfrak{m}}(N)) \cap (t + \mathbb{Z}_{<0})\}$;

(iii) $C^j_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) = C^j_{\mathfrak{m}}(N)$ *for all* $j \geq 2$.

In particular we have

$$C_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \subseteq C_{\mathfrak{m}}(N) \cup (t + \mathbb{Z}_{<0})$$

and

$$\mathcal{C}_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \subseteq \mathcal{C}_{\mathfrak{m}}(N) \cup (t + \mathbb{Z}_{\leq 0}).$$

Proof. First, it is clear that $C^0_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \subseteq C^0_{\mathfrak{m}}(N) \cap (t + \mathbb{Z}_{\geq 0})$. Take $C := N/N_{t+\mathbb{Z}_{\geq 0}}$. By Lemma 1.2.1 we have exact sequence

$$0 \to H^0_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \to H^0_{\mathfrak{m}}(N) \to C \to H^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \to H^1_{\mathfrak{m}}(N) \to 0$$

and isomorphisms $H^j_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) \simeq H^j_{\mathfrak{m}}(N)$ for all $j \geq 2$ whence immediately follow $C^1_{\mathfrak{m}}(N) \subseteq C^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}})$ and item (*iii*). It is also immediate that from the exact sequence we must have $C^0_{\mathfrak{m}}(N) \cap (t + \mathbb{Z}_{\geq 0}) \subseteq C^0_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}})$. Now, given $\gamma \in t + \mathbb{Z}_{<0}$ we have $H^0_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}})_\gamma = (N_{t+\mathbb{Z}_{\geq}})_\gamma = 0$ so we have exact sequence

$$0 \to (N/H^0_{\mathfrak{m}}(N))_\gamma \to H^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}})_\gamma \to H^1_{\mathfrak{m}}(N)_\gamma \to 0$$

and thus $C^1_{\mathfrak{m}}(N) \cup \{\mathrm{Supp}_{\mathbb{Z}}(N/H^0_{\mathfrak{m}}(N)) \cap (t + \mathbb{Z}_{<0})\} \subseteq C^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}})$. On the other hand, since $C^1_{\mathfrak{m}}(N_{t+\mathbb{Z}_{\geq 0}}) = C^1_{\mathfrak{m}}(N) \cup X$ with $X = \mathrm{Supp}_{\mathbb{Z}}(\mathrm{coker}(H^0_{\mathfrak{m}}(N) \to C)) \subseteq \mathrm{Supp}_{\mathbb{Z}}(C) \subseteq t + \mathbb{Z}_{<0}$, we have again the exact sequence above with $\gamma \in X$ and then $X \subseteq \mathrm{Supp}_{\mathbb{Z}}(N/H^0_{\mathfrak{m}}(N)) \cap (t + \mathbb{Z}_{<0})$. ■

Proposition 3.3.24. *Let M be a graded R-module, $\mu = (\mu_1, \mu') \in \mathbb{Z} \times \mathbb{Z}^{k-1}$. If $M_{\mu+\mathbb{Z}^k_{\geq 0}}$ is the truncation of M in μ, then*

$$C_{B_1}(M_{\mu+\mathbb{Z}^k_{\geq 0}}) \subseteq (C_{B_1}(M) \cap (\mu + \mathbb{Z}^k_{\geq 0})) \cup (\mu_1 + \mathbb{Z}_{<0}) \times (\mu' + \mathbb{Z}^{k-1}_{\geq 0})$$

and

$$\mathcal{C}_{B_1}(M_{\mu+\mathbb{Z}^k_{\geq 0}}) \subseteq (\mathcal{C}_{B_1}(M) \cap \mathbb{Z} \times (\mu' + \mathbb{Z}^{k-1}_{\geq 0})) \cup (\mu_1 + \mathbb{Z}_{\leq 0}) \times \mathbb{Z}^{k-1}.$$

In particular, $(\mu_1 + 1) \times \mathbb{Z}^{k-1} \cap \mathcal{C}_{B_1}(M) = \emptyset$ implies $\mathrm{reg}_{B_1}(M_{(\mu_1,*)+\mathbb{Z}^k_{\geq 0}}) \notin \mu_1 + \mathbb{Z}_{>0}$, with $\mathrm{reg}_{B_1}(N) := \sup_{\gamma \in \mathbb{Z}^{k-1}}\{\mathrm{reg}_{B_1}(N_{*,\gamma})\}$ for any graded R-module N.

Proof. First note that for any graded R-module N and $j \geq 0$,

$$C^j_{B_1}(N) = \bigcup_{\gamma' \in \mathbb{Z}^{k-1}} C^j_{B_1}(N_{*,\gamma'}) \times \{\gamma'\}$$

so that, by Lemma 3.3.23,

$$\begin{aligned}
C^j_{B_1}(M) \cap (\mu + \mathbb{Z}^k_{\geq 0}) &= \bigcup_{\gamma' \in \mathbb{Z}^{k-1}} \left(C^j_{B_1}(M_{*,\gamma'}) \times \{\gamma'\} \cap (\mu + \mathbb{Z}^k_{\geq 0})\right) \\
&= \bigcup_{\theta \in \mathbb{Z}^{k-1}_{\geq 0}} C^j_{B_1}(M_{*,\mu'+\theta}) \cap (\mu_1 + \mathbb{Z}_{\geq 0}) \times \{\mu' + \theta\} \\
&= \bigcup_{\theta \in \mathbb{Z}^{k-1}_{\geq 0}} C^j_{B_1}((M_{*,\mu'+\theta})_{\mu_1 + \mathbb{Z}_{\geq 0}}) \cap (\mu_1 + \mathbb{Z}_{\geq 0}) \times \{\mu' + \theta\} \\
&= \bigcup_{\theta \in \mathbb{Z}^{k-1}_{\geq 0}} \left(C^j_{B_1}((M_{*,\mu'+\theta})_{\mu_1 + \mathbb{Z}_{\geq 0}}) \times \{\mu' + \theta\} \cap (\mu + \mathbb{Z}^k_{\geq 0})\right) \\
&= \left(\bigcup_{\theta \in \mathbb{Z}^{k-1}_{\geq 0}} C^j_{B_1}((M_{\mu + \mathbb{Z}^k_{\geq 0}})_{*,\mu'+\theta}) \times \{\mu' + \theta\}\right) \cap (\mu + \mathbb{Z}^k_{\geq 0}) \\
&= \left(\bigcup_{\gamma' \in \mathbb{Z}^{k-1}} C^j_{B_1}((M_{\mu + \mathbb{Z}^k_{\geq 0}})_{*,\gamma'}) \times \{\gamma'\}\right) \cap (\mu + \mathbb{Z}^k_{\geq 0}) \\
&= C^j_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \cap (\mu + \mathbb{Z}^k_{\geq 0})
\end{aligned}$$

for all $j \geq 0$. Therefore

$$C_{B_1}(M) \cap (\mu + \mathbb{Z}^k_{\geq 0}) = C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \cap (\mu + \mathbb{Z}^k_{\geq 0}).$$

Now, since

$$\begin{aligned}
C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) &= (C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \cap (\mu + \mathbb{Z}^k_{\geq 0})) \cup (C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \setminus (\mu + \mathbb{Z}^k_{\geq 0})) \\
&= (C_{B_1}(M) \cap (\mu + \mathbb{Z}^k_{\geq 0})) \cup (C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \setminus (\mu + \mathbb{Z}^k_{\geq 0}))
\end{aligned}$$

and $C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) = \bigcup_{j \geq 0} \bigcup_{\nu \in \mu' + \mathbb{Z}^{k-1}_{\geq 0}} C^j_{B_1}((M_{*,\nu})_{\mu_1 + \mathbb{Z}_{\geq 0}}) \times \{\nu\}$, we have

$$C_{B_1}(M_{\mu + \mathbb{Z}^k_{\geq 0}}) \subseteq (C_{B_1}(M) \cap (\mu + \mathbb{Z}^k_{\geq 0})) \cup (\mu_1 + \mathbb{Z}_{<0}) \times (\mu' + \mathbb{Z}^{k-1}_{\geq 0}).$$

Finally, note that for any graded R-module N and $j \geq 0$,

$$\mathcal{C}^j_{B_1}(N) = C^j_{B_1}(N)^* + \mathcal{E}^1_j = \bigcup_{\gamma \in \mathbb{Z}^{k-1}} \mathcal{C}^j_{B_1}(N_{*,\gamma}) \times (\gamma + \mathbb{Z}^{k-1}_{\leq 0})$$

and

$$\mathcal{C}_{B_1}(N) = \bigcup_{\gamma \in \mathbb{Z}^{k-1}} \mathcal{C}_{B_1}(N_{*,\gamma}) \times (\gamma + \mathbb{Z}^{k-1}_{\leq 0}).$$

By Lemma 3.3.23 again,

$$\begin{aligned}
\mathcal{C}_{B_1}(M_{\mu+\mathbb{Z}^k_{\geq 0}}) &= \bigcup_{\gamma\in\mathbb{Z}^{k-1}} \mathcal{C}^j_{B_1}((M_{\mu+\mathbb{Z}^k_{\geq 0}})_{*,\gamma}) \times (\gamma+\mathbb{Z}^{k-1}_{\leq 0}) \\
&= \bigcup_{\nu\in\mu'+\mathbb{Z}^{k-1}_{\geq 0}} \mathcal{C}_{B_1}((M_{*,\nu})_{\mu_1+\mathbb{Z}_{\geq 0}}) \times (\nu+\mathbb{Z}^{k-1}_{\leq 0}) \\
&\subseteq \left(\left(\bigcup_{\gamma\in\mathbb{Z}^{k-1}} \mathcal{C}_{B_1}(M_{*,\gamma}) \times (\gamma+\mathbb{Z}^{k-1}_{\leq 0})\right) \cap \mathbb{Z}\times(\mu'+\mathbb{Z}^{k-1}_{\geq 0})\right) \cup (\mu_1+\mathbb{Z}_{\leq 0})\times\mathbb{Z}^{k-1} \\
&= (\mathcal{C}_{B_1}(M)\cap\mathbb{Z}\times(\mu'+\mathbb{Z}^{k-1}_{\geq 0})) \cup (\mu_1+\mathbb{Z}_{\leq 0})\times\mathbb{Z}^{k-1}.
\end{aligned}$$

The particular case follows from Remark 3.3.22. ■

It should be noticed that we can fix any other entry of $\mu = (\mu_1,\ldots,\mu_k)$ instead of μ_1 in Proposition 3.3.24 and obtain a similar result. The next corollary is a consequence of this fact.

Corollary 3.3.25. *Let M be a graded R-module with $\mathrm{Supp}_{\mathbb{Z}^k}(M) \subseteq \mu+\mathbb{Z}^k_{\geq 0}$ for some $\mu\in\mathbb{Z}^k$. Then*

$$\hat{T}(M_{\mu+\mathbb{Z}^k_{\geq 0}})^* \subseteq \bigcap_{i\geq 0}\left((C_{B_i}(M)\cap(\mu+\mathbb{Z}^k_{\geq 0})) \cup \prod_{j=1}^{i-1}\mu_j+\mathbb{Z}_{\geq 0}\times(\mu_i+\mathbb{Z}_{<0})\times\prod_{j=i+1}^{k}\mu_j+\mathbb{Z}_{\geq 0}\right)^*.$$

Proof. The result follows by applying Theorem 3.3.15, Corollary 3.3.20 and Proposition 3.3.24:

$$\begin{aligned}
\hat{T}(M_{\mu+\mathbb{Z}^k_{\geq 0}})^* &\subseteq \bigcap_{i\geq 0}\hat{T}^i(M_{\mu+\mathbb{Z}^k_{\geq 0}})^* = \bigcap_{i\geq 0} C_{B_i}(M_{\mu+\mathbb{Z}^k_{\geq 0}})^* \\
&\subseteq \bigcap_{i\geq 0}\left((C_{B_i}(M)\cap(\mu+\mathbb{Z}^k_{\geq 0})) \cup \prod_{j=1}^{i-1}\mu_j+\mathbb{Z}_{\geq 0}\times(\mu_i+\mathbb{Z}_{<0})\times\prod_{j=i+1}^{k}\mu_j+\mathbb{Z}_{\geq 0}\right)^*.
\end{aligned}$$

■

3.3.2 Castelnuovo-Mumford regularity

In this section we consider the same notation and definitions as those of Section 1.2.9. We begin this section by applying Proposition 3.3.7 to regularity. We consider again R being graded by an arbitrary abelian group G.

Proposition 3.3.26. *Suppose S is Noetherian, let M be a graded R-module and $\mathbf{x} = x_1,...,x_d$ a maximal M-regular sequence contained in a graded ideal $\mathfrak{n}$ where x_i is a homogeneous element of degree γ_j for some j. The following assertions hold true.*

(i) If $\gamma = \gamma_i$ for all $i = 1, ..., k$ then

$$\mathrm{reg}_{\mathfrak{n}}(M/(x_1, ..., x_l)M) \subseteq \mathrm{reg}^l_{\mathfrak{n}}(M)$$

for all $l = 1, ..., d$.

(ii) If M is Cohen-Macaulay and $\underline{x}$ is contained in $\mathfrak{m}$ then

$$\mathrm{reg}_{\mathfrak{m}}(M) \subseteq \mathrm{reg}_{\mathfrak{m}}(M/(x_1, ..., x_l)M).$$

for all $l = 1, ..., d$. The equality holds in case of $\gamma = \gamma_i$ for all $i = 1, ..., k$.

Proof. (i) Since $\mathcal{F}_i = \{i\gamma\}$ for all i and $\mathcal{F}_{i-1} = l\gamma + \mathcal{F}_{i-l-1}$ for all $l = 1, ..., d$ (see Definition 1.2.8), then

$$\begin{aligned}\bigcup_{i\geq l} \mathrm{Supp}_G(H^i_{\mathfrak{n}}(M)) + \mathcal{F}_{i-1} &= \bigcup_{i\geq 0} \mathrm{Supp}_G(H^{i+l}_{\mathfrak{n}}(M)) + l\gamma + \mathcal{F}_{i-1} \\ &\subseteq \bigcup_{i\geq 0} \mathrm{Supp}_G(H^i_{\mathfrak{n}}(M/(x_1, ..., x_l)M)) + \mathcal{F}_{i-1}.\end{aligned}$$

(ii) The inclusion follows from

$$\begin{aligned}\mathrm{Supp}_G(H^{d-l}_{\mathfrak{m}}(M/(x_1, ..., x_l)M)) + \mathcal{F}_{d-l-1} &= \mathrm{Supp}_G(H^d_{\mathfrak{m}}(M)) + \sum_{i=1}^{l} \gamma_i + \mathcal{F}_{d-l-1} \\ &\subseteq \mathrm{Supp}_G(H^d_{\mathfrak{m}}(M)) + \mathcal{F}_{d-1}\end{aligned}$$

meanwhile the equality follows by noticing that $\mathcal{F}_{d-1} = \mathcal{F}_l + \mathcal{F}_{d-l-1}$ for all $l = 1, ..., d$.

■

Example 3.3.27. *By taking $m = n = 1$ in Proposition 3.1.1 we have*

$$H^3_B(R) \simeq H^4_{\mathfrak{m}}(R)$$

and

$$H^2_B(R) \simeq H^2_{B_1}(R) \oplus H^2_{B_2}(R).$$

Hence

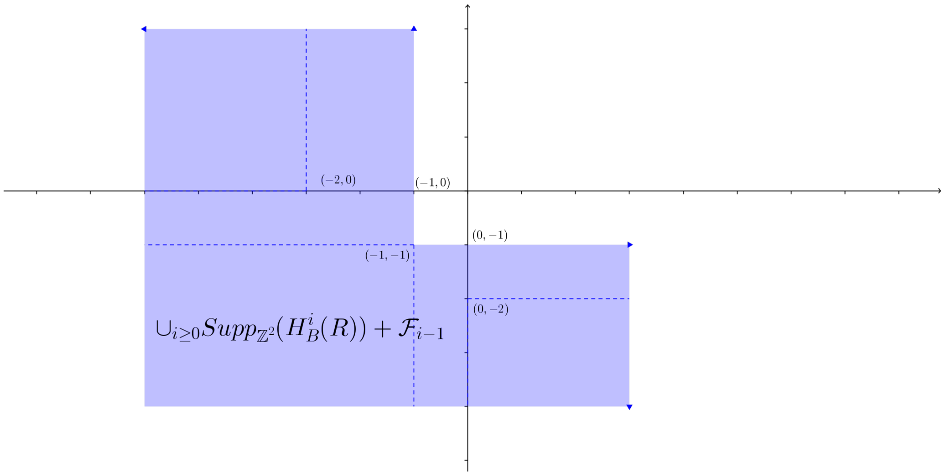
$$\bigcup_{i\geq 0} C^i_B(R) + \mathcal{F}_{i-1} = (C_{B_1}(R) + \mathcal{F}_1) \cup (C_{B_2}(R) + \mathcal{F}_1) \cup (C_{\mathfrak{m}}(R)) + \mathcal{F}_2)$$

and by Example 1.2.6 we obtain

$$\begin{aligned}\bigcup_{i\geq 0} C^i_B(R) + \mathcal{F}_{i-1} = (&\mathbb{Z}_{\leq -1} \times \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{\leq -2} \times \mathbb{Z}_{\geq 1}) \cup (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\leq -2} \cup \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq -1}) \\ &\cup (\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\leq -2} \cup \mathbb{Z}^2_{\leq -1} \cup \mathbb{Z}_{\leq -2} \times \mathbb{Z}_{\leq 0}).\end{aligned}$$

The complement of the union above is exactly $\mathbb{Z}^2_{\geq 0}$ which is trivially stable under itself and therefore

$$\operatorname{reg}_B(R) = \mathbb{Z}^2_{\geq 0}.$$



Remark 3.3.28. *In the same way as with $\operatorname{reg}_{\mathfrak{m}}(R)$ (see Example 1.2.10), the regularity $\operatorname{reg}_B(R)$ is not determined by a single element in general. For instance, in the Hirzebruch surface $\mathbb{F}_2$ its coordinate ring R is such that*

$$\operatorname{reg}_B(R) = ((1,0) + \mathbb{Z}^2_{\geq 0}) \cup ((0,1) + \mathbb{Z}^2_{\geq 0}).$$

(See [69, Example 1.2] for details.)

In examples 1.2.10 and 3.3.27 we note that

$$\operatorname{reg}_B(R) \subseteq \operatorname{reg}_{\mathfrak{m}}(R).$$

It inspires us to look for relations between regularity concerning different finitely generated graded ideals.

Proposition 3.3.29. *Given integer $l \geq 0$, if $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ are two finitely generated graded ideals of R then*

$$\operatorname{reg}^l_{\mathfrak{a}+\mathfrak{b}}(M) \cap \operatorname{reg}^l_{\mathfrak{a}}(M) \cap \operatorname{reg}^l_{\mathfrak{b}}(M) \subseteq \operatorname{reg}^l_{\mathfrak{a}\cap\mathfrak{b}}(M).$$

Proof. It follows directly from the Mayer-Vietoris long exact sequence 2.3.1

$$\ldots \to H^i_{\mathfrak{a}+\mathfrak{b}}(M) \to H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{b}}(M) \to H^i_{\mathfrak{a}\cap\mathfrak{b}}(M) \to H^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M) \to \ldots$$

∎

From now on, consider $B_1, ..., B_k \subseteq \mathfrak{m}$ finitely generated graded ideals and write $\mathfrak{n} = B_1 + ... + B_k$ and $B = B_1 \cap ... \cap B_k$.

Theorem 3.3.30. *For every $l > 1$ one has*

$$\operatorname{reg}^l_{\mathfrak{n}}(M) \cap \left(\bigcap_{p \leq k-2} \bigcap_{i_0 < \ldots < i_p} \operatorname{reg}^l_{B_{i_0 \ldots i_p}}(M) \right) \subseteq \operatorname{reg}^l_B(M).$$

Proof. Consider the Mayer-Vietoris spectral sequence 2.2.1

$$E_1^{p,q} = \bigoplus_{i_0 < \ldots < i_p} H^q(U_{B_{i_0 \ldots i_p}}, \widetilde{M}) \Rightarrow_p H^{p+q}(U_{\mathfrak{n}}, \widetilde{M}).$$

Fix $\gamma \in \operatorname{reg}^l_{\mathfrak{n}}(M) \cap \left(\bigcap_{p \leq k-2} \bigcap_{i_0 < \ldots < i_p} \operatorname{reg}^l_{B_{i_0 \ldots i_p}}(M) \right)$. Given $\gamma' \in \gamma + \mathcal{C}$ one has

$$\left[E_1^{p,i-1}\right]_{\gamma' - \eta} \simeq \bigoplus_{i_0 < \ldots < i_p} \left[H^i_{B_{i_0 \ldots i_p}}(M)\right]_{\gamma' - \eta} = 0$$

for all $i \geq l$ and $\eta \in \mathcal{F}_{i-1}$. Moreover,

$$\left[H^i_B(M)\right]_{\gamma'-\eta} \simeq \left[E_1^{k-1,i-1}\right]_{\gamma'-\eta} \simeq \left[E_\infty^{k-1,i-1}\right]_{\gamma'-\eta} \simeq \left[H^{i+k-1}_{\mathfrak{n}}(M)\right]_{\gamma'-\eta} = 0$$

for $i \geq l$ and $\eta \in \mathcal{F}_{i-1}$. ∎

Proposition 3.3.31. *Consider $k = 3$ and let $l > \operatorname{cd}_B(M)$. The following statements hold true.*

(i) Assuming also that R is Noetherian, one has

$$\operatorname{reg}^l_{\mathfrak{n}}(M) \cap \left(\bigcap_j \operatorname{reg}^l_{B_j}(M) \right) \subseteq \bigcap_{j_0 < j_1} \operatorname{reg}^l_{B_{j_0} + B_{j_1}}(M).$$

(ii) If $l > 1$ then

$$\operatorname{reg}^l_{\mathfrak{n}}(M) \cap \left(\bigcap_j \operatorname{reg}^l_{B_j}(M) \right) \subseteq \bigcap_{j_0 < j_1} \operatorname{reg}^l_{B_{j_0 j_1}}(M).$$

Proof. For (i) consider the spectral sequence B.2.2

$$E_1^{-p,q} = \bigoplus_{i_0 < \ldots < i_p} H^q_{B_{i_0} + \ldots + B_{i_p}}(M) \Rightarrow_p H^{q-p}_B(M)$$

defined in [66]. This spectral degenerates at E_2. The hypothesis implies the exactness of the sequences

$$H^i_{\mathfrak{n}}(M) \longrightarrow \bigoplus_{j_0 < j_1} H^i_{B_{j_0} + B_{j_1}}(M) \longrightarrow \bigoplus_j H^i_{B_j}(M)$$

for $i \geq l$.

In (ii) we consider the Mayer-Vietoris spectral sequence 2.2.1

$$E_1^{p,q} = \bigoplus_{i_0 < \ldots < i_p} H^q(U_{B_{i_0 \ldots i_p}}, \widetilde{M}) \Rightarrow_p H^{p+q}(U_{\mathfrak{n}}, \widetilde{M})$$

which first page has the following shape

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \oplus H^{l+1}_{B_j}(M) \arrow[r] & \oplus H^{l+1}_{B_{j_0 j_1}} \arrow[r] & 0 & \\
0 \arrow[r] & \oplus H^{l}_{B_j}(M) \arrow[r] & \oplus H^{l}_{B_{j_0 j_1}} \arrow[r] & 0 & \\
0 \arrow[r] & \oplus H^{l-2}(U_{B_j}, \widetilde{M}) \arrow[r] & \oplus H^{l-2}(U_{B_{j_0 j_1}}, \widetilde{M}) \arrow[r] & H^{l-2}(U_B, \widetilde{M}) \arrow[r] & 0
\end{tikzcd}$$

The hypothesis assures that

$$\bigoplus_j H^l_{B_j}(M) \longrightarrow \bigoplus_{j_0 < j_1} H^l_{B_{j_0 j_1}}(M)$$

is a surjection and that

$$\bigoplus_j H^{i+1}_{B_j}(M) \longrightarrow \bigoplus_{j_0 < j_1} H^{i+1}_{B_{j_0 j_1}}(M)$$

is an isomorphism in the suitable degrees for $i \geq l$. ■

Theorem 3.3.30 and Proposition 3.3.31 $ii)$ immediately imply the next corollary.

Corollary 3.3.32. *If $l > max\{1, \mathrm{cd}_B(M)\}$ then*

$$\mathrm{reg}^l_{\mathfrak{n}}(M) \cap \left(\bigcap_j \mathrm{reg}^l_{B_j}(M) \right) \subseteq \mathrm{reg}^l_B(M).$$

3.4 Euler characteristic

Let $k \geq 1$ an integer, $G = \mathbb{Z}^k$ and consider the G-graded polynomial ring $R = S[X_{1,0}, \ldots, X_{1,n_1}, \ldots, X_{k,0}, \ldots, X_{k,n_k}]$ with $\deg(X_{i,j}) = e_i$ for all $j = 0, \ldots, n_i$, where e_i denotes the i-th element of the canonical basis of G and S is a commutative unitary ring.

Denote by $\ell_S(N)$ the length of the S-module N.

Definition 3.4.1. *Let M be a G-graded R-module and I a homogeneous ideal of R such that the graded pieces $[H^j_I(M)]_\gamma$ have finite length as S-modules for all $j \geq 0$ and for all $\gamma \in G$. The* Euler characteristic of M with respect to I and $\gamma \in G$ *is defined as*

$$\chi(M, I, \gamma) = \sum_{j \geq 0} (-1)^j \ell_S([H^j_I(M)]_\gamma).$$

Lemma 3.4.2. [35, Proposition A.2.2] *Let $E_r^{-p,q} \Rightarrow_p H^{q-p}$ be a spectral sequence of S-modules. If there exists $r \geq 1$ such that $E_r^{-p,q}$ has finite length for all p, q, then every H^j has finite length and*

$$\sum_{j\in\mathbb{Z}}(-1)^j \ell_S(H^j) = \sum_{j\in\mathbb{Z}}(-1)^j \left(\sum_{q-p=j} \ell_S(E_s^{-p,q})\right)$$

for all $s \geq r$.

Proposition 3.4.3. *Let M be a G-graded R-module and let $B_1, ..., B_r$ be homogeneous ideals of R and write $B = B_1 \cap ... \cap B_r$. If $[H^j_{B_{i_0}+...+B_{i_p}}(M)]_\gamma$ has finite length as S-modules for all $j \geq 0$ and for all $i_0 < ... < i_p$ with $p = 0, ..., r-1$, then $[H^j_B(M)]_\gamma$ has also finite length for all $j \geq 0$ and for all $\gamma \in G$, and*

$$\chi(M, B, \gamma) = \sum_{p=0}^{r-1}(-1)^p \sum_{i_0<...<i_p} \chi(M, B_{i_0} + ... + B_{i_p}, \gamma)$$

for all $\gamma \in G$.

Proof. Let $\gamma \in G$. By considering the γ-th strand of the spectral sequence B.2.2

$$[E_1^{-p,q}]_\gamma = \bigoplus_{i_0<...<i_p} [H^q_{B_{i_0}+...+B_{i_p}}(M)]_\gamma \Rightarrow_p [H^{q-p}_B(M)]_\gamma,$$

Lemma 3.4.2 assures that $[H^j_B(M)]_\gamma$ has finite length for all $j \geq 0$ and

$$\chi(M, B, \gamma) = \sum_{j\in\mathbb{Z}}(-1)^j \left(\sum_{q-p=j}\left(\sum_{i_0<...<i_p} \ell_S([H^q_{B_{i_0}+...+B_{i_p}}(M)]_\gamma)\right)\right).$$

Now we organize this sum.

$$\chi(M, B, \gamma) =$$

$$= \sum_{j\in\mathbb{Z}}(-1)^j \left[\sum_i \ell_S([H^j_{B_i}(M)]_\gamma) + \sum_{i_0<i_1} \ell_S([H^{j+1}_{B_{i_0}+B_{i_1}}(M)]_\gamma) + ... + \ell_S([H^{j+r-1}_{B_1+...+B_r}(M)]_\gamma)\right]$$

$$= \sum_i \sum_{j\in\mathbb{Z}}(-1)^j \ell_S([H^j_{B_i}(M)]_\gamma) - \sum_{i_0<i_1}\sum_{j\in\mathbb{Z}}(-1)^{j+1}\ell_S([H^{j+1}_{B_{i_0}+B_{i_1}}(M)]_\gamma) +$$

$$+ \sum_{i_0<i_1<i_2}\sum_{j\in\mathbb{Z}}(-1)^j \ell_S([H^{j+2}_{B_{i_0}+B_{i_1}+B_{i_2}}(M)]_\gamma) + ... + (-1)^{r-1}\sum_{j\in\mathbb{Z}}(-1)^{j+r-1}\ell_S([H^{j+r-1}_{B_1+...+B_r}(M)]_\gamma)$$

$$= \sum_i \chi(M, B_i, \gamma) - \sum_{i_0<i_1} \chi(M, B_{i_0} + B_{i_1}, \gamma) + ... + (-1)^{r-1}\chi(M, B_1 + ... + B_r, \gamma)$$

$$= \sum_{p=0}^{r-1}(-1)^p \sum_{i_0<...<i_p} \chi(M, B_{i_0} + B_{i_1} + ... + B_{i_p}, \gamma).$$

■

From now on suppose that S is Noetherian.

Proposition 3.4.4. *Let M be a graded finitely generated R-module. If B is the graded ideal of R generated by the variables $X_{i_1}, ..., X_{i_r}$ and A is the polynomial ring in the variables $X_{i_1}^{-1}, ..., X_{i_r}^{-1}$ and X_l for all $l \neq i_1, ..., i_r$ with coefficients in S then $H_B^j(M)$ is a finitely generated A-module for all $j \geq 0$.*

Proof. By considering the spectral sequence B.2.6

$$\mathrm{Tor}_p^R(M, H_B^q(R)) \Rightarrow_p H_B^{q-p}(M),$$

once B is generated by variables this spectral sequence must collapses so that

$$H_B^j(M) \simeq \mathrm{Tor}_{r-j}^R(M, H_B^r(R))$$

for all $j \geq 0$. On the other hand, $\mathrm{Tor}_l^R(M, H_B^r(R))$ can be computed by tensoring a free resolution of M by $H_B^r(R)$, hence it is a subquotient of finitely many copies of $H_B^r(R)$. Therefore, the result will follow by showing that $H_B^r(R)$ is finitely generated over A. But the R-module $H_B^r(R) = S[X_{i_j}^{-1}][X_l : X_l \neq i_1, ..., i_r]$ has a natural structure of ideal over A. Since S is Noetherian, we must have that $H_B^r(R)$ is a finitely generated ideal over A. ■

Lemma 3.4.5. *Let M be a G-graded finitely generated R-module. If each graded piece M_γ has finite length then*

(i) $\mathrm{Supp}_S(M) \subseteq \mathrm{Max}(S)$*;*

(ii) $\mathrm{Supp}_S(M)$ *is finite.*

Proof. From the decomposition $M = \bigoplus_{\gamma \in G} M_\gamma$ one has

$$\mathrm{Supp}_S(M) \subseteq \bigcup_{\gamma \in G} \mathrm{Supp}_S(M_\gamma).$$

Since each M_γ has finite length, $\mathrm{Supp}_S(M_\gamma)$ must be contained in $\mathrm{Max}(S)$, whence item (i).

Once M is finitely generated, there exists a finite set $H \subseteq G$ such that $M = \sum_{\gamma \in H} R \otimes_S M_\gamma$. Hence for all $\eta \in G$ one has $M_\eta = \sum_{\gamma \in H} R_{\eta-\gamma} \otimes_S M_\gamma$ and there is a natural surjection

$$\bigoplus_{\gamma \in H} R_{\eta-\gamma} \otimes_S M_\gamma \twoheadrightarrow M_\eta.$$

Since $R_{\eta-\gamma}$ is a free S-module, it follows that $\mathrm{Supp}_S(M_\eta) \subseteq \bigcup_{\gamma \in H} \mathrm{Supp}_S(M_\gamma)$. Therefore $\mathrm{Supp}_S(M) \subseteq \bigcup_{\gamma \in H} \mathrm{Supp}_S(M_\gamma)$ must be finite. ■

From now on we assume S is also local.

Proposition 3.4.6. *Let M be a finitely generated graded R-module and let B be the graded ideal of R generated by the variables $X_{i_1}, ..., X_{i_r}$. If each graded component of M has finite length, then, for all $\gamma \in G$, the S-module $[H_B^j(M)]_\gamma$ has finite length for all $j \geq 0$.*

Proof. By Proposition 3.4.4, for all $j \geq 0$, the A-module $H_B^j(M)$ is finitely generated. One should be noticed that the graded structure of $H_B^j(M)$ over R is also a graduation of $H_B^j(M)$ over A, so that each of its components $[H_B^j(M)]_\gamma$ are finitely generated over S. Moreover, if $C_B^j(M)$ denotes the j-th Čech module of M supported in B then

$$\mathrm{Supp}_S([H_B^j(M)]_\gamma) \subseteq \mathrm{Supp}_S(H_B^j(M)) \subseteq \mathrm{Supp}_S(C_B^j(M)) \subseteq \mathrm{Supp}_S(M).$$

Lemma 3.4.5 assures us that the component $[H_B^j(M)]_\gamma$ has dimension zero over S. It proves the result. ∎

Theorem 3.4.7. *Let M be a finitely generated graded R-module and let $B_1, ..., B_r$ be ideals of R generated by variables. Write $B = B_1 \cap ... \cap B_r$. If each graded component of M has finite length, then, for all $\gamma \in G$, the S-module $[H_B^j(M)]_\gamma$ has finite length for all $j \geq 0$ and*

$$\chi(M, B, \gamma) = \sum_{p=0}^{r-1} (-1)^p \sum_{i_0 < ... < i_p} \chi(M, B_{i_0} + ... + B_{i_p}, \gamma).$$

Proof. It follows directly from Propositions 3.4.3 and 3.4.6. ∎

Next Corollary follows immediately from last theorem and [50, Theorem 1.3.1].

Corollary 3.4.8. *Suppose S is a field. Give a finitely generated graded R-module M and B a monomial ideal of R, then, for all $\gamma \in G$, the S-module $[H_B^j(M)]_\gamma$ has finite length for all $j \geq 0$. Moreover, if $B = Q_1 \cap ... \cap Q_r$ where each Q_i is generated by pure powers of the variables, that is, Q_i is of the form $(X_{i_1}^{a_1}, ..., X_{i_k}^{a_k})$, then*

$$\chi(M, B, \gamma) = \sum_{p=0}^{r-1} (-1)^p \sum_{i_0 < ... < i_p} \chi(M, B_{i_0} + ... + B_{i_p}, \gamma),$$

for all $\gamma \in G$, where $B_i = \sqrt{Q_i}$ for all $i = 1, ..., r$.

Chapter 4

On deficiency modules

In this chapter we shall look for relations between Bass and Betti numbers of a given module and of its deficiency modules. As Foxby [38] provided the relations above for Cohen-Macaulay modules over a Gorenstein local ring, we furnish the same relations for generalized Cohen-Macaulay canonically Cohen-Macaulay modules of depth at least two over a local ring which is factor of a Gorenstein local ring, see Theorem 4.2.4. Furthermore, theorems 4.2.6 and 4.3.11 shows the same relations for arbitrary finitely generated R-modules when certain homological conditions over its deficiency modules are imposed.

Besides such generalizations, we exhibit bounds for the Bass numbers (Betti numbers) of a module in terms of the Betti numbers (Bass numbers) of its deficiency modules, see theorems 4.2.1 and 4.3.1. They provide several applications that are worked out through this chapter. Three examples of such applications are Corollary 4.2.3, providing the Cohen-Macaulay property of a local ring in terms of homological conditions over deficiency modules, Corollary 4.3.6 furnishing a characterization of the complete intersection property in terms of the first and second Bass numbers of the residue field, and Corollary 4.3.10 that states that the Auslander-Reiten conjecture holds for modules such that its deficiency modules have finite injective dimensions, generalizing then a similar application given quite recently in [39].

Throughout this chapter, we will follow the notation in Section 1.3. Namely, R will always denote a commutative Noetherian local ring with non-zero unity, maximal ideal $\mathfrak{m}$ and residue class field k. Also, R is supposed to be factor of a Gorenstein local ring S of dimension s, i.e., there exists a surjective ring homomorphism $S \to R$. We denote by $M^\vee$ the Matlis dual of a finitely generated R-module.

For an R-module M, $\mathrm{pd}_R M$ and $\mathrm{id}_R M$ denote, respectively, the projective dimension and injective dimension of M. Further, $\beta_i(M) = \dim_k \mathrm{Tor}_i^R(k, M)$ is the i-th Betti number of M, $\mu^i(M) = \dim_k \mathrm{Ext}_R^i(k, M)$ is the i-th Bass number of M and $\mathrm{r}(M) = \dim_k \mathrm{Ext}_R^{\mathrm{depth}_R M}(k, M)$ is its type.

4.1 Generalized Cohen-Macaulay modules

Our main tool in this section is the Foxby spectral sequences B.2.4. It provides interesting relations between generalized Cohen-Macaulay modules and their deficiency modules.

Definition 4.1.1. *A finitely generated R-module M is said to be* generalized Cohen-Macaulay *if $H_{\mathfrak{m}}^j(M)$ is of finite length for all $j < \dim_R M$.*

It should be noticed, due to Matlis duality, that it is equivalent to say that $K^j(M)$ is of finite length for all $j < \dim_R M$.

Theorem 4.1.2. *Let M be a generalized Cohen-Macaulay R-module of dimension t. The following statements hold true.*

(i) There exists isomorphism

$$K^0(K(M)) \simeq \mathrm{Tor}_{-t}^S(M, S);$$

(ii) There exists a five-term type exact sequence

$$\begin{tikzcd} \mathrm{Tor}_{-t+2}^S(M,S) \arrow[r] & K^2(K(M)) \arrow[r] & K^0(K^{t-1}(M)) \arrow[dl] & \\ & \mathrm{Tor}_{-t+1}^S(M,S) \arrow[r] & K^1(K(M)) \arrow[r] & 0 \end{tikzcd}$$

(iii) There exists an exact sequence

$$0 \to K^0(K^0(M)) \to M \to K(K(M)) \to K^0(K^1(M)) \to 0;$$

(iv) If $t \geq 3$, then there exist isomorphisms

$$K^{t-j}(K(M)) \simeq K^0(K^{j+1}(M))$$

for all $1 \leq j \leq t-2$.

Proof. Consider the Foxby spectral sequences B.2.4 by considering M as S-module and $N = P = S$

$$E_2^{p,q} = \operatorname{Ext}_S^p(\operatorname{Ext}_S^q(M,S),S) \Rightarrow_p H^{q-p}$$

and

$$'E_2^{p,q} = \operatorname{Tor}_p^S(M, \operatorname{Ext}_S^q(S,S)) \Rightarrow_p H^{p-q}.$$

Since $'E_2^{p,q} = 0$ for all $q \neq 0$, we have

$$H^j \simeq {}'E_2^{j,0} = \operatorname{Tor}_j^S(M,S)$$

for all $j \geq 0$, and

$$E_2^{p,q} = \operatorname{Ext}_S^p(\operatorname{Ext}_S^q(M,S),S) \Rightarrow_p \operatorname{Tor}_{q-p}^S(M,S).$$

Once $H^j_{\mathfrak{m}}(M)$ being of finite length so is $K^j(M)$ for all $j < t$ and by local duality

$$\operatorname{Ext}_S^p(\operatorname{Ext}_S^q(M,S),S) = \operatorname{Ext}_S^p(K^{s-q}(M),S) = 0$$

for all $q > s - t$ and for all $p \neq s$. Also, Lemma 1.3.2 (i) assures that $\dim_R K(M) = t$. Thus E_2 has the following shape.

$$\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \operatorname{Ext}_S^s(K^0(M),S) & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \operatorname{Ext}_S^s(K^{t-1}(M),S) & 0 \\
0 & K(K(M)) & \operatorname{Ext}_S^{s-(t-1)}(K(M),S) & \cdots & \operatorname{Ext}_S^s(K(M),S) & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

By convergence, there are isomorphisms

$$K^0(K(M)) = \operatorname{Ext}_S^s(K(M),S) \simeq E_\infty^{s,s-t} \simeq \operatorname{Tor}_{-t}^S(M,S),$$

$$K^1(K(M)) = \operatorname{Ext}_S^{s-1}(K(M),S) \simeq E_\infty^{s-1,s-t}$$

and

$$K^0(K^0(M)) = \operatorname{Ext}_S^s(K^0(M),S) \simeq E_\infty^{s,s}.$$

Thus we get item (i) and by applying Matlis dual one has isomorphisms

$$H^1_{\mathfrak{m}}(K(M)) \simeq (E_\infty^{s-1,s-t})^\vee \text{ and } H^0_{\mathfrak{m}}(K^0(M)) \simeq (E_\infty^{s,s})^\vee.$$

Convergence again gives us short exact sequences

$$0 \to E_\infty^{s,s-j} \to \operatorname{Tor}_{-j}^S(M,S) \to E_\infty^{s-(t-j),s-t} \to 0 \tag{4.1.1}$$

for all $j \geq 0$. Further, as we move through the pages of E, differentials between the vertical and horizontal lines in the diagram above come out. In other words, there is an exact sequence

$$0 \to E_\infty^{s-(t-j),s-t} \to \operatorname{Ext}_S^{s-(t-j)}(K(M),S) \to \operatorname{Ext}_S^s(K^{j+1}(M),S) \to E_\infty^{s,s-(j+1)} \to 0 \tag{4.1.2}$$

for all $0 \leq j \leq t-2$.

Item (ii) is exactly the five-term exact sequence of E. For item (iii), by taking $j=0$ in both exact sequences above we have the following exact sequences

$$0 \to \operatorname{Ext}_S^s(K^0(M),S) \to M \to E_\infty^{s-t,s-t} \to 0$$

and

$$0 \to E_\infty^{s-t,s-t} \to K(K(M)) \to \operatorname{Ext}_S^s(K^1(M),S) \to E_\infty^{s,s-1} \to 0.$$

The result follows by splicing these sequences and noticing that $E_\infty^{s,s-1} \subseteq \operatorname{Tor}_{-1}^S(M,S) = 0$.

The exact sequence 4.1.1 assures that $E_\infty^{s-(t-j),s-t} = E_\infty^{s,s-j} = 0$ for all $j > 0$ so that, by the exact sequence 4.1.2,

$$K^{t-j}(K(M)) = \operatorname{Ext}_S^{s-(t-j)}(K(M),S) \simeq \operatorname{Ext}_S^s(K^{j+1}(M),S) = K^0(K^{j+1}(M))$$

for all $1 \leq j \leq t-2$. ■

The concept of *canonically Cohen-Macaulay module* was introduced by Schenzel [77].

Definition 4.1.3. *A finitely generated R-module M is canonically Cohen-Macaulay if its canonical module $K(M)$ is Cohen-Macaulay.*

Corollary 4.1.4. *Let M be a generalized Cohen-Macaulay R-module of dimension t. The following statements hold true.*

(i) If $t > j$ with $j \in \{0,1\}$, then $\operatorname{depth}_R K(M) > j$;

(ii) If $t = 1$, then M is canonically Cohen-Macaulay and there exists the short exact sequence

$$0 \to K^0(K^0(M)) \to M \to K(K(M)) \to 0;$$

(iii) If $t = 2$, then M is canonically Cohen-Macaulay;

(iv) *If* $t \geq 3$*, then* $K(M)$ *is generalized Cohen-Macaulay.*

Proof. Item (i) follows immediately from Theorem 4.1.2 (i) and (ii). For item (ii), item (i) assures that $K(M)$ is Cohen-Macaulay and Theorem 4.1.2 (iii) is the desired exact sequence. As to item (iii), item (i) again assures that $K(M)$ is Cohen-Macaulay. Item (iv) follows directly from item (i) and Theorem 4.1.2 (iv). ■

Corollary 4.1.5. *If* M *is generalized Cohen-Macaulay, then so is* $K(M)$*.*

As Corollary 4.1.4 assures that generalized Cohen-Macaulay of dimension at most two are canonically Cohen-Macaulay, Theorem 4.1.2 (iv) recovers a characterization [16] for the case where the dimension is at least three.

Corollary 4.1.6. ([16, Corollary 2.7]) *Let* M *be a generalized Cohen-Macaulay* R*-module of dimension* $t \geq 3$*. Then the following statements are equivalent*

(i) M *is canonically Cohen-Macaulay;*

(ii) $H^j_{\mathfrak{m}}(M) = 0$ *for all* $j = 2, ..., t-1$*;*

(iii) *The* $\mathfrak{m}$*-transform functor* $\mathrm{D}_{\mathfrak{m}}(M)$ *is a Cohen-Macaulay* R*-module.*

Proposition 4.1.7. *Let* M *be a finitely generated* R*-module of dimension* t*. The following statements hold true.*

(i) *If* M *is generalized Cohen-Macaulay* R*-module with depth at least two, then* $M \simeq K(K(M))$*.*

(ii) *Suppose* M *is equidimensional. If* M *satisfies Serre's condition* S_{k+1} *for some positive integer* k*, then*

$$K^j(K(M)) \simeq \mathrm{Tor}^S_{-t+j}(M, S)$$

for all $t - k + 1 \leq j \leq t$*.*

Proof. Item (i) follows immediately from Theorem 4.1.2 (iii) and from the fact that $K^0(M) = K^1(M) = 0$ whenever $\operatorname{depth}_R M \geq 2$.

For item (ii), consider the Foxby spectral sequence given in Theorem 4.1.2

$$E_2^{p,q} = \mathrm{Ext}^p_S(\mathrm{Ext}^q_S(M, S), S) \Rightarrow_p \mathrm{Tor}^S_{q-p}(M, S).$$

By Lemma 1.3.2 (ii) and local duality, we have in

$$E_2^{s-i,s-j} = \mathrm{Ext}^{s-i}_S(K^j(M), S) = 0$$

for all $0 \leq j \leq k$ and $i \leq 0$. In other words, all modules $E_2^{p,q}$ such that $q \neq s - t$ above the dotted line in the diagram below must be zero.

$$
\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \iddots & \operatorname{Ext}_S^{s}(K^{k+1}(M), S) & 0 \\
0 & \operatorname{Ext}_S^{s-(t-k-2)}(K^{t-1}(M), S) & \cdots & \vdots & 0 \\
\operatorname{Ext}_S^{s-(t-k-1)}(K(M), S) & \operatorname{Ext}_S^{s-(t-k-2)}(K(M), S) & \cdots & \operatorname{Ext}_S^{s}(K(M), S) & 0 \\
0 & \cdots & 0 & 0 & 0
\end{array}
$$

The result follows from convergence. ■

Our results also retrieve the well-known fact that every Cohen-Macaulay module is canonically Cohen-Macaulay, see [76, Theorem 1.14].

Corollary 4.1.8. *If M is Cohen-Macaulay of dimension t, then so is $K(M)$ and $K(K(M)) \simeq M$.*

Proof. There are two immediate ways of proving the desired result. Indeed the result follows directly from Theorem 4.1.2 as well as from Proposition 4.1.7 (ii) too. ■

Proposition 4.1.7 provides a characterization for the Cohen-Macaulay property.

Corollary 4.1.9. *If M is a finitely generated R-module, then M is Cohen-Macaulay if and only if M is equidimensional canonically Cohen-Macaulay satisfying Serre's condition S_{k+1} for some positive integer k.*

Proof. It is well-known that a Cohen-Macaulay module is equidimensional and satisfies Serre's condition S_k for any k. Corollary 4.1.8 assures that such a module is also canonically Cohen-Macaulay. Conversely, by taking $j = t$ in Proposition 4.1.7 (ii), we have the isomorphism $K(K(M)) \simeq M$. Since $K(M)$ is Cohen-Macaulay, Corollary 4.1.8 again assures that $M \simeq K(K(M))$ is Cohen-Macaulay. ■

The next corollary is a extended version of Corollary 4.1.8 for generalized Cohen-Macaulay modules.

Corollary 4.1.10. *If M is a generalized Cohen-Macaulay module with depth at least two, then so is $K(M)$ and $M \simeq K(K(M))$.*

Proof. It follows directly from Theorem 4.1.2, Corollary 4.1.5 and Proposition 4.1.7 (i). ■

4.2 Bounding Bass numbers

The Foxby spectral sequences B.2.4 are again fundamental tools here. They provide the main result of this section.

Theorem 4.2.1. *If M is a finitely generated R-module of depth g and dimension t, then the following inequality holds true for all $j \geq 0$.*

$$\mu^j(M) \leq \sum_{i=g}^{t} \beta_{j+i}(K^i(M)).$$

Moreover, $\mathrm{r}(M) = \beta_0(K^g(M))$ *and*

$$\mu^{g+2}(M) - \mu^{g+1}(M) \leq \beta_2(K^g(M)) - \beta_1(K^g(M)) - \beta_0(K^{g+1}(M)).$$

Proof. Consider the Foxby spectral sequences B.2.4 by taking $S = R, X = k, Y = M$ and $Z = S$

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^q(k, M), S) \Rightarrow_p H^{q-p}$$

and

$${}'E_2^{p,q} = \mathrm{Tor}_p^R(k, \mathrm{Ext}_S^q(M, S)) \Rightarrow_p H^{p-q}.$$

Since $\mathrm{Ext}_R^q(k, M)$ is of finite length we must have $E_2^{p,q} = 0$ for all $p \neq s$ so that

$$H^j \simeq E_2^{s,j+s} = \mathrm{Ext}_S^s(\mathrm{Ext}_R^{j+s}(k, M), S)$$

for all integer j. Once $K^{s-q}(M) = \mathrm{Ext}_S^q(M, S)$ for all $q \geq 0$, we conclude that

$${}'E_2^{p,q} = \mathrm{Tor}_p^R(k, K^{s-q}(M)) \Rightarrow_p \mathrm{Ext}_S^s(\mathrm{Ext}_R^{p-q+s}(k, M), S). \tag{4.2.1}$$

Now, since $\mathrm{Ext}_S^s(k, S)^\vee \simeq k$, where $\_^\vee$ denotes the Matlis dual of R, we have

$$\mathrm{Ext}_S^s(\mathrm{Ext}_R^j(k, M), S) \simeq \mathrm{Ext}_S^s(k, S)^{\mu^j(M)} \simeq k^{\mu^j(M)}$$

as k-vector spaces. Therefore, by the convergence of ${}'E$,

$$\mu^j(M) \leq \sum_{j=p-q+s} \beta_p(K^{s-q}(M)) = \sum_{i=g}^{t} \beta_{j+i}(K^i(M))$$

for all $j \geq 0$.

Now, since $K^i(M) = \mathrm{Ext}_S^{s-i}(M, S) = 0$ for all $i < g$, then ${}'E_2$ has the following corner

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & \mathrm{Tor}_2^R(k, K^{g+1}(M)) & \mathrm{Tor}_2^R(k, K^g(M)) \arrow[ddl] & 0 & \cdots \\
\cdots & \mathrm{Tor}_1^R(k, K^{g+1}(M)) & \mathrm{Tor}_1^R(k, K^g(M)) & 0 & \cdots \\
\cdots & k \otimes_R K^{g+1}(M) & k \otimes_R K^g(M) & 0 & \cdots
\end{tikzcd}$$

Therefore

$$k \otimes_R K^g(M) = {}'E_2^{0,s-g} \simeq H^{g-s} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^g(k,M),S)$$

so that $\mathrm{r}(M) = \beta_0(K^g(M))$ and there exists a five-term-type exact sequence

$$\begin{array}{l} \operatorname{Ext}_S^s(\operatorname{Ext}_R^{g+2}(k,M),S) \longrightarrow \operatorname{Tor}_2^R(k,K^g(M)) \longrightarrow k \otimes_R K^{g+1}(M) \\ \qquad \longrightarrow \operatorname{Ext}_S^s(\operatorname{Ext}_R^{g+1}(k,M),S) \to \operatorname{Tor}_1^R(k,K^g(M)) \to 0 \end{array}$$

whence the desired formula. ■

Corollary 4.2.2. *Let M be a finitely generated R-module of depth g and dimension t. If $\operatorname{pd}_R K^i(M) < \infty$ for all $i = g, ..., t$, then $\operatorname{id}_R M < \infty$.*

Proof. The hypothesis means that $\beta_l(K^i(M)) = 0$ for all $l \gg 0$ and by Theorem 4.2.1 one has

$$\mu^j(M) \leq \sum_{i=g}^{t} \beta_{j+i}(K^i(M)) = 0$$

for $j \gg 0$, i.e., $\operatorname{id}_R M < \infty$. ■

Bass' conjecture [11] was first proved by Peskine-Szpiro in [72] and after in a more general situation by Roberts [73]. It states that a local ring admitting a non-zero module of finite injective dimension must be Cohen-Macaulay. The next corollary provides sufficient conditions in terms of projective dimension for a local ring to be Cohen-Macaulay.

Corollary 4.2.3. *Let M be a finitely generated R-module of depth g and dimension t. If $\operatorname{pd}_R K^i(M) < \infty$ for all $i = g, ..., t$, then R is Cohen-Macaulay.*

Proof. Corollary 4.2.2 assures that $\operatorname{id}_R M < \infty$ and thus the result follows from Bass' conjecture. ■

Theorem 4.2.4. *If M is a generalized Cohen-Macaulay canonically Cohen-Macaulay R-module of dimension t and depth at least two, then*

$$\beta_j(M) = \mu^{j+t}(K(M)) \quad \text{and} \quad \mu^j(M) = \beta_{j-t}(K(M))$$

for all $j \geq 0$. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R K(M) < \infty$ and $\operatorname{id}_R M < \infty$ if and only if $\operatorname{pd}_R K(M) < \infty$.

Proof. By Lemma 1.3.2 (i), $K(M)$ is Cohen-Macaulay of dimension t and by Proposition 4.1.7 (i), $K(K(M)) \simeq M$, that is, $K^i(K(M)) = 0$ for all $i \neq t$ and $K^t(K(M)) \simeq M$. The spectral sequence 4.2.1

$${}'E_2^{p,q} = \operatorname{Tor}_p^R(k, K^{s-q}(K(M))) \Rightarrow_p \operatorname{Ext}_S^s(\operatorname{Ext}_R^{p-q+s}(k,K(M)),S)$$

degenerates so that

$$\operatorname{Tor}_j^R(k, M) \simeq \operatorname{Tor}_j^R(k, K(K(M))) = {}'E_2^{j,s-t} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, K(M)), S)$$

for all $j \geq 0$. Therefore

$$\beta_j(M) = \dim_k \operatorname{Tor}_j^R(k, M) = \dim_k \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, K(M)), S) = \mu^{j+t}(K(M))$$

for all $j \geq 0$. The other equality follows from the fact $K(K(M)) \simeq M$. ■

Theorem 4.2.4 generalizes [38, Corollary 3.6] and improves [39, Corollary 3.3]. We record this in the next corollary.

Corollary 4.2.5. *If M is Cohen-Macaulay R-module of dimension t, then*

$$\beta_j(M) = \mu^{j+t}(K(M)) \quad \textit{and} \quad \mu^j(M) = \beta_{j-t}(K(M))$$

for all $j \geq 0$. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R K(M) < \infty$ and $\operatorname{id}_R M < \infty$ if and only if $\operatorname{pd}_R K(M) < \infty$.

Proof. If $t \geq 2$ then the result follows from Theorem 4.2.4. Otherwise, Corollary 4.1.8 and the spectral sequence argument given in the proof of Theorem 4.2.4 asserts the result. ■

Next theorem is an attempt to extent part of Theorem 4.2.4 to arbitrary modules. In the next section we work on the other part.

Theorem 4.2.6. *Let M be a finitely generated R-module of depth g and dimension t. If $\operatorname{pd}_R K^i(M) < \infty$ for all $g \leq i < t$, then*

$$\mu^j(M) = \beta_{j-t}(K(M))$$

for all $j > \operatorname{depth} R + t$. In particular, $\operatorname{id}_R M < \infty$ if and only if $\operatorname{pd}_R K(M) < \infty$.

Proof. The spectral sequence 4.2.1 is such that ${}'E_2^{p,q} = 0$ for all $p > \operatorname{depth} R$ and $g \leq q < t$ so that

$$\operatorname{Tor}_j^R(k, K(M)) = {}'E_2^{j,s-t} \simeq \operatorname{Ext}_S^s(\operatorname{Ext}_R^{j+t}(k, M), S),$$

whence the result. ■

We derive other consequences of Theorem 4.2.1. In particular, we say exactly when the type of a finite module is one in terms of its deficiency modules.

Corollary 4.2.7. *Let M be a finitely generated R-module of depth g and dimension t. The following statements hold true.*

(i) If M is Cohen-Macaulay of dimension t, then

$$\mu^{t+2}(K(M)) - \mu^{t+1}(K(M)) \geq \beta_2(M) - \beta_1(M).$$

In particular, if $\operatorname{pd}_R M < \infty$ then $\beta_1(M) \geq \beta_2(M)$.

(ii) If $\operatorname{id}_R M < \infty$, then

$$\beta_0(K^{g+1}(M)) \geq \beta_2(K^g(M)) - \beta_1(K^g(M)).$$

In particular, if M is also Cohen-Macaulay, then $\beta_1(K(M)) \geq \beta_2(K(M))$.

(iii) $\mathrm{r}(M) = 1$ if and only if $K^g(M)$ is cyclic.

Proof. Item (*iii*) follows directly from Theorem 4.2.1. Item (*i*) follows from Corollary 4.1.8, Theorem 4.2.1 and Corollary 4.2.5, and item (*ii*) follows from [18, Theorem 3.7], corollaries 4.1.8 and 4.2.5 and item (*i*). ■

The spectral sequence 4.2.1 provides more information when the module involved has only two (possibly) non-zero deficiency modules.

Proposition 4.2.8. *Let M be a finitely generated R-module of depth g and dimension t. Suppose $K^i(M) = 0$ for all $i \neq g, t$. If $\operatorname{id}_R M < \infty$ then $\beta_j(K^g(M)) = \beta_{j+g-t-1}(K(M))$ for all $j > \operatorname{depth} R - g + 1$.*

Proof. Write $t = g + r$. The spectral sequence 4.2.1 has only two vertical lines as the following diagram shows.

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
\cdots & 0 & \operatorname{Tor}^R_{r+1}(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}^R_{r+1}(k, K^g(M)) & 0 & \cdots \\
 & & \vdots & & \iddots & & \vdots & & \\
\cdots & 0 & \operatorname{Tor}^R_{2}(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}^R_{2}(k, K^g(M)) & 0 & \cdots \\
\cdots & 0 & \operatorname{Tor}^R_{1}(k, K(M)) & 0 & \cdots & 0 & \operatorname{Tor}^R_{1}(k, K^g(M)) & 0 & \cdots \\
\cdots & 0 & k \otimes_R K(M) & 0 & \cdots & 0 & k \otimes_R K^g(M) & 0 & \cdots
\end{array}$$

From convergence we obtain an exact sequence

$$\operatorname{Ext}^s_S(\operatorname{Ext}^{j+g}_R(k, M), S) \to \operatorname{Tor}^R_j(k, K^g(M)) \to \operatorname{Tor}^R_{p-r-1}(k, K(M)) \to \operatorname{Ext}^s_S(\operatorname{Ext}^{j+g-1}_R(k, M), S)$$

for all $j \geq 0$. Thus, since $\operatorname{id}_R M = \operatorname{depth} R$ (see [18, Theorem 3.7.1]) we conclude that

$$\operatorname{Tor}^R_j(k, K^g(M)) \simeq \operatorname{Tor}^R_{j-r-1}(k, K(M))$$

for all $j > \operatorname{depth} R - g + 1$, whence the result. ■

4.3 Bounding Betti numbers

In last section we bounded the Bass numbers of a module in terms of the Betti numbers of the deficiency modules. In this section we get a dual version of Theorem 4.2.1 in the following sense.

Theorem 4.3.1. *For a finitely generated R-module M of depth g and dimension t, the following inequality holds true for all $j \geq 0$.*

$$\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M)).$$

Moreover, $\mu^0(K(M)) = \beta_{-t}(M)$ and

$$\beta_{-t+2}(M) - \beta_{-t+1}(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^{t-1}(M)).$$

Proof. By taking $M = k, N = M$ and $P = S$ in the spectral sequence B.2.8 we have the following.

$$E_2^{p,q} = \operatorname{Ext}_R^p(k, \operatorname{Ext}_S^q(M, S)) \Rightarrow_p H^{p+q}$$

and

$$'E_2^{p,q} = \operatorname{Ext}_S^p(\operatorname{Tor}_q^R(k, M), S) \Rightarrow_p H^{p+q}.$$

Since $\operatorname{Tor}_q^R(k, M)$ is of finite length for all $q \geq 0$, due to local duality we must have $'E_2^{p,q} = 0$ for all $p \neq s$ so that

$$H^j \simeq {'E_2^{s,j-s}} = \operatorname{Ext}_S^s(\operatorname{Tor}_{j-s}^R(k, M), S)$$

for all $j \geq 0$. Once $K^{s-q}(M) = \operatorname{Ext}_R^q(M, S)$ for all $q \geq 0$, one has spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p(k, K^{s-q}(M)) \Rightarrow_p \operatorname{Ext}_S^s(\operatorname{Tor}_{p+q-s}^R(k, M), S). \tag{4.3.1}$$

Once $\beta_j(M) = \dim_k \operatorname{Ext}_S^s(\operatorname{Tor}_{(j+s)-s}^R(k, M), S)$, by convergence we conclude that

$$\beta_j(M) \leq \sum_{p+q=j+s} \dim_k \operatorname{Ext}_R^p(k, K^{s-q}(M)) = \sum_{i=g}^{t} \mu^{i+j}(K^i(M)).$$

Now, since $K^i(M) = 0$ for all $i < g$ or $i > t$, then $E_2^{p,q} = 0$ for all $q < s - t$ or $q > s - g$. In particular E_2 has a corner as follows.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\operatorname{Hom}_R(k, K^{t-1}(M)) \arrow[drr] & \operatorname{Ext}_R^1(k, K^{t-1}(M)) & \operatorname{Ext}_R^2(k, K^{t-1}(M)) & \cdots \\
\operatorname{Hom}_R(k, K(M)) & \operatorname{Ext}_R^1(k, K(M)) & \operatorname{Ext}_R^2(k, K(M)) & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Therefore, there exist isomorphism

$$\operatorname{Hom}_R(k, K(M)) = E_2^{0,s-t} \simeq \operatorname{Ext}_S^s(\operatorname{Tor}_{-t}^R(k, M), S)$$

and a five-term-type exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}_R^1(k, K(M)) \arrow[r] & \operatorname{Ext}_S^s(\operatorname{Tor}_{-t+1}^R(k, M), S) \arrow[r] & \operatorname{Hom}_R(k, K^{t-1}(M)) \arrow[dl] \\
& & \operatorname{Ext}_R^2(k, K(M)) \arrow[r] & \operatorname{Ext}_S^s(\operatorname{Tor}_{-t+2}^R(k, M), S)
\end{tikzcd}$$

whence the result. ■

Remark 4.3.2. *It should be noticed that the estimate $\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M))$ is already known, see [76, Theorem 3.2].*

Corollary 4.3.3. *The following statements hold true.*

(i) If $t = 0$, then $\beta_0(M) = \mu^0(K(M))$ and

$$\beta_2(M) - \beta_1(M) \geq \mu^2(K(M)) - \mu^1(K(M)).$$

Otherwise $\operatorname{depth}_R K(M) > 0$*;*

(ii) If $t = 1$, then $\beta_1(M) - \beta_0(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^0(M))$;

(iii) If $t = 2$, then $\beta_0(M) \geq \mu^2(K(M)) - \mu^1(K(M)) - \mu^0(K^1(M))$;

(iv) If $t > 2$, then $\mu^0(K^{t-1}(M)) \geq \mu^2(K(M)) - \mu^1(K(M))$.

Proof. It follows directly from Theorem 4.3.1. ■

Corollary 4.3.4. *If M is a finitely generated Artinian R-module, then*

$$\beta_2(M) - \beta_1(M) = \mu^2(K(M)) - \mu^1(K(M)).$$

Proof. By corollaries 4.2.7 (i) and 4.3.3 (i),

$$\mu^2(K(M)) - \mu^1(K(M)) \geq \beta_2(M) - \beta_1(M) \geq \mu^2(K(M)) - \mu^1(K(M)).$$

■

Lemma 4.3.5. ([47, Proposition 2.8.4]) *Suppose R is d-dimensional with embedding dimension e. Then $\beta_1(R/\mathfrak{m}) = e$ and the following statements are equivalent.*

(i) $\beta_2(R/\mathfrak{m}) = \binom{e}{2} + e - d$;

(ii) R is a complete intersection.

Corollary 4.3.6. *If R is d-dimensional of embedding dimension e, then*

$$\mu^2(k) - \mu^1(k) = \binom{e}{2} - d$$

if and only if R is a complete intersection.

Proof. It follows directly from Corollary 4.3.4 and Lemma 4.3.5. ■

Corollary 4.3.7. *Let M be a finitely generated R-module of depth g and dimension t. If $\operatorname{id}_R K^i(M) < \infty$ for all $i = g, ..., t$, then $\operatorname{pd}_R M < \infty$.*

Proof. By hypothesis we have $\mu^l(K^i(M)) = 0$ for all $l \gg 0$ and by Theorem 4.3.1 one has

$$\beta_j(M) \leq \sum_{i=g}^{t} \mu^{j+i}(K^i(M)) = 0$$

for all $j \gg 0$, whence $\mu^j(M) = 0$ for all $j \gg 0$, that is, $\operatorname{pd}_R M < \infty$. ■

The *Auslander-Reiten conjecture* [7] states the following. Given a finitely generated R-module M, if

$$\operatorname{Ext}^j_R(M, M \oplus R) = 0$$

for all $j > 0$ then M is free. This long-standing conjecture has been largely studied and several positive answers are already known, see for instance [3, 4, 8, 26, 39, 56, 63, 65, 71]. Corollary 4.3.7 provides another positive answer for the Auslander-Reiten conjecture for a class of modules. But first we need a lemma.

Lemma 4.3.8. ([68, Lemma 1 (iii)]) *Let R be a local ring and let M and N be finite R-modules. If $\operatorname{pd}_R M < \infty$ and $N \neq 0$ then*

$$\operatorname{pd}_R M = \sup\{j : \operatorname{Ext}^j_R(M, N) \neq 0\}.$$

Theorem 4.3.9. *Let M be a finitely generated R-module of depth g and dimension t. If $n \leq d$ is a positive integer, then $\operatorname{pd}_R M < n$ provided the following statements hold true.*

(i) $\operatorname{id}_R K^i(M) < \infty$ *for all $i = g, ..., t$;*

(ii) There exists an R-module N such that $\operatorname{Ext}^j_R(M, N) = 0$ for all $j = n, ..., d$.

Proof. It follows directly from Corollary 4.3.7 and Lemma 4.3.8. ■

The next corollary is proves the Auslander-Reiten conjecture for a certain class of modules. It generalizes the case of the conjecture obtained in [39].

Corollary 4.3.10. *The Auslander-Reiten conjecture holds true for finitely generated modules having deficiency modules of finite injective dimension over local rings which are factor of Gorenstein local rings.*

Proof. It follows immediately from Theorem 4.3.9 by taking $n = 1$. ■

In the next theorem, such as Theorem 4.2.6, we furnish another attempt to remove the generalized Cohen-Macaulayness hypothesis from Theorem 4.2.4.

Theorem 4.3.11. *Let M be a finitely generated R-module of depth g and dimension t. If $\operatorname{id}_R K^i(M) < \infty$ for all $g \leq i < t$, then*

$$\beta_j(M) = \mu^{j+t}(K(M))$$

for all $j > s + \operatorname{depth} R - t - g$. In particular, $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R K(M) < \infty$.

Proof. Consider the spectral sequence 4.3.1

$$E_2^{p,q} = \operatorname{Ext}_R^p(k, K^{s-q}(M)) \Rightarrow_p \operatorname{Ext}_S^s(\operatorname{Tor}_{p+q-s}^R(k, M), S).$$

The hypothesis and [18, Theorem 3.7.1] assures that $E_2^{p,q} = 0$ for all $p > \operatorname{depth} R$ and for all $s - t < q \geq s - g$. Therefore, the convergence of E implies that

$$\operatorname{Ext}_R^j(k, K(M)) \simeq \operatorname{Ext}_S^s(\operatorname{Tor}_{j-t}^R(k, M), S)$$

for all $j > s - \operatorname{depth} R - g$, whence the result. ■

The next proposition is an attempt to understand the converse of Corollary 4.3.7.

Proposition 4.3.12. *Assume $K^i(M) = 0$ for all $i \neq g, t$. If $\operatorname{pd}_R M < \infty$, then $\mu^j(K^g(M)) = \mu^{j-g+t+1}(K(M))$ for all $j > \operatorname{pd}_R M + 1$.*

Proof. The spectral sequence 4.3.1 has only two lines as follows.

$$\begin{tikzcd}
0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & & \vdots & \\
\operatorname{Hom}_R(k, K^g(M)) \arrow[ddrrr] & \operatorname{Ext}_R^1(k, K^g(M)) & \cdots & \operatorname{Ext}_R^{p+r+1}(k, K^g(M)) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \\
\operatorname{Hom}_R(k, K(M)) & \operatorname{Ext}_R^1(k, K(M)) & \cdots & \operatorname{Ext}_R^{p+r+1}(k, K(M)) & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & & \vdots &
\end{tikzcd}$$

Such a shape and convergence yields an exact sequence

$$\mathrm{Ext}^s_S(\mathrm{Tor}^R_{j-g}(k,M),S) \to \mathrm{Ext}^j_R(k,K^g(M)) \to \mathrm{Ext}^{j+r+1}_R(k,K(M)) \to \mathrm{Ext}^s_S(\mathrm{Tor}^R_{j-g+1}(k,M),S)$$

for all $j \geq 0$. Thus if $j > \mathrm{pd}_R\, M + 1$ then

$$\mathrm{Ext}^j_R(k,K^g(M)) \simeq \mathrm{Ext}^{j+r+1}(k,K(M))$$

and, in particular, $\mu^j(K^g(M)) = \mu^{j+r+1}(K(M))$. ■

Chapter 5

Finiteness of homological dimensions

In this chapter, we appreciate local cohomology as nice a tool for solving problems in commutative algebra. Here we will notice its relevance in the study of the finiteness of homological dimensions via a cohomology vanishing approach.

In essence, this chapter deals with the interplay between the finiteness of some of the main homological dimensions (see Section 1.5) and the vanishing of cohomology – more precisely, the vanishing of suitable Ext modules. Among such dimensions, we consider the projective dimension, the injective dimension, the Gorenstein injective dimension, and the Gorenstein dimension of (finitely generated) modules over a given (Noetherian, commutative) Cohen-Macaulay local ring having a canonical module.

Our first goal in this chapter is to address some problems concerning the finiteness of projective dimension via the vanishing of Ext modules that have appeared in the literature, the main one being a question raised by D. Jorgensen in [60] about fourteen years ago, see Question 5.1.10. We describe suitable additional conditions under which such questions admit an affirmative answer, see theorems 5.1.4, 5.1.8 and Corollary 5.1.11. Second, we obtain similar results involving other homological dimensions, such as the injective and the Gorenstein injective dimensions, as for instance Theorem 5.2.2. Along the way, we derive several criteria for the freeness of modules. The main tools used in this chapter are generalized local cohomology (see Section 1.4), Suzuki spectral sequences in Appendix B.2.5, Burch ideals, and strongly rigid modules.

Throughout this chapter, we will follow the notation in sections 1.4 and 1.5.

Namely, R will always denote a commutative Noetherian local ring with non-zero unity, maximal ideal $\mathfrak{m}$. We denote by $M^\vee$ the Matlis dual of a finitely generated R-module M.

5.1 Finiteness of projective dimension

5.1.1 Three questions about projective dimension and Ext vanishing

In this part, we are concerned with three problems involving the finiteness of $\operatorname{pd}_R M$ by means of Ext vanishing. The first one, recalled below and raised in [61, Question 4.4], targets projective dimension zero over certain one-dimensional local rings.

Question 5.1.1. *Let R be a Gorenstein local ring of dimension one which is not a complete intersection, and let M be a finitely generated R-module with $\text{CI-dim}_R M < \infty$. If $\operatorname{Ext}^1_R(M, M) = 0$, must M be free?*

It is worth mentioning that this actually fails for one-dimensional complete intersection rings, as the next example shows.

Example 5.1.2. ([61, Example 4.3]) *Consider $R = k[\![x, y]\!]/(xy)$, where k is a field. By splicing the short exact sequences*

$$0 \to xR \to R \to R/xR \to 0,$$

$$0 \to yR \to R \xrightarrow{x} xR \longrightarrow 0$$

and

$$0 \to xR \to R \xrightarrow{y} yR \to 0$$

we obtain a minimal R-free resolution of R/xR:

$$\cdots \to R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/xR \to 0.$$

Thus $\operatorname{pd}_R R/xR = \infty$ and

$$\operatorname{Ext}^1_R(R/xR, R/xR) = \ker(R \xrightarrow{y} R)/\operatorname{im}(R \xrightarrow{x} R) = xR/xR = 0.$$

Our first objective in this subsection is to give equivalent conditions for the freeness of a Cohen-Macaulay module in the setting of Question 5.1.1. To this end, some auxiliary concepts and results are in order. The first notion is that of generalized local cohomology 1.4.1, which plays an important role in this chapter.

Definition 5.1.3. Let $(R, \mathfrak{m})$ be a local ring and let M, N be a pair of finitely generated R-modules. The *cohomological dimension of* M, N *with respect to* $\mathfrak{m}$ is defined as

$$\mathrm{cd}_{\mathfrak{m}}(M, N) := \sup\{i \geq 0 \mid H^i_{\mathfrak{m}}(M, N) \neq 0\}.$$

Note that by taking $M = R$ in the definition above we recover Definition 2.3.2 for $I = \mathfrak{m}$.

Moreover, as a matter of notation, we put

$$e_R(M, N) := \sup\{j \geq 0 \mid \mathrm{Ext}^j_R(M, N) \neq 0\}.$$

Now we are ready to present our approach to Question 5.1.1. Here, the R-module M is assumed to be Cohen-Macaulay, whereas R is taken Cohen-Macaulay but not required to be Gorenstein.

Theorem 5.1.4. *Let R be a Cohen-Macaulay local ring of dimension one, and let M be a Cohen-Macaulay R-module of dimension d (hence d is either 0 or 1). Consider the following assertions:*

(i) *M is free;*

(ii) $e_R(M, H^d_{\mathfrak{m}}(M)) < \infty$;

(iii) $\mathrm{cd}_{\mathfrak{m}}(M, M) < \infty$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Moreover, if* $\text{CI-dim}_R\, M < \infty$ *and* $\mathrm{Ext}^1_R(M, M) = 0$, *then the three assertions are equivalent.*

Proof. Clearly, (i) implies (ii). In order to prove the equivalence between (ii) and (iii), we notice that, by taking $N = M$ in the spectral sequence given in Lemma B.7, it collapses at its second page in such a way that

$$\mathrm{Ext}^p_R(M, H^d_{\mathfrak{m}}(M)) \cong H^{p+d}_{\mathfrak{m}}(M, M).$$

Therefore,

$$e_R(M, H^d_{\mathfrak{m}}(M)) = \mathrm{cd}_{\mathfrak{m}}(M, M) - d,$$

which in particular gives (ii)$\Leftrightarrow$(iii).

Now assume that $\text{CI-dim}_R\, M < \infty, \mathrm{Ext}^1_R(M, M) = 0$, and that (iii) holds. Let us prove (i). Consider the spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m}}(\mathrm{Ext}^q_R(M, M)) \Rightarrow_p H^{p+q}_{\mathfrak{m}}(M, M)$$

given in Lemma B.6. Because $\dim R = 1$, we must have $E_2^{p,q} = 0$ for $p > 1$, so that $E_2^{p,q} = E_\infty^{p,q}$ and hence there are only two columns; by convergence, it follows a short exact sequence

$$0 \to H^1_{\mathfrak{m}}(\mathrm{Ext}^{j-1}_R(M,M)) \to H^j_{\mathfrak{m}}(M,M) \to H^0_{\mathfrak{m}}(\mathrm{Ext}^j_R(M,M)) \to 0$$

for each $j \geq 1$. By hypothesis, $H^j_{\mathfrak{m}}(M,M) = 0$ for all $j \gg 0$, which yields

$$H^0_{\mathfrak{m}}(\mathrm{Ext}^t_R(M,M)) = H^1_{\mathfrak{m}}(\mathrm{Ext}^t_R(M,M)) = 0 \quad \text{for all} \quad t \gg 0.$$

Therefore, since $\dim R = 1$, we necessarily have $\mathrm{Ext}^t_R(M,M) = 0$ for all $t \gg 0$ and then Lemma 1.5.2 ensures that $\mathrm{pd}_R M < \infty$, i.e., $\mathrm{pd}_R M \leq 1$, which gives in fact $\mathrm{Ext}^t_R(M,M) = 0$ for all $t \geq 2$. Since $\mathrm{Ext}^1_R(M,M) = 0$ by hypothesis, we get $e_R(M,M) = 0$. It follows by Lemma 4.3.8 that

$$\mathrm{pd}_R M = e_R(M,M) = 0,$$

as needed. ■

By virtue of a result of Vasconcelos, [82, Theorem 3.1], one more item can be added to Theorem 5.1.4 in the Gorenstein case.

Corollary 5.1.5. *Let R be a Gorenstein local ring of dimension one, and let M be a Cohen-Macaulay R-module of dimension d (hence d is either 0 or 1). Consider the following assertions:*

(i) *M is free;*

(ii) *$\mathrm{Hom}_R(M,M)$ is free;*

(iii) *$e_R(M, H^d_{\mathfrak{m}}(M)) < \infty$;*

(iv) *$\mathrm{cd}_{\mathfrak{m}}(M,M) < \infty$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). *Moreover, if* $\text{CI-dim}_R M < \infty$ *and* $\mathrm{Ext}^1_R(M,M) = 0$*, then the four assertions are equivalent.*

For a finitely generated module M over a local ring, it is usual to denote its module of (first-order) minimal syzygies by ΩM. The following is another question from [61], which is quite connected to – and in fact generalizes the statement of – the third problem to be dealt with later on (see Question 5.1.10).

Question 5.1.6. ([61, Question 4.5]) Let R be a Cohen-Macaulay local ring with canonical module ω_R and positive (co)dimension. For positive integers $n \leq s$, let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$ satisfying $\mathrm{Ext}^j_R(M,M) = \mathrm{Ext}^{j+1}_R(M, M \otimes_R \Omega\omega_R) = 0$ for all $j = n, \ldots, s$. Is it true that $\mathrm{pd}_R M < n$?

We shall give a positive answer to this question as long as the depth of the (possibly trivial) module $\operatorname{Ext}_R^q(M,M)$ is sufficiently high for q in a suitable range. First we invoke some helpful lemmas.

Lemma 5.1.7. ([85, Lemma 2.2]) *Let R be a local ring and M, N be finitely generated R-modules with $\operatorname{pd}_R M < \infty$ and N maximal Cohen-Macaulay. Then, $\operatorname{Tor}_j^R(M,N) = 0$ for all $j \geq 1$.*

Here is our attempt to tackle Question 5.1.6, which indeed solves it in the case $s \geq d := \dim R$. It turns out to be the main result of this section, and moreover it generalizes [61, Theorem 3.1] (which corresponds to the case $s = d$). As usual, we set the depth of the zero module to be $+\infty$.

Theorem 5.1.8. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For positive integers $n \leq s$, let M be a finitely generated R-module with $\operatorname{pd}_R M < \infty$ satisfying*

$$\operatorname{Ext}_R^j(M,M) = \operatorname{Ext}_R^{j+1}(M, M \otimes_R \Omega\omega_R) = 0 \quad \textit{for all} \quad j = n, \ldots, s.$$

In the case of $s < d$ suppose in addition

$$\operatorname{depth}_R \operatorname{Ext}_R^q(M,M) \geq d - s - q \quad \textit{for all} \quad q = 0, \ldots, d-s.$$

Then, $\operatorname{pd}_R M < n$.

Proof. By Lemma 4.3.8 we have $\operatorname{pd}_R M = e_R(M,M)$, so $e_R(M,M) \leq d$. If we first consider the case $d \leq s$, then we must have $e_R(M,M) < n$. Therefore, we may suppose $n \leq s < d$.

Since $\operatorname{pd}_R M < \infty$, Lemma 5.1.7 forces $\operatorname{Tor}_1^R(M, \omega_R) = 0$ and so there is a short exact sequence

$$0 \to M \otimes_R \Omega\omega_R \to M \otimes_R F \to M \otimes_R \omega_R \to 0$$

for some finite free R-module F. Hence, for each $j \geq 0$, we get an exact sequence

$$\operatorname{Ext}_R^j(M, M \otimes_R F) \to \operatorname{Ext}_R^j(M, M \otimes_R \omega_R) \to \operatorname{Ext}_R^{j+1}(M, M \otimes_R \Omega\omega_R)\,.$$

By the hypothesis on the Ext modules, it follows that

$$\operatorname{Ext}_R^j(M, M \otimes_R \omega_R) = 0 \quad \text{for all} \quad j = n, \ldots, s.$$

Now, it should be noticed that taking $\mathfrak{m}$-adic completion (where $\mathfrak{m}$ is the maximal ideal of R) does not affect the conditions present in the statement, i.e., we may suppose that R is complete. Thus, as $\operatorname{pd}_R M < \infty$, there are isomorphisms

$$H_{\mathfrak{m}}^j(M,M) \cong \operatorname{Ext}_R^{d-j}(M, M \otimes_R \omega_R)^\vee = 0 \quad \text{for all} \quad j = d-s, \ldots, d-n$$

by generalized local duality (see Lemma 1.4.3). On the other hand, considering the spectral sequence given in Lemma $B.6$,

$$E_2^{p,q} = H_{\mathfrak{m}}^p(\mathrm{Ext}_R^q(M, M)) \Rightarrow_p H_{\mathfrak{m}}^{p+q}(M, M),$$

the depth hypothesis implies that $E_2^{p,q} = 0$ for all $p < d - s - q$, that is, $p + q < d - s$. By convergence we conclude that $H_{\mathfrak{m}}^j(M, M) = 0$ for all $j < d - s$. Therefore,

$$H_{\mathfrak{m}}^j(M, M) = 0 \quad \text{for all} \quad j \leq d - n$$

and, by Lemma 1.4.2, $\mathrm{depth}_R M > d - n$, so that $\mathrm{pd}_R M = d - \mathrm{depth}_R M < n$. ■

In the sequel we will derive some immediate consequences of Theorem 5.1.8 by taking particular values of n or s. First, we consider the case $n = 1$, i.e., a characterization of freeness.

Corollary 5.1.9. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For a positive integer s, let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$ satisfying*

$$\mathrm{Ext}_R^j(M, M) = \mathrm{Ext}_R^{j+1}(M, M \otimes_R \Omega\omega_R) = 0 \quad \textit{for all} \quad j = 1, \ldots, s.$$

In the case of $s < d$ suppose in addition $\mathrm{depth}_R \mathrm{Ext}_R^q(M, M) \geq d - s - q$ for all $q = 0, \ldots, d - s$. Then, M is free.

Before presenting other special cases of Theorem 5.1.8, we state the third problem we want to tackle in this subsection, which was suggested by Jorgensen in [60]. We point out that Question 5.1.6 in fact recovers the statement of Jorgensen's problem by taking $n = s$ and R a complete intersection.

Question 5.1.10. ([60, Question 1.7]) *Let R be a complete intersection local ring R of positive codimension, and let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$. Does the condition $\mathrm{Ext}_R^n(M, M) = 0$ imply $\mathrm{pd}_R M < n$?*

Note Theorem 5.1.8 detects an additional (depth) condition under which Question 5.1.10 admits a positive answer. On the other hand, we do not require the ring to be a complete intersection. Let us consider the situation where R is Gorenstein. We point out that the Gorenstein case of Theorem 5.1.8 will be recovered later by Corollary 5.2.7; we can put $n = s$ and $M = N$ therein in order to record the following result.

Corollary 5.1.11. *Let R be a Gorenstein local ring of dimension d. Let M be a finitely generated R-module with $\mathrm{pd}_R M < \infty$ satisfying $\mathrm{Ext}_R^n(M, M) = 0$ for some positive integer $n \leq d$ and in addition $\mathrm{depth}_R \mathrm{Ext}_R^q(M, M) \geq d - n - q$ for all $q = 0, \ldots, d - n$. Then, $\mathrm{pd}_R M < n$.*

Over complete intersections we obtain a particularly interesting result (apply Corollary 5.1.11 together with Lemma 1.5.2).

Corollary 5.1.12. *Let R be a complete intersection local ring of dimension d. Let M be a finitely generated R-module satisfying $\operatorname{Ext}_R^n(M, M) = 0$ for some positive even integer $n \leq d$ and in addition $\operatorname{depth}_R \operatorname{Ext}_R^q(M, M) \geq d - n - q$ for all $q = 0, \ldots, d - n$. Then, $\operatorname{pd}_R M < n$.*

Finally, taking $s = d - 1$ in Theorem 5.1.8 we notice that the depth hypothesis is reduced simply to

$$\operatorname{depth}_R \operatorname{Hom}_R(M, M) > 0.$$

Note this occurs whenever $\operatorname{depth} M > 0$ (e.g., $\operatorname{depth} R > 0$ and M is contained in a free R-module), on account of the general bound $\operatorname{depth}_R \operatorname{Hom}_R(M, N) \geq \min\{2, \operatorname{depth}_R N\}$ for all finitely generated R-modules M, N (see [18, Exercise 1.4.19]). As a consequence, with the aid of the Auslander-Buchsbaum formula we see that $\operatorname{depth}_R \operatorname{Hom}_R(M, M) > 0$ if, for instance, $\operatorname{pd}_R M < d$. We thus immediately get the following corollary.

Corollary 5.1.13. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For a positive integer $n \leq d - 1$, let M be a finitely generated R-module with $\operatorname{pd}_R M < d$ satisfying*

$$\operatorname{Ext}_R^j(M, M) = \operatorname{Ext}_R^j(M, M \otimes_R \Omega\omega_R) = 0 \quad \textit{for all} \quad j = n, \ldots, d - 1.$$

Then, $\operatorname{pd}_R M < n$.

Note the Gorenstein case of Corollary 5.1.13 is an immediate consequence of Ischebeck's theorem; see [18, Exercise 3.1.24] or [57, 2.6].

5.1.2 Finite projective dimension via rigid modules

In this part, we make use of the theory of "rigid" modules to establish sufficient conditions for a given module to have finite projective dimension. We shall combine this approach with some of our previous results in order to estimate this invariant and, consequently, provide freeness criteria.

The following concepts, recalled here for the reader's convenience, are collected in [86, p. 3] (some of them have their roots in [5]).

Definition 5.1.14. Let M be a finitely generated R-module.

(a) M is said to be a *test* module if $\operatorname{Tor}_i^R(M,N) \neq 0$ for infinitely many integers i, whenever N is a finitely generated R-module with $\operatorname{pd}_R N = \infty$;

(b) M is *Tor-rigid* provided that $\operatorname{Tor}_j^R(M,N) = 0$ for all $j \geq i$, whenever N is a finitely generated R-module with $\operatorname{Tor}_i^R(M,N) = 0$ for some $i \geq 1$;

(c) M is a *rigid-test* module if M is both test and Tor-rigid;

(d) M is said to be *strongly rigid* if $\operatorname{pd}_R N < \infty$ whenever N is a finitely generated R-module with $\operatorname{Tor}_j^R(M,N) = 0$ for some $j \geq 1$.

Some of the main relations and questions involving these definitions are given in [86, p. 4]. For instance, rigid-test implies strongly rigid (the converse is an open problem) and Tor-rigid as well (the converse fails), while strongly rigid implies test (the converse is false).

Lemma 5.1.15. ([86, Corollary 6.1]) *Let R be a local ring and let M, N be non-zero finitely generated R-modules. Suppose any one of the following conditions:*

(i) N is strongly rigid and $\operatorname{Ext}_R^i(M,N) = 0$ for some $i \geq \operatorname{depth} R$;

(ii) N is rigid-test and $\operatorname{Ext}_R^i(M,N) = 0$ for some $i \geq \operatorname{depth}_R N$.

Then, $\operatorname{pd}_R M = e_R(M,N) < i$.

Our next result combines Theorem 5.1.8 with the classes of modules described in Definition 5.1.14. Note that, in the particular case $s = d$, parts (ii) and (iii) below recover [62, Theorem 2.7].

Theorem 5.1.16. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For positive integers $n \leq s$, let M be a finitely generated R-module satisfying*

$$\operatorname{Ext}_R^j(M,M) = \operatorname{Ext}_R^{j+1}(M, M \otimes_R \Omega\omega_R) = 0 \quad \textit{for all} \quad j = n, \ldots, s.$$

In the case of $s < d$, suppose in addition

$$\operatorname{depth}_R \operatorname{Ext}_R^q(M,M) \geq d - s - q \quad \textit{for all} \quad q = 0, \ldots, d-s.$$

Assume moreover that there exists a non-zero finitely generated R-module N satisfying any one of the following conditions:

(i) N is strongly rigid and $\operatorname{Ext}_R^i(M,N) = 0$ for some $i \geq d$;

(ii) N is rigid-test and $\operatorname{Ext}_R^i(M,N) = 0$ for some $i \geq \operatorname{depth}_R N$;

(iii) N is Tor-rigid, $\operatorname{depth}_R N \geq 1$, *and* $\operatorname{Tor}_{i-1}^R(\mathfrak{m}N, M) = 0$ *for some* $i \geq 2$.

Then, $\operatorname{pd}_R M < \min\{n, i\}$.

Proof. In the cases of (i) and (ii), we readily get $\operatorname{pd}_R M < i$ by Lemma 5.1.15. If (iii) holds then the proof of [62, Theorem 2.7] applies to give $\operatorname{pd}_R M < i$ as well. Now, being $\operatorname{pd}_R M$ finite, Theorem 5.1.8 yields $\operatorname{pd}_R M < n$, whence the result. ■

We derive a couple of immediate corollaries in the Gorenstein case.

Corollary 5.1.17. *Let R be a Gorenstein local ring of dimension d. For positive integers $n \leq s$, let M be a finitely generated R-module satisfying* $\operatorname{Ext}_R^j(M, M) = 0$ *for all $j = n, \ldots, s$. In the case of $s < d$, suppose in addition* $\operatorname{depth}_R \operatorname{Ext}_R^q(M, M) \geq d - s - q$ *for all $q = 0, \ldots, d - s$. Assume moreover that there exists a non-zero finitely generated R-module N satisfying any one of the conditions* (i), (ii), (iii) *described in Theorem 5.1.16. Then,* $\operatorname{pd}_R M < \min\{n, i\}$.

Corollary 5.1.18. *Let R be a Gorenstein local ring of dimension d. For a positive integer $n \leq d - 1$, let M be a finitely generated R-module satisfying* $\operatorname{Ext}_R^j(M, M) = 0$ *for all $j = n, \ldots, d - 1$ and* $\operatorname{depth}_R \operatorname{Hom}_R(M, M) > 0$. *Assume moreover that there exists a non-zero finitely generated R-module N satisfying any one of the conditions* (i), (ii), (iii) *described in Theorem 5.1.16. Then,* $\operatorname{pd}_R M < \min\{n, i\}$.

Finally, we observe that taking $n = 1$ in Theorem 5.1.16 (or any of its corollaries) leads us to a characterization of the freeness of M.

5.2 Finiteness of other homological dimensions

So far in this chapter we have dealt solely with modules of finite projective dimension. In the present section, we add further homological dimensions into our investigation and focus on the interplay between the vanishing of Ext modules and the finiteness of the injective dimension, the Gorenstein injective dimension, and the Gorenstein dimension of a finitely generated module. Applications to prescribed bound on projective dimension and freeness criteria will be given. We maintain the previous notations.

The auxiliary results below will be useful to the main theorem of this section.

Lemma 5.2.1. ([62, Lemma 4.4]) *Let R be a local ring. If M is a maximal Cohen-Macaulay R-module and N is a finitely generated R-module with* $\operatorname{id}_R N < \infty$, *then* $e_R(M, N) = 0$.

The theorem below is the main result of this section. Notice that the case $s = d$ immediately retrieves [62, Theorem 4.6].

Theorem 5.2.2. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For positive integers $n \leq s \leq d$, let M, N be finitely generated R-modules, with $\operatorname{id}_R N < \infty$, satisfying*

$$\operatorname{Ext}_R^j(N, M) = \operatorname{Ext}_R^{j+1}(\operatorname{Hom}_R(\Omega\omega_R, N), M) = 0 \quad \textit{for all} \quad j = n, \dots, s, \quad \textit{and}$$

$$\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q \quad \textit{for all} \quad q = 0, \dots, d - s.$$

Then, $\operatorname{depth}_R N > d - n$. *If in addition* $\operatorname{Gid}_R M < \infty$, *then* $e_R(N, M) < n$.

Proof. By Lemma 5.2.1 we have $\operatorname{Ext}_R^1(\omega_R, N) = 0$, hence there is a short exact sequence

$$0 \to \operatorname{Hom}_R(\omega_R, N) \to \operatorname{Hom}_R(F, N) \to \operatorname{Hom}_R(\Omega\omega_R, N) \to 0$$

for some finite free R-module F. This yields, for each $i \geq 0$, an exact sequence

$$\operatorname{Ext}_R^i(\operatorname{Hom}_R(F, N), M) \to \operatorname{Ext}_R^i(\operatorname{Hom}_R(\omega_R, N), M) \to \operatorname{Ext}_R^{i+1}(\operatorname{Hom}_R(\Omega\omega_R, N), M).$$

Using the hypotheses, we get $\operatorname{Ext}_R^i(\operatorname{Hom}_R(\omega_R, N), M) = 0$ for all $i = n, \dots, s$. Now it should be noticed that R can be assumed to be complete, and therefore Lemma 1.4.4 ensures that

$$H_{\mathfrak{m}}^j(M, N) = 0 \quad \text{for all} \quad j = d - s, \dots, d - n.$$

On the other hand, the spectral sequence given in Lemma B.6

$$E_2^{p,q} = H_{\mathfrak{m}}^p(\operatorname{Ext}_R^q(M, N)) \Rightarrow_p H_{\mathfrak{m}}^{p+q}(M, N)$$

is such that $E_2^{p,q} = 0$ for all $p < d - s - q$. By convergence, it follows that $H_{\mathfrak{m}}^j(M, N) = 0$ for all $j < d - s$. Summing up, we have

$$H_{\mathfrak{m}}^j(M, N) = 0 \quad \text{for all} \quad j < d - n.$$

Thus, Lemma 1.4.2 gives $\operatorname{depth}_R N > d - n$. Finally, if $\operatorname{Gid}_R M < \infty$ then, by Lemma 1.5.6, we conclude that $e_R(N, M) = d - \operatorname{depth}_R N < n$. ■

Remark 5.2.3. By the proof of Theorem 5.2.2 it is clear that the condition

$$\operatorname{Ext}_R^j(\operatorname{Hom}_R(\omega_R, N), M) = 0 \quad \text{for all} \quad j = n, \dots, s$$

suffices to ensure the same conclusions.

As an application we obtain the following criterion for prescribed bound on projective dimension for certain modules of finite injective dimension. It should be compared with Corollary 5.2.5, to be given shortly.

Corollary 5.2.4. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. For positive integers $n \leq s \leq d$, let M, N be finitely generated R-modules, with $\operatorname{id}_R N < \infty$ and $\operatorname{id}_R \operatorname{Hom}_R(\omega_R, N) < \infty$, satisfying*

$$\operatorname{Ext}_R^j(N, M) = 0 \quad \textit{for all} \quad j = n, \dots, s$$

and, in addition, $\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \dots, d - s$. Then, $\operatorname{pd}_R N < n$.

Proof. As $\operatorname{id}_R \omega_R < \infty$, we have $\operatorname{Gid}_R \omega_R < \infty$. Since in addition $\operatorname{id}_R N < \infty$, we can apply [75, Corollary 2.13] to get

$$\operatorname{pd}_R \operatorname{Hom}_R(\omega_R, N) = e_R(N, \omega_R).$$

On the other hand, Lemma 1.5.6 yields $e_R(N, \omega_R) < \infty$. Therefore $\operatorname{pd}_R \operatorname{Hom}_R(\omega_R, N) < \infty$, and since by hypothesis $\operatorname{id}_R \operatorname{Hom}_R(\omega_R, N) < \infty$, we obtain that R must be Gorenstein by a classical fact (see [37, Corollary 4.4]). Now we have $\Omega\omega_R = 0$ and $\operatorname{Gid}_R M < \infty$, so that Theorem 5.2.2 yields $\operatorname{depth}_R N > d - n$. Finally, because R is Gorenstein and $\operatorname{id}_R N < \infty$, we have $\operatorname{pd}_R N < \infty$ (see [18, Exercise 3.1.25]) and then the Auslander-Buchsbaum formula gives $\operatorname{pd}_R N < n$. ■

The result below is a variant (in terms of Gorenstein dimension) of Corollary 5.2.4.

Corollary 5.2.5. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Consider positive integers $n \leq s \leq d$. Let M, N be finitely generated R-modules, with $\operatorname{id}_R N < \infty$ and $\operatorname{G-dim}_R N < \infty$, satisfying*

$$\operatorname{Ext}_R^j(N, M) = 0 \quad \textit{for all} \quad j = n, \dots, s$$

and, in addition, $\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \dots, d - s$. Then, $\operatorname{pd}_R N < n$.

Proof. First, since $\operatorname{id}_R N < \infty$, we must have $\operatorname{G-dim}_R N = \operatorname{pd}_R N$ by Lemma 1.5.4. It follows that $\operatorname{pd}_R N < \infty$, hence R is Gorenstein by [37, Corollary 4.4]. Now the proof of Corollary 5.2.4 applies. ■

A criterion for freeness follows immediately.

Corollary 5.2.6. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Consider a positive integer $s \leq d$. Let M, N be finitely generated R-modules, with $\operatorname{id}_R N < \infty$ and $\operatorname{G-dim}_R N < \infty$, satisfying $\operatorname{Ext}_R^j(N, M) = 0$ for all $j = 1, \dots, s$ and $\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \dots, d - s$. Then, N is free.*

Now we record the Gorenstein case of Corollary 5.2.5, which also recovers the Gorenstein version of Theorem 5.1.8 by taking $N = M$.

Corollary 5.2.7. *Let R be a Gorenstein local ring of dimension d and consider positive integers $n \leq s \leq d$. Let M, N be finitely generated R-modules with $\mathrm{pd}_R N < \infty$ satisfying $\mathrm{Ext}_R^j(N, M) = 0$ for all $j = n, \ldots, s$ and, in addition, $\mathrm{depth}_R \mathrm{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \ldots, d - s$. Then, $\mathrm{pd}_R N < n$.*

From Corollary 5.2.7 we derive immediately two more criteria for the freeness of N, in the situation where R is Gorenstein. The first one is the case $n = 1$, and in the second we take in addition $s = d - 1$ (provided that $d \geq 2$), which thus softens the depth hypothesis by reducing it to the positivity of $\mathrm{depth}_R \mathrm{Hom}_R(M, N)$ – while, on the other hand, more Ext modules are required to vanish.

Corollary 5.2.8. *Let R be a Gorenstein local ring of dimension d and consider a positive integer $s \leq d$. Let M, N be finitely generated R-modules with $\mathrm{pd}_R N < \infty$ satisfying $\mathrm{Ext}_R^j(N, M) = 0$ for all $j = 1, \ldots, s$ and, in addition, $\mathrm{depth}_R \mathrm{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \ldots, d - s$. Then, N is free.*

Corollary 5.2.9. *Let R be a Gorenstein local ring of dimension $d \geq 2$. Let M, N be finitely generated R-modules with $\mathrm{pd}_R N < \infty$ satisfying $\mathrm{Ext}_R^j(N, M) = 0$ for all $j = 1, \ldots, d - 1$ and, in addition, $\mathrm{depth}_R \mathrm{Hom}_R(M, N) > 0$. Then, N is free.*

In the next section we will still be interested in finite projective dimension, but making use of other auxiliary concepts.

5.3 Finite projective dimension via Burch ideals and strongly rigid modules

In this part we establish consequences of Theorem 5.2.2 (more precisely, of Corollary 5.2.7) which deal with finiteness of projective dimension via the existence of either a Burch ideal or a strongly rigid module satisfying suitable hypotheses. We also consider a particular case of the latter that arises from the class of weakly $\mathfrak{m}$-full ideals.

5.3.1 Finite projective dimension via Burch ideals

The following notion was introduced in [27] and further studied in [20].

Definition 5.3.1. ([20, Definition 3.1]) Let R be a local ring and let I be an ideal of R. Then I is called *Burch* provided that

$$I :_R \mathfrak{m} \neq \mathfrak{m}I :_R \mathfrak{m}.$$

It is worth mentioning that if I is a Burch ideal of the local ring R then $\operatorname{depth} R/I = 0$. On the other hand, if an ideal I is such that $\operatorname{depth} R/I = 0$ and I is weakly $\mathfrak{m}$-full (see Definition 5.3.11 below), then I is Burch. We refer to [27, Section 2].

Lemma 5.3.2. ([20, Theorem 3.3]) *Let R be a local ring and let I be a Burch ideal of R. Suppose $\operatorname{Tor}_t^R(N, R/I) = \operatorname{Tor}_{t+1}^R(N, R/I) = 0$ for some finitely generated R-module N and some integer $t \geq 1$. Then, $\operatorname{pd}_R N \leq t$.*

Corollary 5.3.3. *Let R be a Gorenstein local ring of dimension d. Consider positive integers $n \leq s \leq d$. Let M, N be finitely generated R-modules such that $\operatorname{Ext}_R^j(N, M) = 0$ for all $j = n, \ldots, s$ and $\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \ldots, d - s$. If there exists a Burch ideal I of R such that*

$$\operatorname{Tor}_t^R(N, R/I) = \operatorname{Tor}_{t+1}^R(N, R/I) = 0$$

for some $t \geq 1$, then $\operatorname{pd}_R N < \min\{t + 1, n\}$.

Proof. It follows directly from Lemma 5.3.2 and Corollary 5.2.7. ■

Two criteria for the freeness of N are in order.

Corollary 5.3.4. *Let R be a Gorenstein local ring of dimension d. Consider a positive integer $s \leq d$. Let M, N be finitely generated R-modules such that $\operatorname{Ext}_R^j(N, M) = 0$ for all $j = 1, \ldots, s$ and $\operatorname{depth}_R \operatorname{Ext}_R^q(M, N) \geq d - s - q$ for all $q = 0, \ldots, d - s$. If there exists a Burch ideal I of R such that $\operatorname{Tor}_t^R(N, R/I) = \operatorname{Tor}_{t+1}^R(N, R/I) = 0$ for some $t \geq 1$, then N is free.*

Corollary 5.3.5. *Let R be a Gorenstein local ring of dimension $d \geq 2$. Let M, N be finitely generated R-modules such that $\operatorname{Ext}_R^j(N, M) = 0$ for all $j = 1, \ldots, d - 1$ and, in addition, $\operatorname{depth}_R \operatorname{Hom}_R(M, N) > 0$. If there exists a Burch ideal I of R such that $\operatorname{Tor}_t^R(N, R/I) = \operatorname{Tor}_{t+1}^R(N, R/I) = 0$ for some $t \geq 1$, then N is free.*

5.3.2 Finite projective dimension via strongly rigid modules

In this subsection, we are interested in using strongly rigid modules (see Definition 5.1.14(d)) to detect finite projective dimension. We will also consider a particular case related to the notion of weakly $\mathfrak{m}$-full ideal (see Definition 5.3.11 below).

General strongly rigid modules

Let us invoke a couple of preparatory lemmas. First recall that, for a local ring $(R, \mathfrak{m})$, a finitely generated R-module M is said to be of finite projective dimension on the punctured spectrum of R if $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for every $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Moreover, R is said to have an isolated singularity if the ring $R_\mathfrak{p}$ is regular for every $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

Lemma 5.3.6. ([86, Proposition 3.6]) *Let R be a Cohen-Macaulay local ring of dimension d with canonical module, and let N be a finitely generated R-module. Suppose there exists a non-zero strongly rigid R-module M satisfying the following properties:*

(i) M is of finite projective dimension on the punctured spectrum of R (e.g., R has an isolated singularity);

(ii) $\operatorname{Ext}^i_R(M, N) = 0$ for some $i \geq d + 1$.

Then, $\operatorname{id}_R N < \infty$.

Lemma 5.3.7. ([86, Theorem 7.3]) *Let R be a Cohen-Macaulay local ring. If $\operatorname{Gid}_R M < \infty$ for some non-zero strongly rigid R-module M, then R is Gorenstein.*

Corollary 5.3.8. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Let M, N be finitely generated R-modules, with M strongly rigid and $\operatorname{Gid}_R M < \infty$, satisfying the following properties:*

(i) M is of finite projective dimension on the punctured spectrum of R (e.g., R has an isolated singularity);

(ii) $\operatorname{Ext}^i_R(M, N) = 0$ for some $i \geq d + 1$;

(iii) There exist positive integers $n \leq s \leq d$ such that $\operatorname{Ext}^j_R(N, M) = 0$ for all $j = n, \ldots, s$, and in addition $\operatorname{depth}_R \operatorname{Ext}^q_R(M, N) \geq d - s - q$ for all $q = 0, \ldots, d - s$.

Then, $\operatorname{pd}_R N < n$.

Proof. By Lemma 5.3.6, items (i) and (ii) yield that $\operatorname{id}_R N < \infty$, whereas Lemma 5.3.7 ensures that R is Gorenstein. Thus, $\operatorname{pd}_R N < \infty$, and by Corollary 5.2.7 we conclude that $\operatorname{pd}_R N < n$. ∎

Next we record the freeness criterion that follows readily by the case $n = 1$ of Corollary 5.3.8. Note it also follows from a combination of Lemma 5.3.6, Lemma 5.3.7 and Corollary 5.2.8.

Corollary 5.3.9. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Let M, N be finitely generated R-modules, with M strongly rigid and $\operatorname{Gid}_R M < \infty$, satisfying the following properties:*

(i) *M is of finite projective dimension on the punctured spectrum of R (e.g., R has an isolated singularity);*

(ii) $\operatorname{Ext}_R^i(M,N)=0$ *for some* $i \geq d+1$*;*

(iii) *There exists a positive integer $s \leq d$ such that* $\operatorname{Ext}_R^j(N,M)=0$ *for all* $j=1,\ldots,s$*, and in addition* $\operatorname{depth}_R \operatorname{Ext}_R^q(M,N) \geq d-s-q$ *for all* $q=0,\ldots,d-s$*.*

Then, N is free.

Over Gorenstein local rings we also have the following fact.

Corollary 5.3.10. *Let R be a Gorenstein local ring of dimension d and let M,N be finitely generated R-modules, with M strongly rigid, satisfying the following properties:*

(i) $\operatorname{Ext}_R^i(N,M)=0$ *for some* $i \geq d$*;*

(ii) *There exist positive integers $n \leq s \leq d$ such that* $\operatorname{Ext}_R^j(N,M)=0$ *for all* $j=n,\ldots,s$*, and in addition* $\operatorname{depth}_R \operatorname{Ext}_R^q(M,N) \geq d-s-q$ *for all* $q=0,\ldots,d-s$*.*

Then, $\operatorname{pd}_R N < n$*.*

Proof. Lemma 5.1.15(i) gives $\operatorname{pd}_R N < \infty$. The result now follows by Corollary 5.2.7. ∎

Clearly, the case $n=1$ of Corollary 5.3.10 provides yet another freeness criterion.

Weakly $\mathfrak{m}$-full ideals

To conclude the chapter, we consider separately a special class of strongly rigid modules over a Gorenstein local ring $(R,\mathfrak{m})$. It arises from the notion of weakly $\mathfrak{m}$-full ideal, defined as follows.

Definition 5.3.11. ([20, Definition 2.1]) Let $(R,\mathfrak{m})$ be a local ring and let I,J be ideals of R. We say that I is *weakly $\mathfrak{m}$-full with respect to J* provided that

$$I :_R J = \mathfrak{m}I :_R \mathfrak{m}J.$$

In case $J=R$, i.e. if $I=\mathfrak{m}I :_R \mathfrak{m}$, then I is simply said to be *weakly $\mathfrak{m}$-full.*

For example, if $\operatorname{depth} R > 0$ then all integrally closed ideals of R are weakly $\mathfrak{m}$-full with respect to $\mathfrak{m}^s$ for each $s \geq 0$ (see [20, Proposition 2.4]).

Lemma 5.3.12. ([20, Corollary 2.14]) *Let $(R, \mathfrak{m})$ be a non-regular local ring with* $\operatorname{depth} R > 0$*, and let I be an $\mathfrak{m}$-primary ideal of R such that I is weakly $\mathfrak{m}$-full with respect to $\mathfrak{m}^\nu$ and $I \subset \mathfrak{m}^{\nu+1}$, for some $\nu \geq 0$ (note the case $\nu = 0$ means that I is weakly $\mathfrak{m}$-full). In addition, let N be a finitely generated R-module and $t \geq 1$ be an integer. If $\operatorname{Tor}_t^R(R/I, N) = 0$, then $\operatorname{pd}_R N < t$.*

Note Lemma 5.3.12 implies that R/I is a strongly rigid R-module. This will be used in the result below.

Theorem 5.3.13. *Let $(R, \mathfrak{m})$ be a Gorenstein non-regular local ring of dimension d, and let I be an $\mathfrak{m}$-primary ideal of R such that I is weakly $\mathfrak{m}$-full with respect to $\mathfrak{m}^\nu$ and $I \subset \mathfrak{m}^{\nu+1}$, for some $\nu \geq 0$. Consider positive integers $n \leq s \leq d$, and let N be a finitely generated R-module such that $\operatorname{Ext}_R^j(N, R/I) = 0$ for all $j = n, \ldots, s$ and* $\operatorname{depth}_R \operatorname{Ext}_R^q(R/I, N) \geq d - s - q$ *for all $q = 0, \ldots, d - s$. Suppose in addition any one of the following conditions:*

(i) $\operatorname{Tor}_t^R(R/I, N) = 0$ *for some $t \geq 1$;*

(ii) $\operatorname{Ext}_R^i(R/I, N) = 0$ *for some $i \geq d + 1$;*

(iii) $\operatorname{Ext}_R^i(N, R/I) = 0$ *for some $i \geq d$.*

Then, $\operatorname{pd}_R N < n$.

Proof. In the case that (i) takes place, Lemma 5.3.12 yields $\operatorname{pd}_R N < t < \infty$, and hence $\operatorname{pd}_R N < n$ by Corollary 5.2.7. So it remains to prove the result in the other two cases. As already pointed out, R/I is strongly rigid as an R-module. Note $\operatorname{Gid}_R R/I < \infty$ because R is Gorenstein. Moreover, since in particular I is $\mathfrak{m}$-primary, we have $(R/I)_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ and hence, trivially, R/I has finite projective dimension on the punctured spectrum of R. Now if (ii) (resp. (iii)) holds then we get $\operatorname{pd}_R N < n$ by Corollary 5.3.8 (resp. Corollary 5.3.10). ■

Clearly, criteria for the freeness of N can be readily seen by taking $n = 1$ in the above theorem. We close the chapter with a few more comments.

Remark 5.3.14. (a) In the case that (i) holds, the result (together with Lemma 5.3.12) in fact yields $\operatorname{pd}_R N < \min\{t, n\}$.

(b) In order to make Theorem 5.3.13 feasible, an obstruction on the shape of N must be taken into account. Precisely, N cannot be of the form $\mathfrak{m}^k N'$, where N' is any non-zero finitely generated R-module and $k \geq 1$ is any integer. Indeed, suppose by way of contradiction that the module $\mathfrak{m}^k N'$ fits into the hypotheses of the theorem. Then we would get

$$\operatorname{pd}_R \mathfrak{m}^k N' < n < \infty$$

which by [64, Theorem 1.1] is equivalent to R being regular; this violates our choice of R.

(c) The case $n = s = d$ of Theorem 5.3.13 gives that if $\operatorname{Ext}^d_R(N, R/I) = 0$ then $\mathfrak{m}$ contains an N-regular element.

Chapter 6

Questions

In this chapter we gathered some remarks and issues that came up along the development of the work.

6.1 On Chapter 2

We observe that in some sense there is a duality between two spectral sequences in this work. Our Mayer-Vietoris spectral sequence 2.1.8 has its components sheaf cohomologies in partial products of ideals and abuts to the sheaf cohomology in the sum of the ideals, meanwhile the Lyubeznik spectral sequence B.2.2 (also called Mayer-Vietoris spectral sequence) has as components local cohomologies supported in partial sums of ideals and abuts to the local cohomology supported in the product of the given ideals. So we are motivated to ask the following.

Question 6.1.1. *Is there some relation between the spectral sequences 2.1.8 and B.2.2?*

Lyubeznik has raised a question in [66] on the degeneration of his spectral sequence B.2.2. We ask the same about ours.

Question 6.1.2. *Does the spectral sequence 2.1.8 degenerate at second page?*

6.2 On Chapter 3

The Mayer-Vietoris spectral sequence provides in propositions 3.1.1 and 3.2.2 and Corollary 3.2.3 relations between local cohomology modules supported in the irrelevant ideal, in the ideals generated by products of variables and the *maximal ideal. We thus ask the following.

Question 6.2.1. *Does the Mayer-Vietoris spectral sequence 2.2.1 provide relations such as those of propositions 3.1.1 and 3.2.2 for an arbitrary k?*

6.3 On Chapter 4

Corollary 4.1.5 inspire us to ask the following.

Question 6.3.1. *Given a finitely generated R-module M, when is $K(M)$ generalized Cohen-Macaulay?*

Based on sections 4.2 and 4.3, we finish this section by asking the following.

Question 6.3.2. *Let M be a finitely generated R-module of depth g and dimension t. Is it true that*

$$\operatorname{id}_R M < \infty \Leftrightarrow \operatorname{pd}_R K^i(M) < \infty, \forall i = g, ..., t$$

or

$$\operatorname{pd}_R M < \infty \Leftrightarrow \operatorname{id}_R K^i(M) < \infty, \forall i = g, ..., t?$$

6.4 On Chapter 5

As a matter of interest, we reinforce the Miranda-Neto and Jorge-Pérez' questions 5.1.1, 5.1.6 and Jorgensen's question 5.1.10.

Appendix

Appendix A

Second alternative to the second page

The construction given in this appendix is based on [59].

Let $(\Sigma, \leq)$ be a ordered set and for each $\alpha \in \Sigma$ define

$$(-\infty, \alpha] = \{\beta \in \Sigma : \beta \leq \alpha\}.$$

One may be seen that $\{(-\infty, \alpha] : \alpha \in \Sigma\}$ is a basis for a topology of Σ. In particular, if $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_n$ are finite sequences of R and $I_1, I_2, ..., I_n$ are, respectively, the ideals generated by them, then the set

$$\Sigma = \{I_{i_0} \cdot I_{i_1} \cdot ... \cdot I_{i_p} : p \in \{0, ..., n-1\} \text{ and } i_0 < i_1 < ... < i_p\}$$

endowed with the inclusion order turns out to be a topological space. In this way, if R-mod denotes the category of R-modules then one may consider two other categories: the category of the inverse systems on R-mod with Σ as the index set, which is denoted by $R\text{-mod}^{\Sigma}$, and the category of sheaves of R-modules on Σ, denoted by $Sh(\Sigma)$.

Proposition A.1. *$Sh(\Sigma)$ is equivalent to R-mod$^{\Sigma}$.*

Proof. Let $\mathcal{F}$ be a sheaf on Σ. If I, J and K are elements of Σ such that $I \subseteq J \subseteq K$ then $(-\infty, I] \subseteq (-\infty, J] \subseteq (-\infty, K]$ is a chain of open subsets of Σ and the diagram

$$\begin{tikzcd}
\mathcal{F}((-\infty, K]) \arrow[rr] \arrow[dr] & & \mathcal{F}((-\infty, J]) \arrow[dl] \\
& \mathcal{F}((-\infty, I]) &
\end{tikzcd}$$

commutes, where the morphisms are the corresponding restriction morphisms. Moreover, if $\mathcal{F} \xrightarrow{\theta} \mathcal{G}$ is a morphism of sheaves on Σ, since θ commutes with the corresponding restrictions, then it induces a morphism between the inverse systems $\{\mathcal{F}((-\infty, I])\}_{I\in\Sigma}$ and $\{\mathcal{G}((-\infty, I])\}_{I\in\Sigma}$. Therefore we have constructed a mapping

$$\begin{array}{rlcl} \zeta: & Sh(\Sigma) & \to & R\text{-mod}^{\Sigma} \\ & \mathcal{F} & \mapsto & \{\mathcal{F}((-\infty, I])\}_{I\in\Sigma} \end{array}$$

and one may be checked that $\zeta(\delta \circ \theta) = \zeta(\delta) \circ \zeta(\theta)$ and $\zeta(1_{\mathcal{F}}) = \{1_{\mathcal{F}((-\infty,I])}\}_{I\in\Sigma}$, that is, ζ is a functor. Now let $P : \Sigma \to R$-mod be an inverse system. We have to construct a sheaf on Σ from P. Given an open subset U of Σ one may define

$$\mathcal{P}(U) = \varprojlim_{I\in U} P(I).$$

By the universal property of the inverse limit, if V and U are open subsets of Σ such that $V \subseteq U$ then there exists a unique morphism ρ_{VU} such that

$$\begin{tikzcd} \mathcal{P}(U) \arrow[rr, "\rho_{VU}"] \arrow[dr] & & \mathcal{P}(V) \arrow[dl] \\ & P(I) & \end{tikzcd}$$

for all $I \in V$, where the diagonal maps in the diagram are the canonical maps involved. Again, by the universal property of the inverse limit, we conclude that $\mathcal{P}$ is a presheaf on Σ.

Now, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of the open subset U of Σ. Suppose that $s \in \mathcal{P}(U)$ satisfies $s_{|U_\alpha} = 0$ for all $\alpha \in \Lambda$. Given $I \in U$ there exists $\gamma \in \Lambda$ such that $I \in U_\gamma$. Since $\{(-\infty, J]\}_{J\in\Sigma}$ is a basis for the topology of Σ we have that $I \in (-\infty, J] \subseteq U_\gamma$ for some $J \in \Sigma$. It implies the commutativity of the diagram

$$\begin{tikzcd} \mathcal{P}(U) \arrow[rr] \arrow[d] \arrow[dr] & & \mathcal{P}(U_\gamma) \arrow[dl] \\ P(I) & \mathcal{P}((-\infty, J]) \arrow[l] & \end{tikzcd}$$

Once $s_{|U_\gamma} = 0$ we conclude that the image of s by the projection $\mathcal{P}(U) \to P(I)$ equals zero. As it holds for every $I \in U$ we must have $s = 0$.

Let (s_α) be an element in $\prod_\alpha \mathcal{P}(U_\alpha)$ such that $s_{\alpha_{|U_\alpha\cap U_\beta}} = s_{\beta_{|U_\alpha\cap U_\beta}}$ for all $\alpha, \beta \in \Lambda$. Given $I \in U$, if there are $\alpha, \beta \in \Lambda$ such that $I \in U_\alpha \cap U_\beta$ then $(-\infty, I] \subseteq U_\alpha \cap U_\beta$ and

$$s_{\alpha_{|(-\infty,I]}} = (s_{\alpha_{|U_\alpha\cap U_\beta}})_{|(-\infty,I]} = (s_{\beta_{|U_\alpha\cap U_\beta}})_{|(-\infty,I]} = s_{\beta_{|(-\infty,I]}}$$

which implies that the images of s_α and s_β through the morphisms

$$\mathcal{P}(U_\alpha) \longrightarrow \mathcal{P}((-\infty, I]) \longrightarrow P(I) \text{ and } \mathcal{P}(U_\beta) \longrightarrow \mathcal{P}((-\infty, I]) \longrightarrow P(I)$$

respectively, coincide. Let s_I be such image and consider $s = (s_I)_{I\in U}$. Notice that, for any I and J in U such that $I \subseteq J$, we have $J \in U_\alpha$ for some $\alpha \in \Lambda$ and $(-\infty, I] \subseteq (-\infty, J] \subseteq U_\alpha$. Hence s_I and s_J can be chosen as the images of s_α through the morphisms $\mathcal{P}(U_\alpha) \to P(I)$ and $\mathcal{P}(U_\alpha) \to P(J)$, respectively, which implies that s_J is the image of s_I through $P(I) \to P(J)$. We thus conclude that $s \in \mathcal{P}(U)$ and $s_{|U_\alpha} = s_\alpha$. It proves that $\mathcal{P}$ is a sheaf.

Let $P \xrightarrow{\eta} Q$ be a morphism between inverse systems. If U is an open subset of Σ then there exists a unique morphism η_U such that the diagram

$$\begin{tikzcd} \mathcal{P}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{Q}(U) \arrow[d] \\ P(I) \arrow[r, "\eta_I"'] & Q(I) \end{tikzcd}$$

commutes.

Given V and U two open subsets of Σ such that $V \subseteq U$, due to the diagram above and the fact that η is a morphism of inverse systems, we have commutativity in the following diagram

$$\begin{tikzcd} \mathcal{P}(U) \arrow[r, "\eta_U"] \arrow[d] & \mathcal{Q}(U) \arrow[d] \\ \mathcal{P}(V) \arrow[r, "\eta_V"'] & \mathcal{Q}(V) \end{tikzcd}$$

where the vertical morphisms are the respective restriction morphisms. Again we have constructed a mapping

$$\begin{array}{rlcl} \xi: & R\text{-mod}^\Sigma & \to & Sh(\Sigma) \\ & P & \mapsto & \mathcal{P} \end{array}$$

and one may also be checked that $\xi(\eta \circ \mu) = \xi(\eta) \circ \xi(\mu)$ and $\xi(\{1_{P(I))}\}_{I\in X}) = 1_{\mathcal{P}}$, that is, ξ is a functor.

Claim A.2. *If $1_{Sh(\Sigma)}$ and $1_{R\text{-}mod^\Sigma}$ are the identity functors of $Sh(\Sigma)$ and R-mod$^\Sigma$, respectively, then $\zeta \circ \xi \simeq 1_{R\text{-}mod^\Sigma}$ and $\xi \circ \zeta \simeq 1_{Sh(\Sigma)}$.*

Indeed, given a morphism of inverse systems $P \xrightarrow{\eta} Q$, since the set $\{I\}$ is a cofinal subset of $(-\infty, I]$ for every $I \in \Sigma$, we have

$$\mathcal{P}((-\infty, I]) = \varprojlim_{J\in(-\infty,I]} P(J) \simeq P(I)$$

for every $I \in \Sigma$. It implies that the diagram

$$\begin{tikzcd} \mathcal{P}((-\infty, J]) \arrow[r, "\eta_{(-\infty,J]}"] \arrow[d] & \mathcal{Q}((-\infty, J]) \arrow[d] \\ \mathcal{P}((-\infty, I])) \arrow[r, "\eta_{(-\infty,I]}"'] & \mathcal{Q}((-\infty, I]) \end{tikzcd}$$

is isomorphic to the diagram

$$\begin{tikzcd} P(J) \arrow[r, "\eta_J"] \arrow[d] & Q(J) \arrow[d] \\ P(I) \arrow[r, "\eta_I"] & Q(I) \end{tikzcd}$$

for all $I, J \in \Sigma$ such that $I \subseteq J$. It means that $\zeta \circ \xi(\eta) \simeq \eta$.

On the other hand, let $\mathcal{F} \xrightarrow{\theta} \mathcal{G}$ be a morphism of sheaves on Σ. Given an open subset $U \subseteq \Sigma$ there exists a unique morphism φ such that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi"] \arrow[d] & \varprojlim_{I \in U} \mathcal{F}((-\infty, I]) \arrow[dl] \\ \mathcal{F}((-\infty, I]) & \end{tikzcd}$$

commutes for all $I \in U$. Since $\varphi(x) = (x_{|_{(-\infty,I]}})$ for all $x \in \mathcal{F}(U)$, $\{(-\infty, I]\}_{I \in U}$ is an open cover of U and $\mathcal{F}$ is a sheaf we conclude that φ is an isomorphism. It turns out to be a functorial isomorphism, that is, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\theta_U"] \arrow[d, "\varphi"'] & \mathcal{G}(U) \arrow[d, "\varphi"] \\ \varprojlim_{I \in U} \mathcal{F}((-\infty, I]) \arrow[r, "(\xi \circ \zeta(\theta))_U"'] & \varprojlim_{I \in U} \mathcal{G}((-\infty, I]) \end{tikzcd}$$

commutes for every open subset $U \subseteq \Sigma$. Therefore $\xi \circ \zeta(\theta) \simeq \theta$. ■

An immediate consequence is the following result.

Corollary A.3. *The global sections functor $\Gamma(\Sigma, \_)$ on $Sh(\Sigma)$ is isomorphic to the inverse limit functor $\varprojlim_{I \in \Sigma}$ on R-mod$^{\Sigma}$.*

Let $\mathcal{F}$ be a sheaf of R-modules on Σ, consider the open cover $\mathcal{U} = \{(-\infty, I]\}_{I \in \Sigma}$ of Σ and denote by $\check{H}^p(\mathcal{U}, \mathcal{F})$ the p-th Čech cohomology of $\mathcal{U}$ with coefficients in $\mathcal{F}$. [45, Lemma 4.1 chapter III] and Corollary A.3 give a functorial isomorphism

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \simeq \varprojlim_{I \in \Sigma} \mathcal{F}((-\infty, I]).$$

Lemma A.4. *[46] The Čech cohomologies $\{\check{H}^p(\mathcal{U}, \_)\}_{p \geq 0}$ form a δ-functor.*

Proof. Firstly note that for a given element $I \in \Sigma$ the set $\{(-\infty, I]\}$ is cofinal in the set of neighborhoods of I with the reverse order given by inclusion. This implies that if $\mathcal{F}$ is a sheaf on Σ then

$$\mathcal{F}_I = \varinjlim_{I \in U} \mathcal{F}(U) \simeq \mathcal{F}((-\infty, I])$$

for any $I \in \Sigma$. Therefore an exact sequence of sheaves on Σ

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

induces exact sequences of R-modules

$$0 \to \mathcal{F}'((-\infty, I]) \to \mathcal{F}((-\infty, I]) \to \mathcal{F}''((-\infty, I]) \to 0$$

for all $I \in \Sigma$. By taking suitable direct products one has exact sequence of Čech complexes

$$0 \to \check{C}(\mathcal{U}, \mathcal{F}') \to \check{C}(\mathcal{U}, \mathcal{F}) \to \check{C}(\mathcal{U}, \mathcal{F}'') \to 0.$$

One sees that a morphism of short exact sequences of sheaves on Σ induces a morphism of short exact sequences of the Čech complexes involved. This give us the result. ■

[74, Propositions 6.72 and 6.73] says that every sheaf on Σ can be embedded in a flasque sheaf. [45, Proposition 4.3 chapter III] says that flasque sheaves are $\varprojlim_{I\in\Sigma}$-acyclic and [45, Proposition 1.2A chapter III] says that the derived functors of $\varprojlim_{I\in\Sigma}$ can be computed by the Čech cohomology of $\mathcal{U}$. In other words, $\{\check{H}^p(\mathcal{U}, \_)\}_{p\geq 0}$ is a universal δ-functor and

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq \varprojlim_{I\in\Sigma}{}^{(p)} \mathcal{F}((-\infty, I]).$$

Let $q \geq 0$. We may see sheaf cohomology groups as R-modules of the form $\mathcal{F}(U)$ where $\mathcal{F}$ is an object in $Sh(\Sigma)$ and U is an open subset of Σ. Indeed, let M be an R-module and, if I and J are two ideals in Σ such that $I \subseteq J$, then the canonical inclusion $I \hookrightarrow J$ induces a morphism $H^q(U_J, \widetilde{M}) \to H^q(U_I, \widetilde{M})$. It is immediate to see that it defines an object $H^q(U_\bullet, \widetilde{M})$ in R-mod^Σ. Furthermore, since $\{I\}$ is cofinal in the set $(-\infty, I]$, if $\mathcal{H}^q(U_\bullet, \widetilde{M})$ is the sheaf in $Sh(\Sigma)$ associated to $H^q(U_\bullet, \widetilde{M})$ then

$$\mathcal{H}^q(U_\bullet, \widetilde{M})((-\infty, I]) = \varprojlim_{J\in(-\infty, I]} H^q(U_J, \widetilde{M}) \simeq H^q(U_I, \widetilde{M})$$

for all $I \in \Sigma$.

It follows that

$$\check{H}^p(\mathcal{U}, \mathcal{H}^q(U_\bullet, \widetilde{M})) \simeq \varprojlim_{I\in\Sigma}{}^{(p)} H^q(U_I, \widetilde{M})$$

for all integer p.

The horizontal lines of the first page of the Mayer-Vietoris spectral sequence 2.1.8 define complexes

$$\mathcal{H}^q(M)^\bullet : 0 \longrightarrow \mathcal{H}^q(M)^0 \longrightarrow \mathcal{H}^q(M)^1 \longrightarrow \cdots$$

where

$$\mathcal{H}^q(M)^p = \bigoplus_{i_0<\ldots<i_p} H^q(U_{I_{i_0}\ldots I_{i_p}}, \widetilde{M}).$$

One may see that the complex $\mathcal{H}^q(M)^\bullet$ is isomorphic to the Čech complex of $\mathcal{U}$ with coefficients in $\mathcal{H}^q(U_\bullet, \widetilde{M})$, and from the last isomorphisms, we conclude that, for every $p \geq 0$,

$$H^p(\mathcal{H}^q(M)^\bullet) \simeq \varprojlim_{I\in\Sigma}{}^{(p)} H^q(U_I, \widetilde{M}).$$

Therefore, the second page of the Mayer-Vietoris spectral sequence 2.1.8 is

$$E_2^{p,q} = \varprojlim_{I\in\Sigma}{}^{(p)} H^q(U_I, \widetilde{M}).$$

Appendix B

Spectral sequences

We devote this appendix to describe some important generalities on spectral sequences and some examples which are vastly used through the literature, and, in particular, in this thesis. The reader not familiarized with the general theory of spectral sequences can found [70, 74, 83] as nice introductory textbooks to get acquainted with this important homological tool.

B.1 Spectral sequences arising from double complexes

In this section, we follow the textbook [83].

Let $\mathcal{A}$ be an abelian category.

Definition B.1. *[83, Definition 5.2.1] A spectral sequence in $\mathcal{A}$ consists of the following data:*

i) A family $\{E_r^{p,q}\}$ of objects of $\mathcal{A}$ defined for all integers p, q and $r \geq 0$;

ii) Maps $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that are differentials in the sense that $dd = 0$;

iii) Isomorphisms between $E_{r+1}^{p,q}$ and the cohomology of E_r at the spot $E_r^{p,q}$:

$$E_{r+1}^{p,q} \simeq \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1}).$$

If for each p and q there exists r_0 such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq r_0$ then we write $E_\infty^{p,q}$ for this stable value of $E_r^{p,q}$.

Let $\mathcal{C}$ be a double complex in $\mathcal{A}$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r] \arrow[u] & C^{p,q+1} \arrow[r] \arrow[u] & C^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r] \arrow[u] & C^{p,q} \arrow[r, "d"] \arrow[u, "d'"] & C^{p+1,q} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[u] & C^{p,q-1} \arrow[r] \arrow[u] & C^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Definition B.2. *[83, Definition 1.2.6] The total complex $Tot(\mathcal{C})$ of $\mathcal{C}$ is defined as*

$$Tot^n(\mathcal{C}) = \bigoplus_{p+q=n} C^{p,q}$$

with differential $Tot^n(\mathcal{C}) \to Tot^{n+1}(\mathcal{C})$ defined by $d'^{p,q} + (-1)^p d^{p,q}$.

$Tot(\mathcal{C})$ is indeed a complex and we may naturally define two filtrations for this complex. First, we define the filtration by columns.

Definition B.3. *[83, Definition 5.6.1] For each p the sequence $F^pTot(\mathcal{C})$ defined as*

$$F^pTot^n(\mathcal{C}) = \bigoplus_{p\geq n} C^{p,n-p}$$

is a subcomplex of $Tot(\mathcal{C})$.

$\{F^pTot(\mathcal{C})\}_p$ defines a filtration of $Tot(\mathcal{C})$ so that it induces a spectral sequence E (see [83, page 141]). The first and second pages are well known. The first page is given by

$$E_1^{p,q} = H^q(\mathcal{C}^{p,\bullet}).$$

Since passing cohomology is a functorial operation, $E_1^{\bullet,q}$ is a complex and the objects in the second page of E coincides exactly with these homologies, so we use a suggestive notation:

$$E_2^{p,q} = H^pH^q(\mathcal{C}).$$

The second filtration of $Tot(\mathcal{C})$ is given by its rows.

Definition B.4. *[83, Definition 5.6.2] For each* q *the sequence* $'F^qTot(\mathcal{C})$ *defined as*

$$'F^qTot^n(\mathcal{C}) = \bigoplus_{q \geq n} C^{n-q,q}$$

is a subcomplex of $Tot(\mathcal{C})$.

In a similar way, $\{'F^qTot(\mathcal{C})\}_q$ defines a filtration of $Tot(\mathcal{C})$ and so we have another spectral sequence $'E$. Its first and second pages are given by

$$'E_1^{p,q} = H^q(\mathcal{C}^{\bullet,p}) \text{ and } 'E_2^{p,q} = H^qH^p(\mathcal{C}).$$

Definition B.5. *[83, 5.2.11] A spectral sequence* E *converges to a graded object* H*, denoted by*

$$E_2^{p,q} \Rightarrow_p H^n,$$

if for each n *there exists a decreasing filtration*

$$0 = F^{n+1}H^n \subset F^nH^n \subset ... \subset F^1H^n \subset F^0H^n = H^n$$

such that, for $p+q=n$*,*

$$E_\infty^{p,q} \simeq F^pH^n/F^{p+1}H^n.$$

Theorem B.6. *[83, Theorem 5.51] If* $\mathcal{C}$ *is a first quadrant double complex then the filtrations* $\{F^p\}_p$ *and* $\{'F^p\}_p$ *of* $Tot(\mathcal{C})$ *are both bounded and*

$$E_2^{p,q} \Rightarrow_p H^{p+q}(Tot(\mathcal{C}))$$

and

$$'E_2^{p,q} \Rightarrow_p H^{p+q}(Tot(\mathcal{C})).$$

B.2 Examples

B.2.1 Čech spectral sequence

In this section, we follow [19].

Let $\mathcal{F}$ be a sheaf of abelian groups on X and let $\mathcal{U}$ be an open cover of X. Pick up an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$, and form the double complex $C^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & & & \vdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \to & C^0(\mathcal{U},\mathcal{I}^m) & \to & C^1(\mathcal{U},\mathcal{I}^m) & \to & \cdots & \to & C^n(\mathcal{U},\mathcal{I}^m) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \vdots & & \vdots & & \iddots & & \vdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \to & C^0(\mathcal{U},\mathcal{I}^1) & \to & C^1(\mathcal{U},\mathcal{I}^1) & \to & \cdots & \to & C^n(\mathcal{U},\mathcal{I}^1) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \to & C^0(\mathcal{U},\mathcal{I}^0) & \to & C^1(\mathcal{U},\mathcal{I}^0) & \to & \cdots & \to & C^n(\mathcal{U},\mathcal{I}^0) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & 0 & & 0 & & & & 0
\end{array}
$$

It induces two spectral sequences. Let E be the spectral sequence associated with the filtration by columns. Its first page is

$$E_1^{p,q} = H^q(C^p(\mathcal{U},\mathcal{I}^\bullet)) = \prod_{i_0<\ldots<i_p} H^q(U_{i_0\ldots i_p},\mathcal{F}).$$

We can compute the second page of E as follows. For each $q \geq 0$ consider the presheaf

$$\mathcal{H}^q(\mathcal{F})(U) := H^q(U,\mathcal{F})$$

where U is an open subset of X. Thus the first page can be rewritten as $E_1^{p,q} = C^p(\mathcal{U},\mathcal{H}^q(\mathcal{F}))$ so that $E_1^{\bullet,q}$ is the Čech complex of the open cover $\mathcal{U}$ with coefficients in $\mathcal{H}^q(\mathcal{F})$. Therefore

$$E_2^{p,q} \simeq H^p(\mathcal{U},\mathcal{H}^q(\mathcal{F})).$$

Let $'E$ be the spectral sequence associated with the filtration by rows. Its first page is

$$'E_1^{p,q} = H^q(C^\bullet(\mathcal{U},\mathcal{I}^p)) = H^q(\mathcal{U},\mathcal{I}^p).$$

Since the sheaves $\mathcal{I}^p$ are flasque (see [19, Lemma 4.20]), we have $'E_1^{p,q} = 0$ for $q > 0$ and $'E_1^{p,0} = \Gamma(X,\mathcal{I}^p)$. Hence $'E_2^{p,q} = 0$ for $q > 0$ and

$$H^p(X,\mathcal{F}) = {}'E_2^{p,q} \simeq {}'E_\infty^{p,q}.$$

Theorem B.1 (Čech Spectral Sequence)**.** *[19, Theorem 5.32] Let $\mathcal{U}$ be an open cover of a topological space X, and let $\mathcal{F}$ be a sheaf of abelian groups on X. There is a spectral sequence E whose first and second pages are given by*

$$E_1^{p,q} = \prod_{i_0<\ldots<i_p} H^q(U_{i_0\ldots i_p},\mathcal{F}) \text{ and } E_2^{p,q} = H^p(\mathcal{U},\mathcal{H}^q(\mathcal{F}))$$

and converges to the sheaf cohomology $H^\bullet(X,\mathcal{F})$.

B.2.2 Another type of the Mayer-Vietoris spectral sequence

The principal tool in [66] is another spectral sequence, which they call by Mayer-Vietoris spectral sequence as well. It is a generalization of the spectral sequence that appears first in [1]. Here we give a sketch of the construction of this spectral sequence. For the details see [66].

Recall that for an ideal J of a Noetherian ring R, $\Gamma_J(M)$ denotes the submodule of the finitely generated R-module M consisting of the elements of M annihilated by some power of J, see for example [17, 58]. If $J' \subseteq J$, we let $\gamma_{J,J'} : \Gamma_J(M) \hookrightarrow \Gamma_{J'}(M)$ be the natural inclusion. Hence, given ideals $I_1, ..., I_n \subseteq R$, the sequence

$$\Gamma^\bullet(M): \qquad 0 \longrightarrow \Gamma^{-n+1}(M) \xrightarrow{d^{-n+1}} \Gamma^{-n+2}(M) \xrightarrow{d^{-n+2}} \cdots \xrightarrow{d^{-1}} \Gamma^0(M) \longrightarrow 0$$

where $\Gamma^{-p}(M) = \bigoplus_{i_0<...<i_p} \Gamma_{I_{i_0}+...+I_{i_p}}(M)$ and $d^{-p}(x) = ((-1)^j \gamma_{J,J_j}(x))_j$ for every element $x \in \Gamma_J(M) \subseteq \Gamma^{-p}(M)$, where $J = I_{i_0} + ... + I_{i_p}$ and $J_j = I_{i_0} + ... + I_{j-1} + I_{j+1} + ... + I_{i_p}$.

It may be seen that $\Gamma^\bullet(\_)$ defines a functor from the category of R-modules to the category of complexes of R-modules. Moreover, if M is injective, then

$$H^p(\Gamma^\bullet(M)) = \begin{cases} \Gamma_{I_1\cap...\cap I_n}(M), & \text{if } p = 0 \\ 0, & \text{else.} \end{cases}$$

Hence, if $M \to E^\bullet$ is a injective resolution of M, then the third quadrant double complex $\Gamma^\bullet(E^\bullet)$ yields a spectral sequence collapsing at its second page, with the modules $H^q_{I_1\cap...\cap I_n}(M)$ at the spot $(0, q)$. Meanwhile, since homology commutes with direct sums, the other spectral sequence is given by $E_1^{-p,q} = \bigoplus_{i_0<...<i_p} H^q_{I_{i_0}+...+I_{i_p}}(M)$. Convergence asserts the following result.

Theorem B.2. *[66, Theorem 2.1] Suppose R Noetherian and let $I_1, ..., I_n \subset R$ be ideals and let M be an R-module. There exists a spectral sequence*

$$E_1^{-p,q} = \bigoplus_{i_0<...<i_p} H^q_{I_{i_0}+...+I_{i_p}}(M) \Rightarrow_p H^{q-p}_{I_1\cap...\cap I_n}(M).$$

As [66] says, if $n = 2$, i.e., there are just two ideals, this spectral sequence becomes the standard Mayer-Vietoris long exact sequence. Furthermore, if $n \leq 3$, then the Mayer-Vietoris spectral sequence degenerates at E_2.

B.2.3 Koszul-Čech spectral sequence

Assume R Noetherian. Let $\mathfrak{a} = a_1, ..., a_m$ be a finite sequence of R and let I be the ideal generated by $\mathfrak{a}$. Let $\mathbf{x} = x_1, ..., x_n$ also be a sequence of elements of R. Consider the Čech complex of R associated to $\mathfrak{a}$, see Definition 1.1.1, $C_{\mathfrak{a}}^{\bullet}(R)$, and the Koszul complex of R associated to $\mathbf{x}$, $K_{\bullet}(\mathbf{x})$, see [18]. Let M be an R-module. We may consider the first quadrant double complex $C_{\mathfrak{a}}^{\bullet}(R) \otimes_R K_{\bullet}(\mathbf{x}) \otimes_R M$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{a}}^0(R) \otimes_R K_n(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & C_{\mathfrak{a}}^1(R) \otimes_R K_n(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{a}}^m(R) \otimes_R K_n(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{a}}^0(R) \otimes_R K_1(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & C_{\mathfrak{a}}^1(R) \otimes_R K_1(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{a}}^m(R) \otimes_R K_1(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{a}}^0(R) \otimes_R K_0(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & C_{\mathfrak{a}}^1(R) \otimes_R K_0(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{a}}^m(R) \otimes_R K_0(\mathbf{x}) \otimes_R M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}
$$

It induces two spectral sequences that converge to the same module. Let E be the spectral sequence induced by passing cohomology on horizontal and let $'E$ be the spectral sequence considering vertical homology. Since both Koszul and Čech complexes considered are composed by free R-modules, one has

$$E_1^{p,q} = H_I^q(M) \otimes_R K_p(\mathbf{x}) \text{ and } E_2^{p,q} = H_p(\mathbf{x}; H_I^q(M))$$

and

$$'E_1^{p,q} = C_{\mathfrak{a}}^p(R) \otimes_R H_q(\mathbf{x}; M) \text{ and } 'E_2^{p,q} = H_I^p(H_q(\mathbf{x}; M))$$

where $H_q(\mathbf{x}; \_)$ denotes the q-th Koszul homology of the sequence $\mathbf{x}$. Both spectral sequences converge to a graded module H in such a way that

$$E_2^{p,q} = H_p(\mathbf{x}; H_I^q(M)) \Rightarrow_p H^{p-q}$$

and

$$'E_2^{p,q} = H_I^p(H_q(\mathbf{x}; M)) \Rightarrow_p H^{q-p}.$$

These spectral sequence are called Koszul-Čech spectral sequence. Interesting recent applications for them can be found in [14] and [35].

Now we consider two cases for these spectral sequences that come out to be useful in this thesis. First, let M be a finitely generated R-module and suppose that $\mathbf{x}$ forms a M-regular sequence. Hence $'E_2^{p,q} = H_I^p(H_q(\mathbf{x}; M)) = 0$ for all $q > 0$ so that

$$H_I^p(M/\mathbf{x}M) = {'E_2^{p,0}} \simeq {'E_\infty^{p,0}} \simeq H^{-p}$$

for all $p \geq 0$. Therefore there is spectral sequence

$$E_2^{p,q} = H_p(\mathbf{x}; H_I^q(M)) \Rightarrow_p H_I^{q-p}(M/\mathbf{x}M).$$

For the second case, suppose that R is a $\mathbb{Z}^k$-graded *local ring, $\mathfrak{a}$ is a sequence of homogeneous ideal of positive degree and $\mathbf{x}$ is an R-regular sequence generating the *local maximal ideal. It follows that $H_q(\mathbf{x}; M)$ is annihilated by the *local maximal ideal meanwhile every homogeneous element a_i acts as an invertible element of $'E_1^{p,q}$ for all $p > 0$ so that $'E_1^{p,q} = C_{\mathfrak{a}}^p(R) \otimes_R H_q(\mathbf{x}; M) = 0$ for all $p > 0$ and

$$\mathrm{Tor}_q^R(M, R/(\mathbf{x})) \simeq H_q(\mathbf{x}; M) = {'E_1^{0,q}} = {'E_\infty^{0,q}} \simeq H^q$$

for all $q \geq 0$. Therefore there is a spectral sequence

$$E_2^{p,q} = H_p(\mathbf{x}; H_I^q(M)) \Rightarrow_p \mathrm{Tor}_{p-q}^R(M, R/(\mathbf{x})).$$

Notice that one may suppose R local, $\mathbf{x}$ a generating set of the maximal ideal and I a proper ideal of R. An alike argument applies to this case.

B.2.4 Foxby spectral sequences

Foxby spectral sequences make up a quite useful homological tool. They were named after their first use in Foxby's work [38]. They have several applications; for example, they can be used to prove the local duality theorem (see [17, Theorem 12.1.20] or [58, Theorem 11.44]) or to get relations between Bass numbers and the minimal number of generators of certain modules, Bass numbers and Betti numbers, injective dimension and depth (Ischebeck's formula). See for instance [18, Exercises 3.1.24, 3.1.25, 3.3.26, Proposition 3.3.11]. It should be noticed that all the discussion in this section can be used in the graded case.

Let R and S be Noetherian rings and consider $R \to S$ a ring homomorphism.

Lemma B.3. *If M is a finitely generated R-module, N a S-module and E an injective S-module, then there exists isomorphism*

$$M \otimes_R \mathrm{Hom}_S(N, E) \simeq \mathrm{Hom}_S(\mathrm{Hom}_R(M, N), E)$$

which is functorial in M.

Proof. The functor $\mathrm{Hom}_S(\mathrm{Hom}_R(\_, N), E)$ is a right exact functor that commutes with finite direct sums defined from the category of finitely generated R-modules into the category of abelian groups. Therefore

$$\mathrm{Hom}_S(\mathrm{Hom}_R(\_, N), E) \simeq \_ \otimes_R \mathrm{Hom}_S(\mathrm{Hom}_R(R, N), E) \simeq \_ \otimes_R \mathrm{Hom}_S(N, E).$$

■

Theorem B.4. *If M is a finitely generated R-module, and N and P are S-modules, then there are two first quadrant spectral sequences*

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^q(M, N), P) \text{ and } {}'E_2^{p,q} = \mathrm{Tor}_p^R(M, \mathrm{Ext}_S^q(N, P)).$$

Furthermore, they converge to the same limit H whenever either M has finite projective dimension or P has finite injective dimension (as S-module):

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^q(M, N), P) \Rightarrow_p H^{q-p}$$

and

$${}'E_2^{p,q} = \mathrm{Tor}_p^R(M, \mathrm{Ext}_S^q(N, P)) \Rightarrow_p H^{p-q}.$$

Proof. Let $F_\bullet$ be a free resolution of M and let $E^\bullet$ be a injective resolution of P (as S-module). By Lemma B.3 one has isomorphism of first quadrant double complexes

$$F_\bullet \otimes_R \mathrm{Hom}_S(N, E^\bullet) \simeq \mathrm{Hom}_S(\mathrm{Hom}_R(F_\bullet, N), E^\bullet).$$

By using the exactness of the functors $F_i \otimes_R \_$ and $\mathrm{Hom}_S(\_, E^j)$, the right hand side gives rise to the spectral sequence E whereas the left hand side gives rise to the spectral sequence ${}'E$ as follows.

$$E_1^{p,q} = \mathrm{Hom}_S(\mathrm{Ext}_R^q(M, N), E^p) \text{ and } E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Ext}_R^q(M, N), P),$$

and

$${}'E_1^{p,q} = F_p \otimes_R \mathrm{Ext}_S^q(N, P) \text{ and } E_2^{p,q} = \mathrm{Tor}_p^R(M, \mathrm{Ext}_S^q(N, P)).$$

The convergence follows from Theorem B.6. ■

As already mentioned, local duality is an immediate consequence of the convergence of the two spectral sequences above. Indeed, suppose $(R, \mathfrak{m})$ is Cohen-Macaulay local of dimension d with canonical module ω_R, let E be the injective hull of the residue field of R and denote $\_^{\vee} = \operatorname{Hom}_R(\_, E)$. By taking $R = S, N = \omega_R$ and $P = E$ in Theorem B.4, both spectral sequences degenerate so that

$$H_{\mathfrak{m}}^{d-i}(M) \simeq \operatorname{Tor}_i^R(M, \omega_R^{\vee}) \simeq {}'E_2^{i,0} \simeq H^i \simeq E_2^{0,i} \simeq \operatorname{Ext}_R^i(M, \omega_R)^{\vee}$$

for all $i \geq 0$.

B.2.5 Suzuki spectral sequences

The spectral sequences in this section were first used by Suzuki in [80]. They are quite useful in the study of generalized cohomology modules, see for example [29, 48, 80]. Here we suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$.

Lemma B.5. [80, Theorem 1.4] *Let M and N be finitely generated R-modules, $\underline{x} = x_1, ..., x_n$ in $\mathfrak{m}$ generating a $\mathfrak{m}$-primary ideal, and for each $m \geq 0$, $K_\bullet(\underline{x}^m; R)$ denotes the Koszul complex of R with respect to $\underline{x}^m = x_1^m, ..., x_n^m$ and $F_\bullet$ be a free resolution of M. If C^m denotes the total complex associated to the double complex $K_\bullet(\underline{x}^m; R) \otimes_R F_\bullet$, then $H_{\mathfrak{m}}^i(M, N) \simeq \varinjlim_m H^i(\operatorname{Hom}_R(C^m, N))$.*

Lemma B.6. ([80, Proposition 1.8]) *Let R be a local ring. If M, N are finitely generated R-modules then there exists a first quadrant spectral sequence*

$$H_{\mathfrak{m}}^p(\operatorname{Ext}_R^q(M, N)) \Rightarrow_p H_{\mathfrak{m}}^{p+q}(M, N).$$

Proof. First let $F_\bullet$ be a free resolution of M and notice that, for each $m \geq 0$, from the hom-tensor adjunction, there is an isomorphism of double complexes

$$\operatorname{Hom}_R(K_\bullet(\underline{x}^m; R) \otimes_R F_\bullet, N) \simeq \operatorname{Hom}_R(K_\bullet(\underline{x}^m; R), \operatorname{Hom}_R(F_\bullet, N))$$

and by the lemma above the total complex associated to this double complex is isomorphic to $\operatorname{Hom}_R(C^m, N)$. It assures the existence of a spectral sequence

$$H^p(\operatorname{Hom}_R(K_\bullet(\underline{x}^m; R), \operatorname{Ext}_R^q(M, N)) \Rightarrow_p H^{p+q}(\operatorname{Hom}_R(C^m, N)).$$

It consists of a direct system of spectral sequences and thus the result follows by applying $\varinjlim_m$ to it. ■

Lemma B.7. ([80, Proposition 1.7]) *Let R be a local ring. If M, N are finitely generated R-modules then there exists a first quadrant spectral sequence*

$$\operatorname{Ext}_R^p(M, H_{\mathfrak{m}}^q(N)) \Rightarrow_p H_{\mathfrak{m}}^{p+q}(M, N).$$

Proof. The construction of the desired spectral sequence is completely analogous to that of Lemma B.7 by considering the other spectral sequence arising from the double complex

$$\mathrm{Hom}_R(K_\bullet(\underline{x}^m;R)\otimes_R F_\bullet, N) \simeq \mathrm{Hom}_R(F_\bullet, \mathrm{Hom}_R(K_\bullet(\underline{x}^m;R),N)).$$

instead. ■

It should be observed that both spectral sequences above can be constructed for any ideal I. Indeed, it can be proved that for two finitely generated M and N over a (non-necessarily local) ring R there exist two Grothendieck spectral sequences

$$H_I^p(\mathrm{Ext}_R^q(M,N)) \Rightarrow_p H_I^{p+q}(M,N)$$

and

$$\mathrm{Ext}_R^p(M, H_I^q(N)) \Rightarrow_p H_I^{p+q}(M,N).$$

See [74] for Grothendieck spectral sequences.

B.2.6 Cohomological dimension estimate

Theorem B.8. *Let $\mathfrak{a} = a_1, ..., a_n$ be a finite sequence of elements of R, and let M and N be two R-modules. If I is the ideal generated by $\mathfrak{a}$ then there exist a graded R-module H and two spectral sequences*

$$E_2^{-p,q} = \mathrm{Tor}_p^R(M, H_I^q(N)) \Rightarrow_p H^{q-p}$$

and

$$'E_2^{p,-q} = H_I^p(\mathrm{Tor}_q^R(M,N)) \Rightarrow_p H^{p-q}.$$

Proof. Let $F_\bullet$ be a free resolution of M and consider the second quadrant double complex $F_\bullet \otimes_R C_{\mathfrak{a}}^\bullet(R) \otimes_R N$:

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
\cdots \arrow[r] & F_2 \otimes_R C^n_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_1 \otimes_R C^n_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_0 \otimes_R C^n_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
\cdots \arrow[r] & F_2 \otimes_R C^1_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_1 \otimes_R C^1_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_0 \otimes_R C^1_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & 0 \\
\cdots \arrow[r] & F_2 \otimes_R C^0_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_1 \otimes_R C^0_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & F_0 \otimes_R C^0_{\mathfrak{a}}(R) \otimes_R N \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

By passing homology on horizontal one gets spectral sequence whose first and second pages are

$$'E_1^{p,-q} = \operatorname{Tor}_q^R(M,N) \otimes_R C^p_{\mathfrak{a}}(R) \text{ and } 'E_2^{p,-q} = H_I^p(\operatorname{Tor}_q^R(M,N)).$$

Since $F_\bullet$ is a resolution, the other spectral sequence E is such that

$$E_1^{-p,q} = F_p \otimes_R H_I^q(N) \text{ and } E_2^{-p,q} = \operatorname{Tor}_p^R(M, H_I^q(N)).$$

The convergence follows from Theorem B.6. ■

This spectral sequence is useful for estimating cohomological dimension of any module in terms of the cohomological dimension of the ring. Precisely, by considering the hypothesis of the theorem above, $\operatorname{cd}_I(M) \leq \operatorname{cd}_I(R)$. Indeed, write $\operatorname{cd}_I(R) = t$ and suppose $N = S$ is a faithfully flat ring extension of R. Thus $H_I^i(M \otimes_R S) \simeq H_I^i(M) \otimes_R S$ for all integer i and, from the theorem above, the spectral sequence $'E$ is such that $'E_2^{p,-q} = 0$ for all $q > 0$ so that $H^p \simeq H_I^p(M) \otimes_R S$ for all integer p. Therefore, there exists spectral sequence

$$E_2^{-p,q} = \operatorname{Tor}_p^R(M, H_I^q(S)) \Rightarrow_p H_I^{q-p}(M) \otimes_R S.$$

Since $H_I^i(S) \simeq H_I^i(R) \otimes_R S$ for all integer i, one has $E_2^{-p,q} = 0$ for all $q > t$. By convergence we conclude that

$$M \otimes_R H_I^t(S) \simeq H_I^t(M) \otimes_R S$$

and that

$$H_I^j(M) \otimes_R S \simeq M \otimes_R H_I^j(S) = 0$$

for all $j > t$. Since S is faithfully flat one has $H_I^j(M) = 0$, that is, $\operatorname{cd}_I(M) \leq t$.

B.2.7 Local cohomology modules supported in I, J and $I+J$

The spectral sequence in this section has also been used in [22] and [28].

Theorem B.9. *Let $\mathfrak{a} = a_1, ..., a_r$ and $\mathfrak{b} = b_1, ..., b_s$ be two finite sequences of R and let M be an R-module. If I and J are the ideals generated by $\mathfrak{a}$ and $\mathfrak{b}$, respectively, then there exists spectral sequence*

$$E_2^{p,q} = H_I^p(H_J^q(M)) \Rightarrow_p H_{I+J}^{p+q}(M).$$

Proof. Consider the first quadrant double complex $C_{\mathfrak{a}}^{\bullet}(R) \otimes_R C_{\mathfrak{b}}^{\bullet}(R) \otimes_R M$:

$$\begin{tikzcd}
 & 0 & 0 & & 0 & \\
0 \arrow[r] & C_{\mathfrak{b}}^s(R) \otimes_R C_{\mathfrak{a}}^0(R) \otimes_R M \arrow[r] \arrow[u] & C_{\mathfrak{b}}^s(R) \otimes_R C_{\mathfrak{a}}^1(R) \otimes_R M \arrow[r] \arrow[u] & \cdots \arrow[r] & C_{\mathfrak{b}}^s(R) \otimes_R C_{\mathfrak{a}}^r(R) \otimes_R M \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] & \\
0 \arrow[r] & C_{\mathfrak{b}}^1(R) \otimes_R C_{\mathfrak{a}}^0(R) \otimes_R M \arrow[r] \arrow[u] & C_{\mathfrak{b}}^1(R) \otimes_R C_{\mathfrak{a}}^1(R) \otimes_R M \arrow[r] \arrow[u] & \cdots \arrow[r] & C_{\mathfrak{b}}^1(R) \otimes_R C_{\mathfrak{a}}^r(R) \otimes_R M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & C_{\mathfrak{b}}^0(R) \otimes_R C_{\mathfrak{a}}^0(R) \otimes_R M \arrow[r] \arrow[u] & C_{\mathfrak{b}}^0(R) \otimes_R C_{\mathfrak{a}}^1(R) \otimes_R M \arrow[r] \arrow[u] & \cdots \arrow[r] & C_{\mathfrak{b}}^0(R) \otimes_R C_{\mathfrak{a}}^r(R) \otimes_R M \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

By passing homology on horizontal one gets spectral sequence E whose second page is given by

$$E_2^{p,q} = H_I^p(H_J^q(M)).$$

The result follows by noticing that $C_{\mathfrak{a}}^{\bullet}(R) \otimes_R C_{\mathfrak{b}}^{\bullet}(R) \otimes_R M \simeq C_{\mathfrak{a},\mathfrak{b}}^{\bullet}(R) \otimes_R M$. (It can be proved, for instance, using the characterization of Čech complexes as direct limit of Koszul complexes, see [22].) ■

B.2.8 Tensor-Hom adjunction

Let $S \to R$ be a rings homomorphism.

Theorem B.10. *Given M and N R-modules and P a S-module, there exist graded R-module and spectral sequences*

$$E_2^{p,q} = \operatorname{Ext}_R^p(M, \operatorname{Ext}_S^q(N, P)) \Rightarrow_p H^{p+q}$$

and

$${}'E_2^{p,q} = \operatorname{Ext}_S^p(\operatorname{Tor}_q^R(M, N), P) \Rightarrow_p H^{p+q}.$$

Proof. By taking a free resolution $F_\bullet$ of M (as R-module) and an injective resolution $E^\bullet$ of P, the Tensor-Hom adjunction gives us isomorphism of first quadrant double complexes

$$\mathrm{Hom}_R(F_\bullet, \mathrm{Hom}_S(N, E^\bullet)) \simeq \mathrm{Hom}_S(F_\bullet \otimes_R N, E^\bullet).$$

This isomorphism gives rise to two spectral sequences E and $'E$ converging to H^{p+q} and such that

$$E_1^{p,q} = \mathrm{Hom}_R(F_p, \mathrm{Ext}_S^q(N, P)) \text{ and } E_2^{p,q} = \mathrm{Ext}_R^p(M, \mathrm{Ext}_S^q(N, P))$$

and

$$'E_1^{p,q} = \mathrm{Hom}_S(\mathrm{Tor}_q^R(M, N), E^p) \text{ and } 'E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(M, N), P)$$

whence the result. ■

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