Local superlinearity for elliptic systems involving parameters

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Abstract

We study the existence, non–existence, and multiplicity of positive solutions for a class of systems of second–order ordinary differential equations using the fixed–point theorem of cone expansion/compression type, the upper–lower solutions method, and degree arguments. We apply our abstract results to study semilinear elliptic systems in bounded annular domains with non–homogeneous boundary conditions. Here the nonlinearities satisfy local superlinear assumptions.

Keywords and phrases: Elliptic system, annular domains, positive radial solution, multiplicity upper and lower solutions, fixed–points, degree theory .

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1 Introduction

We deal with systems of second–order ordinary differential equations which have the form

$$\begin{array}{rcll} -u'' & = & f(t,u,v,a,b) & \text{in} \quad (0,1) \\ -v'' & = & g(t,u,v,a,b) & \text{in} \quad (0,1)\,, \end{array} \tag{$P_{a,b}$}$$

with boundary conditions

$$\begin{array}{rcl} u(0) & = & u(1)=0\,, \\ v(0) & = & v(1)=0\,. \end{array} \tag{BC}$$

where the nonlinearities f and g are superlinear at the origin as well as at infinity, and a, b are non–negative constants. We show that there exists a continuous curve Γ which splits the positive quadrant of the $(a,b)-$ plane into two disjoint sets $\mathcal{S}$ and $\mathcal{R}$ such that the System $(P_{a,b})$, with boundary conditions (BC), has at least two positive solutions in $\mathcal{S}$, has at least one positive solution on the boundary of $\mathcal{S}$, and has no positive solutions in $\mathcal{R}$. Our approach is based on fixed–point theorems of cone expansion/compression type, the upper–lower solutions method, and degree arguments.

In what follows, we will impose the following.

(H_0) The functions $f,g:[0,1]\times[0,+\infty)^4\to[0,+\infty)$ are continuous and non–decreasing in the last four variables. In other words,

$$f(t,u_1,v_1,a_1,b_1)\le f(t,u_2,v_2,a_2,b_2) \text{ and } g(t,u_1,v_1,a_1,b_2)\le g(t,u_2,v_2,a_2,b_2)$$

whenever $(u_1,v_1,a_1,b_1)\le(u_2,v_2,a_2,b_2)$, where the inequality is understood inside every component.

(H_1) There exists a subset $\Upsilon_1\subset(0,1)$ of positive Lebesgue measure such that for all fixed $a,b>0$,

$$\lim_{|(u,v)|\to 0}\frac{\ell(t,u,v,a,b)}{|(u,v)|}=+\infty \quad \text{uniformly for almost everywhere } t\in\Upsilon_1, \tag{1.1}$$

for either $\ell=f$ or $\ell=g$. Here we use the notation $|(x_1,\ldots,x_m)|=|x_1|+\ldots+|x_m|\,.$

(H_2) There exist subsets $\Upsilon_2,\Upsilon_3\subset(0,1)$ of positive Lebesgue measure such that

$$\lim_{|(u,v)|\to\infty}\frac{f(t,u,v,0,0)}{|(u,v)|}=+\infty \quad \text{uniformly for almost everywhere } t\in\Upsilon_2 \tag{1.2}$$

and

$$\lim_{|(u,v)|\to\infty}\frac{g(t,u,v,0,0)}{|(u,v)|}=+\infty \quad \text{uniformly for almost everywhere } t\in\Upsilon_3\,. \tag{1.3}$$

It is not difficult to see that there exists $\tau^\star(f) = \tau^\star(f, \xi, \eta, a, b)$ such that

$$\int_0^1 \tau f(\tau, \xi, \eta, a, b)\, d\tau = \int_{\tau^\star}^1 f(\tau, \xi, \eta, a, b) d\tau\,.$$

Similarly, there exists $\tau^\star(g) = \tau^\star(g, \xi, \eta, a, b)$ such that

$$\int_0^1 \tau g(\tau, \xi, \eta, a, b)\, d\tau = \int_{\tau^\star}^1 g(\tau, \xi, \eta, a, b) d\tau\,.$$

The next assumptions are related to the numbers $\tau^\star(f)$ and $\tau^\star(g)$, both denoted $\tau^\star$ for simplicity.

(H_3) There exist $R_0, a_0, b_0, s_0 > 0$ such that

$$\int_0^{\tau^\star} \tau f(\tau, R_0, R_0, a_0, b_0)\, d\tau \le s_0 R_0$$

and

$$\int_0^{\tau^\star} \tau g(\tau, R_0, R_0, a_0, b_0)\, d\tau \le (1 - s_0) R_0\,.$$

Also, we assume that there exists a subset $\Upsilon \subset (0,1)$ such that $f(t, 0, 0, a_0, b_0) > 0$ and $g(t, 0, 0, a_0, b_0) > 0$, for all $t \in \Upsilon$.

When $|(a,b)|$ is sufficiently large, the next hypothesis together with (H_2) give a non–existence result for the System $(P_{a,b})$.

(H_4) There exists a subset $\Upsilon_4 \subset (0,1)$ of positive Lebesgue measure such that

$$\lim_{|(a,b)| \to +\infty} h(t, u, v, a, b) = +\infty \quad \text{uniformly for } t \in \Upsilon_4 \text{ and all } u, v \ge 0\,,$$

for either $h = f$ or $h = g$.

Remark 1 Observe the local character of the assumptions $(H_1), (H_2)$, and (H_4) in the variable t. We further note that the sets $\Upsilon_1, \Upsilon_2, \Upsilon_3$, and Υ_4 may, in general, be different and that hypothesis (H_3) is verified for instance when

$$\lim_{|z| \to 0} \frac{f(t,z)}{|z|} = \lim_{|z| \to 0} \frac{g(t,z)}{|z|} = 0 \quad \text{uniformly for almost everywhere } t \in [0,1] \tag{1.4}$$

as well as when for some $i, j \in \{3, 4\}$, we have that

$$\lim_{z_i \to 0^+} f(t,z) = \lim_{z_j \to 0^+} g(t,z) = 0 \quad \text{uniformly for almost everywhere } t \in [0,1] \tag{1.5}$$

where $z = (z_1, z_2, z_3, z_4)$.

Our main result is Theorem 1.1, which will be proved in Section 3.

Theorem 1.1 *Suppose that the pair of functions consisting of $f(t,u,a,b)$ and $g(t,u,a,b)$ satisfies conditions (H_0) through (H_4). Then there exist a positive constant $\overline{a}$ and a continuous function $\Gamma : [0,\overline{a}] \to [0,+\infty)$ such that for all $a \in [0,\overline{a}]$, the System $(P_{a,b})$ with boundary conditions (BC):*

(i) has at least one positive solution if $0 \le b \le \Gamma(a)$*;*

(ii) has no solution if $b > \Gamma(a)$*;*

(iii) has a second positive solution if $0 < b < \Gamma(a)$ *.*

Applications. As a main application of Theorem 1.1, and indeed as a principal motivation for Theorem 1.1 itself, we can prove the existence and multiplicity of positive radial solutions for the following class of semilinear elliptic system in annular domains. In fact, let $0 < r_1 < r_2$, and let $A(r_1,r_2) = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$, with $N \ge 3$, be an annulus. Consider the system

$$\begin{array}{rcll} -\Delta u &=& h(|x|,u,v) & \text{in} \quad A(r_1,r_2)\,, \\ -\Delta v &=& k(|x|,u,v) & \text{in} \quad A(r_1,r_2)\,, \\ (u,v) &=& (0,0) & \text{on} \quad |x| = r_1\,, \\ (u,v) &=& (a,b) & \text{on} \quad |x| = r_2\,, \end{array} \qquad (E_{a,b})$$

where a,b are non–negative parameters, and the nonlinearities h and k satisfy the next four conditions.

(A_0) The functions $h,k : [0,1] \times [0,+\infty)^2 \to [0,+\infty)$ are continuous and non–decreasing in the last two variables.

(A_1) There exist a set $\Lambda_1 \subset (r_1,r_2)$ of positive Lebesgue measure, and a function ℓ such that either $\ell = h$ or $\ell = k$ and such that $\ell(r,u,v) > 0$, for almost everywhere $r \in \Lambda_1$ and all $u,v > 0$.

(A_2) There exist subsets $\Lambda_2, \Lambda_3 \subset (r_1,r_2)$ of positive Lebesgue measure such that

$$\lim_{|(u,v)|\to\infty} \frac{h(r,u,v)}{|(u,v)|} = +\infty \quad \text{uniformly for almost everywhere } r \in \Lambda_2 \tag{1.6}$$

and

$$\lim_{|(u,v)|\to\infty} \frac{k(r,u,v)}{|(u,v)|} = +\infty \quad \text{uniformly for almost everywhere } r \in \Lambda_3\,. \tag{1.7}$$

(A_3) $$\lim_{|(u,v)|\to 0} \frac{h(r,u,v)}{|(u,v)|} = \lim_{|(u,v)|\to 0} \frac{k(r,u,v)}{|(u,v)|} = 0 \quad \text{uniformly for almost everywhere } r \in (r_1,r_2). \tag{1.8}$$

Note that performing the change of variable

$$t = Ar^{2-N} + B\,, \quad \text{where } A = \frac{(r_1 r_2)^{N-2}}{r_2^{N_2} - r_1^{N-2}} \quad \text{and} \quad B = \frac{r_2^{N-2}}{r_2^{N_2} - r_1^{N-2}}\,,$$

we see that the System $(E_{a,b})$ is equivalent to the system

$$\begin{array}{rcl} -u'' & = & f(t,u,v,a,b) \;\text{ in }\; (0,1)\,, \\ -v'' & = & g(t,u,v,a,b) \;\text{ in }\; (0,1)\,, \\ u(0) = u(1) & = & v(0) = v(1) = 0\,, \end{array}$$

where here the nonlinearities f and g are given by

$$\begin{array}{rcl} f(t,u,v,a,b) & = & d(t)h((\frac{A}{B-t})^{1/(N-2)}, u+ta, v+tb)\,, \\ g(t,u,v,a,b) & = & d(t)k((\frac{A}{B-t})^{1/(N-2)}, u+ta, v+tb)\,, \\ d(t) & = & (1-N)^2 \frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}}\,. \end{array}$$

Now it is easy to see that f and g satisfy assumptions (H_0) through (H_4), and hence the following is an immediate consequence of Theorem 1.1.

Theorem 1.2 *Under the assumptions* (A_0) *through* (A_3)*, there exists a continuous function* $\Gamma : [0,\overline{a}] \to [0,+\infty)$ *such that for all* $a \in [0,\overline{a}]$*, we have:*

(i) If $0 \le b \le \Gamma(a)$*, then the System* $(E_{a,b})$ *has at least one positive radial solution.*

(ii) If the inequalities above are strict, or in other words if $0 < b < \Gamma(a)$*, then the System* $(E_{a,b})$ *has at least two positive radial solutions.*

(iii) When $b > \Gamma(a)$*, the System* $(E_{a,b})$ *has no positive radial solutions.*

We next give three typical examples of nonlinearities that satisfy the hypotheses of Theorem 1.2.

Example 1.3 Let $h,k : [r_1,r_2] \times [0,+\infty)^2 \to [0,+\infty)$ be nonlinearities given by

$$h(r,u,v) = u^p + v^q \quad \text{and} \quad k(r,u,v) = d_1(r)(u^p + v^q) + d_2(r)u^p v^q$$

where $p\,,q > 1$ and $d_1,d_2 : [r_1,r_2] \to \mathbb{R}$ are non–trivial, non–negative continuous functions such that $d_1(r) > 0$ and $d_2(r) = 0$ in some sub–interval J of $[r_1,r_2]$. For instance, we may assume, Λ_1 is any sub–interval with $l = h$, Λ_2 is also any sub–interval and $\Lambda_3 = J$.

Example 1.4 Let $h,k : [r_1,r_2] \times [0,+\infty)^2 \to [0,+\infty)$ be nonlinearities given by

$$h(r,u,v) = u^p + v^q \quad \text{and} \quad k(r,u,v) = (d_1(r)(u^p + v^q) + 1)\arctan(d_2(r)(u^p + v^q))$$

where $p\,,q > 1$ and $d_1,d_2 : [r_1,r_2] \to \mathbb{R}$ are non–negative continuous functions which are positive in some subinterval J of $[r_1,r_2]$.

Example 1.5 Let $f_0, g_0 : [0,+\infty)^2 \to [0,+\infty)$ be continuous functions such that $f_0(u,v) > 0,\ g_0(u,v) > 0$ for all $u, v > 0$, and such that

$$\begin{array}{rclcl} \displaystyle\lim_{|(u,v)|\to 0} \frac{f_0(u,v)}{|(u,v)|} & = & \displaystyle\lim_{|(u,v)|\to 0} \frac{g_0(u,v)}{|(u,v)|} & = & 0\,, \\ \displaystyle\lim_{|(u,v)|\to +\infty} \frac{f_0(u,v)}{|(u,v)|} & = & \displaystyle\lim_{|(u,v)|\to +\infty} \frac{g_0(u,v)}{|(u,v)|} & = & +\infty\,. \end{array}$$

Take $h, k : [r_1, r_2] \times [0,+\infty)^2 \to [0,+\infty)$ defined by

$$h(r,u,v) = d_1(r) f_0(u,v) \quad \text{and} \quad k(u,v) = d_2(r) g_0(u,v)\,,$$

where $d_1, d_2 : [r_1, r_2] \to \mathbb{R}$ are non–trivial, non–negative continuous functions.

In recent years, the study of semilinear elliptic problems in annular domains has received considerable attention. We first refer to the progress made on the study of single equations involving non–homogeneous boundary conditions. These problems have been studied by C. Bandle and L.A. Peletier, M.G. Lee and S.S. Lin, D.D. Hai, among others. In [1], Bandle and Peletier consider the problem

$$\begin{array}{rclcll} -\Delta u & = & u^{(N+2)/(N-2)} & \text{and} & u > 0 \quad \text{in} & A(r_1, r_2)\,, \\ u = 0 & \text{for} & |x| = r_2 & \text{and} & u = b \quad \text{for} & |x| = r_1\,, \end{array} \tag{1.9}$$

where $N \geq 3$. They show the existence of a positive constant b_0 such that the Problem (1.9) has one solution for $b < b_0$ and no solutions for $b > b_0$. In [9], this result is extended to nonlinearities f which are convex and superlinear at zero and infinity. Also, uniqueness and multiplicity questions are discussed. Hai [7] extends the results of [1] and [9] to nonlinearities locally Lipschitz continuous and superlinear at zero and infinity. More recently, Naito and Tanaka [10] have used Shooting Methods together with Sturm's comparison theorem to obtain nodal solutions.

In the context of elliptic systems in annular domains, we mention the works of Dunninger and Wang on homogeneous Dirichlet boundary conditions, as well as that of Lee on nonhomogeneous Dirichlet boundary conditions. (See [4], [5], as well as [8], and the references therein.) This work is more related to results of [8]. In fact, in [8], among other problems, the following elliptic system is considered.

$$\begin{array}{rclll} -\Delta u & = & \lambda k_1(|x|) f(u,v) & \text{in} & A(r_1, r_2)\,, \\ -\Delta v & = & \mu k_2(|x|) g(u,v) & \text{in} & A(r_1, r_2)\,, \\ (u,v) & = & (0,0) & \text{on} & |x| = r_1\,, \\ (u,v) & = & (\overline{a}, \overline{b}) & \text{on} & |x| = r_2\,, \end{array} \tag{1.10}$$

where $\overline{a}, \overline{b} \in (0,+\infty)$, $(\lambda, \mu) \in [0,+\infty)^2 \setminus \{(0,0)\}$. Further, the following conditions are imposed:

(h) $k_i \in C([r_1, r_2], [0,+\infty))$ does not vanish identically on any subinterval of $[r_1, r_2]$;

(h') $k_i \in C([r_1, r_2], (0,+\infty))$ so that $k_i > 0$ on $[r_1, r_2]$;

(h_0) $f, g \in C([0,+\infty)^2, (0,+\infty))$ so that $f(0,0) > 0$ and $g(0,0) > 0$;

(h_0') $f, g \in C([0,+\infty)^2, [0,+\infty))$ with $f(0,0) = 0$ and $g(0,0) = 0$;

(h_1) f and g are non–decreasing on $[0,+\infty)^2$;

(h_1') f and g are increasing on $[0,+\infty)^2$;

(h_2) $\lim_{(u,v)\to\infty} \frac{f(u,v)}{u+v} = \lim_{(u,v)\to\infty} \frac{g(u,v)}{u+v} = \infty$.

More precisely, under the conditions (h), (h_0), (h_1), and (h_2), the existence of a continuous curve Γ is established, which splits the region $[0,+\infty)^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the System (1.10) with $\overline{a} = \overline{b} = 0$, has at least two (respectively at least one, no) positive radial solutions for $(\lambda,\mu) \in \mathcal{O}_1$ (respectively Γ, $\mathcal{O}_2$). On the other hand, under the conditions (h'), (h_0'), (h_1'), and (h_2), the existence of both a continuous curve Γ, which splits the region $[0,+\infty)^2 \setminus \{(0,0)\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$, and a subset $\mathcal{O} \subseteq \mathcal{O}_1$ is established such that the System (1.10) has at least two (respectively at least one, no) positive radial solutions for $(\lambda,\mu) \in \mathcal{O}$ (respectively $(\mathcal{O}_1 \setminus \mathcal{O}) \cup \Gamma$, $\mathcal{O}_2$).

Note that the System (1.10) is equivalent to the system

$$\begin{aligned} -u'' &= \lambda \overline{d}_1(t) f(u + t\overline{a}, v + t\overline{b}) \quad \text{in} \quad A(r_1,r_2)\,, \\ -v'' &= \mu \overline{d}_2(t) g(u + t\overline{a}, v + t\overline{b}) \quad \text{in} \quad A(r_1,r_2)\,, \\ u(0) &= u(1) = v(0) = v(1) = 0\,, \end{aligned}$$

where $\overline{d}_i(t) = d(t)k_i((\frac{A}{B-t})^{1/(N-2)})$, with $i = 1,2$.

Take $\delta > 0$, $\lambda = a + \delta$ and $\mu = b + \delta$, with $a, b \in [0,+\infty)$. Consider the nonlinearities

$$\begin{aligned} f_1(t,u,v,a,b) &= (a+\delta)\overline{d}_1(t) f(u + t\overline{a}, v + t\overline{b}) \\ g_1(t,u,v,a,b) &= (b+\delta)\overline{d}_2(t) g(u + t\overline{a}, v + t\overline{b})\,. \end{aligned}$$

Taking $\delta > 0$ sufficiently small, Theorem 1.2 allows us to improve the results of [8], since the coefficients k_i may vanish in parts of the interval (r_1, r_2), and since the hypotheses $(h_0'), (h_1')$ and (h_2) imply the multiplicity results above for $\mathcal{O} = \mathcal{O}_1$. Observe that, in [8], the coefficients k_i are considered positive in the interval (r_1, r_2) because the System (1.10) is compared with another one with constant coeffifients that is studied using Shooting Methods (see Lemma 4.4 and 4.5 in [8]).

This paper is organized as follows. Section 2 contains preliminary results. Section 3 is devoted to proving our main result, Theorem 1.1.

Notation Summary. Here is a brief summary of some notation:
$B(p,R)$: the open ball with radius R centered at the point p.
C, C_0, C_1, C_2, ... : positive (possibly different) constants.
$i(F, C_r, C) = 1$: the fixed–point index of F with respect to the cone C.
$deg(F, A, y)$: mapping degree for the equation $F(x) = y$, for $x \in A$.

2 Preliminary Results

It is not difficult to show that if the pair (u, v) is a solution of System $(P_{a,b})$, then for all $t \in [0, 1]$,

$$\begin{aligned} u(t) &= \int_0^1 K(t,\tau) f(\tau, u(\tau), v(\tau), a, b)\ d\tau, \\ v(t) &= \int_0^1 K(t,\tau) g(\tau, u(\tau), v(\tau), a, b)\ d\tau, \end{aligned} \tag{$S_{a,b}$}$$

where $K(t, \tau)$ is the Green's function

$$K(t,s) := \begin{cases} t(1-s) & \text{if} \quad t \le s, \\ s(1-t) & \text{if} \quad t > s. \end{cases} \tag{2.11}$$

Let

$$\begin{aligned} A(u,v)(t) &:= \int_0^1 K(t,\tau) f(\tau, u(\tau), v(\tau), a, b)\ d\tau, \\ B(u,v)(t) &:= \int_0^1 K(t,\tau) g(\tau, u(\tau), v(\tau), a, b)\ d\tau, \\ F(u,v) &:= (A(u,v), B(u,v)). \end{aligned}$$

Therefore, System $(S_{a,b})$ is equivalent to the fixed point equation

$$F(u, v) = (u, v)$$

in the usual Banach space $X = C([0, 1]; \mathbb{R}) \times C([0, 1]; \mathbb{R})$ endowed with the norm $||(u, v)|| := ||u||_\infty + ||v||_\infty$, where $||w||_\infty := \sup_{t \in [0,1]} |w(t)|$.

The proof of the existence of the first positive solution of $(P_{a,b})$ will be based on the following fixed-point theorem of cone expansion/compression type. One may refer to [2, 3, 6] for proofs and further discussion of the fixed point index.

Lemma 2.1 *Let X be a Banach space with norm $|\cdot|$, and let $C \subset X$ be a cone in X. For $r > 0$, define $C_r = C \cap B[0, r]$ where $B[0, r] = \{x \in X : |x| \le r\}$ is the closed ball of radius r centered at origin of X. Assume that $F : C_r \to C$ is a compact map such that $Fx \neq x$, for all $x \in \partial C_r = \{x \in C : |x| = r\}$. Then:*

1. *If $|x| \le |Fx|$ for all $x \in \partial C_r$, then $i(F, C_r, C) = 0$.*
2. *If $|x| \ge |Fx|$ for all $x \in \partial C_r$, then $i(F, C_r, C) = 1$.*

Let us consider the cone C in X defined by

$$C = \{(u, v) \in X \ : \ (u, v)(0) = (u, v)(1) = 0, \ and \ u, v \ are\, concave\, functions\}.$$

Lemma 2.2 *$F : X \to X$ is completely continuous and $F(C) \subset C$.*

Proof. We only give the main ideas of the proof. The Arzela-Ascoli theorem implies that $F : X \to X$ is completely continuous. Is is easy to see that F_1 and F_2 (the coordinates functions of $F(u, v)$) are twice differentiable on $(0, 1)$ with $F_1'' \leq 0$ and $F_2'' \leq 0$. This implies that $F(C) \subset C$. ∎

Remark 2 *For each subset Υ_i, $i = 1, 2, 3, 4$, there exist $1 - \epsilon_i > \delta_i > 0$ and subsets of positive measure $\hat{\Upsilon}_i \subset \Upsilon_i \cap (\delta_i, 1 - \epsilon_i)$ such that for all $u, v \in C$, we have*

$$\inf_{t \in \hat{\Upsilon}_i} (u(t) + v(t)) \geq \delta_i (1 - \epsilon_i) ||(u, v)||. \tag{2.12}$$

3 Proof of Theorem 1.1

3.1 The first positive solution for System (P_{a_0,b_0})

Using that $f, g : [0, 1] \times [0, +\infty)^4 \to [0, +\infty)$ are continuous and non–decreasing in the second and third variables, and assumptions (H_1) and (H_3) we apply the Lemma 2.1 to prove the existence of a first positive solution for System (P_{a_0,b_0}), where (a_0, b_0) is given in assumption (H_3).

Lemma 3.1 *Assume condition (H_3), then for all $(u, v) \in C_{R_0}$,*

$$||F(u, v)|| \leq ||(u, v)||.$$

Proof. Given $(u, v) \in C_{R_0}$,

$$\begin{aligned}
A(u, v)(t) &= \int_0^1 K(t, \tau) f(\tau, u(\tau), v(\tau), a_0, b_0)\, d\tau \\
&\leq \int_0^1 K(t, \tau) f(\tau, R_0, R_0, a_0, b_0)\, d\tau \\
&\leq \int_0^1 K(\tau^\star, \tau) f(\tau, R_0, R_0, a_0, b_0)\, d\tau \\
&= \int_0^{\tau^\star} \tau f(\tau, R_0, R_0, a_0, b_0)\, d\tau \\
&\leq s_0 R_0.
\end{aligned}$$

Similarly, we can prove that

$$B(u, v)(t) \leq (1 - s_0) R_0.$$

Hence, for all $(u, v) \in C_{R_0}$,

$$||F(u, v)|| = ||A(u, v)||_\infty + ||B(u, v)||_\infty \leq R_0 = ||(u, v)||.$$

∎

Lemma 3.2 *Assume the hypothesis (1.1). Then there exists* $R_1 \in (0, R_0)$ *such that for all* $(u,v) \in C_{R_1}$,

$$||F(u,v)|| \geq ||(u,v)||.$$

Proof. Using assumption (H_1) with $\ell = f$, and according Remark 2, given $M > 0$ there exists $R_1 = R_1(M) \in (0, R_0)$ such that for all $(u,v) \in [0, R_1]^2$ and almost every $\tau \in \hat{\Upsilon}_1$,

$$f(\tau, u, v, a_0, b_0) \geq M|(u,v)|.$$

Thus, for all $(u,v) \in C_{R_1}$,

$$\begin{aligned} ||A(u,v)||_\infty &\geq \int_0^1 K(1/2,\tau) f(\tau, u(\tau), v(\tau), a_0, b_0)\, d\tau \\ &\geq \int_{\hat{\Upsilon}_1} K(1/2,\tau) f(\tau, u(\tau), v(\tau), a_0, b_0)\, d\tau \\ &\geq M \int_{\hat{\Upsilon}_1} K(1/2,\tau)[u(\tau) + v(\tau)]\, d\tau \\ &\geq \delta_1(1-\epsilon_1) M ||(u,v)|| \int_{\hat{\Upsilon}_1} K(1/2,\tau)\, d\tau, \end{aligned}$$

where in the last inequality we have used (2.12). Finally, taking $M > 0$ sufficiently large such that

$$\delta_1(1-\epsilon_1) M \int_{\hat{\Upsilon}_1} K(1/2,\tau)\, d\tau > 1,$$

we get

$$||F(u,v)|| \geq ||(u,v)||.$$

An analogous estimate holds if we use assumption (H_1) with $\ell = g$. ■

Now, in view of Lemmas 3.1 and 3.2, as a direct consequence of Lemma 2.1, we have the following result.

Theorem 3.3 *F has a fixed point* $(u,v) \in C$ *such that* $R_1 < ||(u,v)|| < R_0$.

Therefore, the pair (u,v) *is a positive solution of System* (P_{a_0,b_0}).

Using a combination of the maximum principle and hypothesis (H_3) we obtain that both u and v are positive functions.

3.2 A priori estimate

Next, as a consequence of assumption (H_2) we have the following a priori estimate for positive solutions of System $(P_{a,b})$.

Lemma 3.4 *There exists* $C_0 > 0$ *independent of* a *and* b *such that* $||(u,v)|| \leq C_0$, *for all positive solution* (u,v) *of System* $(P_{a,b}$ *) with boundary condition (BC).*

Proof. Assume by contradiction that there exists a sequence of solution $(u_n, v_n) \in X$ of System $(P_{a,b})$ such that $||(u_n, v_n)|| \to \infty$. Without loss of generality we may assume that $||u_n||_\infty \to \infty$. From assumption (H_2) we can take a sequence of real numbers $\alpha_n \nearrow +\infty$ such that for almost every $\tau \in \Upsilon_2$ and $a,\ b \geq 0$,

$$\frac{f(\tau, u_n(\tau), v_n(\tau), a, b)}{u_n(\tau) + v_n(\tau)} \geq \alpha_n. \tag{3.13}$$

Thus, using the fact that u_n is concave together with Remark 2,

$$\begin{aligned} ||u_n||_\infty \geq u_n(t) &= \int_0^1 K(t,\tau) f(\tau, u_n(\tau), v_n(\tau), a, b)\ d\tau \\ &\geq \int_{\hat{\Upsilon}_2} K(t,\tau) \frac{f(\tau, u_n(\tau), v_n(\tau), a, b)}{u_n(\tau) + v_n(\tau)} u_n(\tau)\ d\tau \\ &\geq \delta_2(1-\epsilon_2)||u_n||_\infty \int_{\hat{\Upsilon}_2} K(t,\tau) \frac{f(\tau, u_n(\tau), v_n(\tau), a, b)}{u_n(\tau) + v_n(\tau)}\ d\tau, \end{aligned}$$

which together with (3.13), implies that

$$\frac{1}{\alpha_n} \geq \delta_2(1-\epsilon_2) \int_{\hat{\Upsilon}_2} K(t,\tau)\ d\tau$$

which is a contradiction. ■

3.3 Lower and upper solutions

Now, we will establish the classical lower and upper solutions method for our class of problems. To do this, consider the system

$$\begin{array}{rclll} -u'' &=& f_0(t,u,v) & \text{in} & (0,1) \\ -v'' &=& g_0(t,u,v) & \text{in} & (0,1) \\ u(0) &=& u(1) = v(0) &=& v(1) = 0 \end{array} \tag{S}$$

Where f_0 and g_0 are nonnegative continuous functions which are nondecreasing in the variables u and v.

As usual, we say that (u, v) is a lower solution for System (S) when (u, v) verify the following inequations

$$\begin{array}{rclll} -u'' &\leq& f_0(t,u,v) & \text{in} & (0,1) \\ -v'' &\leq& g_0(t,u,v) & \text{in} & (0,1) \\ (u,v) &\leq& 0 & \text{on} & \{0,1\} \end{array} \tag{T}$$

Similarly we define the upper solution of System (S) putting "greater or equal" instead of "lest or equal".

Lemma 3.5 *Let $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ be a lower and upper solution respectively of System (S) such that*

$$(0,0) \leq (\underline{u}, \underline{v}) \leq (\bar{u}, \bar{v}).$$

Then System (S) has a nonnegative solution (u, v) verifying

$$(\underline{u}, \underline{v}) \leq (u, v) \leq (\bar{u}, \bar{v}).$$

Proof. Let

$$\begin{aligned} M(u,v)(t) &:= \int_0^1 K(t,\tau) f_0(\tau, u(\tau), v(\tau))\, d\tau, \\ N(u,v)(t) &:= \int_0^1 K(t,\tau) g_0(\tau, u(\tau), v(\tau))\, d\tau, \\ G(u,v) &:= (M(u,v), N(u,v)). \end{aligned}$$

Therefore, System (S) is equivalent to the fixed point equation

$$G(u,v) = (u,v)$$

in the Banach space $X = C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ endowed with the norm $||(u,v)|| := ||u||_\infty + ||v||_\infty$.

Now, we need to introduce the following auxiliary operator $\tilde{G}$ defined as follows

$$\tilde{G}(u,v) := (\tilde{M}(u,v), \tilde{N}(u,v)),$$

where

$$\begin{aligned} \tilde{M}(u,v)(t) &:= \int_0^1 K(t,\tau) f_0(\tau, \xi(t,u), \zeta(\tau,v))\, d\tau, \\ \tilde{N}(u,v)(t) &:= \int_0^1 K(t,\tau) g_0(\tau, \xi(t,u), \zeta(\tau,v))\, d\tau \end{aligned}$$

and

$$\xi(t,u) := \max\{\underline{u}(t), \min\{u, \overline{u}(t)\}\} \text{ and } \zeta(t,v) := \max\{\underline{v}(t), \min\{v, \overline{v}(t)\}\}$$

It is easy to see that the operator $\tilde{G}$ has the following properties:

(a) $\tilde{G}$ is a bounded and completely continuous operator;

(b) if the pair $(u,v) \in X$ is a fixed point of $\tilde{G}$, then (u,v) is a fixed point of G with $(\underline{u}, \underline{v}) \leq (u,v) \leq (\overline{u}, \overline{v})$;

(c) if $(u,v) = \lambda \tilde{G}(u,v)$ with $0 \leq \lambda \leq 1$ then $||(u,v)||_1 \leq C_3$, where C_3 does not depend on λ, and $(u,v) \in X$.

Thus using the topological degree of Leray-Schauder we obtain a fixed point of the operator G. Then the lemma is proved. ■

Lemma 3.6 *Assume that* (P_{a_2,b_2}) *has a nonnegative solution and*

$$(0,0) \le (a_1, b_1) \le (a_2, b_2),$$

then (P_{a_1,b_1}) *has a nonnegative solution.*

Proof. Let the pair (u_2, v_2) be a nonnegative solution of System (P_{a_2,b_2}). Since the functions f, g are increasing functions in the last two variables, we have that (u_2, v_2) is a super-solution and $(0,0)$ is a sub-solution for for System (P_{a_1,b_1}). Thus using the lemma above we have complete the proof of Lemma 3.6. ■

3.4 Nonexistence

Next we establish the following nonexistence result

Lemma 3.7 *Suppose the hypotheses* (H_2) *and* (H_4)*. Then there exist* $C > 0$ *such that for all* (a, b) *with* $|(a, b)| > C$ *the System* $(P_{a,b})$ *has no solutions.*

Proof. Assume by contradiction that there exists a sequence (a_n, b_n) with $|(a_n, b_n)| \to +\infty$ such that for each n, System (P_{a_n,b_n}) possesses a positive solution $(u_n, v_n) \in C$.

By assumption (H_4), given $M > 0$, there exists $C > 0$ such that for all (a, b) with $|(a, b)| \ge C$, without lost of generality and according to Remark 2, we have

$$f(t, u, v, a, b) \ge M \text{ for all } t \in \hat{\Upsilon}_4 \text{ and } u, v \ge 0. \tag{3.14}$$

Thus,

$$\begin{aligned} u_n(t) &= \int_0^1 K(t,\tau) f(\tau, u_n(\tau), v_n(\tau), a_n, b_n)\, d\tau \\ &\ge \int_{\hat{\Upsilon}_4} K(t,\tau) f(\tau, u_n(\tau), v_n(\tau), a_n, b_n)\, d\tau, \end{aligned} \tag{3.15}$$

which together with (3.14) implies that for n sufficiently large we obtain

$$u_n(t) \ge M \int_{\hat{\Upsilon}_4} K(t,\tau)\, d\tau.$$

Hence

$$||u_n||_\infty \ge M \sup_{t \in \Upsilon_4} \int_{\hat{\Upsilon}_4} K(t,\tau)\, d\tau.$$

Since we can choose M in (3.14) arbitrarily large, we conclude that (u_n) is an unbounded sequence in X.

On the other hand, by using assumption (H_2), we have that given $M > 0$ there exits $R > 0$ such that for all $u \ge R$,

$$f(t, u, v, a, b) \ge Mu, \text{ for all } t \in \hat{\Upsilon}_2 \text{ and } a, b \ge 0. \tag{3.16}$$

Using again the estimates (2.12) and (3.15), for n sufficiently large, we get

$$u_n(t) \geq M \int_{\hat{\Upsilon}_2} K(t,\tau) u_n(\tau)\ d\tau \geq M(1-\varepsilon_2)\delta_2 ||u_n||_\infty \int_{\hat{\Upsilon}_2} K(t,\tau)\ d\tau.$$

Hence

$$1 \geq M(1-\varepsilon_2)\delta_2 \sup_{t\in\Upsilon_2} \int_{\hat{\Upsilon}_2} K(t,\tau)\ d\tau,$$

which it is a contradiction with the fact that M can be chosen arbitrary large. The proof of Lemma 3.7 is now complete. ■

Let us define

$$\overline{a} := \sup\{a > 0 :\ (P_{a,b}) \text{ has a positive solution for some } b > 0\}$$

From Lemma 3.7 it follows immediately that

$$0 < \overline{a} < \infty.$$

It is easy to see, using the sub- and super solutions methods that for all $a \in (0, \overline{a})$ there exists $b > 0$ such that System $(P_{a,b})$ has a solution. Furthermore, using Lemma 3.7 and the Arzelá-Ascoli Theorem, we can prove that exists $b > 0$ such that $(P_{\overline{a},b})$ has a positive solution.

Now, we introduce the following function

$$\Gamma(a) := \sup\{b > 0 : (P_{a,b}) \text{ has a positive solution}\}.$$

As a consequence of Lemma 3.6, we see that $\Gamma : (0, \overline{a}) \to \mathbb{R}$ is a continuous and non increasing function.

We would like to observe that until now, we have proved that System $(P_{a,b})$ has at least one solution when $0 \leq b \leq \Gamma(a)$ and has no solutions when $b > \Gamma(a)$.

3.5 The second positive solution

In this section we shall use the degree theory to prove the existence of a second positive solution for System $(P_{a,b})$ in the region of the plane

$$\mathcal{S} := \{(a,b) \in \mathbb{R}^2\ :\ 0 < a < \overline{a} \text{ and }\ 0 < b < \Gamma(a)\}. \tag{3.17}$$

Let $(a,b) \in \mathcal{S}$ and let $(u_1, v_1) \in X$ be a positive solution of System $(P_{a,b})$ and $(\overline{u}, \overline{v}) \in X$ be a positive solution of System $(P_{(a,\Gamma(a))})$. Using that f, g are monotone increasing functions in the variables u, v, a, b and using the maximum-principle argument we may suppose

$$\begin{array}{lcccl} (0,0) & \leq & (u_1(t), v_1(t)) & \leq & (\overline{u}(t), \overline{v}(t)), \\ (0,0) & < & (u_1'(0), v_1'(0)) & < & (\overline{u}'(0), \overline{v}'(0)), \\ (0,0) & > & (u_1'(1), v_1'(1)) & > & (\overline{u}'(1), \overline{v}'(1)). \end{array}$$

Now we consider the Banach space

$$X_1 = \{(u,v) \in C^1([0,1],\mathbb{R}) \times C^1([0,1],\mathbb{R}) \ : \ (u,v)(0) = (u,v)(1) = (0,0)\}$$

endowed with the norm

$$||(u,v)||_1 := ||u||_\infty + ||v||_\infty + ||u'||_\infty + ||v'||_\infty.$$

Let $\rho_1 > 0$ such that $||(u_1,v_1)||_1 < \rho_1$. We also consider the open subset $\mathcal{A}$ of X_1 contained (u_1,v_1) given by

$$\mathcal{A} := \{(u,v) \in X_1 \text{ satisfying conditions (i)-(iv) below } \}$$

(i) $(0,0) < (u(t),v(t)) < (\overline{u}(t),\overline{v}(t))$ for all $t \in (0,1)$;

(ii) $(0,0) < (u'(0),v'(0)) < (\overline{u}'(0),\overline{v}'(0))$;

(iii) $(0,0) > (u'(1),v'(1)) > (\overline{u}'(1),\overline{v}'(1))$;

(vi) $||(u,v)||_1 < \rho_1$.

Let $\mathcal{G} : X_1 \to X_1$ such that $\mathcal{G} = F|_{X_1}$. The existence of our second positive solution of System $(P_{a,b})$ will be a consequence of the following basic result.

Lemma 3.8 *Let $(a,b) \in \mathcal{S}$. Then using the notation above, we have:*

(i) $deg(Id - \mathcal{G}_{(a,b)}, \mathcal{A}, 0) = 1$

(ii) There exists $\rho_2 > \rho_1$ such that $deg(Id - \mathcal{G}_{(a,b)}, B(0,\rho_2), 0) = 0$.

Proof. Let us consider the auxiliary operator $\overline{\mathcal{G}}_{(a,b)} : X_1 \to X_1$ given by

$$\overline{\mathcal{G}}_{(a,b)}(u,v) := (\overline{A}(u,v), \overline{B}(u,v))$$

where

$$\begin{aligned} \overline{A}(u,v)(t) &:= \int_0^1 K(t,\tau)\overline{f}(\tau,u(\tau),v(\tau),a,b)\ d\tau, \\ \overline{B}(u,v)(t) &:= \int_0^1 K(t,\tau)\overline{g}(\tau,u(\tau),v(\tau),a,b)\ d\tau \end{aligned}$$

and

$$\overline{f}(t,u,v,a,b) := \begin{cases} f(t,\xi_0(t,u),\zeta_0(t,v),a,b) & \text{if } \ 0 \le u \ \text{ and } \ 0 \le v, \\ 0 & \text{if } \ u < 0 \ \text{ or } \ v < 0, \end{cases}$$

$$\overline{g}(t,u,v,a,b) := \begin{cases} g(t,\xi_0(t,u),\zeta_0(t,v),a,b) & \text{if } \ 0 \le u \ \text{ and } \ 0 \le v, \\ 0 & \text{if } \ u < 0 \ \text{ or } \ v < 0, \end{cases}$$

with

$$\xi_0(t,u) := \min\{u,\overline{u}(t)\} \text{ and } \zeta_0(t,v) := \min\{v,\overline{v}(t)\}$$

As in the proof of Lemma 3.5 it is easy to see that the operator $\overline{\mathcal{G}}_{(a,b)}$ satisfies the following properties:

(a) $\overline{\mathcal{G}}_{(a,b)}$ is a completely continuous operator;

(b) if the pair $(u,v)\in X_1$ is a fixed point of $\overline{\mathcal{G}}_{(a,b)}$, then (u,v) is a fixed point of $\mathcal{G}_{(a,b)}$ with $(0,0)\leq(u,v)\leq(\overline{u},\overline{v})$;

(c) if $(u,v)=\lambda\overline{\mathcal{G}}_{(a,b)}(u,v)$ with $0\leq\lambda\leq 1$ then $||(u,v)||_1\leq C_3$, where C_3 does not depends of λ and $(u,v)\in X_1$.

Using the a priori estimate propriety established in assertion (c), we have that there exists $\rho_2>\rho_1$ such that

$$deg(Id-\overline{\mathcal{G}}_{(a,b)},B((u_1,v_1),\rho_2),0)=1. \tag{3.18}$$

By the Maximum Principle, the operator $\overline{\mathcal{G}}_{(a,b)}$ has no fixed point in $\overline{B((u_1,v_1),\rho_2)}\backslash\mathcal{A}$. Now, if $\overline{\mathcal{G}}_{(a,b)}$ has a fixed point on $\partial\mathcal{A}$, then we have a second positive solution of System $(P_{a,b})$. Otherwise, we have that the topological degree of Leray-Schauder is defined for the equation $(Id-\overline{\mathcal{G}}_{(a,b)})(Z)=0,\ Z\in\mathcal{A}$. Then by using (3.18) and the excision property of mapping degree we have

$$deg(Id-\overline{\mathcal{G}}_{(a,b)},\mathcal{A},0)=1.$$

Since $\mathcal{G}_{(a,b)}(u,v)=\overline{\mathcal{G}}_{(a,b)}(u,v),\ (u,v)\in\partial\mathcal{A}$, the part (i) of Lemma 3.8 is now complete.

Next, according to (H_2) the Lemma 3.4 allow us to obtain a priori estimate ρ_2 for solutions of the equation

$$(u,v)=\mathcal{G}_{(a,b)}(u,v),\ (u,v)\in X_1, \tag{3.19}$$

which does not depends of the parameters a and b. Let $(\overline{a},\overline{b})$ such that $|(\overline{a},\overline{b})|$ is sufficiently large such that System $(P_{(\overline{a},\overline{b})})$ has no positive solutions (see Lemma 3.4). Thus

$$deg(Id-\mathcal{G}_{(\overline{a},\overline{b})},B(0,\rho_2),0)=0.$$

Hence, by the homotopy invariance property of the mapping degree we have

$$deg(Id-\mathcal{G}_{(a,b)},B(0,\rho_2),0))=0.$$

The proof of Lemma 3.8 is now complete.

■

Finally, the Lemma 3.8 and the excision property of the topological degree imply

$$deg(Id-\mathcal{G}_{(a,b)},B((u_1,u_2),\rho_2)\setminus\overline{\mathcal{A}},0)=-1,$$

hence we have a second solution of System $(P_{a,b})$ and the proof of Theorem 1.1 is complete.

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