Multiple solutions for a class of quasilinear elliptic problems

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Abstract

We deal with a class of p–Laplacian Dirichlet boundary value problems where the combined effects of "sublinear" and "superlinear" growths allow us to establish the existence of at least two positive solutions.

1 Introduction

The objective of this paper is to establish the existence of two radial solutions for the quasilinear boundary value problem

$$\begin{array}{rcccl} -\Delta_p u & = & f(u) & \text{in} & \Omega\,, \\ u & > & 0 & \text{in} & \Omega\,, \\ u & = & 0 & \text{on} & \partial\Omega \end{array} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a ball of radius b, and where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p–Laplacian with $1 < p < N$. We will assume that the function $f : [0, +\infty) \to [0, +\infty)$ is a given continuous function satisfying the following two conditions:

(H_1) $\lim_{t\to 0} f(t)/t^{p-1} = +\infty\,,$

(H_2) $\lim_{t\to +\infty} f(t)/t^{p-1} = +\infty\,.$

It follows from the assumptions (H_1) and (H_2) that there exists $R > 0$ such that

$$\frac{f(R)}{R^{p-1}} = \min_{t>0} \frac{f(t)}{t^{p-1}}\,.$$

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Let $\overline{R}$ be a point where f attains its maximum on the interval $(0, R]$. We will assume the following two further conditions:

(H_3) $\quad f(\overline{R})/\overline{R}^{p-1} < \eta = (p/(p-1))^{p-1} N/b^p$.

(H_4) There exist increasing functions $g_1,\ g_2 \in C([0, +\infty), [0, +\infty))$ and positive constants $\delta,\ \eta$, with $\delta \in (0, 1)$, such that for all $t > 0$

$$\begin{aligned} g_2(t) &\leq \eta g_1(\delta t) \quad \text{and} \\ g_1(t) &\leq f(t) \leq g_2(t)\,. \end{aligned}$$

Our main result is Theorem 1.1, which will be proved in Section 3 using fixed point techniques.

Theorem 1.1 *Under the assumptions* (H_1) *through* (H_4)*, the problem* (1.1) *has at least two radial solutions.*

Our study was motivated by some recent work on elliptic problems with concave–convex nonlinearities (see [1], [2],[3], [9], [11], [12]).

Ambrosetti et al.[1] study the second order elliptic problem

$$\begin{aligned} -\Delta u &= \lambda u^s + u^r \quad \text{in } \Omega\,, \\ u &> 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (for $N \geq 3$) with smooth boundary $\partial\Omega$, Δ is the Laplace operator, λ is a positive real parameter, and $0 < s < 1 < r$. They prove that there exists a positive real constant Λ such that, for all $0 < \lambda < \Lambda$, the problem (1.2) has a solution, which is found using sub as well as supersolution methods. Here the essential term is u^s while the exponent r may be arbitrary. Using variational methods, a second solution of (1.2) is found. In this case, the term u^r plays a fundamental role, where r must satisfy $r \leq (N+2)/(N-2)$. Among others, the following question is left open: Suppose that $r > (N+2)/(N-2)$ and that Ω is a ball. Does the problem (1.2) have two positive solutions for λ small enough? In [12], R. Ma proves that the assertion is true.

Difficulties arise while extending the study of the problem (1.2) to the p–Laplacian operator. Many known techniques and results for the Laplacian no longer apply for the p–Laplacian due to its nonlinear nature. Using a radial setting, a priori estimates, and topological arguments, Ambrosetti et al.[2] obtain a global multiplicity result for elliptic problems of the form

$$\begin{aligned} -\Delta_p u &= \lambda u^s + u^r \quad \text{in } \Omega\,, \\ u &> 0 \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.3}$$

More precisely, they prove that there is $\Lambda > 0$ such that there exists at least two positive solutions of the problem (1.3) in the interval $(0, \Lambda)$, where Ω is a ball and the following hypotheses are satisfied $0 < s < p-1 < r < p^* = Np/(N-p)$, with $p < N$. In [3] the authors study the critical case considering the following restrictive assumptions on p : $2N/(N+2) < p < 3$ or $p \geq 3$ and $p-1 > s > (p^*+1) - 2/(p-1)$. Related results may be found in [4], [8]. For global multiplicity results on a general bounded domain, in the subcritical case see [9]. When $1 \leq s < p-1 < r \leq p^* - 1$, which includes the critical case, see [11].

Observe that we improve those results for the p–Laplacian operator which involve concave and convex nonlinearities because there are no restrictions on $p \in (1, N)$ nor on the growth of the nonlinearities which may have a subcritical, or critical, or supercritical growth. Note that the nonlineartities we consider are sublinear at 0 and superlinear at $+\infty$, hence contain the concave and convex nonlinearities above. We point out that our result is an improvement even in the case studied in [12] because we consider more general nonlinearities. For instance, let $g_1(t) = a_1 t^s + b_1 t^r \leq g_2(t) = a_2 t^s + b_2 t^r$, where $0 < s < p-1 < r$, and where a_1, b_1, a_2 and b_2 are positive constants. Assume that $g_1(t) = a_1 t^s + b_1 t^r \leq f(t) \leq g_2(t) = a_2 t^s + b_2 t^r$. It is easy to see that f satisfies the hypotheses of Theorem 1.1 . Finally, note that, in [7], D. De Figueiredo and P. L. Lions studied the Laplacian operator with subcritical nonlinearities that satisfy a sublinearity condition at zero and a superlinearity condition at infinity.

The paper is organized as follows: Section 2 contains preliminary results. Section 3 is devoted to proving our main result, Theorem 1.1.

2 Preliminary Results

We will establish radial solutions of the problem (1.1). In fact, we will obtain solutions $u = u(r)$ of the ordinary equation

$$\begin{array}{rclcl} -(r^{N-1}\phi(u'))' & = & r^{N-1}f(u) & \text{in} & (0,b)\,, \\ u & > & 0 & \text{in} & (0,b)\,, \\ u(b) & = & u'(0) & = & 0\,, \end{array} \tag{2.1}$$

where $\phi(t) = |t|^{p-2}t$. Performing the change of variable $t = a(r)$, define $z(t) = u(r(t))$ where $a : [0, b) \to [0, +\infty)$ is given by

$$a(r) = \frac{p-1}{N-p}\left[r^{(p-N)/(p-1)} - b^{(p-N)/(N-1)} \right].$$

Thus (2.1) can be rewritten as

$$\begin{array}{rclcl} -(\phi(z'(t)))' & = & r^{(N-1)p/(p-1)}(-t)f(z(t)) & \text{in} & (0,+\infty)\,, \\ z & > & 0 & \text{in} & (0,+\infty)\,, \\ z(0) & = & z'(+\infty) & = & 0\,. \end{array} \tag{2.2}$$

Integrating the equation of (2.2) and using the boundary conditions we obtain

$$\phi(z'(t)) = \int_t^{+\infty} r^{(N-1)p/(p-1)}(-\tau) f(z(\tau)) d\tau \,,$$

which is equivalent to

$$z'(t) = \left[\int_t^{+\infty} r^{(N-1)p/(p-1)} f(z(\tau)) d\tau \right]^{1/(p-1)} .$$

Integrating once again we obtain

$$z(t) \;\; = \;\; \int_0^t \left[\int_t^{+\infty} G(\tau) f(z(\tau)) d\tau \right]^{1/(p-1)} ds \tag{2.3}$$

where

$$G(\tau) = \left(b^{(p-N)/(p-1)} + \tau \frac{N-p}{p-1} \right)^{p(1-N)/(N-p)} . \tag{2.4}$$

Consequently, we will solve (2.1) using fixed point techniques. For this, we state the following well known abstract result without proof (compare [5], [6], [10]).

Lemma 2.1 *Let X be a Banach space with norm $|\cdot|$, and let $K \subset X$ be a cone in X. For $r > 0$, define $K_r = K \cap B[0,r]$ where $B[0,r] = \{x \in X : |x| \leq r\}$ is the closed ball of radius r centered at origin of X. Assume that $F : K_r \to K$ is a compact map such that $Fx \neq x$, for all $x \in \partial K_r = \{x \in K : |x| = r\}$.*
Then:

1. *If $|x| \leq |Fx|$ for all $x \in \partial K_r$, then $\imath(F, K_r, K) = 0$.*
2. *If $|x| \geq |Fx|$ for all $x \in \partial K_r$, then $\imath(F, K_r, K) = 1$.*

Now we consider the space

$$X = \{z : [0, +\infty) \to \mathbb{R} : \; z \text{ is a bounded, continuous function}\}$$

endowed with the sup norm $|z|_\infty = \sup\{|z(t)| : t \in [0, +\infty)\}$. Let $A : K_1 \to X$ be the operator defined by

$$(Az)(t) \;\; = \;\; \int_0^t \left[\int_s^{+\infty} G(\tau) f(z(\tau)) d\tau \right]^{1/(p-1)} ds \,, \tag{2.5}$$

where K_1 is the cone defined by

$$K_1 = \{z \in X : \; z \text{ is nonnegative, concave and z(0)=0 } \} .$$

Note that the elements of K_1 are increasing functions.

Lemma 2.2 *A is well defined, $A(K_1) \subset K_1$, and A is a completely continuous operator.*

Proof. For all $s \geq 0$, note that

$$\int_s^{+\infty} G(\tau)d\tau = \frac{1}{N} G(s)^{N(p-1)/p(N-1)}$$

and that

$$\int_0^{+\infty} \left[\int_s^{+\infty} G(\tau)d\tau\right]^{1/(p-1)} ds = \eta^{1/(1-p)} .$$

Hence A is well defined.

Also, note that the function $(Az)(t)$ is of class C^2 whose derivatives are given by

$$\begin{aligned} \frac{d}{dt}(Az)(t) &= \left[\int_t^{+\infty} G(\tau)f(z(\tau)d\tau\right]^{1/(p-1)} \\ \frac{d^2}{dt^2}(Az)(t) &= \frac{1}{1-p}G(t)[\frac{d}{dt}(Az)(t)]^{p-2}f(z(t)) . \end{aligned}$$

Thus $(Az)(t)$ is increasing and concave. Therefore, $A(K_1) \subset K_1$.

It remains to prove that A is a completely continuous operator. Let $\mid z_n \mid_\infty \leq C_0$, and let $M_1 = \max\{f(t) : t \in [0, C_0]\}$. It follows that

$$\begin{aligned} \mid (Az_n)(t) \mid &\leq M_1^{1/(p-1)} \int_0^{+\infty} \left[\int_s^{+\infty} G(\tau)d\tau\right]^{1/(p-1)} ds \\ \mid \frac{d}{dt}(Az_n)(t) \mid &\leq \left[M_1 \int_0^{+\infty} G(\tau)d\tau\right]^{1/(p-1)} . \end{aligned}$$

By the Arzelá–Ascoli compactness criterion for uniform convergence, there exists a uniformly convergent subsequence $(Az_{n_k}) \subset (Az_n)$ on compact subsets of $[0, +\infty)$. To prove that there exists uniformly convergent subsequence of (Az_n) it suffices to recall that given $\epsilon > 0$, there is $T = T(\epsilon)$ such that

$$\int_T^{+\infty} \left[\int_s^{+\infty} G(\tau)d\tau\right]^{1/(p-1)} ds < \epsilon.$$

We now verify that A is continuous. Let $(z_n) \in X$ such that $\mid z_n - z_0 \mid_\infty \to 0$ as $n \to \infty$. Thus

$$\mid (Az_n)(t) - (Az_0)(t) \mid \leq \int_0^{+\infty} \mid \Gamma_n(s) - \Gamma_0(s) \mid ds$$

where

$$\Gamma_n(s) = \int_s^{+\infty} G(\tau)f(z_n(\tau))d\tau \text{ and } \Gamma_0(s) = \int_s^{+\infty} G(\tau)f(z_0(\tau))d\tau.$$

It follows from $| z_n - z_0 |_\infty \to 0$ that $\Gamma_n(s) \to \Gamma_0(s)$ and that $\Gamma_n(s) \leq C/NG(s)^{N(p-1)/p(N-1)}$ for all $s \in [0, +\infty)$. By the Lebesgue dominated convergence theorem,

$$| Az_n - Az_0 |_\infty \to 0,$$

which implies that A is continuous. ■

Given $\omega \in K_1$, there clearly exists a unique $\tau = \tau(\omega)$ such that $2\omega(\tau) = | \omega |_\infty$.

Define

$$\tau* = \sup\{\tau(A(z)) : z \in K_1\}$$

and

$$K = \{z \in K_1 : 2 \inf_{t \geq \tau*} z(t) \geq | z |_\infty\}.$$

Lemma 2.3 *$\tau*$ is a positive real number and K is a cone invariant by A.*

The proof is based on the following Assertion.

Assertion 1 *$\{\omega / | \omega |_\infty : \omega \in A(K_1) \setminus \{0\}\}$ is a relatively compact subset of X.*

Proof. Since $\{Az / | Az |_\infty : z \in K_1 \text{ and } Az \neq 0\}$ is a bounded subset of X, it suffices to prove that

$$\{[Az]' / | Az |_\infty : z \in K_1 \text{ and } Az \neq 0\}$$

is also a bounded subset of X.

Integrating by parts we have

$$\begin{aligned}
\left[\frac{[Az]'(t)}{| Az |_\infty}\right]^{p-1} &= \frac{\int_t^{+\infty} G(\tau) f(z(\tau) d\tau}{\left[\int_0^{+\infty} \left[\int_s^{+\infty} G(\tau) f(z(\tau)) d\tau\right]^{1/(p-1)} ds\right]^{p-1}} \\
&= \frac{(p-1)^{p-1} \int_t^{+\infty} G(\tau) f(z(\tau)) d\tau}{\left[\int_0^{+\infty} \left[\int_s^{+\infty} G(\tau) f(z(\tau)) d\tau\right]^{(2-p)/(p-1)} sG(s) f(z(s)) ds\right]^{p-1}} \cdot
\end{aligned} \tag{2.6}$$

We consider two cases.

Case 1. $1 < p < 2$. In this case, it follows from condition (H_4) that

$$\begin{aligned}
\left[\frac{[Az]'(t)}{| Az |_\infty}\right]^{p-1} &\leq \frac{(p-1)^{p-1} \int_0^{+\infty} G(\tau) g_2(z(\tau)) d\tau}{\left[\int_0^{+\infty} \left[\int_s^{+\infty} G(\tau) g_1(z(\tau)) d\tau\right]^{(2-p)/(p-1)} sG(s) g_1(z(s)) ds\right]^{p-1}} \\
&\leq \frac{(p-1)^{p-1} \int_0^{+\infty} G(\tau) g_2(z(\tau)) d\tau}{\left[\int_0^{+\infty} \left[\int_s^{+\infty} G(\tau) d\tau\right]^{(2-p)/(p-1)} sG(s) g_1(z(s))^{1/(p-1)} ds\right]^{p-1}} \\
&\leq I_1 + I_2,
\end{aligned}$$

where I_1 and I_2 are given by

$$I_1 = \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))d\tau}{\left[\int_0^1\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}ds\right]^{p-1}}$$

and

$$I_2 = \frac{(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))ds}{\left[\int_1^{+\infty}\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}ds\right]^{p-1}}\,.$$

We estimate each integral separately.
To estimate I_1, we use condition (H_4) to obtain

$$\begin{aligned}
I_1 &= \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))d\tau}{\left[\int_0^1\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}ds\right]^{p-1}}\\
&\le \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))d\tau}{\left[\int_\delta^1\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_1(z(\delta))^{1/(p-1)}ds\right]^{p-1}}\\
&\le \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(1))d\tau}{\left[\int_\delta^1\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_2(\delta z(1))^{1/(p-1)}ds\right]^{p-1}}\\
&\le \frac{(p-1)^{p-1}\int_0^1 G(\tau)d\tau}{\left[\int_\delta^1\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)ds\right]^{p-1}}\,.
\end{aligned}$$

To estimate I_2 we note that

$$\begin{aligned}
\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)(g_1(z(s)))^{1/(p-1)} &\ge N^{(p-2)/(p-1)}[G(s)g_1(z(s))]^{1/(p-1)}\\
&\ge \frac{N^{(p-2)/(p-1)}}{\eta^{1/(p-1)}}[G(s)g_2(z(s))]^{1/(p-1)}\,,
\end{aligned}$$

which implies

$$
\begin{aligned}
I_2 &= \frac{(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))ds}{\left[\int_1^{+\infty}\left[\int_s^{+\infty} G(\tau)d\tau\right]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}ds\right]^{p-1}} \\
&\leq \frac{(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))ds}{N^{p-2}\left[\int_1^{+\infty}[sG(s)g_1(z(s))]^{1/(p-1)}ds\right]^{p-1}} \\
&\leq \frac{(p-1)^{p-1}\int_1^{+\infty} s\frac{1}{s}G(s)g_2(z(s))ds}{N^{p-2}\left[\int_1^{+\infty}[sG(s)g_1(z(s))]^{1/(p-1)}ds\right]^{p-1}} \\
&\leq \frac{(p-1)^{p-1}\|1/s\|_{L^{1/(2-p)}[1,+\infty)}}{N^{p-2}}\,.
\end{aligned}
$$

Case 2. $p \geq 2$**.** In this case, in accordance to conditions (2.6) and (H_4)

$$
\begin{aligned}
\frac{[Az]'(t)}{\mid Az\mid_\infty} &\leq \frac{(p-1)\int_0^{+\infty} G(s)f(z(s))ds}{\int_0^{+\infty} G(s)sf(z(s))ds} \\
&\leq (p-1)\left[\frac{\int_0^1 G(s)f(z(s))ds}{\int_0^1 G(s)sf(z(s))ds}+1\right] \\
&\leq (p-1)\left[\frac{\int_0^1 G(s)g_2(z(s))ds}{\int_0^1 G(s)sg_1(z(s))ds}+1\right] \\
&\leq (p-1)\left[\frac{\int_0^1 G(s)g_2(z(s_M))ds}{\int_\delta^1 sG(s)g_1(z(s_m))ds}+1\right]
\end{aligned}
$$

where $z(s_M) = \max\{z(s) : s \in [0,1]\}$ and $z(s_m) = \min\{z(s) : s \in [\delta,1]\}$. It now follows from the fact that $z(s_m) \geq \delta z(s_M)$ and condition (H_4) that

$$
\frac{[Az]'(t)}{\mid Az\mid_\infty} \leq (p-1)\left[\eta\frac{\int_0^1 G(s)ds}{\int_\delta^1 sG(s)ds}+1\right].
$$

The result follows by the Arzelá–Ascoli compactness criterion. ■

Proof of Lemma 2.3 We first show that $\tau*$ is a positive real number. Suppose to the contrary that $\tau* = +\infty$. Then there must exist a sequence $(z_n) \subset K_1 \setminus \{0\}$ such that $(\tau(z_n/\mid z_n\mid_\infty))$ is a strictly increasing sequence of positive real numbers converging to $+\infty$. By assertion 1, there exists a subsequence of $(z_n/\mid z_n\mid_\infty)$ which we denote the same way, such that $(z_n/\mid z_n\mid_\infty)$ converges to some ω_0 in X. Hence $\mid \omega_0\mid_\infty = 1$ and, for large n, we must have

$$
\tau(z_n/\mid z_n\mid) > \tau(\omega_0)\,.
$$

Note that $\omega_0(t) \leq 1/2$, for all $t \in [0, \tau(\omega_0)]$. On the other hand, given $t > \tau(\omega_0)$, we have $t < \tau(z_n / \mid z_n \mid)$ for large n. It follows that $\omega_0(t) = \lim_{n\to+\infty} z_n(t) / \mid z_n \mid_\infty \leq 1/2$, for $t > \tau(\omega_0)$. We conclude that $\omega_0(t) \leq 1/2$, for all $t \geq 0$. But this is impossible, since $\mid \omega_0 \mid_\infty = 1$.

That K is a cone invariant by A is clear. The proof of the lemma is now complete. ■

Lemma 2.4 *We have* $\imath(A, K_R, K) = 1$.

Proof. According to condition (H_3), for $u \in \partial K_R$,

$$\begin{aligned}
|Az|_\infty &= \max_{t\geq 0} \int_0^t \left[\int_s^{+\infty} G(\tau) f(z(\tau)) d\tau \right]^{1/(p-1)} ds \\
&\leq \int_0^{+\infty} \left[\int_s^{+\infty} G(\tau) f(\overline{R}) d\tau \right]^{1/(p-1)} ds \\
&= \frac{f(\overline{R})^{1/(p-1)} (p-1) b^{p/(p-1)})}{p N^{1/(p-1)}} \\
&< \overline{R}.
\end{aligned}$$

Since $\overline{R} \leq R$, we have $|Az|_\infty < R = |z|_\infty$. The result now follows from part 2. of Lemma 2.1. ■

Lemma 2.5 *There is* $r_1 \in (0, R)$ *such that* $\imath(A, K_{r_1}, K) = 0$.

Proof. According to condition (H_1), given $M > 0$ there exists $r_1 \in (0, R)$ such that

$$f(t) \geq M t^{p-1}, \quad \text{for all } \; t \in [0, r_1].$$

Thus for $z \in \partial K_{r_1}$,

$$\begin{aligned}
(Az)(\tau^*) &= \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) f(z(\tau)) d\tau \right]^{1/(p-1)} ds \\
&\geq \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) M z(\tau)^{p-1} d\tau \right]^{1/(p-1)} ds \\
&\geq \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) M z(\tau)^{p-1} d\tau \right]^{1/(p-1)} ds \\
&\geq \left[\int_{\tau^*}^{+\infty} G(\tau) d\tau \right]^{1/(p-1)} \frac{\tau^* M^{1/(p-1)}}{2} \mid z \mid_\infty .
\end{aligned}$$

Choosing $M > 0$ such that

$$\tau^* G(\tau^*)^{N/p(N-1)} \left[\frac{M}{N} \right]^{1/(p-1)} > 2, \tag{2.7}$$

we have that $| Az |_\infty > | z |_\infty$, for all $z \in \partial K_{r_1}$. The result now follows from part 1. of Lemma 2.1. ■

Lemma 2.6 *There is* $r_2 > R$ *such that* $\imath(A, K_{r_2}, K) = 0$.

Proof. It follows from condition (H_2) that there exists $r_3 > R$ such that

$$f(t) \geq Mt^{p-1}, \quad \text{for all } t \geq r_3\,.$$

Note that for $z \in \partial K_{2r_3}$ we have

$$2 \min_{t \geq \tau *} z(t) \geq | z |_\infty = 2r_3\,,$$

which implies

$$f(z(t)) \geq Mz(t)^{p-1}, \quad \text{for all } t \geq \tau *\,.$$

Thus

$$\begin{aligned} (Az)(\tau^*) &= \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) f(z(\tau) d\tau \right]^{1/(p-1)} \\ &\geq \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) f(z(\tau)) d\tau \right]^{1/(p-1)} ds \\ &\geq \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) M z(\tau)^{p-1} d\tau \right]^{1/(p-1)} \\ &\geq \tau^* G(\tau^*)^{N/p(N-1)} \left[\frac{M}{N} \right]^{1/(p-1)} \frac{| z |_\infty}{2}\,. \end{aligned}$$

Define the number $r_2 = 2r_3$. By (2.7), we have $| Az |_\infty > | z |_\infty$, for $z \in \partial K_{r_2}$, and the result now follows from part 1. of Lemma 2.1. ■

3 Proof of the Main Result

Proof of theorem 1.1 It follows from Lemmas 2.4 through 2.6 and the additivity of the fixed point index that

$$\imath(A, K_R \setminus K_{r_1}, K_{r_1}) = 1$$

and that

$$\imath(A, K_{r_2} \setminus K_R, K_R) = -1\,.$$

Consequently, the operator A has two fixed points, namely z_1 in $K_R \setminus K_{r_1}$ and z_2 in $K_{r_2} \setminus K_R$.

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