A Multiplicity Result for a Class of Superquadratic Hamiltonian Systems

João Marcos do Ó[a*] and Pedro Ubilla[b†]

[a]Departamento de Matemática - Univ. Fed. Paraíba

58059 900 João Pessoa, PB, Brazil

[b]Universidade de Santiago de Chile

Cassilla 307, Correo2, Santiago, Chile

Abstract

We establish the existence of two nontrivial solutions for certain semilinear elliptic systems with superquadratic and subcritical growth rates of the form

$$\begin{cases} -\Delta v &= \lambda f(u), \quad \text{in} \quad \Omega, \\ -\Delta u &= g(v), \quad \text{in} \quad \Omega, \\ u &= v = 0, \quad \text{on} \quad \partial\Omega. \end{cases}$$

for a small positive parameter λ and where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded domain. The first solution is obtained applying Ambrosetti and Rabinowitz's mountain-pass theorem while the second by local minimization.

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1 Introduction

The main purpose of this paper is to establish by using a variational approach the existence of two nontrivial solutions for a class of elliptic problems. In particular we

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consider the following problem for Hamiltonian systems:

$$\left\{\begin{array}{rcll} -\Delta v & = & \lambda f(u), & \text{in} \quad \Omega \\ -\Delta u & = & g(v), & \text{in} \quad \Omega \\ u & = & v=0, & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a smooth bounded domain, λ a small positive parameter, and f, $g : \Omega \to \mathbb{R}$ are given continuous functions satisfying the following hypotheses:

(H1) there exists a positive constant C such that

$$\mid f(t) \mid \leq C(1+ \mid t \mid^r), \text{ for all } t \in \mathbb{R};$$

(H2) g is an increasing odd function such that $g(0) = 0$ and

$$\lim_{t\to+\infty} \frac{g(t)}{t^s} = 1$$

where $r \geq 0$, $s > 0$ and

$$\frac{1}{r+1} + \frac{1}{s+1} > \frac{N-2}{N}, \tag{2}$$

when $N \geq 3$. For $N = 1, 2$ there is no restriction.

The last inequality expresses the subcritical character of system (1). Its superquadratic behavior is given by the next assumption.

(H3) There are positive constants μ and R, with $(\mu - 1)s > 1$, such that for all $\mid u \mid\geq R$ we have

$$0 < \mu F(u) \leq uf(u).$$

In recent years, various results on the existence of solutions for superlinear elliptic systems have been obtained. Among others, de Figueiredo and Felmer in [7] and Hulshof and van der Vorst in [9] study these problems by means of a variational approach that considers a Lagrangian formulation with strongly indefinite quadratic part and uses the generalized mountain-pass theorem in its infinite dimensional setting due to Benci-Rabinowitz [2].

In system (1) we isolate v in the second equation to obtain the following fourth order quasilinear scalar problem

$$\left\{\begin{array}{rcll} \Delta(g^{-1}(\Delta u) & = & \lambda f(u) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \\ \Delta u & = & 0 & \text{nn} \quad \partial\Omega \end{array}\right. \tag{3}$$

which has a variational structure and whose solutions may be obtained through the framework of the Critical Point Theory as was first developed by Ambrosetti-Rabinowitz [1]. Namely, by now classical mountain-pass theorem and, by local minimization as a consequence of the Ekeland's variational principle.

We study the existence of solutions of (3) understood as critical points of the associated functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega A(\mid \Delta u \mid^p)dx - \lambda \int_\Omega F(u)dx \tag{4}$$

defined on the reflexive Banach space $E = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm $\| u \| = \| \Delta u \|_{L^p}$ and with Fréchet derivative given by

$$\langle I'_\lambda(u), \phi\rangle = \int_\Omega a(\mid \Delta u \mid^p) \mid \Delta u \mid^{p-2} \Delta u \Delta\phi dx - \lambda \int_\Omega f(u)\phi dx, \qquad \phi \in E, \tag{5}$$

where

$$A(t) = \int_0^t a(s)ds, \qquad F(t) = \int_0^t f(s)ds$$

and the function a is defined by

$$a(\mid t \mid^p) \mid t \mid^{p-2} t = g^{-1}(t) \tag{6}$$

where $p = (s+1)/s$. Using (H1) and (H2), usual arguments give that the expressions in (4) and (5) are well defined, as well as that the functional I_λ is of class C^1 (see [6] and [11], for example). Note that the subcritical condition for equation (3) that is given by $r < p^{**} = Np/(N-2p)$, is equivalent to condition (2).

We now consider a technical condition concerned with the regularity of critical points of the functional I_λ, namely,

(H4) Assume either

$$\begin{cases} s \leq 2, \\ \dfrac{3N-2}{2N}\dfrac{1}{r+1} + \dfrac{1}{s+1} \geq \dfrac{N-2}{N} + \dfrac{1}{(r+1)(s+1)} \end{cases}$$

and that g is a differentiable function such that its derivative g' is a Lipschitz continuous function, or

$$\begin{cases} s > 2, \\ \dfrac{1}{r+1} + \dfrac{1}{s+1} \leq \dfrac{N-2}{N} + \dfrac{1}{(r+1)(s+1)} \end{cases}$$

and that g is of class C^2 with $g''(t) = O(|\, t\,|^{s-2})$ at infinity.

We now state our main result.

Theorem 1 *Assume that hypothesis (H1)-(H3) hold. Furthermore, suppose*
(H5) there are positive constants r_0 *and* s_0 *such that* $r_0 s_0 < 1$,

$$\lim_{t\to 0} \frac{f(t\sigma)}{f(t)} = \sigma^{r_0} \qquad \textit{and} \qquad \lim_{t\to 0} \frac{g(t\sigma)}{g(t)} = \sigma^{s_0}.$$

Then there exists a positive constant λ^* *such that, for any* $0 < \lambda < \lambda^*$, *there exist at least two nontrivial critical points* $u_{\lambda,1}$, $u_{\lambda,2} \in E$ *of the functional* I_λ *such that* $\|\, u_{\lambda,1} \,\|_E \to +\infty$ *and* $\|\, u_{\lambda,2} \,\|_E \to 0$ *as* $\lambda \to 0$. *Moreover, if we assume that (H4) holds, then we have that* $(u_{\lambda,1}, g^{-1}(\Delta u_{\lambda,1}))$ *and* $(u_{\lambda,2}, g^{-1}(\Delta u_{\lambda,2}))$ *are strong solutions of system (1).*

The superquadratic behavior of system (1) expresses by assumption (H3) takes into account the coupling of the system. It does not imply that both equations in (1) are superquadratic. A similar remark is valid for the subcritical character expressed by assumptions (H1) and (H2).

It should be mentioned that this idea of isolating one variable of the system to obtain a scalar equation is similar in spirit to that used in [4], to derive some results concerning the existence of positive periodic and of homoclinic solutions to a class of Hamiltonian system. There it was considered systems where the nonlinearity $g(v)$ is a power and consequently the functional analytic framework is different that we have considered here.

We remark that some authors have studied superlinear elliptic systems with the help of the a priori estimates and degree theory argument see, for example [3, 12]) and also the references therein. Other results on systems with a Hamiltonian form are discussed by Costa and Magalhães in [5].

This paper is composed of three sections. In Section 2 we have an abstract framework where we establish an abstract critical point theorem which is used in Section 3 to prove the main result.

2 The Abstract Framework

In this section we establish an abstract critical point theorem which will be used in the next section to prove our main result.

We start by recalling some standard notations and definitions. Let X be a reflexive Banach space equipped with norm $\| \, . \, \|$. By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X and its dual X^*. We denote the weak convergence in X by " $\rightharpoonup$ " and strong convergence by " $\to$ ".

As usual, we say that a mapping $T : X \to X^*$ satisfies condition (S_+) if for every sequence $(u_n) \subset X$ with $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle T(u_n), u_n - u \rangle \leq 0$ we have $u_n \to u$ in X.

Let $I \in C^1(X, \mathbb{R})$. We say I satisfies the Palais-Smale condition, denoted by (PS), if every Palais-sequence of I, that is, $(u_n) \subset X$ such that $(I(u_n))$ is bounded and $I'(u_n) \to 0$ in the dual space X^*, is relatively compact.

Lemma 2 *Let $\Phi, \Psi : X \to \mathbb{R}$ be C^1 functionals satisfying*

$$\begin{array}{rcl} \mu\Phi(u) - \langle \Phi'(u), u \rangle & \geq & M \parallel u \parallel^p - N \quad \textit{and} \\ \mu\Psi(u) - \langle \Psi'(u), u \rangle & \leq & Q \end{array} \tag{7}$$

for all $u \in X$, where $\mu > p > 1$ and M, N and Q are positive constants. Then every Palais-sequence of the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ is bounded.

Proof. Let $(u_n) \subset X$ be a Palais-Smale sequence; i.e.,

$$\Phi(u_n) - \lambda\Psi(u_n) \to c, \tag{8}$$

and

$$| \langle \Phi'(u_n), v \rangle - \lambda \langle \Psi'(u_n), v \rangle | \leq \epsilon_n \parallel v \parallel \tag{9}$$

where $\epsilon_n \to 0$ as $n \to +\infty$. Multiplying (8) by μ, subtracting (9), with $v = u_n$, from the expression obtained, using (7) we conclude that

$$\begin{array}{rcl} 1 + \mu c + \epsilon_n \parallel u_n \parallel & \geq & \mu\Phi(u_n) - \Phi'(u_n)u_n + \lambda(\Psi'(u_n)u_n - \mu\Psi(u_n)) \\ & \geq & M \parallel u_n \parallel^p - N - \lambda Q. \end{array}$$

Consequently, (u_n) is bounded in X, since $p > 1$. ■

Lemma 3 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying the hypotheses of Lemma 2 such that Φ' belongs to the class $(S)_+$ and Ψ' is such that for every sequence (u_n) in X with $u_n \rightharpoonup u$, we have $\lim_{n\to\infty} \langle \Psi'(u_n), u_n - u \rangle = 0$. Then the functional $I : X \to \mathbb{R}$ given by $I(u) = \Phi(u) - \lambda\Psi(u)$ satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset X$ be a Palais-Smale sequence. By Lemma 2, (u_n) is bounded in X, thus we may take a subsequence, again denoted by (u_n), such that $u_n \rightharpoonup u$ for some u in X. Now by $I'(u_n) \to 0$ in X^*,

$$|\langle \Phi'(u_n), u_n - u\rangle - \lambda\langle \Psi'(u_n), u_n - u\rangle| \leq \epsilon_n \| u_n - u \|$$

where $\epsilon_n \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle = 0$, since $\lim_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle = 0$. Therefore, using that Φ' belongs to the class $(S)_+$, we conclude that $u_n \to u$ in X. ∎

Next we have the main result of this section.

Theorem 4 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying the hypotheses of Lemma 3. Furthermore suppose*
(I1) $C_1 \| u \|^p \leq \Phi(u) \leq C_2 \| u \|^p + C_3$, *for all* $u \in X$;
(I2) $C_4 \| u \|^\mu - C_5 \leq \Psi(u) \leq C_6 \| u \|^r + C_7$, *for all* $u \in X$;
(I3) *there is* $v \in X - \{0\}$ *such that*

$$\lim_{t\to 0} \frac{\Psi(tv)}{\Phi(tv)} = +\infty,$$

where $r > \mu > p > 1$ and $C_1, \ldots, C_7$ are positive constants. Then there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, there exist two nontrivial critical points $\{u_\lambda, v_\lambda\}$ of the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ such that $\| u_\lambda \| \to +\infty$ and $\| v_\lambda \| \to 0$ as $\lambda \to 0$.

The proof of Theorem 4 follows the proof of Lemma 7 below.

Lemma 5 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying*

$$\begin{aligned} \Phi(u) &\geq C_1 \| u \|^p, \\ \Psi(u) &\leq C_6 \| u \|^r + C_7, \end{aligned} \tag{10}$$

for all $u \in X$, where $r > p > 1$ and C_1, C_6 and C_7 are positive constants. Then there exist positive constants α_λ, ρ_λ such that

$$\lim_{\lambda\to 0^+} \rho_\lambda = +\infty$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) > \alpha_\lambda, \; if \; \| u \| = \rho_\lambda.$$

Proof. By assumptions (10), we obtain

$$I_\lambda(u) \geq C_1 \parallel u \parallel^p - \lambda C_6 \parallel u \parallel^r - \lambda C_7, \text{ for all } u \in X.$$

Now, choosing $u \in X$ such that

$$\parallel u \parallel = \lambda^{-s}, \quad \text{with} \quad 0 < s(r-p) < 1, \tag{11}$$

and setting $\rho_\lambda = \lambda^{-s}$, we obtain

$$I_\lambda(u) \geq C_1 \lambda^{-sp} - C_6 \lambda^{1-sr} - \lambda C_7.$$

Finally, taking $\alpha_\lambda = C_1 \lambda^{-sp} - C_6 \lambda^{1-sr} - \lambda C_7$, we have completed the proof of the lemma, since $0 < s(r-p) < 1$. ■

Lemma 6 *Let $\Phi, \Psi : X \to \mathbb{R}$ be functionals satisfying*

$$\begin{array}{rcl} \Phi(u) & \leq & C_2 \parallel u \parallel^p + C_3, \\ \Psi(u) & \geq & C_4 \parallel u \parallel^\mu - C_5, \end{array} \tag{12}$$

for all $u \in X$, where $\mu > p > 1$ and C_2, C_3, C_4 and C_5 are positive constants. Then $I_\lambda(tu) = \Phi(tu) - \lambda\Psi(tu) \to -\infty$ as $t \to +\infty$, for all $u \in X - \{0\}$.

Proof. Using (12) we conclude easily that, for all $t > 0$,

$$I_\lambda(tu) \leq C_2 t^p \parallel u \parallel^p + C_3 - C_4 t^\mu \lambda \parallel u \parallel^\mu + C_5, \text{ for all } u \in X.$$

Consequently, we obtain $I(tu) \to -\infty$ as $t \to +\infty$, for all $u \in X - \{0\}$, since $\mu > p$. ■

Lemma 7 *Let $\Phi, \Psi : X \to \mathbb{R}$ be C^1 functionals satisfying (PS) condition and (10). Furthermore, assume that there exits $v \in X - \{0\}$ such that*

$$\lim_{t \to 0} \frac{\Psi(tv)}{\Phi(tv)} = +\infty. \tag{13}$$

Then there exits $\tilde{\lambda} > 0$ such that, for all $\lambda \in (0, \tilde{\lambda}\ )$, the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ has a nontrivial critical point v_λ such that $\parallel v_\lambda \parallel \to 0$ as $\lambda \to 0$, provided that $I_\lambda(0) = 0$.

Proof. Since for all $u \in X$,

$$I_\lambda(u) \geq C_1 \parallel u \parallel^p - \lambda C_6 \parallel u \parallel^r - \lambda C_7.$$

Now, choosing $\alpha \in \mathbb{R}$ such that $\alpha p < 1$ and $u \in X$, with $\parallel u \parallel = \lambda^\alpha$, we obtain $\tilde{\lambda} > 0$ such that, for all $\lambda \in (0, \tilde{\lambda})$, we have

$$I_\lambda(u) \geq C_1 \lambda^{\alpha p} - C_6 \lambda^{1+\alpha r} - \lambda C_7 \geq 0.$$

Now, using the first inequality in (10) and (13), we can prove that there is $\delta > 0$ such that $\Psi(tv) > 0$, for all $\mid t \mid \leq \delta$. Therefore, given $\lambda \in (0, \tilde{\lambda})$, we have $t_\lambda \in (-\delta, \delta)$ such that

$$\begin{aligned} I_\lambda(t_\lambda v) &= \Phi(t_\lambda v) - \lambda \Psi(t_\lambda v) \\ &= \Psi(t_\lambda v) \left(\frac{\Phi(t_\lambda v)}{\Psi(t_\lambda v)} - \lambda \right) < 0. \end{aligned}$$

Then we conclude that the infimum of the functional I_λ in $B_X[0, \lambda^\alpha]$ is negative, where $B_X[0, R]$ denotes the closed ball with radius R centered at origin of X. Applying Ekeland's variational principle we obtain a sequence $(u_n) \subset B_X[0, \lambda^\alpha]$ such that $I_\lambda(u_n) \to \inf_{B_X[0,\lambda^\alpha]} I_\lambda$ and $I'_\lambda(u_n) \to 0$. Finally, using that I_λ satisfies the (PS) condition and $I_\lambda(0) = 0$, we find a nontrivial minimizer u_λ. This proves Lemma 7. ■

2.1 The proof of Theorem 4

By Lemma 3 the functional I_λ satisfies the (PS) condition. Now, in view of Lemmas 5 and 6, we may apply the mountain-pass theorem, hence it follows that there exists $\widehat{\lambda} > 0$ such that, for all $\lambda \in (0, \widehat{\lambda})$, the functional I_λ has a critical point u_λ such that $I_\lambda(u_\lambda) > \alpha_\lambda > 0$ and $\parallel u_\lambda \parallel \geq \rho_\lambda = \lambda^{-s} \to +\infty$ as $\lambda \to 0$. Finally, since the functional I_λ satisfies the (PS) condition, using Lemma 7, we can take a suitable small λ^* such that, for all $\lambda \in (0, \lambda^*)$, the functional I_λ has another critical point v_λ such that $I_\lambda(v_\lambda) < 0$ and $\parallel v_\lambda \parallel \leq \lambda^\alpha \to 0$ as $\lambda \to 0$. The proof of Theorem 4 is now complete.

2.2 On the (S_+) condition

The next two lemmas concern the condition (S_+) and they are crucial to our minimax argument.

Lemma 8 *Let $L : D(L) \to (L^p(\Omega))^k$ be a continuous and injective linear operator defined on the Banach space $D(L)$ endowed with the norm given by*

$$\| u \|^p = \int | Lu |^p,$$

where $|\ |$ denote a norm of $\mathbb{R}^k$. Then, the derivative of the functional $J : D(L) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\int_\Omega | Lu |^p \, dx$$

belongs to class (S_+).

Proof. Let $(u_n) \subset D(L)$ such that

$$u_n \rightharpoonup u \text{ in } D(L) \text{ and } \lim_{n\to\infty} \sup\langle J'(u_n), u_n - u\rangle \leq 0.$$

Note that

$$\mathfrak{I}_n \doteq \frac{1}{p}\int_\Omega | Lu |^{p-2} Lu(Lu_n - Lu)dx \to 0, \tag{14}$$

since $u_n \rightharpoonup u$ in $D(L)$ and L is continuous. Also, using the Hölder inequality and the elementary inequality

$$| x - y |^p \leq c(p)\{(| x |^{p-2} x - | y |^{p-2} y)(x - y)\}^{s/2}\{| x |^p + | y |^p\}^{1-s/2},$$

where $s = p$ if $p \in (1, 2)$, $s = 2$ if $p \geq 2$ and $c(p)$ is a positive constant depending only on p, we obtain

$$\begin{array}{rcl} \langle J'(u_n), u_n - u\rangle & = & \int_\Omega | Lu_n |^{p-2} Lu_n(Lu_n - Lu)dx \\ & = & \int_\Omega [| Lu_n |^{p-2} Lu_n - | Lu |^{p-2} Lu]L(u_n - u)dx + \mathfrak{I}_n \\ & \geq & C\left\{\int_\Omega | Lu_n |^p + | Lu |^p\right\}^{s/2-1} \int_\Omega | L(u_n - u) |^p + \mathfrak{I}_n. \end{array} \tag{15}$$

Finally, (14) and (15) together with the fact that

$$\lim_{n\to\infty} \sup\langle J'(u_n), u_n - u\rangle \leq 0$$

imply that

$$\lim_{n\to\infty} \int_\Omega | L(u_n - u) |^p \, dx = 0,$$

which completes the proof of the lemma. ∎

Let $a \in C(\mathbb{R}^+, \mathbb{R})$ and $A(s) \doteq \int_0^s a(t)dt$ such that:

(A1) the function $h(t) \doteq A(|\ t\ |^p)$ is strictly convex;

(A2) there are positive constants c_0, c_1, c_2 and c_4 such that

$$c_0 t - c_1 \leq A(t) \leq c_2 t - c_3, \quad \text{for all } t > 0.$$

Now, consider the functional $\Phi : D(L) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{p} \int_\Omega A(|\ Lu\ |^p) dx.$$

Φ is well defined and it is a C^1 functional with Fréchet derivative given by

$$\langle \Phi'(u), v \rangle = \int_\Omega a(|\ Lu\ |^p) \mid Lu \mid^{p-2} LuLvdx.$$

These statements are standard (see [11]). Also, the following result holds:

Lemma 9 *The derivative of the functional Φ belongs to the class (S_+).*

Proof. Let $(u_n) \subset D(L)$ such that

$$u_n \rightharpoonup u \text{ in } D(L) \quad \text{and} \quad \lim_{n\to\infty} \sup \langle \Phi'(u_n), u_n - u \rangle \leq 0.$$

We will to prove that

$$\lim_{n\to\infty} \sup \int_\Omega \mid Lu_n \mid^{p-2} Lu_n . (Lu_n - Lu) dx = 0,$$

and the proof follows from Lemma 8. First we note that

$$\langle \Phi'(u_n), u_n - u \rangle =$$
$$\int_\Omega [a(|\ Lu_n\ |^p) \mid Lu_n \mid^{p-2} Lu_n - a(|\ Lu\ |^p) \mid Lu \mid^{p-2} Lu](Lu_n - Lu)dx + \mathfrak{K}_n$$

where

$$\mathfrak{K}_n \doteq \int_\Omega a(|\ Lu\ |^p) \mid Lu \mid^{p-2} Lu](Lu_n - Lu)dx \to 0,$$

since $u_n \rightharpoonup u$ in $D(L)$ and, a and L are continuous functions. Also, since the function $h(t) = A(|\ t\ |^p)$ is strictly convex and $\lim_{n\to\infty} \sup \langle \Phi'(u_n), u_n - u \rangle \leq 0$, we conclude that $Lu_n(x) \to Lu(x)$, almost everywhere in Ω. Next we remark that, if $x \in \Omega$ and satisfies

$$Lu_n(x)(Lu_n(x) - Lu(x)) \leq 0,$$

we have

$$Lu_n(x)Lu_n(x) \leq Lu_n(x)Lu(x) \leq \mid Lu_n(x) \mid\mid Lu(x) \mid,$$

which implies that

$$\mid Lu_n(x) \mid \leq \mid Lu(x) \mid .$$

On the other hand, if $x \in \Omega$ and satisfies

$$Lu_n(x)(Lu_n(x) - Lu(x)) \geq 0,$$

then

$$a(\mid Lu_n \mid^p) \mid Lu_n \mid^{p-2} Lu_n(Lu_n - Lu) \geq C \mid Lu_n \mid^{p-2} Lu_n(Lu_n - Lu),$$

by (A2) if $\mid Lu_n(x) \mid > M$ for some positive constants C and M. Now, we set

$$\eta_n(x) = \mid Lu_n(x) \mid^{p-2} Lu_n(x)(Lu_n(x) - Lu(x))$$

and consider the sets

$$\begin{array}{rcl} \mathcal{A}_n & = & \{x \in \Omega : \mid Lu_n(x) \mid \leq M\}\,; \\ \mathcal{B}_n & = & \{x \in \Omega : \mid Lu_n(x) \mid > M\}\,; \\ \mathcal{C}_n & = & \{x \in \Omega : \eta_n(x) \geq 0\}\,; \\ \mathcal{D}_n & = & \{x \in \Omega : \eta_n(x) < 0\}\,. \end{array}$$

We have

$$\int_\Omega \eta_n(x)dx = \int_\Omega \eta_n \chi_{\mathcal{A}_n} \chi_{\mathcal{C}_n} dx + \int_\Omega \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{C}_n} dx + \int_\Omega \eta_n \chi_{\mathcal{D}_n} dx,$$

where $\chi_{\mathcal{U}}$ denotes the characteristic function of the a set $\mathcal{U} \subset \mathbb{R}^N$. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega \eta_n \chi_{\mathcal{A}_n} \chi_{\mathcal{C}_n} dx = \int_\Omega \eta_n \chi_{\mathcal{D}_n} dx = 0. \tag{16}$$

On the other hand,

$$\begin{array}{rcl} \int_\Omega \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{C}_n} dx & \leq & C \int_\Omega a(\mid Lu_n \mid^p) \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{C}_n} dx \\ & = & C \int_\Omega a(\mid Lu_n \mid^p) \eta_n \chi_{\mathcal{B}_n} (1 - \chi_{\mathcal{D}_n}) dx \\ & = & C \int_\Omega a(\mid Lu_n \mid^p) \eta_n \chi_{\mathcal{B}_n} dx - C \int_\Omega a(\mid Lu_n \mid^p) \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{D}_n} dx \\ & = & C \int_\Omega a(\mid Lu_n \mid^p) \eta_n (1 - \chi_{\mathcal{A}_n}) dx - C \int_\Omega a(\mid Lu_n \mid^p) \eta_n \chi_{\mathcal{B}_n} \chi_{\mathcal{D}_n} dx. \end{array}$$

This estimates together with the Lebesgue dominated convergence theorem and the fact that $\lim_{n\to\infty}\sup\langle\Phi'(u_n), u_n - u\rangle \le 0$, imply that

$$\lim_{n\to\infty}\sup\int_\Omega \eta_n \chi_{\mathcal{B}_n}\chi_{\mathcal{C}_n}\,dx = 0. \tag{17}$$

From (16) and (17) we obtain

$$\lim_{n\to\infty}\sup\int_\Omega \mid Lu_n\mid^{p-2} Lu_n.(Lu_n - Lu)dx = 0,$$

which together with $u_n \rightharpoonup u$ in $D(L)$ imply that $u_n \to u$ in $D(L)$. The proof of Lemma 9 is complete. ■

3 Proof of Theorem 1

3.1 Existence of critical point for functional I_λ in (4)

This part of the proof of Theorem 1 is an application of Theorem 4. Consider the functionals

$$\Phi(u) = \frac{1}{p}\int_\Omega A(\mid \Delta u\mid^p)dx \quad \text{and} \quad \Psi(u) = \int_\Omega F(u)dx$$

defined on the Banach space $E = W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Next we check that the conditions of Theorem 4 are satisfied. By our assumption (H2) and (6), it is standard to prove that assumptions (A1) and (A2) of Lemma 9 hold with $c_1 = 0$ and, of course, the condition (I1) of Theorem 4 holds with $L = \Delta$. Thus, by Lemma 9, the derivative of the functional Φ belongs to the class (S_+). Furthermore, from assumptions (H1), (H2) and (H3) it is easy to see that conditions (7) and (I2) hold, and using the Sobolev imbedding theorem we have

$$\lim_{n\to\infty}\langle\Psi'(u_n), u_n - u\rangle = \lim\int_\Omega f(x, u_n)(u_n - u) = 0,$$

for every sequence (u_n) in E such that $u_n \rightharpoonup u$.

Finally, using assumption (H4), we show that condition (I3) holds, i.e., there exists $v \in E$ such that

$$\lim_{t\to 0}\frac{\Psi(tv)}{\Phi(tv)} = +\infty. \tag{18}$$

First, note that by assumption (H5),

$$\lim_{t\to 0}\frac{F(t)}{A(t^p)}=+\infty. \tag{19}$$

Let $v\in C_0^\infty(\Omega,[0,1])\backslash\{0\}$ such that $0\leq \Delta v\leq 1$. We have

$$\frac{\Psi(tv)}{\Phi(tv)}=\frac{p\int_\Omega \dfrac{F(tv)}{F(t)}dx}{\int_\Omega \dfrac{A(t^p\mid \Delta v\mid^p)}{A(t^p)}dx}\frac{F(t)}{A(t^p)}. \tag{20}$$

Also, from (H5), by the Lebesgue dominated convergence theorem we get

$$\lim_{t\to 0^+}\int_\Omega \frac{F(tv)}{F(t)}dx=\int_\Omega v^{r_0+1}dx$$

and

$$\lim_{t\to 0^+}\int \frac{A(t^p\mid \Delta v\mid^p)}{A(t^p)}dx=\int_\Omega \mid \Delta v\mid^p dx.$$

Therefore, passing to the limit in (20) and using (19) we obtain (18).

3.2 Regularity and the existence of solutions for system (1)

Here we prove that critical points of functional I_λ are indeed strong solutions of problem (3).

Proposition 10 *Let $u\in E=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ be a critical point of functional I_λ. Then $u\in W^{4,l}(\Omega)$ for some $l>1$.*

As a consequence of Proposition 10 we see that $u\in W^{4-(1/l),l}(\partial\Omega)$. Hence, integration by parts shows that u satisfies the second boundary condition of problem (3). Therefore the pair $(u,g^{-1}(\Delta u))$ is a strong solution of system (1).

Proof of Proposition 10. Assume that $N>2p$, the other case is easiest. Using the continuous imbedding $W^{2,p}(\Omega)\hookrightarrow L^q(\Omega)$ for $q=Np/(N-2p)$ and assumption (H1) we see that $f(u)\in L^{q/r}(\Omega)$. Thus, from standard regularity argument we get that $v=g^{-1}(\Delta u)$ belongs to $W^{2,q/r}(\Omega)$ which is continuously immersed in the Sobolev space $W^{1,r_1}(\Omega)$ with $r_1=Nq/(Nr-q)$ (see [8]).

Claim: $g(v) \in W^{2,l}(\Omega)$ for some $l > 1$.

Notice that from Claim, using standard regularity argument we get that $u \in W^{4,l}(\Omega)$.

Verification of Claim: For that matter it is enough to verify that $g'(v)\partial v/\partial x_i \in W^{1,l}(\Omega)$ for some $l > 1$. We separate the proof into two cases.

Case 1. Assume that

$$\begin{cases} s \leq 2, \\ \dfrac{3N-2}{2N}\dfrac{1}{r+1} + \dfrac{1}{s+1} \geq \dfrac{N-2}{N} + \dfrac{1}{(r+1)(s+1)} \end{cases} \tag{21}$$

and that g is a differentiable function such that its derivative g' is a Lipschitz continuous function. In this case, it is well known that $g'(v) \in W^{1,r_1}(\Omega)$. This fact together with (21) imply that $g'(v)\partial v/\partial x_i \in W^{1,l}(\Omega)$ for some $l > 1$. Hence, Proposition 10 is proved in case 1.

Case 2. Assume that

$$\begin{cases} s > 2, \\ \dfrac{1}{r+1} + \dfrac{1}{s+1} \leq \dfrac{N-2}{N} + \dfrac{1}{(r+1)(s+1)} \end{cases} \tag{22}$$

and that g is of class C^2 with $g''(t) = O(\mid t \mid^{s-2})$ at infinity.

Next, we use the following result concerns superposition mapping on Sobolev space due to Marcus and Mizel [10].

Let $\mathfrak{M}(\Omega)$ denote the space of real measurable functions in Ω. Given a Borel function $h : \mathbb{R} \to \mathbb{R}$ we define the superposition mapping $T_h : \mathfrak{M}(\Omega) \to \mathfrak{M}(\Omega)$ by $T_h u \doteq h \circ u$.

Proposition 11 *Assume that η, ξ are two numbers such that $1 < \eta \leq \xi < N$. Then T_h maps $W^{1,\xi}(\Omega)$ into $W^{1,\eta}(\Omega)$ if and only if the following conditions hold*

1. *h is locally Lipschitz in $\mathbb{R}$;*
2. *the first order derivative of h satisfies the inequality*

$$\mid h'(t) \mid \leq C(1+ \mid t \mid^R) \quad \textit{almost everywere in } \mathbb{R},$$

where C is a positive constant and

$$R = \frac{N(\xi - \eta)}{\eta(N - \xi)}.$$

We know that $v \in W^{1,\xi}(\Omega)$ where $\xi = Nq/(Nr - q)$, thus using that g is of class C^2 and $g''(t) = O(| t |^{s-2})$ at infinity, we get that $g'(v) \in W^{1,\eta}(\Omega)$ where

$$s - 2 = \frac{N(\xi - \eta)}{\eta(N - \xi)},$$

thus

$$\eta = \frac{N\xi}{(N - \xi)(s - 2) + N}.$$

We must have that $\eta \leq \xi$, i.e.,

$$(N - \xi)(s - 2) \geq 0,$$

since $s > 2$ we see that we must have

$$N \geq \xi = \frac{Nq}{Nr - q},$$

that is,

$$\frac{N - 2}{N} + \frac{1}{(r + 1)(s + 1)} \geq \frac{1}{r + 1} + \frac{1}{s + 1}.$$

This completes the proof of Proposition 10

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