Taken from ".Analysis" by E. H. Lieb and M. Loss

REMARK. In case $f^j(x)$ converges to $f(x)$ for almost every $x \in \Omega$ the lemma says that

$$\liminf_j \int_\Omega f^j(x)\mu(\mathrm{d}x) \geq \int_\Omega f(x)\mu(\mathrm{d}x).$$

Even in this case the inequality can be strict. To give an example, consider on $\mathbb{R}$ the sequence of functions $f^j(x) = 1/j$ for $|x| \leq j$ and $f^j(x) = 0$ otherwise. Obviously $\int_\mathbb{R} f^j(x)\,\mathrm{d}x = 2$ for all j but $f^j(x) \to 0$ pointwise for all x.

● So far we have only considered the interchange of limits and integrals for nonnegative functions. The following theorem, again due to Lebesgue, is the one that is usually used for applications and takes care of this limitation. It is one of the most important theorems in analysis. It is equivalent to the monotone convergence theorem in the sense that each can be simply derived from the other.

1.8 THEOREM (Dominated convergence)

Let $f^1, f^2, \ldots$ be a sequence of complex-valued summable functions on (Ω, Σ, μ) and assume that these functions converge to a function f pointwise a.e. If there exists a summable, nonnegative function $G(x)$ on (Ω, Σ, μ) such that $|f^j(x)| \leq G(x)$ for all $j = 1, 2, \ldots,$ then $|f(x)| \leq G(x)$ and

$$\lim_{j\to\infty} \int_\Omega f^j(x)\mu(\mathrm{d}x) = \int_\Omega f(x)\mu(\mathrm{d}x).$$

▶ *Caution*: The existence of the dominating G is crucial!

PROOF. It is obvious that the real and imaginary parts of f^j, R^j and I^j, satisfy the same assumptions as f^j itself. The same is true for the positive and negative parts of R^j and I^j. Thus it suffices to prove the theorem for nonnegative functions f^j and f. By Fatou's lemma

$$\liminf_{j\to\infty} \int_\Omega f^j \geq \int_\Omega f.$$

Again by Fatou's lemma

$$\liminf_{j\to\infty} \int_\Omega (G(x) - f^j(x))\mu(\mathrm{d}x) \geq \int_\Omega (G(x) - f(x))\mu(\mathrm{d}x),$$

since $G(x) - f^j(x) \geq 0$ for all j and all $x \in \Omega$. Summarizing these two inequalities we obtain

$$\liminf_{j\to\infty} \int_\Omega f^j(x)\mu(\mathrm{d}x) \geq \int_\Omega f(x)\mu(\mathrm{d}x) \geq \limsup_{j\to\infty} \int_\Omega f^j(x)\mu(\mathrm{d}x),$$

which proves the theorem. ■

REMARK. The previous theorem allows a slight, but useful, generalization in which the dominating function $G(x)$ is replaced by a *sequence* $G^j(x)$ with the property that there exists a summable G such that

$$\int_\Omega |G(x) - G^j(x)|\mu(\mathrm{d}x) \to 0 \quad \text{as } j \to \infty$$

and such that $0 \leq |f^j(x)| \leq G^j(x)$. Again, if $f^j(x)$ converges pointwise a.e. to f the limit and the integral can be interchanged, i.e.,

$$\lim_{j\to\infty} \int_\Omega f^j(x)\mu(\mathrm{d}x) = \int_\Omega f(x)\mu(\mathrm{d}x).$$

To see this assume first that $f^j(x) \geq 0$ and note that

$$\int (G - f^j)_+ \to \int (G - f)_+ \quad \text{as } j \to \infty$$

since $(G - f^j)_+ \leq G$, using dominated convergence. Next observe that

$$\int (G - f^j)_- = \int (G - G^j + G^j - f^j)_- \leq \int (G - G^j)_-$$

since $G^j - f^j \geq 0$. See 1.5(5). The last integral however tends to zero as $j \to \infty$, by assumption. Thus we obtain

$$\lim_{j\to\infty} \int (G - f^j) = \int (G - f)_+ = \int (G - f)$$

since clearly $f(x) \leq G(x)$. The generalization in which f takes complex values is straightforward.

● Theorem 1.8 was proved using Fatou's lemma. It is interesting to note that Theorem 1.8 can be used, in turn, to prove the following generalization of Fatou's lemma. Suppose that f^j is a sequence of nonnegative functions that converges pointwise to a function f. As we have seen in the Remark after Lemma 1.7, limit and integral cannot be interchanged since, intuitively, the sequence f^j might 'leak out to infinity'. The next theorem taken from [Brézis-Lieb] makes this intuition precise and provides us with a correction term that changes Fatou's lemma from an inequality to an equality. While it is not going to be used in this book, it is of intrinsic interest as a theorem in measure theory and has been used effectively to solve some problems in the calculus of variations. We shall state a simple version of the theorem; the reader can consult the original paper for the general version in which, among other things, $f \mapsto |f|^p$ is replaced by a larger class of functions, $f \mapsto j(f)$.

1.9 THEOREM (Missing term in Fatou's lemma)

Let f^j be a sequence of complex-valued functions on a measure space that converges pointwise a.e. to a function f (which is measurable by the remarks in 1.5). Assume, also, that the f^j's are uniformly p^{th} power summable for some fixed $0 < p < \infty$, i.e.,

$$\int_\Omega |f^j(x)|^p \mu(\mathrm{d}x) < C \quad \text{for } j = 1, 2, \ldots$$

and for some constant C. Then

$$\lim_{j\to\infty} \int_\Omega \big||f^j(x)|^p - |f^j(x) - f(x)|^p - |f(x)|^p\big| \mu(\mathrm{d}x) = 0. \tag{1}$$

REMARKS. (1) By Fatou's lemma, $\int |f|^p \le C$.

(2) By applying the triangle inequality to (1) we can conclude that

$$\int |f^j|^p = \int |f|^p + \int |f - f^j|^p + o(1), \tag{2}$$

where $o(1)$ indicates a quantity that vanishes as $j \to \infty$. Thus the correction term is $\int |f - f^j|^p$, which measures the 'leakage' of the sequence f^j. One obvious consequence of (2), for all $0 < p < \infty$, is that if $\int |f - f^j|^p \to 0$ and if $f^j \to f$ a.e., then

$$\int |f^j|^p \to \int |f|^p.$$

(In fact, this can be proved directly under the sole assumption that $\int |f - f^j|^p \to 0$. When $1 \le p < \infty$ this a trivial consequence of the triangle inequality in 2.4(2). When $0 < p < 1$ it follows from the elementary inequality $|a + b|^p \le |a|^p + |b|^p$ for all complex a and b.) Another consequence of (2), for all $0 < p < \infty$, is that if $\int |f^j|^p \to \int |f|^p$ and $f^j \to f$ a.e., then

$$\int |f - f^j|^p \to 0.$$

PROOF. Assume, for the moment, that the following family of inequalities, (3), is true: For any $\varepsilon > 0$ there is a constant C_ε such that for all numbers $a, b \in \mathbb{C}$

$$\big||a+b|^p - |b|^p\big| \le \varepsilon |b|^p + C_\varepsilon |a|^p. \tag{3}$$

Next, write $f^j = f + g^j$ so that $g^j \to 0$ pointwise a.e. by assumption. We claim that the quantity

$$G_\varepsilon^j = \left(\big||f+g^j|^p - |g^j|^p - |f|^p\big| - \varepsilon |g^j|^p\right)_+ \tag{4}$$

satisfies $\lim_{j\to\infty} \int G_\varepsilon^j = 0$. Here $(h)_+$ denotes as usual the positive part of a function h. To see this, note first that

$$\begin{aligned}&\big||f+g^j|^p - |g^j|^p - |f|^p\big| \\ &\quad \le \big||f+g^j|^p - |g^j|^p\big| + |f|^p \le \varepsilon |g^j|^p + (1+C_\varepsilon)|f|^p\end{aligned}$$

and hence $G_\varepsilon^j \le (1+C_\varepsilon)|f|^p$. Moreover $G_\varepsilon^j \to 0$ pointwise a.e. and hence the claim follows by Theorem 1.8 (dominated convergence). Now

$$\int \big||f+g^j|^p - |g^j|^p - |f|^p\big| \le \varepsilon \int |g^j|^p + \int G_\varepsilon^j.$$

We have to show $\int |g^j|^p$ is uniformly bounded. Indeed,

$$\int |g^j|^p = \int |f - f^j|^p \le 2^p \int (|f|^p + |f^j|^p) \le 2^{p+1} C.$$

Therefore,

$$\limsup_{j\to\infty} \int \big||f+g^j|^p - |g^j|^p - |f|^p\big| \le \varepsilon D.$$

Since ε was arbitrary the theorem is proved.

It remains to prove (3). The function $t \mapsto |t|^p$ is convex if $p > 1$. Hence $|a+b|^p \le (|a|+|b|)^p \le (1-\lambda)^{1-p}|a|^p + \lambda^{1-p}|b|^p$ for any $0 < \lambda < 1$. The choice $\lambda = (1+\varepsilon)^{-1/(p-1)}$ yields (3) in the case where $p > 1$. If $0 < p \le 1$ we have the simple inequality $|a+b|^p - |b|^p \le |a|^p$ whose proof is left to the reader. ■

● With these convergence tools at our disposal we turn to the question of proving Fubini's theorem, 1.12. Our strategy to prove Fubini's theorem in full generality will be the following: First, we prove the 'easy' form in Theorem 1.10; this will imply 1.5(9). Then we use a small generalization of Theorem 1.10 to establish the general case in Theorem 1.12.