THE BREZIS-NIRENBERG PROBLEM ON $\mathbb{H}^n$

Existence and Uniqueness of Solutions

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ABSTRACT. We consider the equation $\Delta_{\mathbb{H}^n} u + \lambda u + u^{\frac{n+2}{n-2}} = 0$ in a domain D' in hyperbolic space $\mathbb{H}^n$, $n \geq 3$ with Dirichlet boundary conditions. For different values of λ we search for positive solutions. Existence holds for $\lambda^* < \lambda < \lambda_1$, where we can compute the value of λ^* exactly if D' is a geodesic ball. In particular it turns out that - like in the Euclidean space - the case $n = 3$ is different from the case $n \geq 4$ and has to be studied separately.

1 Introduction

We consider the problem

$$\begin{aligned} \Delta_{\mathbb{H}^n} u + \lambda u + u^{2^*-1} &= 0 \quad \text{in } D' \\ u &> 0 \quad \text{in } D' \\ u &= 0 \quad \text{on } \partial D' \end{aligned} \tag{BN}$$

where D' is a domain in hyperbolic space $\mathbb{H}^n$, $n \geq 3$, $\lambda \in \mathbb{R}$ and $2^* = \frac{2n}{n-2}$ the critical Sobolovexponent. We want to know for which values of λ there exists a solution $u \in W_0^{1,2}(D')$.

The same problem for balls in Euclidean space was solved in 1983 by Brezis and Nirenberg [BN] and in the following years a lot of extensions of this problem appeared.
In spaces of constant curvature it has been studied by Bandle, Brillard and Flucher [BBF]. The special case of $\mathbb{S}^3$ has been treated in [BB]. Our aim is now to extend the problem to domains in hyperbolic space. It turns out that the results are very similar to the results in the Euclidean case.

After a brief introduction in the hyperbolic space we will discuss the existence of nontrivial solutions for the two cases $n \geq 4$ (section 3) and $n = 3$ (section 4). In the special case $n = 3$ we will make further remarks on properties of solutions.

2 The hyperbolic space

The hyperbolic space $\mathbb{H}^n$ is defined as a subset of $\mathbb{R}^{n+1}$ by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}.$$

We can use the stereographic projection to map $\mathbb{H}^n$ into $\mathbb{R}^n$. This is done by mapping a point P' in $\mathbb{H}^n$ to a point $P \in \mathbb{R}^n$. P is the intersection of the line between P' and the point $(0, \ldots, 0, -1)$ and $\mathbb{R}^n$. In particular, the space $\mathbb{H}^n$ is mapped into $B(0,1) \subset \mathbb{R}^n$.

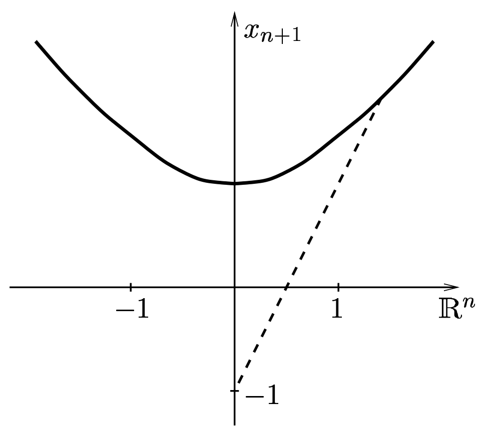

Figure 1: The stereographic projection from $\mathbb{H}^n$ into $B(0,1) \subset \mathbb{R}^n$

Change of coordinates transforms the line element of $\mathbb{H}^n$ into

$$ds = p(x)|dx|, \qquad \text{with } p(x) = \frac{2}{1 - |x|^2}.$$

The gradient, the Dirichlet integral and the Laplace-Beltrami operator corresponding to this metric are

$$\begin{aligned} \nabla_{\mathbb{H}^n} u &= \frac{\nabla u}{p} \\ Du &= \int_{D'} |\nabla_{\mathbb{H}^n} u|^2 \, ds = \int_D |\nabla u|^2 p^{n-2} \, dx \\ \Delta_{\mathbb{H}^n} u &= p^{-n} \operatorname{div}(p^{n-2} \nabla u) \end{aligned}$$

Here is $D' \subset \mathbb{H}^n$ and D its stereographic projection into $\mathbb{R}^n$.

The first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary conditions will be denoted by λ_1.

3 The case $n \geq 4$

The main result of this section is

Theorem 1 (Existence of solutions for $n \geq 4$)
Let D' be a bounded domain in $\mathbb{H}^n, n \geq 4$. Then the following statements are true:

i) *For $\lambda \geq \lambda_1$ the problem (BN) has no nontrivial solution.*

ii) *For $\lambda \leq \frac{n(n-2)}{4}$ and if D' is starshaped, the problem (BN) has no nontrivial solution.*

iii) *If $\lambda \in \left(\frac{n(n-2)}{4}, \lambda_1\right)$ there exists a nontrivial solution of the problem (BN).*

Remarks • Statement *i)* and *ii)* of Theorem 1 remain true if $n = 3$. They can be proved in the same way.
• If D' is a geodesic ball in $\mathbb{H}^n$ we may assume without loss of generality that D' is centered at $(0, \ldots, 0, 1) \in \mathbb{H}^n$. Then the stereographic projection D of D' is a ball in $\mathbb{R}^n$, centered at the origin with radius $0 < R < 1$. We can illustrate the statements of Theorem 1 in the following picture

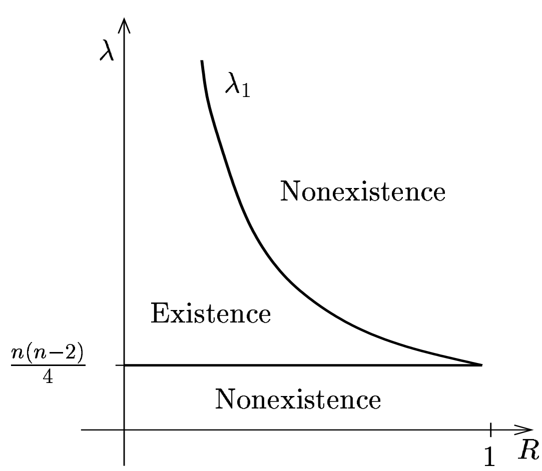

Figure 2: Existence of solutions for $n \geq 4$

Proof of Theorem 1 We shall sketch the proof and refer to [St] for more details.
Denote by φ_1 the eigenfunction of $-\Delta_{\mathbb{H}^n}$ corresponding to the eigenvalue λ_1 on D' with $\varphi_1 > 0$ in D'. Assume that u is a solution of (BN). Then

$$\int_{D'} \Delta_{\mathbb{H}^n} u \; \varphi_1 \; ds + \lambda \int_{D'} u \; \varphi_1 \; ds + \int_{D'} u^{2^*-1} \varphi_1 \; ds \;=\; 0$$

This is equivalent to

$$\underbrace{\int_{D'} (\lambda - \lambda_1) \; u \; \varphi_1 \; ds}_{\geq 0} \;=\; - \underbrace{\int_{D'} u^{2^*-1} \varphi_1 \; ds}_{\geq 0}.$$

The equality above only holds if $u \equiv 0$. This completes the proof of the first statement.

To show the second claim we assume that u is a nontrivial solution of (BN) and we define

$$v(x) := p^{\frac{n-2}{2}}(x)\, u(x)$$

The function v is a nontrivial solution of

$$\begin{aligned} \Delta v + \underbrace{\Big(\lambda - \frac{n(n-2)}{4}\Big)}_{=:\mu} p^2 v + v^{2^*-1} &= 0 \quad \text{in } D \\ v &> 0 \quad \text{in } D \\ v &= 0 \quad \text{on } \partial D \end{aligned} \qquad \text{(BN*)}$$

where $D \subset \mathbb{R}^n$ is the stereographic projection of D' into $\mathbb{R}^n$. Notice that D is also starshaped.

We now use the classical Pohozaev inequality. Multiplying the equation $(BN*)$ by $x\nabla v$ we get

$$(-\Delta v)(x\nabla v) = (v^{2^*-1} + \mu p^2 v)(x\nabla v)$$

This equation is equivalent to

$$\begin{aligned} -\nabla\Big(\nabla v(x\nabla v) - x\frac{|\nabla v|^2}{2} + x\big(\frac{v^{2^*}}{2^*} + \frac{\mu}{2}p^2v^2\big)\Big) \\ = \frac{n-2}{2}(|\nabla v|^2 - v^{2^*}) - \frac{n}{2}\mu p^2 v^2 - x\frac{\mu}{2}v^2\nabla p^2 \end{aligned}$$

Integration over D yields

$$\begin{aligned} \frac{1}{2}\int_{\partial D}\Big|\frac{\partial v}{\partial \nu}\Big|^2 (x\cdot\nu)dS &= \mu\int_D (p^2 + \frac{x}{2}\nabla p^2)v^2\ dx \\ \Leftrightarrow \frac{1}{2}\int_{\partial D}\Big|\frac{\partial v}{\partial \nu}\Big|^2 (x\cdot\nu)dS &= \mu\int_D v^2p^2(1+p|x|^2)\ dx \end{aligned}$$

Because D is starshaped, the left hand side of the equation is strictly positiv if v is a nontrivial solution. On the other hand, the right hand side is negativ if $\lambda \leq \frac{n(n-2)}{4}$ which is a contradiction. We conclude that $v \equiv 0$ in D and $u \equiv 0$ in D' and the second statement is proved.

Existence of solutions of problem (BN) will be shown by the concentration-compactness alternative ([Li1], [Li2]; for a summary see [B]).

We have to prove that there exists a function $u \in W^{1,2}(D, p\,dx)$ so that the value of the quotient

$$Q_{\lambda,p}(u) = \frac{\int_D |\nabla u|^2 p^{n-2}\ dx - \lambda\int_D u^2p^n\ dx}{\left(\int_D u^{2^*}p^n\ dx\right)^{2/2^*}}$$

is smaller then S^* where S^* denotes the best Sobolev constant of the embedding of $W_0^{1,2}(D)$ into $L^{2^*}(D)$. As trial functions we choose

$$u_\varepsilon(x) = p^{-\frac{n-2}{2}}(x)\,\frac{\varphi(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}}$$

with φ a smooth function, $\varphi \equiv 1$ near 0 and $\varphi = 0$ on ∂D and estimate the quotient:

$$Q_{\lambda,p}(u_\varepsilon) \leq \begin{cases} S^* + O(\varepsilon^{\frac{n-2}{2}}) + c\left(\frac{n(n-2)}{4} - \lambda\right)\varepsilon & \text{if } n \geq 5 \\ S^* + O(\varepsilon) + c\left(\frac{n(n-2)}{4} - \lambda\right)\varepsilon \ln \varepsilon & \text{if } n = 4 \end{cases}$$

for positive ε and with some constant $c > 0$. We conclude $Q_{\lambda,p}(u_\varepsilon) < S^*$ if ε is small enough.
In view of the concentration-compactness alternative there exists a minimizer of the quotient if $\lambda > \frac{n(n-2)}{4}$ and this minimizer is a solution of problem (BN) if $\lambda < \lambda_1$. ❐

4 The case $n = 3$

It turns out that in this case the value of λ^* depends on the geometry of D'. We will give a complete picture of existence of solutions for geodesic balls. Without loss of generality we can assume that this ball has his center in $(0, 0, 0, 1) \in \mathbb{H}^3$.

Our main result is

Theorem 2 (Existence of solutions for $n = 3$)
Let D' be a geodesic ball in $\mathbb{H}^3$ with center at $(0, 0, 0, 1)$, and $D = B(0, R)$ with $0 < R < 1$ the stereographic projection of D' into $\mathbb{R}^3$. Put

$$\lambda^* = 1 + \frac{\pi^2}{16\,\text{arctanh}^2 R}.$$

Then the following statements are true:

i) For $\lambda \leq \lambda^*$ and $\lambda \geq \lambda_1$ the problem (BN) has only the trivial solution.

ii) If $\lambda \in (\lambda^*, \lambda_1)$ there exists a nontrivial solution of the problem (BN).

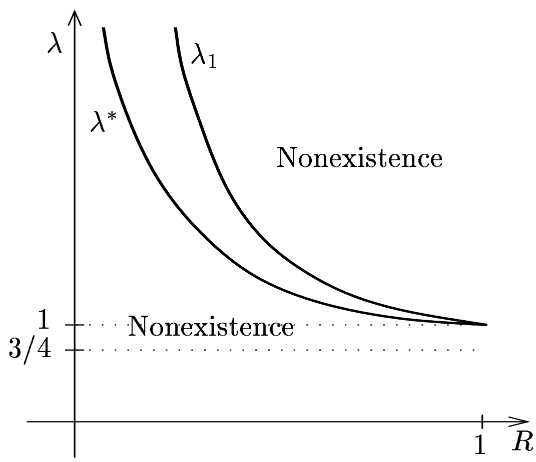

Figure 3: Existence of solutions for $n = 3$

Proof of Theorem 2 Again we sketch the proof and refer to [St] for details.

The nonexistence results for $\lambda \geq \lambda_1$ and $\lambda \leq \frac{3}{4}$ can be shown in the same way as in the case $n \geq 4$.

To show the nonexistence of a nontrivial solution for $\frac{3}{4} < \lambda \leq \lambda^*$ we use again a Pohozaev argument with special test functions.

By the moving plane method we know that if a nontrivial solution exists it is radial. Problem (BN) is equivalent to

$$\begin{aligned} u'' + \frac{p}{r}u' + p^2\lambda u + p^2 u^5 &= 0 \quad \text{in } (0, R) \\ u &> 0 \quad \text{in } (0, R) \\ u(R) &= 0 \end{aligned} \tag{BNR}$$

Testing the equation (BNR) with $r^2 f(r) u'$ where

$$f(r) = \begin{cases} \sinh(2\sqrt{1-\lambda}\, g(r)) \cdot \cosh(2\sqrt{1-\lambda}\, g(r)) & \text{if } \frac{3}{4} < \lambda < 1 \\ g(r) & \text{if } \lambda = 1 \\ \sin(2\sqrt{\lambda - 1}\, g(r)) \cdot \cos(2\sqrt{\lambda - 1}\, g(r)) & \text{if } 1 < \lambda \leq \lambda^* \end{cases}$$

and $g(r) = \operatorname{arctanh} r$

gives us after some computations an integral equality for the solution u which can only be valid if $u \equiv 0$. So the first statement is proved.

To prove the second part of Theorem 2 we must again estimate the quotient $Q_{\lambda,p}$. Assuming φ is a smooth function, $\varphi(0) = 1$, $\varphi'(0) = 0$,

$\varphi(R) = 0$ and

$$u_\varepsilon(x) = \frac{\varphi(|x|)}{(\varepsilon + |x|^2)^{1/2}}$$

we get

$$Q_{\lambda,p}(u_\varepsilon) = S^* + \frac{\sqrt{\varepsilon}}{(2\pi^2)^{1/3}} F(\varphi, \lambda) + O(\varepsilon)$$

with

$$F(\varphi, \lambda) = 4\pi \int_0^R \varphi'^2 p \; dr + 4\pi \int_0^R \varphi^2 p^2 \; dr - 4\pi\lambda \int_0^R \varphi^2 p^3 \; dr.$$

Now choose $\varphi(r) = \varphi_1(r) = (1 - r^2) \cdot \cos(\pi/2 \cdot \operatorname{arctanh} r / \operatorname{arctanh} R)$. This function satisfies the assumptions above and is a ground state of the eigenvalue problem

$$\begin{aligned} -(p\varphi_1')' + p^2\varphi_1 - \lambda^* p^3 \varphi_1 = 0 \quad & \text{in } (0, R) \\ \varphi_1 > 0 \quad & \text{in } (0, R) \\ \varphi_1(R) = 0, \quad & \varphi_1'(0) = 0 \end{aligned}$$

In particular

$$\frac{\int_0^R \varphi_1'^2 p \; dr + \int_0^R \varphi_1^2 p^2 \; dr}{\int_0^R \varphi_1^2 p^3 \; dr} = \lambda^*$$

and

$$\int_0^R \varphi'^2 p \; dr + \int_0^R \varphi^2 p^2 \; dr \geq \lambda^* \int_0^R \varphi^2 p^3 \; dr$$

for all admissible functions φ. We deduce

$$F(\varphi, \lambda) \geq 4\pi \Big(\lambda^* - \lambda\Big) \int_0^R p^3 \varphi^2 \; dr$$

and $F(\varphi, \lambda) < 0$ if $\lambda > \lambda^*$. If ε is small enough it follows that $Q_{\lambda,p}(u_\varepsilon) < S^*$. Again we use the concentration-compactness alternative to conclude that there exists a minimizer and if $\lambda < \lambda_1$ this minimizer is a solution of our problem (BN). ❐

Remarks For $n = 3$ the following properties of nontrivial solutions of the problem (BN) are known:

- By the moving plane method it can be shown that all solutions of problem BN are radial and by [KwL] we know that a radially symmetric solution is unique.

- Suppose that u_λ is a solution of equation (BN) for $\lambda \in (\lambda^*, \lambda_1)$.

  If λ tends to λ_1 the solution u_λ belonging to λ tends to 0 pointwise.

  If λ tends to λ^* the radially symmetric solution concentrates at the origin.

  (see [B] for references)

- Suppose that $D' \subset \mathbb{H}^3$ is a geodesic ball with center at $c := (0,0,0,1) \in \mathbb{R}^4$ and G is Green's function of $\Delta_{\mathbb{H}^n} + \lambda$ on D' with Dirichlet boundary conditions.

  After changing to radial symmetric coordinates in Euclidean space we can compute G with singularity in 0.

$$G(r) = \frac{1}{2} \cdot \frac{1-r^2}{r} \Big(-\cos(4\sqrt{\lambda-1}\,\mathrm{arctanh}\, R - 2\sqrt{\lambda-1}\,\mathrm{arctanh}\, r) + \cos(2\sqrt{\lambda-1}\,\mathrm{arctanh}\, r) \Big)$$

  Denote the regular part of G by $h_\lambda(r)$. $h_\lambda(r)$ is a monotone decreasing function in r and is strictly negativ if $\lambda < \lambda^*$. If $\lambda \geq \lambda^*$ the function $h_\lambda(r)$ changes sign in $(0, R)$. In particular $h_{\lambda^*}(0) = 0$.

  This supports the conjecture of Budd and Humphries ([BuH], confirm also [B]) as for balls in $\mathbb{S}^3$.

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