CHAPTER I

SOME NONSTANDARD PROPERTIES OF SOBOLEV MAPS

We start with some preliminary properties about Sobolev functions and maps which will often be used in the sequel. We assume throughout that Ω is a smooth, bounded, open connected set in $\mathbb{R}^N$. (The assumption that Ω is smooth is to be on the "safe side"; for most purposes a Lipschitz boundary would suffice.)

1.1. Lifting of Sobolev maps with values into $\mathbf{S}^1$

Here, we assume in addition that Ω is simply connected. It is well-known (see e.g. Gilbarg-Trudinger [1], Theorem 7.8 or Brezis [1], Proposition IX.5) that if $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f' \in L^\infty(\mathbb{R})$ and if $\varphi \in W^{1,p}(\Omega; \mathbb{R})$, $1 \leq p < \infty$, then $f \circ \varphi \in W^{1,p}(\Omega; \mathbb{R})$. Moreover we have

$$\frac{\partial}{\partial x_i}(f \circ \varphi) = f'(\varphi)\frac{\partial \varphi}{\partial x_i}.$$

In particular, if $\varphi \in W^{1,p}(\Omega; \mathbb{R})$, then

$$u = e^{i\varphi} \in W^{1,p}(\Omega; \mathbb{C})$$

and in addition $|u| = 1$ a.e., that is, $u \in S^1$ a.e.

We are concerned with the converse. Namely, set

$$W^{1,p}(\Omega; S^1) = \{u \in W^{1,p}(\Omega; \mathbb{C}); |u| = 1 \text{ a.e. on } \Omega\}.$$

We ask the question whether **any** $u \in W^{1,p}(\Omega; S^1)$ can be written as

$$u = e^{i\varphi} \quad \text{for some } \varphi \in W^{1,p}(\Omega; \mathbb{R}).$$

If $N = 1$ the answer is positive for any p, $1 \leq p < \infty$. This is left as an exercise. (Hint: since u is continuous it can be written as $u = e^{i\varphi}$ for some continuous φ; locally we have $\varphi = f \circ u$ where f is the local inverse of the map $t \mapsto e^{it}$, i.e., $\varphi = -i \log u$ and thus $\varphi \in W^{1,p}$ by the above considerations.)

When $N \geq 2$, it is quite surprising that the answer is positive only for $p \geq 2$. This is the content of the following theorem and Remark 1.1.

Theorem 1.1. *Assume $u \in W^{1,p}(\Omega; S^1)$ with $p \geq 2$. Then there exists some $\varphi \in W^{1,p}(\Omega, \mathbb{R})$ such that*

$$u = e^{i\varphi}.$$

Theorem 1.1 is due to Bethuel and Zheng [1]. The proof we present, due to Bourgain, Brezis and Mironescu [1], is simpler than the original one (see also Carbou [1]).

Proof. Assume first that φ exists and let us derive some consequences. Write

$$u = u_1 + iu_2 \quad \text{with } u_1 = \cos\varphi \text{ and } u_2 = \sin\varphi.$$

We have

$$\nabla u_1 = -(\sin\varphi)\nabla\varphi = -u_2\nabla\varphi$$

and

$$\nabla u_2 = (\cos\varphi)\nabla\varphi = u_1\nabla\varphi.$$

Hence

$$\nabla\varphi = u_1\nabla u_2 - u_2\nabla u_1. \tag{1}$$

The strategy is now to find φ by solving (1) with the help of a generalized form of Poincaré's lemma,

Lemma 1.1. *Let $1 \leq p < \infty$ and let $f \in L^p(\Omega; \mathbb{R}^N)$. The following properties are equivalent:*

a) there is some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$ such that

$$f = \nabla\varphi,$$

b) one has

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \qquad \forall i, j, \ \ 1 \leq i, j \leq N, \tag{2}$$

in the sense of distributions, i.e.,

$$\int f_i \frac{\partial\psi}{\partial x_j} = \int f_j \frac{\partial\psi}{\partial x_i} \qquad \forall\psi \in C_0^\infty(\Omega).$$

We emphasize that the assumption that Ω is simply connected is needed in this lemma.

Proof of Lemma 1.1. The implication $a) \Rightarrow b)$ is obvious. To prove the converse, let $\bar{f}$ be the extension of f by 0 outside Ω and let $\bar{f}_\varepsilon = \rho_\varepsilon \star \bar{f}$ where (ρ_ε) is a sequence of mollifiers. The $\bar{f}_\varepsilon$'s satisfy (2) on every compact subset of Ω (for ε sufficiently small). In particular, on every smooth simply connected domain $\omega \subset \Omega$ with compact closure in Ω there is a function ψ_ε such that

$$\nabla \psi_\varepsilon = \bar{f}_\varepsilon \quad \text{in} \quad \omega.$$

(Here we have used the standard Poincaré lemma). Passing to the limit we obtain some $\psi \in W^{1,p}(\omega)$ such that $\nabla\psi = f$ in ω. Finally we write Ω as an increasing union of ω_n as above and obtain a corresponding sequence ψ_n. In the limit we find some $\varphi \in L^1_{\text{loc}}(\Omega)$ with $\nabla\varphi = f$ in Ω. Using the regularity of Ω and a standard property of Sobolev spaces (see e.g. Maz'ja [1], Corollary in Section 1.1.11) we conclude that $\varphi \in W^{1,p}(\Omega)$.

Proof of Theorem 1.1 completed. We will first verify condition *b)* of the lemma for

$$f = u_1 \nabla u_2 - u_2 \nabla u_1 \tag{3}$$

i.e.,

$$f_i = u_1 \frac{\partial u_2}{\partial x_i} - u_2 \frac{\partial u_1}{\partial x_i}.$$

Formally, this is clear. Indeed, assume u_1 and u_2 are smooth; then,

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 2\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} - \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right).$$

On the other hand, if we differentiate the relation

$$|u|^2 = u_1^2 + u_2^2 = 1,$$

we find

$$u_1 \frac{\partial u_1}{\partial x_i} + u_2 \frac{\partial u_2}{\partial x_i} = 0 \qquad \forall i = 1, 2, \dots, N. \tag{4}$$

Thus, in $\mathbb{R}^2$, the vector $\left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}\right)$ is orthogonal to (u_1, u_2). It follows that the vectors $\left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}\right)$ and $\left(\frac{\partial u_1}{\partial x_j}, \frac{\partial u_2}{\partial x_j}\right)$ are colinear and therefore

$$\det\begin{pmatrix} \dfrac{\partial u_1}{\partial x_i} & \dfrac{\partial u_2}{\partial x_i} \\ \dfrac{\partial u_1}{\partial x_j} & \dfrac{\partial u_2}{\partial x_j} \end{pmatrix} = \frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j} - \frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i} = 0. \tag{5}$$

Hence (2) holds. To make this argument rigorous we rely on the density of smooth functions in the Sobolev space $W^{1,p}(\Omega;\mathbb{R})$ (see e.g. Adams [1], Chap. III or Brezis [1], Chap. IX): there are sequences (u_{1n}) and (u_{2n}) in $C^\infty(\overline{\Omega},\mathbb{R})$ such that $u_{1n}\to u_1$ and $u_{2n}\to u_2$ in $W^{1,p}(\Omega;\mathbb{R})$ and $\|u_{1n}\|_{L^\infty}\le 1, \|u_{2n}\|_{L^\infty}\le 1$.

(**Warning**: We do not claim that $u_n=(u_{1n},u_{2n})$ takes its values in S^1. The density of $C^\infty(\overline{\Omega};N)$ in $W^{1,p}(\Omega;N)$, where N is a compact manifold without boundary, e.g. $N=S^1$, is a delicate matter which has been extensively studied by Bethuel [1]. As we will see in Remark 1.30, Theorem 1.1 can be used to prove the density of $C^\infty(\overline{\Omega};S^1)$ in $W^{1,p}(\Omega;S^1)$ for $p\ge 2$.)

Set

$$f_n = u_{1n}\nabla u_{2n} - u_{2n}\nabla u_{1n},$$

so that

$$f_n\to f \qquad \text{in} \quad L^p$$

and

$$\frac{\partial f_{in}}{\partial x_j}-\frac{\partial f_{jn}}{\partial x_i} = 2\left(\frac{\partial u_{1n}}{\partial x_j}\frac{\partial u_{2n}}{\partial x_i}-\frac{\partial u_{1n}}{\partial x_i}\frac{\partial u_{2n}}{\partial x_j}\right) \tag{6}$$

converges in $L^{p/2}$ to $2\left(\dfrac{\partial u_1}{\partial x_j}\dfrac{\partial u_2}{\partial x_i}-\dfrac{\partial u_1}{\partial x_i}\dfrac{\partial u_2}{\partial x_j}\right)$. Multiplying (6) by $\psi\in C_0^\infty(\Omega)$, integrating by parts and passing to the limit (**using the fact that** $p\ge 2$) we obtain

$$-\int_\Omega\left(f_i\frac{\partial\psi}{\partial x_j}-f_j\frac{\partial\psi}{\partial x_i}\right) = 2\int_\Omega\left(\frac{\partial u_1}{\partial x_j}\frac{\partial u_2}{\partial x_i}-\frac{\partial u_1}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right)\psi.$$

On the other hand (4) holds a.e. (even for any $u\in W^{1,p}(\Omega;S^1)$, $1\le p<\infty$). It follows that f satisfies *b)* of Lemma 1.1, and therefore there is some $\varphi\in W^{1,p}(\Omega;\mathbb{R})$ such that

$$f=\nabla\varphi.$$

We will now prove that this φ is essentially the one in the conclusion of Theorem 1.1.

Recall that (see e.g. Brezis [1], Chap. IX) if $g,h\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $1\le p<\infty$ then $gh\in W^{1,p}$ and

$$\frac{\partial}{\partial x_i}(gh) = g\frac{\partial h}{\partial x_i}+h\frac{\partial g}{\partial x_i}.$$

Set

$$v=ue^{-i\varphi},$$

so that $v \in W^{1,p}$ and

$$\begin{aligned}\nabla v &= e^{-i\varphi}(\nabla u - iu\nabla\varphi) = ue^{-i\varphi}(\bar{u}\nabla u - i\nabla\varphi) \\ &= ue^{-i\varphi}(\bar{u}\nabla u - if) = ue^{-i\varphi}(u_1\nabla u_1 + u_2\nabla u_2) = 0 \quad \text{by (4).}\end{aligned}$$

We deduce that v is a constant and since $|v| = 1$ we may write $v = e^{iC}$ for some constant $C \in \mathbb{R}$. Hence $u = e^{i(\varphi + C)}$ and the function $\varphi + C$ has the desired properties.

Remark 1.1. The conclusion of Theorem 1.1 fails when $p < 2$ (in any dimension $N \geq 2$). To see this assume first that $N = 2$ and, for simplicity, that $0 \in \Omega$. Set

$$u(x) = \frac{x}{|x|}.$$

Clearly, $u \in W^{1,p}(\Omega; S^1)$ for every $1 \leq p < 2$ (but $u \notin W^{1,p}$ for $p \geq 2$). We claim that there exists **no** $\varphi \in W^{1,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Suppose that such φ exists. As in the proof of Theorem 1.1 we find that (1) holds and since $u \in C^\infty(\Omega\backslash\{0\})$ we see that $\varphi \in C^\infty(\Omega\backslash\{0\})$. Fix $r > 0$ so small that

$$S_r = \{x; |x| = r\} \subset \Omega.$$

On S_r we have

$$e^{i\varphi(r,\theta)} = e^{i\theta}$$

and thus

$$\varphi(r,\theta) = \theta + 2\pi k(\theta)$$

for some $k(\theta) \in \mathbb{Z}$. By continuity k is constant on $(0, 2\pi)$. Hence

$$\lim_{\substack{\theta\to 2\pi \\ \theta<2\pi}} \varphi(r,\theta) - \lim_{\substack{\theta\to 0 \\ \theta>0}} \varphi(r,\theta) = 2\pi.$$

This is impossible since φ is singlevalued and continuous. The same conclusion can also be reached using degree theory (see Section 1.3).

When $N \geq 3$, we assume, as above, that $0 \in \Omega$ and consider

$$u(x) = \frac{(x_1, x_2)}{(x_1^2 + x_2^2)^{1/2}}.$$

Clearly $u \in W^{1,p}(\Omega; S^1)$ for every p, $1 \leq p < 2$ and moreover $u \in C^\infty(\omega)$ where $\omega = \Omega\backslash\{x; x_1 = x_2 = 0\}$. If $\varphi \in W^{1,p}(\Omega; \mathbb{R})$ exists we see, using (1), that $\varphi \in C^\infty(\omega)$. Since $u = e^{i\varphi}$ a.e. in Ω we have by continuity $u = e^{i\varphi}$ **everywhere** in ω.

Let $S_r = \{x\,; x_1^2 + x_2^2 = r^2$ and $x_3 = \cdots = x_N = 0\}$ and fix $r > 0$ so small that $S_r \subset \omega$. Since $u = e^{i\varphi}$ on S_r we obtain the same contradiction as in the case $N = 2$.

The question of lifting of Sobolev maps for **fractional** Sobolev spaces $W^{s,p}$ is much more delicate.

Recall the definition of fractional Sobolev spaces (see e.g. Adams [1]). Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded simply connected domain. For $0 < \sigma < 1$ and $1 < p < \infty$, let

$$W^{\sigma,p}(\Omega) = \left\{ f \in L^p(\Omega); \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+\sigma p}} dxdy < \infty \right\}.$$

We will often use the standard semi-norm

$$||f||^p_{W^{\sigma,p}} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+\sigma p}} dxdy.$$

To work with a norm it suffices to add $\;|\int_\Omega f|$ to$||f||_W\sigma, p.$

If $0 < s < \infty$ is a real number and s is not an integer, write $s = m + \sigma$ where $m = [s]$ is the integer part of s and $\sigma = s - m$. Then set

$$W^{s,p}(\Omega) = \{f \in W^{m,p}(\Omega);\;\; D^\alpha f \in W^{\sigma,p}(\Omega),\;\; \forall \alpha, |\alpha| = m\}.$$

As above we set

$$W^{s,p}(\Omega; S^1) = \{u \in W^{s,p}(\Omega; \mathbb{C}); |u| = 1 \text{ a.e. on } \Omega\}$$

and ask the question whether any $u \in W^{s,p}(\Omega; S^1)$ may be written as

$$u = e^{i\varphi} \quad \text{for some} \quad \varphi \in W^{s,p}(\Omega; \mathbb{R}).$$

Here is a summary of the main results from Bourgain, Brezis and Mironescu [1]:

Case 1: $\mathbf{N = 1}$, $\mathbf{0 < s < \infty}$, $\mathbf{1 < p < \infty}$,

Theorem 1.2.*Assume $N = 1$, $0 < s < \infty$ and $1 < p < \infty$. Then, every $u \in W^{s,p}(\Omega; S^1)$ may be written as*

$$u = e^{i\varphi} \quad \text{for some} \quad \varphi \in W^{s,p}(\Omega; \mathbb{R}).$$

Case 2: $\mathbf{N \geq 2}$, $\mathbf{0 < s < 1}$, $\mathbf{1 < p < \infty}$.

Theorem 1.3. *Assume $N \geq 2$, $0 < s < 1$ and $1 < p < \infty$. The answer to the lifting question is:*

a) positive when $sp \geq N$,

b) negative when $1 \leq sp < N$,
c) positive when $sp < 1$.

Case 3: $\mathbf{N \geq 2}$, $\mathbf{1 \leq s < \infty}$, $\mathbf{1 < p < \infty}$.

Theorem 1.4. *Assume* $N \geq 2$, $1 \leq s < \infty$ *and* $1 < p < \infty$. *The answer to the lifting question is :*

a) positive when $sp \geq 2$,
b) negative when $1 < sp < 2$.

Here "positive" means that every $u \in W^{s,p}(\Omega; S^1)$ may be written as:

$$u = e^{i\varphi} \text{ for some } \varphi \in W^{s,p}(\Omega; \mathbb{R}),$$

"negative" means that for some u's in $W^{s,p}(\Omega; S^1)$ there is no $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$.

In case $p = 2$ the above theorems for $H^s = W^{s,2}$ can be summarized as follows:

Corollary 1.1. *When* $N = 1$ *the answer to the lifting problem in* H^s *is always positive.*
When $N \geq 2$ *the answer to the lifting problem in* H^s *is:*
a) positive when $0 < s < 1/2$,
b) negative when $1/2 \leq s < 1$,
c) positive when $s \geq 1$.

So far, this concerns the question of existence. Turning to the question of **uniqueness** one may ask whether the difference $\varphi_1 - \varphi_2$ of two solutions is of the form $2\pi k$ for some $k \in \mathbb{Z}$. Clearly $\frac{1}{2\pi}(\varphi_1 - \varphi_2)$ takes its values into $\mathbb{Z}$, but this does **not** imply that it is constant (since $\varphi_1 - \varphi_2$ need not be continuous). We will prove in Section 1.2 that uniqueness holds if $sp \geq 1$ (but not if $sp < 1$).

We present here partial proofs of the above results and we refer the reader to Bourgain, Brezis and Mironescu [1] for complete proofs.

First, the easy case where u is continuous:

Theorem 1.5. *Assume* $N \geq 1$, $0 < s < \infty$, $1 < p < \infty$ *and* $sp > N$ *(or* $p = 1$ *and* $s = N$*). Then any* $u \in W^{s,p}(\Omega; S^1)$ *may be lifted as* $u = e^{i\varphi}$ *with* $\varphi \in W^{s,p}(\Omega; \mathbb{R})$.

Proof. By the Sobolev imbedding theorem, u is continuous and, locally, we may consider $\varphi = -i \log u$, which is well defined and singlevalued. To conclude we need a lemma about composition.

Lemma 1.2. *Assume* $N \geq 1$, $0 < s < \infty$ *and* $1 < p < \infty$. *Assume* $v \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$ *and let* $\Phi \in C^\infty$. *Then* $\Phi \circ v \in W^{s,p}(\Omega)$.

The proof is very simple when $0 < s < 1$ (using the definition of $W^{s,p}$ and the fact that Φ is Lipschitz on the range of v). This lemma is also well-known when s is an integer, with the help of the Gagliardo-Nirenberg inequality. When $s > 1$ is not an integer the argument is more delicate, we refer to Escobedo [1], Alinhac and Gérard [1] and Lemma A.1.1 in Appendix A.1.1.

Case 1: $\mathbf{N = 1,\ 0 < s < \infty,\ 1 < p < \infty}$. **Proof of Theorem 1.2 when** $\mathbf{sp \geq 1}$.

Set I= (0,1). In view of Theorem 1.5 it suffices to consider the case $s = 1/p$. By standard trace theory there is some $\widetilde{u} \in W^{s+1/p,p}(I^2; \mathbb{R}^2)$ such that

$$\widetilde{u}(x, 0) = u(x).$$

Since u takes its values into S^1 one may expect that, near $I \times \{0\}$, $\widetilde{u}$ takes its values "close" to S^1. This is not true for a general extension $\widetilde{u}$. However **special** extensions have that property. For example

$$\widetilde{u}(x, y) = \frac{1}{2y} \int_{x-y}^{x+y} u(t)dt$$

has the property that $\widetilde{u} \in W^{s+1/p,p}$ (see e.g. Appendix A.1.4) and moreover,$|\widetilde{u}(x, y)| \to 1$ **uniformly** in x as $y \to 0$. This is a consequence of the fact that $W^{s,p} \subset$ VMO in the limiting case of the Sobolev imbedding (see e.g. Boutet de Monvel-Berthier, Georgescu and Purice [1], [2], Brezis and Nirenberg [1] and also Lemmas 1.9 and 1.10 below. Similarly, any harmonic extension $\widetilde{u}$ of u in I^2 has also the same property (see Brezis and Nirenberg [2], Appendix 3). If we consider $v = \widetilde{u}/|\widetilde{u}|$ in a neighborhood ω of $I \times \{0\}$ we have an extension v of u such that

$$v \in W^{s+1/p,p}(\omega; S^1).$$

Here, we have used again Lemma 1.2.

Let us now explain how to complete the proof of the theorem when $p = 2$, i.e., $u \in H^{1/2}(I; S^1)$. From the above discussion we have some extension

$$v \in H^1(\omega; S^1).$$

Applying Theorem 1.1 we may write

$$v = e^{i\psi}$$

for some $\psi \in H^1(\omega; \mathbb{R})$ and then $\varphi = \psi_{|I}$ has the required properties.

We now turn to the general case. Here, we shall use the following lemma about products in fractional Sobolev spaces. Its proof is given in Appendix A.1.3 when $\Omega = \mathbb{R}^N$. The case of a smooth domain Ω follows by extending the functions to $\mathbb{R}^N$.

Lemma 1.3. *Assume $s \geq 1$ and $1 < p < \infty$. Let*

$$f, g \in W^{s,p}(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. Then

$$f \nabla g \in W^{s-1,p}(\Omega).$$

Moreover, if s is an integer we may also take $p = 1$.

Remark 1.2. In Lemma 1.3 there is no relation between s, p and N.

Proof of Theorem 1.2 completed when $s = 1/p$. We recall that there is a neighborhood Q of $I \times \{0\}$ in $\mathbb{R}^2$ and an extension v of u such that

$$v \in W^{s+1/p,p}(Q; S^1).$$

Applying once more the same construction we find some

$$w \in W^{s+2/p,p}(U; S^1)$$

where U is a neighborhood of $Q \times \{0\}$ in $\mathbb{R}^3$. (This construction is possible since $(s+1/p)p = 2$, so that we are again in a limiting case for the Sobolev imbedding and thus $v \in$ VMO). Iterating this construction we find some

$$\zeta \in W^{s+(k/p),p}(G; S^1)$$

where G is a neighborhood of $I \times \{0\} \times \ldots \times \{0\}$ in $\mathbb{R}^{k+1}$. Consider the first integer $k \geq 1$ such that

$$s + (k/p) \geq 1.$$

This choice of k implies that

$$s + \frac{j}{p} < 1, \quad j = 0, 1, \ldots, k-1,$$

so that at, each previous step, the standard trace theory applies (recall that a function in $W^{s,p}$ has an extension in $W^{s+1/p,p}$ provided s is not an integer).

From the Sobolev imbedding we have

$$\zeta \in W^{1,k+1}(G; S^1).$$

By Theorem 1.1 we may write

$$\zeta = e^{i\psi}$$

for some $\psi \in W^{1,k+1}(G;\mathbb{R})$. Moreover, by (1) and (4) we have

$$\nabla\psi = -i\bar{\zeta}\nabla\zeta.$$

By Lemma 1.3 we have

$$\nabla\psi \in W^{s+(k/p)-1,p}(G).$$

Hence

$$\psi \in W^{s+(k/p),p}(G).$$

Taking traces we have

$$\varphi = \psi_{|I\times\{0\}} \in W^{s,p}(I)$$

and

$$u = e^{i\varphi}.$$

Remark 1.3. As we have just seen, every $u \in H^{1/2}(I;S^1)$ admits a lifting $\varphi \in H^{1/2}(I;\mathbb{R})$. Moreover, this lifting is unique modulo an integer multiple of 2π (see Section 1.2 below) and the map $u \mapsto \varphi$ is continous from $H^{1/2}$ into $H^{1/2}$ (this can be established using the same argument as in Step 7 of the proof of Theorem 4 in Brezis and Nirenberg [1]). Surprisingly, there is **no bound** whatsoever for $||\varphi||_{H^{1/2}}$ in terms of $||u||_{H^{1/2}}$. Here is an example of a sequence (φ_n) such that $||\varphi_n||_{H^{1/2}} \to +\infty$ while $||e^{i\varphi_n}||_{H^{1/2}} \le C$. Consider the sequence φ_n defined by

$$\varphi_n(x) = \begin{cases} 0 & \text{for } 0 < x < 1/2 \\ 2\pi n(x-1/2) & \text{for } 1/2 < x < (1/2)+(1/n) \\ 2\pi & \text{for } (1/2)+(1/n) < x < 1. \end{cases}$$

Clearly $||\varphi_n||_{H^{1/2}} \to +\infty$ (Since $\varphi_n \to \varphi = \chi_{(1/2,1)}$ in L^2, where χ_A denotes the characteristic function of a set A, and φ does not belong to $H^{1/2}$). On the other hand, the reader will easily check (for example by scaling) that $||e^{i\varphi_n} - 1||_{H^{1/2}}$ remains bounded. The same lack of estimate holds when $H^{1/2}$ is replaced by $W^{1/p,p}$ with any $p, 1 < p < \infty$. Curiously this is the only case with lack of estimate. When $sp > 1$ one may control $||\varphi||_{W^{s,p}}$ in terms of $||u||_{W^{s,p}}$. This may be easily derived from the fact that $||u||_{W^{s,p}} \le K$ implies a uniform modulus of continuity; one may then lift u successively on a partition of I into small intervals.

We also call the attention of the reader to the fact that (in dimension one) there is an estimate for φ in the space $H^{1/2} + W^{1,1}$ (equipped with its usual norm) in terms of $||e^{i\varphi}||_{H^{1/2}}$; see Bourgain, Brezis and Mironescu [1] [2] [3] for further developments.

Sketch of the proof of Theorem 1.2 when $sp < 1$.

We present the principal ingredients and refer to Bourgain, Brezis and Mironescu [1] for detailed arguments. The main tool is a characterization of fractional Sobolev spaces when $sp < 1$. It is originally due to Bourdaud [1] (see also an earlier work of Devore and Popov [1]); the interested reader will find new proofs in Bourgain, Brezis and Mironescu [1] which yield sharp constants.

Set $I = (0.1)$. For each integer $j \geq 0$, consider the partition $\mathcal{P}_j$ of I into 2^j intervals of length 2^{-j}. Denote by $\mathcal{E}_j$ the space of functions from I into $\mathbb{C}$ which are constant on each interval of $\mathcal{P}_j$. Given a function $f \in L^p(I)$ consider the function $f_j = E_j(f) \in \mathcal{E}_j$ defined as follows:

Any $x \in I$ belongs exactly to one interval of the partition $\mathcal{P}_j$, say $Q_j(x)$. Set

$$f_j(x) = E_j(f)(x) = \fint_{Q_j(x)} f = 2^j \int_{Q_j(x)} f. \tag{7}$$

Clearly we have

$$||E_j(f)||_{L^p(I)} \leq ||f||_{L^p(I)} \qquad \forall j, \tag{8}$$

and

$$E_j(f) \to f \text{ in } L^p \text{ and a.e. as } j \to \infty. \tag{9}$$

Lemma 1.4. *For any $0 < s < 1$ and any $1 < p < \infty$ we have*

$$\sum_{j \geq 0} 2^{spj} ||f - f_j||^p_{L^p(I)} \leq C ||f||^p_{W^{s,p}(I)} \tag{10}$$

where C depends only on s and p.

On the other hand, we have

Lemma 1.5. *Assume $sp < 1$ and let $(g_j)_{j=0,1,\ldots}$ be a sequence of functions on I such that*

$$g_j \in \mathcal{E}_j \qquad \forall j = 0, 1, \ldots \tag{11}$$

and

$$\sum_{j \geq 1} 2^{spj} ||g_j - g_{j-1}||^p_{L^p(I)} < \infty. \tag{12}$$

Then (g_j) converges in $L^p(I)$ to some $g \in W^{s,p}(I)$ with

$$||g||^p_{W^{s,p}} \leq C \sum_{j \geq 1} 2^{spj} ||g_j - g_{j-1}||^p_{L^p(I)}. \tag{13}$$

where the constant C depends only on s, p and blows up as $sp \to 1$.

Combining Lemmas 1.4 and 1.5 (applied with $g_j = E_j(f)$) we obtain, for $sp < 1$, the norm - equivalence,

$$||f||^p_{W^{s,p}(I)} \sim \sum_{j \geq 1} 2^{spj} ||E_j(f) - E_{j-1}(f)||^p_{L^p(I)}, \tag{14}$$

which can be regarded as a characterization of $W^{s,p}$ functions in terms of their components in a Haar (or wavelet) basis; see Bourdaud [1].

Assuming the two lemmas we now proceed as follows. Let $u \in W^{s,p}(I; S^1)$; for each integer $j \geq 0$ define u_j as in (7) and

$$U_j(x) = \begin{cases} \dfrac{u_j(x)}{|u_j(x)|} & \text{if } u_j(x) \neq 0 \\ 1 & \text{if } u_j(x) = 0. \end{cases}$$

Clearly $U_j \to u$ a.e. For each $j \geq 0$ we construct a function $\varphi_j : I \to \mathbb{R}$ in $\mathcal{E}_j$, such that

$$e^{i\varphi_j} = U_j \quad \text{on} \quad I \tag{15}$$

$$|\varphi_j - \varphi_{j-1}| \leq C|U_j - U_{j-1}|, \; j = 1, 2, ... \tag{16}$$

Note that (16) can be achieved by induction on j, for example with $C = \pi/2$.

On the other hand, observe that for every $\xi, \eta, \zeta \in \mathbb{C}$ with $|\zeta| = 1$, we have

$$\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \leq 4(|\zeta - \xi| + |\zeta - \eta|) \tag{17}$$

with the convention that $\frac{0}{0} = 1$ (consider separately the case where $|\xi|, |\eta| \geq 1/2$ and the case where either $|\xi| < 1/2$ or $|\eta| < 1/2$).

Applying (17) to $\xi = E_j(u)(x), \eta = E_{j-1}(u)(x)$ and $\zeta = u(x)$ we obtain a.e. on Ω

$$|U_j - U_{j-1}| \leq 4(|u - E_j(u)| + |u - E_{j-1}(u)|). \tag{18}$$

Combining this with (16) yields

$$|\varphi_j - \varphi_{j-1}| \le C(|u - E_j(u)| + |u - E_{j-1}(u)|) \tag{19}$$

and thus

$$\sum_{j\ge 1} 2^{spj}||\varphi_j - \varphi_{j-1}||^p_{L^p} \le C \sum_{j\ge 0} 2^{spj}||u - E_j(u)||^p_{L^p}. \tag{20}$$

In view of Lemmas 1.4 and 1.5 we conclude that $\varphi_j \to \varphi$ in L^p with $\varphi \in W^{s,p}, e^{i\varphi} = u$ and

$$||\varphi||_{W^{s,p}(I)} \le C||u||_{W^{s,p}(I)}$$

We may always assume (by adding to φ an integer multiple of 2π) that

$$|\int_I \varphi| \quad \le \quad 2\pi.$$

Thus we have constructed a function φ in $W^{s,p}$ such that $e^{i\varphi} = u$ and

$$||\varphi||_{L^p(I)} + ||\varphi||_{W^{s,p}(I)} \le C\ (1 + ||u||_{W^{s,p}(I)}). \tag{21}$$

Note that there is an interesting contrast between estimate (21) and the lack of estimate when $s = 1/p$ (see Remark 1.3).

Remark 1.4. The proof of Lemma 1.4 is easy, but the proof of Lemma 1.5 is quite technical which makes the complete proof of Theorem 1.2, when $sp < 1$ rather involved. It would be interesting to find a simpler proof, even in case $p = 2$ (i.e., in $H^s = W^{s,2}$) when $s < 1/2$; see O P1 in Section 1.8.

Remark 1.5. The function φ constructed above also belongs to every L^q, $q < \infty$. This may be seen by observing that $u \in W^{s,p} \cap L^\infty \subset W^{\sigma,q}$ for every $\sigma < s$ with $\sigma q = sp$ (by the Gagliardo-Nirenberg inequality, see Appendix A.1.3). Since the construction of φ is independent of s and p, this φ belongs to every such $W^{\sigma,q}$. Choosing σ close to zero we obtain a q which is arbitrarily large. We do not know if this φ belongs to L^∞ (or even to BMO); see also OP2 in Section 1.8.

Case 2: $\mathbf{N \geq 2, 0 < s < 1, 1 < p < \infty}$.

This case is covered by Theorem 1.3. We examine separately the assertions *a), b), c)* in the theorem.

Proof of Theorem 1.3 a). When $sp > N$ the conclusion follows from Theorem 1.5. When $sp = N$ the argument is exactly the same as in the proof of Theorem 1.2 (with $sp = 1$).

Proof of Theorem 1.3 b). Without loss of generality we may assume that Ω is the unit ball. Recall that s and p are given with $0 < s < 1$, $1 < p < \infty$ and $1 \leq sp < N$. Let

$$\psi(x) = \frac{1}{|x|^\alpha} \quad \text{with} \quad \frac{N - sp}{p} \leq \alpha < \frac{N - sp}{sp}$$

and let

$$u = e^{i\psi}.$$

We claim that

$$u \in W^{s,p}(\Omega; S^1). \tag{22}$$

Indeed, it is easy to check that

$$\psi \in W^{1,q}, \qquad \forall q < \frac{N}{\alpha + 1}.$$

Thus

$$u \in W^{1,q}, \qquad \forall q < \frac{N}{\alpha + 1}$$

and consequently

$$u \in W^{\sigma,q}, \qquad \forall \sigma < 1, \qquad \forall q < \frac{N}{\alpha + 1}.$$

Since $u \in L^\infty$ we also have

$$u \in W^{t,r}, \qquad \forall t \in (0,1), \quad \forall r \in (1,\infty) \text{ with } \quad tr < \frac{N}{\alpha + 1}. \tag{23}$$

This is a consequence of the Gagliardo-Nirenberg type inequality for fractional Sobolev spaces:

Lemma 1.6. *Assume $f \in W^{\sigma,q} \cap L^\infty$ where $0 < \sigma < \infty$ and $1 < q < \infty$. Then*

$$f \in W^{t,r}, \qquad \forall t \in (0, \sigma) \quad \text{with } \ r = \sigma q/t.$$

This is the well-known Gagliardo-Nirenberg inequality when both σ and t are integers. It is easy to verify (see Appendix A.1.3) when both σ and t are not integers (This is the case used above). The more delicate case, where one of the reals σ, t is an integer and the other is not, is discussed in Appendix A.1.3.

In particular, in (23), we may take $t = s$ and $r = p$, i.e., (22) holds.

Next we claim that there is no $\varphi \in W^{s,p}(\Omega;\mathbb{R})$ such that $u = e^{i\varphi}$. Assume, by contradiction, that such φ exists. Set

$$\eta = \frac{1}{2\pi}(\varphi - \psi).$$

Clearly η takes its value in $\mathbb{Z}$ and

$$\eta \in W^{s,p}_{\text{loc}}(\Omega\backslash\{0\};\mathbb{Z})$$

(because ψ is smooth on $\Omega\backslash\{0\}$). Since $sp \geq 1$ and $\Omega\backslash\{0\}$ is connected we conclude, using Theorem 1.6 below that η is a constant. Hence $\psi \in W^{s,p}(\Omega;\mathbb{R})$. But this is not true: note that, by scaling,

$$A(r) = \int_{B_r}\int_{B_r} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}dxdy$$

satisfies $A(1) = r^\beta A(r)$ with $\beta = (\alpha+s)p - N \geq 0$ (by assumption on α). If $A(1) < \infty$, then $A(1) = 0$ (by letting $r \to 0$). But this is impossible, Thus $A(1) = \infty$, i.e., $\psi \notin W^{s,p}$.

A topological obstruction. Here is an alternative example of nonexistence in the special case $N = 2$, $s = 1/2$ and $p = 2$. Set

$$u(x) = \frac{x}{|x|}.$$

It belongs to $H^{1/2}(\Omega;S^1)$ (in fact to $W^{1,p}(\Omega)$ for every $p < 2$). There is no $\varphi \in H^{1/2}(\Omega;\mathbb{R})$ such that $u = e^{i\varphi}$. Assume, by contradiction, that such φ exists and let $\overline{\varphi}(x,y)$ be some harmonic extension to $Q = \Omega \times (0,1)$ with $\overline{\varphi} \in H^1(Q)$. Let $\overline{u} = e^{i\overline{\varphi}} \in H^1(Q;S^1)$. Let $\Sigma_r = \{(x,y); |x| = r \text{ and } y \in (0,1)\}$. For a.e. r the restriction of $\overline{u}$ to Σ_r belongs to $H^1(\Sigma_r)$ and its trace on $C_r \times \{0\}$ is $u_{|C_r}$. Fix any such r. As $y \to 0$, $\overline{u}(x,y)$ tends to $u(x)$ in $H^{1/2}(C_r)$.

On the other hand, for each $y \in (0,1)$, $\deg(\overline{u}(\cdot,y), C_r) = 0$. Indeed $\overline{u}$ is smooth on Q and $\overline{u}_t(\sigma) = \overline{u}(t\sigma, y)$, $\sigma \in S^1$, $t \in [0,r]$ is a S^1-valued homotopy connecting $\overline{u}(\cdot,y)$ to a constant.

We conclude using the stability of degree under $H^{1/2}$ convergence (see Theorem 1.11) that

$$\deg(u, C_r) = 0$$

but this degree is one since $u(x) = \dfrac{x}{|x|}$.

When $N \geq 3$, the same construction as above with

$$u(x) = \frac{(x_1,x_2)}{(x_1^2+x_2^2)^{1/2}} \qquad , x = (x_1, x_2, ..., x_N)$$

provides and example of a function $u \in W^{s,p}(\Omega; S^1)$ for every $s \in (0,1)$ and every $p \in (1,\infty)$ with $sp < 2$ and which has no lifting when $sp \geq 1$. However this example does not reach the optional condition $sp < N$ when $N \geq 3$ – see Remark 1.6 below.

Remark 1.6. The lack of lifting (within $W^{s,p}$) is in fact more "dramatic". For simplicity, consider first the case $N = 2$. The topological example described above provides an example of a function

$$u \in W^{s,p}(\Omega; S^1) \qquad \forall p \in (1,\infty), \qquad \forall s < 2/p,$$

which has **no** lifting $\varphi \in W^{1/p,p}$ (but it does have a lifting $\varphi \in W^{(1/p)-\varepsilon,p} \quad \forall \varepsilon > 0$). Since s is arbitrarily close to $2/p$, this means that the operation of lifting may induce a "loss of $1/p$ derivative" in the $W^{\sigma,p}$ scale.

When $N \geq 3$ the precise loss of regularity in lifting is not well understood. For simplicity, consider the case $N = 3$ and $p = 4$. First a summary of the known results:

(a) If $s < 1/4$, any $u \in W^{s,4}$ has a lifting in $W^{s,4}$.
(b) If $s \geq 3/4$, any $u \in W^{s,4}$ has a lifting in $W^{s,4}$.
(c) If $1/4 \leq s < 3/4$ some u's in $W^{s,4}$ have no lifting in $W^{s,4}$.
(d) The topological example provides an example of a function $u \in W^{s,4}$ $\forall s < 1/2$, and this u has no lifting even in $W^{1/4,4}$.

It would be interesting to understand what happens when $1/2 \leq s < 3/4$, see OP3 in Section 1.8.

Proof of Theorem 1.3 c). The existence of a lifting $\varphi \in W^{s,p}$ when $sp < 1$ is proved using the same argument as in the case $N = 1$. (The statements of Lemma 1.4 and 1.5 hold without any change when the interval I is replaced by a cube; see Bourgain, Brezis and Mironescu [1]).

Remark 1.7. When $sp < 1$ the existence of a lifting $\varphi \in W^{s,p}$ comes with an estimate

$$||\varphi||_{W^{s,p}} \leq C(1 + ||u||_{W^{s,p}}). \tag{24}$$

The constant C depends on s, p and blows up as $sp \to 1$. Indeed, if the constant C would remain bounded we would reach a contradiction with Remark 1.3 when $N = 1$ and a contradiction with Theorem 1.3 b) when $N \geq 2$. It is of interest to study how the best constant in (24) behaves as $sp \to 1$. For example, when $p = 2$, the best constant is of the order of $(1 - 2s)^{1/2}$ as $s \to 1/2$; see Bourgain, Brezis and Mironescu [1], [3], where the reader will also find the motivation for the study of the best constant.

Case 3: $\mathbf{N \geq 2}$, $\mathbf{s \geq 1}$, $\mathbf{1 < p < \infty}$.

Our main positive result is a slight improvement of Theorem 1.4 *a*):

Theorem 1.4'*a). Assume $N \geq 2$, $s \geq 1$, $1 < p < \infty$ and $sp \geq 2$. Then any $u \in W^{s,p}(\Omega; S^1)$ may be lifted as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})$.*

Proof. Observe that

$$W^{s,p} \cap L^\infty \subset W^{1,sp}$$

by the Gagliardo-Nirenberg inequality. Since $sp \geq 2$ we may apply Theorem 1.1 and write $u = e^{i\varphi}$ for some $\varphi \in W^{1,sp}(\Omega; \mathbb{R})$. Using Lemma 1.3 we find that

$$\nabla\varphi = -i\overline{u}\,\nabla u \in W^{s-1,p}.$$

so that $\varphi \in W^{s,p}$.

Remark 1.8. If s is an **integer**, $s \geq 2$ and $1 \leq p < \infty$, then any $u \in W^{s,p}(\Omega; S^1)$ may be lifted as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(\Omega; \mathbb{R})$. In view of Theorem 1.4'a, it is only the case $p = 1$ which is new. This case is treated noting that Lemma 1.3 still holds when $p = 1$ and s is an integer.

We complete this section with the

Proof of Theorem 1.4b). Consider the example of Remark 1.1. Assume $N = 2$ (the case $N \geq 3$ is handled as in Remark 1.1). First one checks (as in the proof of Lemma A.1.XXX) that

$$u(x) = \frac{x}{|x|}$$

belongs to $W^{s,p}(\Omega; S^1)$ for any $s \in (0, \infty)$ and any $p \in (1, \infty)$ with $sp < 2$. We claim that there exists **no** $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Suppose that such φ exists. Fix a disc $\omega \subset \Omega$ with $0 \notin \overline{\omega}$. We have

$$\frac{1}{2\pi}(\varphi - \theta) \in \mathbb{Z} \quad \text{on} \quad \omega.$$

Moreover, the function θ is smooth on ω. Thus $\frac{1}{2\pi}(\varphi - \theta) \in W^{s,p}(\omega; \mathbb{Z})$. Since $sp \geq 1$ we may apply Theorem 1.6 below to conclude that

$$\varphi = \theta + 2k\pi \quad \text{for some} \quad k \in \mathbb{Z}.$$

Hence $\varphi \in C^\infty(\Omega \backslash \{0\})$ and we obtain a contradiction as in Remark 1.1.

Remark 1.9. Finally, we point out that the question of lifting can be raised for a variety of function spaces. In particular it has been studied for VMO in Coifman and Meyer [1] and in Brezis-Nirenberg [1], where it is established that the answer is positive.

1.2. Sobolev functions with values into $\mathbb{Z}$; connectedness of the essential range

A continuous function on a connected space with values into $\mathbb{Z}$ must be constant. Sobolev functions in $W^{s,p}$ which are not necessarily continuous have that property under some appropriate conditions on s and p. The main result is the following

Theorem 1.6. *Assume Ω is a connected open set in $\mathbb{R}^N$, $N \geq 1$. Let $0 < s < \infty$ and $1 < p < \infty$ be such that*

$$sp \geq 1. \tag{1}$$

Assume

$$u \in W^{s,p}(\Omega; \mathbb{Z}).$$

Then u is constant.

Remark 1.10. It is surprising that the conclusion of Theorem 1.6 holds under condition (1) which is much weaker than the condition $sp > N$ needed for the injection of $W^{s,p}$ into continuous functions. Assumption (1) is **optimal**. Indeed, the characteristic function, χ_ω, of any smooth domain ω compactly contained in Ω belongs to $W^{s,p}(\Omega)$ for any s, p with $sp < 1$. It suffices to check that

$$\int_\omega \int_{\Omega \setminus \omega} \frac{dxdy}{|x-y|^{N+\delta}} < \infty \quad \text{for all} \quad \delta < 1.$$

After localization we may take

$$\Omega = \{(x_1, x') \quad \text{with } x_1 \in \mathbb{R}, |x_1| < 1 \quad \text{and } x' \in \mathbb{R}^{N-1}, |x'| < 1\}$$

and

$$\omega = \{(x_1, x') \quad \text{with } 0 < x_1 < 1 \text{ and } |x'| < 1\}.$$

One has to verify that

$$I = \int_B \int_B \int_0^1 \int_0^1 \frac{dx'dy'dx_1dy_1}{(|x'-y'| + (x_1+y_1))^{N+\delta}} < \infty,$$

where B denotes the unit ball in $\mathbb{R}^{N-1}$. Changing the variable y' into $Y' = y' - x'$ we see that

$$I \leq |B| \int_{2B} \int_0^1 \int_0^1 \frac{dY'dx_1dy_1}{(|Y'| + (x_1+y_1))^{N+\delta}} \leq C \int_0^1 \frac{r^N dr}{r^{N+\delta}} < \infty.$$

Remark 1.11. There is a very simple proof of Theorem 1.6 when $s \geq 1$. It suffices to check that any function $u \in W^{1,1}(\Omega; \mathbb{Z})$ is a constant. Indeed, let $k \in \mathbb{Z}$ be such that

$|\{x; u(x) = k\}| > 0$. Let $f \in C^\infty(\mathbb{R})$ be such that $f(t) \equiv 1$ near $t = k$ and $f \equiv 0$ outside a small neighborhood of k. Then $\nabla(f \circ u) = f'(u)\nabla u = 0$ a.e. since $f'(\ell) = 0 \quad \forall \ell \in \mathbb{Z}$. Hence $f \circ u$ is constant, so $f \circ u \equiv 1$. It follows that $u = k$ a.e.

Alternatively, let $A_k = \{x \in \Omega;\ u(x) = k\}$ so that $\Omega = \bigcup_{k \in \mathbb{Z}} A_k$. By a well-known result of Stampacchia [1] (see also Gilbarg and Trudinger [1]), $\nabla u = 0$ a.e. on A_k and therefore $\nabla u = 0$ a.e. on Ω. Hence u is a constant.

Remark 1.12. There is also a simple proof of Theorem 1.6 when $0 < s < 1$ and $sp \geq N$. We may always assume that $sp = N$, otherwise, if $sp > N$, u is continuous by the Sobolev imbedding theorem and the conclusion is obvious. If $sp = N$ and $u \in W^{s,p}$ then $u \in$ VMO, i.e.,

$$⨍_Q |u - ⨍_Q u| \to 0 \qquad \text{as } |Q| \to 0$$

where Q is a cube or a ball in Ω (see e.g. Brezis and Nirenberg [1]). Set

$$u_\varepsilon(x) = ⨍_{B_\varepsilon(x)} u(y)dy.$$

If there is a closed set $F \subset \mathbb{R}$ such that $u(x) \in F$ for a.e. $x \in \Omega$ then $\operatorname{dist}(u_\varepsilon(x), F) \to 0$ as $\varepsilon \to 0$ uniformly in x (in a compact set $K \subset \Omega$). In particular, when $F = \mathbb{Z}$, since u_ε is continuous, there is, for small ε, some $k_\varepsilon \in \mathbb{Z}$ such that $\|u_\varepsilon - k_\varepsilon\|_{L^\infty(K)} \to 0$ as $\varepsilon \to 0$. Since $\int_K u_\varepsilon$ converges, it follows that k_ε converges as $\varepsilon \to 0$, hence is constant for small ε. Thus $u = k$ a.e. on K.

Remark 1.13. Bethuel and Demengel [1] (Lemma A.1 in their Appendix) have stated Theorem 1.6 when $sp > 1$ with a sketch of proof. Here is their main idea, for example, when $\Omega = (0,1)^2$. For a.e. y, the function $x \mapsto u(x,y)$ belongs to $W^{s,p}(0,1)$. Hence it is continuous and thus constant. Similarly the function $y \mapsto u(x,y)$ is constant. Therefore u is a constant a.e. on Ω. All these claims have to be carefully justified; this is done in the first proof below. Hardt, Kinderlehrer and Lin[2] have also stated Theorem 1.6 when $s = 1/2$ and $p = 2$ with a sketch of proof. Their method is similar, when $N = 1$, to the second proof we present below.

First proof of Theorem 1.6. (Following Brezis, Li, Mironescu and Nirenberg [1]).

It suffices to prove that u is locally constant and thus we may assume that $\Omega = (0,1)^N$. Recall that if $u \in W^{s,p}(\Omega; \mathbb{R})$ (any s and any p), then for each $1 \leq i \leq N$ and for a.e. $x' = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_N) \in (0,1)^{N-1}$, the function

$$a \mapsto v(a) = u(x_1, x_2, ..., x_{i-1}, a, x_{i+1}, ..., x_N)$$

belongs to $W^{s,p}((0,1); \mathbb{R})$ (see e.g. Bethuel and Demengel [1]). On the other hand recall that if $sp \geq 1$

$$W^{s,p}(0,1) \subset VMO(0,1)$$

(see Brezis and Nirenberg [1] or Lemma 1.9 below) Combining these two facts we deduce, as in Remark 1.12, that for a.e. $x' \in (0,1)^{N-1}$ the function v is constant a.e. To complete the argument we rely on a purely measure theoretical.

Lemma 1.7. *Let u be a real-valued measurable function on $(0,1)^N$ such that for each $1 \leq i \leq N$ and for a.e. $x' \in (0,1)^{N-1}$ the function*

$$a \mapsto v(a) = u(x_1, x_2, ...x_{i-1}, a, x_{i+1}, ..., x_n)$$

is constant a.e. on $(0,1)$. Then u is constant a.e. on $(0,1)^N$.

Proof. We may always assume that u is also bounded (and thus integrable) since otherwise we may replace u by Arctan u. By the triangle inequality, with

$$\lambda = (\lambda_1, ...\lambda_N) \text{ and } \mu = (\mu_1, ...\mu_N),$$

we have

$$\begin{aligned} |u(\lambda) - u(\mu)| \leq & |u(\lambda_1, \lambda_2, ...\lambda_{N-1}, \lambda_N) - u(\lambda_1, \lambda_2, ...\lambda_{N-1}, \mu_N)| \\ & + |u(\lambda_1, \lambda_2, ...\lambda_{N-1}, \mu_N) - u(\lambda_1, \lambda_2, ...\mu_{N-1}, \mu_N)| \\ & + ... + |u(\lambda_1, \mu_2, ...\mu_{N-1}, \mu_N) - u(\mu_1, \mu_2, ...\mu_{N-1}, \mu_N)|. \end{aligned}$$

It follows from the assumption that

$$\int_{(0,1)^N} \int_{(0,1)^N} |u(\lambda) - u(\mu)| d\lambda d\mu = 0.$$

Consequently, $u(\lambda) - u(\mu) = 0$ a.e. on $(0,1)^N \times (0,1)^N$ which implies that $u(\lambda)$ is constant a.e. on $(0,1)^N$

Second proof of Theorem 1.6. The main tool is the following

Lemma 1.8. *Let Ω be a connected open set in $\mathbb{R}^N$. Let A be a measurable subset of Ω. Assume*

$$\int_A \int_{\Omega \setminus A} \frac{dx\,dy}{|x-y|^{N+1}} < \infty \tag{2}$$

then either $|A| = 0$ or $|\Omega \setminus A| = 0$.

Assuming the lemma, we may now present the

Proof of Theorem 1.6. As we have just observed the conclusion is obvious when $s \geq 1$. Thus we may assume that $0 < s < 1$. Without loss of generality we may also assume that Ω is bounded. Let $k \in \mathbb{Z}$ be such that $A = \{x \,;\, u(x) = k\}$ has a positive measure. Then

$$\infty > \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \geq C \int_A dx \int_{\Omega \setminus A} \frac{dy}{|x-y|^{N+1}}$$

and by Lemma 1.8, $|\Omega\backslash A| = 0$ i.e., $u = k$ a.e. on Ω.

When A is an open set with smooth boundary the proof of Lemma 1.8 is straightforward: it may easily be reduced to the case where $\Omega = (0,1)^N$ and $A = (0,1)^{N-1} \times (0, 1/2)$ and the fact that

$$\int_0^{1/2} \int_{1/2}^1 \frac{dxdy}{|x-y|^2} = \infty.$$

To handle the case where A has a rough boundary it is natural to introduce a smoothing of the characteristic function of A:

Proof of Lemma 1.8. Set $B = \Omega\backslash A$ and let (ρ_ε) be a sequence of mollifiers, i.e., $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon)$ where ρ is a smooth function with support in the unit ball, $\rho \geq 0$ and $\int \rho = 1$.

Set $f_\varepsilon = \rho_\varepsilon \star \chi_A$ and $g_\varepsilon = \rho_\varepsilon \star \chi_B$, so that $f_\varepsilon \geq 0$, $g_\varepsilon \geq 0$, $f_\varepsilon \to \chi_A, g_\varepsilon \to \chi_B$ a.e. and in $L^1_{\text{loc}}(\mathbb{R}^N)$.

Fix any open set $\omega \subset \Omega$ with compact closure in Ω. Note that

$$f_\varepsilon + g_\varepsilon = 1 \quad \text{in } \omega \text{ for } \varepsilon < \varepsilon_0 = \text{dist}(\omega, \partial\Omega). \tag{3}$$

For $\varepsilon < \varepsilon_0$, set

$$Z_\varepsilon = \{x \in \omega; \frac{1}{3} < f_\varepsilon(x) < \frac{2}{3}\} = \{x \in \omega; \frac{1}{3} < g_\varepsilon(x) < \frac{2}{3}\}.$$

Claim: Under assumption (2) we have

$$|Z_\varepsilon| = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0. \tag{4}$$

Proof of Claim. We have

$$\begin{aligned}\int_{Z_\varepsilon \cap A} g_\varepsilon(x)dx &= \int_{Z_\varepsilon \cap A} dx \int_{|y-x|<\varepsilon} \varepsilon^{-N}\rho(\frac{x-y}{\varepsilon})\chi_B(y)dy \\ &\leq \varepsilon\|\rho\|_\infty \int_{Z_\varepsilon \cap A} dx \int_B \frac{dy}{|x-y|^{N+1}}\end{aligned}$$

and therefore

$$|Z_\varepsilon \cap A| \leq 3\varepsilon\|\rho\|_\infty \int_{Z_\varepsilon \cap A} dx \int_B \frac{dy}{|x-y|^{N+1}}. \tag{5}$$

In particular,

$$|Z_\varepsilon \cap A| \leq \varepsilon C \int_A \int_B \frac{dxdy}{|x-y|^{N+1}} = O(\varepsilon),$$

by assumption (2). Using (5) and (2) once more we see that

$$|Z_\varepsilon \cap A| = o(\varepsilon).$$

Similarly,

$$|Z_\varepsilon \cap B| = o(\varepsilon)$$

and consequently

$$|Z_\varepsilon| = |Z_\varepsilon \cap A| + |Z_\varepsilon \cap B| = o(\varepsilon).$$

We may now complete the proof of Lemma 1.8. Let $h : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function such that $h(t) = 0$ for $t \leq 1/3$ and $h(t) = 1$ for $t \geq 2/3$. We have

$$\int_\omega |\nabla(h \circ f_\varepsilon)| = \int_\omega |h'(f_\varepsilon)|\, |\nabla f_\varepsilon| \leq C \int_{Z_\varepsilon} |\nabla f_\varepsilon|. \tag{6}$$

But $\nabla f_\varepsilon = (\nabla \rho_\varepsilon) \star \chi_A$ and therefore

$$\|\nabla f_\varepsilon\|_\infty \leq \|\nabla \rho_\varepsilon\|_1 \|\chi_A\|_\infty \leq C/\varepsilon. \tag{7}$$

Combining (4), (6) and (7) we are led to

$$\int_\omega |\nabla(h \circ f_\varepsilon)| \to 0 \quad \text{as } \varepsilon \to 0. \tag{8}$$

On the other hand $h \circ f_\varepsilon \to h \circ \chi_A = \chi_A$ in $L^1(\omega)$. Together with (8) this yields

$$\nabla(\chi_A) = 0 \quad \text{in the sense of distributions in } \omega.$$

Since ω is an arbitrary open set in Ω with compact closure in Ω we conclude that

$$\nabla(\chi_A) = 0 \quad \text{in the sense of distributions in } \Omega.$$

Therefore χ_A is a constant in Ω (recall that Ω is connected). That constant is either 0 (and then $|A| = 0$) or 1 (and then $|\Omega \backslash A| = 0$).

There is a version of Lemma 1.8 which involves functions instead of sets and which may be easily derived from Lemma 1.8:

Lemma 1.8′. *Let Ω be a connected open set in $\mathbb{R}^N$. Let f, g be nonnegative measurable functions on Ω such that*

$$f + g \geq 1 \quad \textit{a.e. on } \Omega$$

and

$$\int_\Omega \int_\Omega \frac{f(x)g(y)}{|x-y|^{N+1}} dxdy < \infty.$$

Then either $f = 0$ a.e. on Ω or $g = 0$ a.e. on Ω.

Proof. Set $\tilde{f} = \min\{f, 1\}$ and $\tilde{g} = \min\{g, 1\}$, so that we still have $\tilde{f} + \tilde{g} \geq 1$ a.e., in Ω and

$$\int_\Omega \int_\Omega \frac{\tilde{f}(x)\tilde{g}(y)}{|x-y|^{N+1}} dxdy < \infty. \tag{9}$$

As above, let $\omega \subset \Omega$ be an open set with compact closure in Ω. For $\varepsilon < \varepsilon_0 = \operatorname{dist}(\omega, \partial\Omega)$ and for $x \in \omega$, set

$$\tilde{g}_\varepsilon(x) = ⨍_{B_\varepsilon(x)} \tilde{g}(y)dy,$$

so that $\tilde{g}_\varepsilon \to \tilde{g}$ a.e. in ω as $\varepsilon \to 0$.

From (9) we have

$$\int_\omega \tilde{f}(x)\tilde{g}_\varepsilon(x)dx \leq C\varepsilon \iint\limits_{\substack{\omega\times\omega \\ |x-y|<\varepsilon}} \frac{\tilde{f}(x)\tilde{g}(y)}{|x-y|^{N+1}} dxdy \leq C\varepsilon. \tag{10}$$

Passing to the limit in (10), as $\varepsilon \to 0$, we obtain

$$\int_\omega \tilde{f}(x)\tilde{g}(x)dx = 0$$

and thus $\tilde{f}\tilde{g} = 0$ a.e. on ω. Since this holds for every ω we deduce that $\tilde{f}\tilde{g} = 0$ a.e. on Ω. It follows easily that $\tilde{f} = \chi_A$ and $\tilde{g} = \chi_{\Omega\setminus A}$ for some measurable set $A \subset \Omega$, so that we are reduced to the setting of Lemma 1.8.

Remark 1.14. Assume $u \in W^{s,p}(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega, \mathbb{R}^d)$ with $sp \geq 1$, where Ω is connected. Then $essR(u)$ is connected. [Here, the essential range, $essR(u)$, is the smallest closed set F in $\mathbb{R}^d$ such that $u(x) \in F$ a.e. (see e.g. Brezis and Nirenberg [1]).] This property follows easily from Lemma 1.8. The restriction $u \in L^\infty(\Omega; \mathbb{R}^d)$ is important when $s = 1/p$ and $d \geq 2$. For example, on $\Omega = (0, 1)$, the function

$$t \mapsto \left(t, \left|\log|t-1/2|\right|^\alpha\right), \quad 0 < \alpha < 1/2$$

belongs to $H^{1/2}(\Omega; \mathbb{R}^2)$ and its range is disconnected.

However, if $sp > 1$ and $u \in W^{s,p}(\Omega; \mathbb{R}^d)$, then $essR(u)$ is connected. This is clear when $N = 1$ and the general case is done by induction.

Remark 1.15. Lemma 1.6 is also a direct consequence of a characterization of BV functions due to Bourgain, Brezis and Mironescu [3]: for every $f \in L^1(\Omega)$,

$$||f||_{BV} \quad \leq C \liminf_{\varepsilon \to 0} \; \varepsilon \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{N+1-\varepsilon}} dxdy$$

where C depends only on N (with the convention that, if f is not BV then $||f||_{BV} = \infty$). Applying this result to $f = \chi_A$ we see that, under assumption (2), $\chi_A \in BV$ and $\nabla(\chi_A) = 0$. Thus χ_A is a constant which is either 0 (and then $|A| = 0$) or 1 (and then $|\Omega \backslash A| = 0$).

1.3. Degree, traces and lifting

We assume here that $\Omega \subset \mathbb{R}^2$ is a smooth bounded connected domain. Consider first the case where Ω is simply connected.

We start with some preliminaries about the degree. Assume $g \in C^0(\partial\Omega; S^1)$. Consider a direct parametrization $p(t) : [0,1] \to \partial\Omega$. Since $g \circ p$ is continuous on $[0,1]$ there is a continuous function φ on $[0,1]$ such that $g(p(t)) = e^{i\varphi(t)}$.

Note that $g(p(0)) = g(p(1))$ and thus $\varphi(1) - \varphi(0) = 2\pi k$ with $k \in \mathbb{Z}$. The number k is by definition the degree of g on $\partial\Omega$. We denote it by $\deg(g, \partial\Omega)$ or simply $\deg g$ if there is no ambiguity. It is independent of the choice of p. Moreover, if $g_n \to g$ uniformly then $\deg g_n = \deg g$ for n sufficiently large. As a consequence the degree is constant along a continuous homotopy, i.e., if $H \in C^0(\partial\Omega \times [0,1]; S^1)$ then $\deg(H(\cdot, 0)) = \deg(H(\cdot, 1))$.

Note that if $g \in C^0(\partial\Omega; S^1)$ is such that $\deg g = 0$, we may write $g = e^{i\varphi_0}$ for some $\varphi_0 \in C^0(\partial\Omega; \mathbb{R})$. We emphasize that φ_0 is a singlevalued function. Conversely, if $g = e^{i\varphi_0}$ for some $\varphi_0 \in C^0(\partial\Omega; \mathbb{R})$, then $\deg g = 0$.

It is clear that if $g, h \in C^0(\partial\Omega; S^1)$ then

$$\deg(gh) = \deg g + \deg h \tag{1}$$

and

$$\deg(g/h) = \deg g - \deg h.$$

Here gh means the usual product of the two complex functions (it is not the scalar product of vectors in $\mathbb{R}^2$).

When $g \in C^1(\partial\Omega; S^1)$ there is a convenient formula which we shall often use:

$$\deg g = \frac{1}{2\pi} \int_{\partial\Omega} (g \wedge g_\tau) ds, \tag{2}$$

where $a \wedge b = a_1 b_2 - a_2 b_1$ and g_τ denotes the derivative of g with respect to τ, the positively oriented unit tangent vector at $\partial\Omega$.

Proof of (2). We have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial\Omega} g \wedge g_\tau ds &= \frac{1}{2\pi}\int_0^1 g(p(t)) \wedge \left(\frac{1}{|p'(t)|}\frac{d}{dt}(g(p(t))\right)|p'(t)|dt \\ &= \frac{1}{2\pi}\int_0^1 e^{i\varphi(t)} \wedge \frac{d}{dt}(e^{i\varphi(t)})dt = \frac{1}{2\pi}\int_0^1 \varphi'(t)dt \\ &= \frac{1}{2\pi}[\varphi(1) - \varphi(0)] = \deg g.\end{aligned}$$

Remark 1.16. For the reader familiar with Complex Analysis, we point out that (2) is the index formula for a function $g \in C^1(\partial\Omega; \mathbb{C}\backslash\{0\})$, namely

$$\deg g = \frac{1}{2i\pi}\int_{\partial\Omega} \frac{g_\tau}{g}.$$

In the special case where $|g| = 1$, this becomes

$$\deg g = \frac{1}{2i\pi}\int_{\partial\Omega} \overline{g} g_\tau.$$

But $g \cdot g_\tau = 0$ and thus $\overline{g} g_\tau = i(g \wedge g_\tau)$ (note that $a \cdot b = 0$ implies $\overline{a} b = i(a \wedge b)$ for $a, b \in \mathbb{R}^2$).

There is a "cousin" of formula (2), namely if $u \in C^2(\overline{\Omega}; \mathbb{R}^2)$ is such that $u = g$ on $\partial\Omega$, then

$$\deg g = \frac{1}{\pi}\int_\Omega u_x \wedge u_y dxdy. \tag{3}$$

Proof of (3). We have

$$\begin{aligned}\int_\Omega u_x \wedge u_y dxdy &= \int_\Omega \frac{1}{2}\left[(u \wedge u_y)_x + (u_x \wedge u)_y\right] \\ &= \frac{1}{2}\int_{\partial\Omega} (u \wedge u_y)n_x + (u_x \wedge u)n_y = \frac{1}{2}\int_{\partial\Omega} u \wedge (-u_x n_y + u_y n_x) \\ &= \frac{1}{2}\int_{\partial\Omega} u \wedge (u_x \tau_x + u_y \tau_y) = \frac{1}{2}\int_{\partial\Omega} u \wedge u_\tau.\end{aligned}$$

Here $n = (n_x, n_y)$ denotes the outward normal vector and we have used the fact that $n_x = \tau_y$ and $n_y = -\tau_x$.

Notation. Given a boundary condition g, set

$$H^1_g(\Omega;\mathbb{R}^2)=\{u\in H^1(\Omega;\mathbb{R}^2);u=g\text{ on }\partial\Omega\}$$

and

$$H^1_g(\Omega;S^1)=\{u\in H^1_g(\Omega;\mathbb{R}^2);|u|=1\text{ a.e.}\}.$$

We have already seen (Theorem 1.1) that

$$H^1(\Omega;S^1)=\{u=e^{i\varphi}\quad\text{with }\varphi\in H^1(\Omega;\mathbb{R})\}.$$

We now give a similar result for $H^1_g(\Omega;S^1)$:

Theorem 1.7. *Let $g\in H^{1/2}(\partial\Omega;S^1)\cap C^0(\partial\Omega;S^1)$ be such that $\deg g=0$. Then*

a) $g=e^{i\varphi_0}$ *for some* $\varphi_0\in H^{1/2}(\partial\Omega;\mathbb{R})\cap C^0(\partial\Omega;\mathbb{R})$*;*

b) $H^1_g(\Omega;S^1)=\{u=e^{i\varphi};\varphi\in H^1_{\varphi_0}(\Omega;\mathbb{R})\}$.

Proof.

a) Since $\deg g=0$ there is some $\varphi_0\in C^0(\partial\Omega;\mathbb{R})$ such that $g=e^{i\varphi_0}$.

This φ_0 belongs to $H^{1/2}$ since locally $\varphi_0=-i\log g$ and thus it is of the form $\varphi_0=\Phi(g)$ where Φ is smooth.

b) The inclusion $\supset$ is clear and we turn to $\subset$.

Let $u\in H^1_g(\Omega;S^1)$. By Theorem 1.1 there is a $\varphi\in H^1(\Omega;\mathbb{R})$ such that $u=e^{i\varphi}$. Then $e^{i\varphi_{|\partial\Omega}}=u_{|\partial\Omega}=g=e^{i\varphi_0}$, so that $\dfrac{1}{2\pi}(\varphi_{|\partial\Omega}-\varphi_0)\in H^{1/2}(\partial\Omega;\mathbb{Z})$. It follows from Theorem 1.8 that, for some $k\in\mathbb{Z},\varphi_{|\partial\Omega}=\varphi_0+2k\pi$ a.e. Hence, $u=e^{i(\varphi-2k\pi)}$, where $\varphi-2k\pi\in H^1_{\varphi_0}(\Omega;\mathbb{R})$.

The next result is an extension of formula (3) to H^1 maps:

Theorem 1.8. *Let $u\in H^1(\Omega;\mathbb{R}^2)$ with $u_{|\partial\Omega}=g\in C^0(\partial\Omega;S^1)$. Then*

$$\deg g=\frac{1}{\pi}\int_\Omega u_x\wedge u_y. \tag{4}$$

Proof. We first claim that if $u,v\in H^1(\Omega;\mathbb{R}^2)$ are such that $u_{|\partial\Omega}=v_{|\partial\Omega}$, then

$$\int_\Omega u_x\wedge u_y=\int_\Omega v_x\wedge v_y. \tag{5}$$

Indeed let $w = v - u$. Then $w \in H_0^1(\Omega;\mathbb{R}^2)$ and

$$\int_\Omega v_x \wedge v_y = \int_\Omega u_x \wedge u_y + \int_\Omega w_x \wedge u_y + \int_\Omega u_x \wedge w_y + \int_\Omega w_x \wedge w_y.$$

It suffices to prove that

$$\int w_x \wedge f_y = \int w_y \wedge f_x, \quad w \in H_0^1(\Omega;\mathbb{R}^2),\ f \in H^1(\Omega;\mathbb{R}^2). \tag{6}$$

Now, if $w \in C_0^\infty$ then

$$\int w_x \wedge f_y = \int (w_x \wedge f)_y - \int w_{xy} \wedge f = -\int w_{xy} \wedge f$$

and

$$\int w_y \wedge f_x = \int (w_y \wedge f)_x - \int w_{xy} \wedge f = -\int w_{xy} \wedge f,$$

so that (6) holds for $w \in C_0^\infty$. For general w's, (6) follows via approximation.

We now turn to the proof of (4). Let (h_n) be a sequence in $C^3(\partial\Omega)$ such that $h_n \to g$ uniformly and in $H^{1/2}(\partial\Omega)$. Then $g_n = \dfrac{h_n}{|h_n|}$ is well-defined for large n, and, also for large n, we have $\deg g_n = \deg g$. Moreover, $g_n \to g$ in $H^{1/2}(\partial\Omega)$ (see Lemma A.1.5). Let u_n be the harmonic extension of g_n and u the harmonic extension of g. By (3), we have

$$\deg g = \deg g_n = \frac{1}{\pi}\int (u_n)_x \wedge (u_n)_y \to \frac{1}{\pi}\int u_x \wedge u_y.$$

Corollary 1.2. *Let $g \in C^0(\partial\Omega;S^1) \cap H^{1/2}(\partial\Omega;S^1)$.*

Then

$$H_g^1(\Omega;S^1) \neq \phi \quad \textit{if and only if } \deg g = 0.$$

Indeed, recall that for $u \in H^1(\Omega;S^1)$ we have $u_x \wedge u_y = 0$ a.e. (see (5) in the proof of Theorem 1.1).

Consider now the case of a multi-connected domain Ω, which we write as

$$\Omega = G \setminus \bigcup_{i\in I} \overline{\omega}_i$$

where I is finite and G, ω_i are open, smooth and simply connected in $\mathbb{R}^2$ and $\overline{\omega}_i \subset G$, $\overline{\omega}_i \cap \overline{\omega}_j = \phi$ for $i \neq j$.

It will be convenient to introduce "reference" points and "reference" maps. For each $j = 1, 2, \ldots$, fix some $a_j \in \omega_j$ and set

$$w_j(z) = \frac{z - a_j}{|z - a_j|} \in C^\infty(\mathbb{R}^2 \backslash \{a_j\}; S^1).$$

Given the integers $d_j \in \mathbb{Z}$, let

$$w = \prod_j w_j^{d_j}.$$

We claim that

(7) $\deg(w_j, \partial\omega_j) = 1,$

(8) $\deg(w_j, \partial\omega_k) = 0 \quad \text{if } k \neq j,$

(9) $\deg(w_j, \partial G) = 1,$

and

(10) $\deg(w, \partial G) = \sum_j d_j.$

Proof of (7) and (9). Fix some $\rho > 0$ so small that $\overline{B}_\rho(a_j) \subset \omega_j$. Since $|w_j| = 1$ we have

$$(w_j)_x \wedge (w_j)_y = 0 \quad \text{on } \mathbb{R}^2 \backslash \{a_j\}.$$

In particular

$$I = \frac{1}{\pi} \int_{\omega_j \backslash B_\rho(a_j)} (w_j)_x \wedge (w_j)_y = 0$$

and

$$J = \frac{1}{\pi} \int_{G \backslash B_\rho(a_j)} (w_j)_x \wedge (w_j)_y = 0.$$

Integrating by parts as in the proof of (3) we obtain

$$\begin{aligned} I &= \deg(w_j, \partial\omega_j) - \deg(w_j, \partial B_\rho(a_j)) \\ &= \deg(w_j, \partial\omega_j) - 1 = 0, \end{aligned}$$

i.e., (7) holds and similarly for (9).

Proof of (8). By (3) we have

$$\deg(w_j, \partial\omega_k) = \frac{1}{\pi} \int_{\omega_k} (w_j)_x \wedge (w_j)_y = 0.$$

Proof of (10). We have, using (1),

$$\deg(w, \partial G) = \sum_j \deg(w_j^{d_j}, \partial G) = \sum_j d_j.$$

The next result is an analogue of Theorem 1.1 for multi-connected domains

Theorem 1.9. *Let $u \in H^1(\Omega; S^1)$. Assume that*

$$u_{|\partial\Omega} \in H^{1/2}(\partial\Omega) \cap C^0(\partial\Omega).$$

Let

$$d = \deg(u, \partial G) \quad \text{and } d_j = \deg(u, \partial\omega_j).$$

Then

a) $d = \sum_j d_j$;

b) *there exists some $\psi \in H^1(\Omega; \mathbb{R})$ such that*

(11) $$u = we^{i\psi}.$$

Proof of a). For each j, let v_j be the harmonic extension in ω_j of $u_{|\partial\omega_j}$. Thus $v_j \in H^1(\omega_j; \mathbb{R}^2) \cap C^0(\overline{\omega}_j; \mathbb{R}^2)$ and

(12) $$d_j = \frac{1}{\pi} \int_{\omega_j} (v_j)_x \wedge (v_j)_y$$

by Theorem 1.8.

On the other hand, let

$$\widetilde{u} = \begin{cases} u & \text{in } \Omega \\ v_j & \text{in } \omega_j \end{cases}$$

so that $\widetilde{u} \in H^1(G; \mathbb{R}^2)$ and $\widetilde{u}_{|\partial G} = u_{|\partial G}$. Then

$$\deg(\widetilde{u}, \partial G) = \frac{1}{\pi} \int_G \widetilde{u}_x \wedge \widetilde{u}_y = \sum_j \frac{1}{\pi} \int_{\omega_j} (v_j)_x \wedge (v_j)_y$$

since $u_x \wedge u_y = 0$ on Ω.

Therefore

$$d = \deg(\widetilde{u}, \partial G) = \sum_j d_j.$$

Proof of b). Set

$$v = uw^{-1}.$$

Clearly, $v \in H^1(\Omega; S^1)$ and $v_{|\partial\Omega} \in H^{1/2}(\partial\Omega) \cap C^0(\partial\Omega)$. Moreover, one has

$$\deg(v, \partial\omega_j) = 0 \qquad \forall j$$

by (1), (7) and (8).

According to Theorem 1.7, for each j, there is some $v_j \in H^1(\omega_j; S^1)$ such that $v_{j|\partial\omega_j} = v_{|\partial\omega_j}$. Now let

$$\widetilde{u} = \begin{cases} v & \text{in } \Omega \\ v_j & \text{in } \omega_j. \end{cases}$$

Clearly, $\widetilde{u} \in H^1(G; S^1)$, so that $\widetilde{u} = e^{i\varphi}$ for some $\varphi \in H^1(G; \mathbb{R})$. It follows that $u = we^{i\psi}$, where $\psi = \varphi_{|\Omega}$.

We next present a variant of Theorem 1.7 *b)* for multi-connected domains.

Let $g_0 \in H^{1/2}(\partial G; S^1) \cap C^0(\partial G; S^1)$ and $g_j \in H^{1/2}(\partial\omega_j; S^1) \cap C^0(\partial\omega_j; S^1)$. Set

$$d_0 = \deg(g_0, \partial G)$$

$$d_j = \deg(g_j, \partial\omega_j), \quad j = 1, 2, \dots .$$

Assume that

$$\sum_{j \geq 1} d_j = d_0. \tag{13}$$

For each $k = 0, 1, 2, \dots$ write,

$$g_k = we^{i\psi_k} \tag{14}$$

with $\psi_k \in H^{1/2} \cap C^0$. (Here we use Theorem 1.9 *a)* and, for $k = 0$, assumption (13).)

Consider the map $g : \partial\Omega \to S^1$ defined by

$$g = \begin{cases} g_0 & \text{on } \partial G \\ g_j & \text{on } \partial\omega_j \end{cases}$$

and recall the notation

$$H^1_g(\Omega; S^1) = \{u \in H^1(\Omega; S^1); u = g \quad \text{on } \partial\Omega\}.$$

Theorem 1.10. *Under assumption (13), $H^1_g(\Omega;S^1)$ is not empty and more precisely*

$$H^1_g(\Omega;S^1) = \left\{ u = we^{i\psi} \middle| \begin{array}{l} \psi \in H^1(\Omega;\mathbb{R}), \psi = \psi_0 \, on\ \partial G \ and \\ \\ \psi - \psi_j = 2\pi k_j \, on\ \partial\omega_j \ for\ some\ integers\ k_j \end{array} \right\}.$$

Proof. The inclusion $\supset$ is clear. So we have only to prove $\subset$. Let $u \in H^1_g(\Omega;S^1)$. Then

$$\deg(u,\partial G) = \deg(g_0,\partial G) = d$$

and

$$\deg(u,\partial\omega_j) = \deg(g_j,\partial\omega_j) = d_j.$$

Thus, by Theorem 1.9 *b*) there is some $\psi \in H^1(\Omega;\mathbb{R})$ such that

$$u = we^{i\psi}.$$

Taking traces yields

$$g = w_{|\partial\Omega} e^{i\psi_{|\partial\Omega}}$$

i.e.,

$$g_0 = w_{|\partial G} e^{i\psi_{|\partial G}} = w_{|\partial G} e^{i\psi_0} \qquad \text{by (14)}$$

and

$$g_j = w_{|\partial\omega_j} e^{i\psi_{|\partial\omega_j}} = w_{|\partial\omega_j} e^{i\psi_j} \qquad \text{by (14).}$$

Hence $\frac{1}{2\pi}(\psi_{|\partial G} - \psi_0) \in H^{1/2}(\partial G;\mathbb{Z})$; thus it is a constant by Theorem 1.6. Similarly for $\psi_{|\partial\omega_j} - \psi_j$. Finally, we may assume that one of these constants is zero.

1.4. Degree for $H^{1/2}$ maps

This is a continuation of Section 1.3. We will show that all results from Section 1.3 are still valid without continuity assumptions on the boundary values. There, we made a continuity assumption in order to be able to talk about (standard) degree. In 1985 L. Boutet de Monvel and O. Gabber observed that maps in the Sobolev class $H^{1/2}(\partial\Omega, S^1)$ have a well-defined degree. Their motivation also came from the study of the Ginzburg-Landau theory and their argument is presented as an appendix in Boutet de Monvel-Berthier, Georgescu and Purice [1]. Their main observation is that, in the formula (see Remark 1.16)

$$\deg g = \frac{1}{2i\pi} \int_{\partial\Omega} \overline{g} g_\tau \tag{1}$$

giving the degree for smooth maps, the integral on the right-hand side makes sense if g is merely in $H^{1/2}$. Indeed g_τ belongs to $H^{-1/2}$ and the integral may be interpreted as a scalar product in the duality between $H^{1/2}$ and $H^{-1/2}$. One may wonder whether the resulting quantity is an integer (or even just a real number). This is indeed true, but far from obvious. A key ingredient is the fact that $C^\infty(\overline{\Omega}; M)$ is dense in the fractional Sobolev space $W^{s,p}(\Omega; M)$ where $\Omega \subset \mathbb{R}^N$, M is a smooth manifold without boundary and $sp \geq N$; in the special case where $s = 1$, $p = 2$ and $N = 2$ this is due to Schoen and Uhlenbeck [2].

We will discuss here various other definitions of degree for $H^{1/2}$ maps and prove that they are all equivalent. Moreover, this degree enjoys all the standard properties of degree. There is still a wider class of maps, the VMO (= vanishing mean oscillation) maps, for which one may define a degree. We will briefly describe it and refer to Brezis and Nirenberg [1] and Brezis [4] for further details.

For simplicity, assume that Ω is the unit disc. Let

$$g \in H^{1/2}(S^1; S^1).$$

Consider the function $f : \mathbb{R} \to S^1$ defined by

$$f(t) = g(e^{it}).$$

Clearly, $f \in H^{1/2}_{\mathrm{loc}}(\mathbb{R})$ and $f(t+2\pi) = f(t)$ a.e. By Theorem 1.2 there is some φ in $H^{1/2}_{\mathrm{loc}}(\mathbb{R})$ such that

$$f(t) = e^{i\varphi(t)}.$$

Therefore

$$\frac{1}{2\pi}\left(\varphi(t+2\pi) - \varphi(t)\right) \in \mathbb{Z}, \text{ a.e.}$$

Using Theorem 1.6, we obtain a constant $k \in \mathbb{Z}$ such that

$$\varphi(t+2\pi) - \varphi(t) = 2\pi k \quad \text{a.e.}$$

We claim that the integer k is independent of the choice of φ. Indeed, consider another $\widetilde{\varphi} \in H^{1/2}_{\mathrm{loc}}(\mathbb{R})$ and a corresponding integer $\widetilde{k}$. We have

$$\frac{1}{2\pi}\left(\varphi(t) - \widetilde{\varphi}(t)\right) \in \mathbb{Z} \qquad \text{a.e.}$$

Hence there is some integer $\ell \in \mathbb{Z}$ such that

$$\varphi(t) - \widetilde{\varphi}(t) = 2\pi\ell \qquad \text{a.e.}$$

Therefore $k = \widetilde{k}$.

Definition 1. Set

$$\deg_1 g = k.$$

It is straightforward that $\deg_1 g$ coincides with the standard degree when g belongs to $H^{1/2} \cap C^0$.

In view of formula (4) in Section 1.3 there is another natural definition. Fix any

$$u \in H^1_g(\Omega; \mathbb{R}^2).$$

Definition 2. Set

$$\deg_2 g = \frac{1}{\pi} \int_\Omega u_x \wedge u_y.$$

It follows from (5) in Section 1.3 that $\deg_2 g$ is independent of the choice of u. In contrast with $\deg_1$, it is not clear that $\deg_2$ is an integer; this a consequence of Theorem 1.11 below.

Next, we return to (1) to present another definition of degree for $H^{1/2}$ maps. We will use the idea already mentioned but we will translate it in the language of Fourier series instead of the $H^{1/2}, H^{-1/2}$ duality. Consider the Fourier series associated with g,

$$g(e^{i\theta}) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}.$$

Then, for smooth g,

$$\deg g = \frac{1}{2i\pi} \int_{S^1} \overline{g} g_\theta = \sum_{n=-\infty}^{+\infty} n|a_n|^2.$$

Definition 3. Set

$$\deg_3 g = \sum_{n=-\infty}^{+\infty} n|a_n|^2.$$

Recall that $g \in H^{1/2}$ if and only if

$$\sum_{n=-\infty}^{+\infty} |n|\, |a_n|^2 < \infty.$$

Moreover,

$$\|g\|^2_{H^{1/2}} \sim |a_0|^2 + \sum_{n=-\infty}^{+\infty} |n|\ |a_n|^2.$$

It then follows that $\deg_3 g$ is well-defined for any $g \in H^{1/2}$. But, again, it is not clear that this number is an integer.

Finally, a definition of degree which requires more work (but extends to a larger class of functions, and also to higher dimensions). For $\varepsilon > 0$ set

$$\overline{g}_\varepsilon(z) = ⨍_{A_\varepsilon(z)} g, \quad z \in S^1$$

where $A_\varepsilon(z) = S^1 \cap B_\varepsilon(z)$.

The main observation is that if $g \in H^{1/2}(S^1; S^1)$ then

(2) $$|\overline{g}_\varepsilon(z)| \to 1 \quad \textbf{uniformly} \text{ on } S^1 \text{ as } \varepsilon \to 0.$$

This is a consequence of the following two lemmas

Lemma 1.9. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be smooth; let $s > 0$, $1 < p < \infty$ with $sp \geq N$. Then, any function $f \in W^{s,p}(\Omega)$ belongs to VMO, i.e.,*

(3) $$⨍_{B_\varepsilon(x)} \big|f(y) - \overline{f}_\varepsilon(x)\big| dy \longrightarrow 0 \quad \textit{uniformly in } x, \textit{ as } \varepsilon \to 0,$$

where

$$\overline{f}_\varepsilon(x) = ⨍_{B_\varepsilon(x)} f.$$

Proof. Consider two cases.

Case 1: s is an integer.

We may always assume that $s = 1$ since, when $sp \geq N$ and $s \geq 1$,

$$W^{s,p} \subset W^{1,sp}$$

by the Sobolev imbedding theorem. From the Poincaré inequality we have

$$⨍_{B_\varepsilon(x)} |f(y) - \overline{f}_\varepsilon(x)| dy \leq C|B_\varepsilon(x)|^{-1+1/N} \int_{B_\varepsilon(x)} |\nabla f| \leq C \left(\int_{B_\varepsilon(x)} |\nabla f|^N \right)^{1/N}$$

and the right-hand side tends to zero uniformly in x as $\varepsilon \to 0$.

Case 2: s is not an integer.

We may always assume that $0 < s < 1$ (if $s > 1$, note that $W^{s,p} \subset W^{1,sp}$).

We have

$$\fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} |f(y)-f(z)|dydz \leq \fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} \frac{|f(y)-f(z)|}{|y-z|^{2N/p}}(2\varepsilon)^{2N/p}$$
$$\leq C\Big[\int_{B_\varepsilon(x)}\int_{B_\varepsilon(x)} \frac{|f(y)-f(z)|^p}{|y-z|^{2N}}\Big]^{1/p}.$$

Finally,

$$|f(y) - \fint_{B_\varepsilon(x)} f(z)dz| \leq \fint_{B_\varepsilon(x)} |f(y)-f(z)|dz$$

and thus

$$\fint_{B_\varepsilon(x)} |f(y) - \overline{f}_\varepsilon(x)|dy \leq \fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} |f(y)-f(z)|dydz,$$

and the right-hand side tends to zero, uniformly in x, as $\varepsilon \to 0$, by the above computation, since $f \in W^{s,p}$.

Lemma 1.10. *Assume $f \in VMO$ is an S^1-valued function. Then*

$$|\overline{f}_\varepsilon| \to 1 \qquad \textit{uniformly as } \varepsilon \to 0.$$

Proof. We have clearly $|\overline{f}_\varepsilon| \leq 1$ and on the other hand, we have

$$1 - |\overline{f}_\varepsilon(x)| = \text{dist}\left(\overline{f}_\varepsilon(x), S^1\right) \leq \fint_{B_\varepsilon(x)} |f(y) - \overline{f}_\varepsilon(x)|dy$$

and the conclusion follows from the definition of VMO.

Returning to the definition of degree, consider $g \in H^{1/2}(S^1; S^1)$ and set

$$g_\varepsilon(z) = \frac{\overline{g}_\varepsilon(z)}{|\overline{g}_\varepsilon(z)|}. \tag{4}$$

By (2) this is well defined for ε sufficiently small, say $\varepsilon < \varepsilon_0$, depending on g. Moreover $g_\varepsilon \in C^0(S^1; S^1)$.

Definition 4. Set

$$\deg_4 g = \deg g_\varepsilon \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}$$

Note that this degree is independent of ε, since g_ε and $g_{\varepsilon'}$ can be connected via the continuous homotopy $g_{\varepsilon t + \varepsilon'(1-t)}$.

Remark 1.17. Some of the above definitions make sense for a class more general than $H^{1/2}$. $\mathrm{Deg}_1 g$ could be defined for $g \in W^{s,p}$ provided $sp \geq 1$. (In fact $\deg_1$ also makes sense for $g \in \mathrm{VMO}$ but this is much more delicate. It uses the existence of lifting in VMO, and also the connectedness of the essential range for VMO maps; see Theorem 3 and Section I.5 in Brezis and Nirenberg [1].)

Theorem 1.11. *For any $g \in H^{1/2}(S^1; S^1)$, all the above definitions of degree coincide and we set*

$$\deg g = \deg_j g \qquad j = 1, 2, 3, 4.$$

This is the same as the usual degree when $g \in H^{1/2} \cap C^0$. Moreover the map $g \mapsto \deg g$ is continuous from $H^{1/2}$ into $\mathbb{Z}$ and property (1) in Section 1.3 still holds.

A basic ingredient in the proof is

Lemma 1.11. *For every $g \in H^{1/2}(S^1; S^1)$, g_ε (defined by (4)) tends to g in $H^{1/2}$ as $\varepsilon \to 0$. Moreover,*

$$C^\infty(S^1; S^1) \quad \text{is dense in } H^{1/2}(S^1; S^1). \tag{5}$$

Proof of Lemma 1.8. It is standard that $\overline{g}_\varepsilon \to g$ in $H^{1/2}$. Next, note that $g_\varepsilon = \Phi(\overline{g}_\varepsilon)$ where $\Phi(\xi) = \xi/|\xi|$ and apply Lemma A.1.5 in Appendix A.1.1.

To prove (5) fix $g \in H^{1/2}(S^1; S^1)$ and $\delta > 0$. For $\varepsilon > 0$ sufficiently small $\|g_\varepsilon - g\|_{H^{1/2}} < \delta$. Since $g_\varepsilon \in H^{1/2} \cap C$, we may find, by standard smoothing arguments, some $h \in C^\infty(S^1; S^1)$ such that $\|h - g_\varepsilon\|_{H^{1/2}} < \delta$.

Proof of Theorem 1.11. We split the proof into 5 steps.

Step 1: The maps $g \mapsto \deg_2 g$ and $g \mapsto \deg_3 g$ are continuous from $H^{1/2}$ into $\mathbb{Z}$.

Given $g, h \in H^{1/2}(S^1; S^1)$, consider their harmonic extensions u, v in $\Omega = B_1$. We have

$$\left| \int_\Omega u_x \wedge u_y - \int_\Omega v_x \wedge v_y \right| \leq 2\|u - v\|_{H^1} \left(\|u\|_{H^1} + \|v\|_{H^1}\right) \leq C\|g - h\|_{H^{1/2}} \left(\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}\right)$$

and therefore $\deg_2$ is continuous on $H^{1/2}$.

Similarly, for $\deg_3$, we have,

$$\left||a_n|^2 - |b_n|^2\right| \le |a_n - b_n| \left(|a_n| + |b_n|\right)$$

and thus

$$|\deg_3 g - \deg_3 h| \le \sum |n|\; |a_n - b_n|(|a_n| + |b_n|) \le C\|g-h\|_{H^{1/2}} \left(\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}\right).$$

Since $\deg_2$ and $\deg_3$ are continuous from $H^{1/2}$ into $\mathbb{R}$, they are integers by Lemma 1.11.

Step 2: $\deg_2 g = \deg_3 g, \quad \forall g \in H^{1/2}$.

This is clear for smooth g. The general case follows by density via Lemma 1.11.

Step 3: If $g \in H^{1/2}$ is such that $\deg_1 g = 0$, we may write

$$g = e^{i\psi} \quad \text{for some } \psi \in H^{1/2}(S^1;\mathbb{R}).$$

We already know that $g(e^{it}) = e^{i\varphi(t)}$ for some $\varphi \in H^{1/2}_{\text{loc}}(\mathbb{R})$ and $\varphi(t+2\pi) = \varphi(t)$ a.e. Hence we may consider $\psi : S^1 \to \mathbb{R}$ such that

$$\psi(e^{it}) = \varphi(t) \qquad \text{a.e.}$$

Clearly, this ψ belongs to $H^{1/2}(S^1)$ and satisfies the desired property.

Step 4: We have $\deg_1 g = \deg_2 g \quad \forall g \in H^{1/2}$.

First note that $H^{1/2}(S^1;S^1)$ is an algebra (this is an obvious consequence of the definition of $H^{1/2}$). Moreover if $g_n \to g$ in $H^{1/2}$ and $h_n \to h$ in $H^{1/2}$, then $g_n h_n \to gh$ in $H^{1/2}$ (all functions are S^1-valued). This is a consequence of Lemma A.1.5 in Appendix A.1.1 applied to $\Phi(x,y) = xy$ which is globally Lipschitz on $S^1 \times S^1$.

Next we claim that

$$\deg_2(gh) = \deg_2 g + \deg_2 h, \quad \forall g,h \in H^{1/2}. \tag{6}$$

This is standard for smooth functions (see (1) in Section 1.3). The general case follows from Step 1, the observation above and Lemma 1.11.

On the other hand (6) also holds for $\deg_1$; this is just a consequence of the definition of $\deg_1$.

Finally, set $k = \deg_1 g$, so that $\deg_1(gz^{-k}) = 0$. We claim that $\deg_2(gz^{-k}) = 0$. (This will imply $0 = \deg_2(gz^{-k}) = \deg_2 g - k = \deg_2 g - \deg_1 g$.) To prove the claim, consider $\psi \in H^{1/2}(S^1;\mathbb{R})$ such that

$$gz^{-k} = e^{i\psi} \qquad \text{(see Step 3).}$$

We have

$$gz^{-k} = \left(e^{\frac{i\psi}{n}}\right)^n$$

so that

$$\deg_2(gz^{-k}) = n\ \deg_2(e^{\frac{i\psi}{n}}).$$

If $\deg_2(gz^{-k}) \neq 0$ we would have $\deg_2(e^{\frac{i\psi}{n}}) \neq 0$ and thus $|\deg_2(gz^{-k})| \geq n, \quad \forall n$, which is impossible.

Step 5: We have $\deg_4 g = \deg_2 g, \quad \forall g \in H^{1/2}$.

Fix some $g \in H^{1/2}$. For ε sufficiently small $\deg_4 g = \deg g_\varepsilon$. Since $g_\varepsilon \in C^0 \cap H^{1/2}$ $\deg_2 g_\varepsilon = \deg g_\varepsilon$. Finally recall that $g_\varepsilon \to g$ in $H^{1/2}$ (see Lemma 1.11) and that $\deg_2$ is continuous under $H^{1/2}$ convergence (Step 1). The conclusion follows.

An immediate consequence of Step 3 and (6) is

Corollary 1.3. *Any $g \in H^{1/2}(S^1; S^1)$ of degree d may be written as*

$$g(z) = z^d e^{i\varphi(z)},\ z \in S^1$$

for some $\varphi \in H^{1/2}(S^1; \mathbb{R})$.

Remark 1.18. Definition 3 of degree suggests an interesting development connected to a question of I. M. Gelfand. A general map $g \in C^0(S^1; S^1)$ need not belong to $H^{1/2}(S^1; S^1)$ and thus the series

$$\sum_{n=-\infty}^{+\infty} |n|\ |a_n|^2$$

may be divergent. It could happen that

$$\deg g = \sum_{n=+1}^{+\infty} n|a_n|^2 + \sum_{n=-\infty}^{-1} n|a_n|^2 = +\infty - \infty$$

has no meaning. However, $\deg g$ makes sense for any $g \in C^0(S^1; S^1)$. This indicates that there is some cancellation of the two infinite quantities, expressing the degree as a kind of "principal value". It would be very interesting to understand what summation process (if any) may be used to compute

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2$$

for a general $g \in C^0(S^1; S^1)$. The most natural summation methods are, for example,

$$\lim_{j\to\infty} \sum_{n=-j}^{+j} n|a_n|^2$$

or

$$\lim_{r \uparrow 1} \sum_{n=-\infty}^{+\infty} n|a_n|^2 r^{|n|}.$$

Recently, J. Korevaar [1] has constructed a continuous function g for which these two summation processes fail to converge to the degree. See also OP5 and OP6 in Section 1.8.

Remark 1.19. An amusing consequence of Theorem 1.11 is the following. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers satisfying

$$\sum_{n=-\infty}^{+\infty} |n|\, |a_n|^2 < \infty, \tag{7}$$

$$\sum_{n=-\infty}^{+\infty} |a_n|^2 = 1, \tag{8}$$

and

$$\sum_{n=-\infty}^{+\infty} a_n \overline{a}_{n+k} = 0 \qquad \forall k \neq 0. \tag{9}$$

Then

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2 \in \mathbb{Z}. \tag{10}$$

Indeed, under the above assumptions, the map

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n z^n, \quad |z| = 1$$

belongs to $H^{1/2}(S^1; S^1)$. (Note that $|g(z)| = 1,\ \forall z$ since

$$\int |g(e^{i\theta})|^2 e^{ik\theta} = \int \sum_{m,n} a_n \overline{a}_m e^{i(n-m+k)\theta} d\theta = 2\pi \sum_n a_n \overline{a}_{n+k}.)$$

Is there an elementary proof of the fact that (7), (8), (9) imply (10)?

Remark 1.20. As we have pointed out the map $g \mapsto \deg g$ is continuous from $H^{1/2}$ into $\mathbb{Z}$. However, it is not continuous under weak $H^{1/2}$ convergence. Consider for example the sequence

$$g_k(e^{i\theta}) = \begin{cases} e^{ik\theta} & \text{if } 0 < \theta < \dfrac{2\pi}{k}, \\ 1 & \text{if } \dfrac{2\pi}{k} \le \theta \le 2\pi. \end{cases}$$

One easily checks that $\|g_k\|_{H^{1/2}} \le C$, and that g_k converges weakly in $H^{1/2}$ to $g \equiv 1$. However, $\deg g_k = 1 \ \forall k$ and $\deg g = 0$.

Remark 1.21. There is still another approach to define the degree for $H^{1/2}$ maps. Let $g \in H^{1/2}(S^1; S^1)$. There exists some $u \in H^1(\omega; S^1)$ where $\omega = \{z; 1/2 < |z| < 1\}$ with $u = g$ for $|z| = 1$ (see Theorem 1.14 or simply take $u(re^{i\theta}) = g_\varepsilon(e^{i\theta})$ with g_ε as in (4) and $\varepsilon = \varepsilon_0(1-r)$. Let $\varphi \in C^\infty(\overline{\omega}; \mathbb{R})$ with $\varphi \equiv 1$ on $|z| = 1$ and $\varphi \equiv 0$ on $|z| = 1/2$. Then

$$\deg g = \frac{1}{2\pi} \int_\omega \big[(u \wedge u_y)\varphi_x + (u_x \wedge u)\varphi_y\big]. \tag{11}$$

Indeed, let (u_n) be a sequence of smooth maps from $\overline{\omega}$ into S^1 which converges to u in H^1 (see e.g. Theorem 1.16). Let $g_n = u_{n|S^1}$, so that $g_n \to g$ in $H^{1/2}$.

We have $(u_n)_x \wedge (u_n)_y = 0$ and thus

$$\begin{aligned} 0 &= \frac{1}{\pi} \int_\omega (u_n)_x \wedge (u_n)_y \varphi = \frac{1}{2\pi} \int_\omega \big[(u_n \wedge (u_n)_y)_x + ((u_n)_x \wedge u_n)_y\big]\varphi \\ &= \frac{1}{2\pi} \int_{S^1} \big[(u_n \wedge (u_n)_y) n_x + ((u_n)_x \wedge u_n)\, n_y\big]\varphi - \frac{1}{2\pi} \int_\omega (u_n \wedge (u_n)_y)\varphi_x + ((u_n)_x \wedge u_n)\varphi_y \\ &= \frac{1}{2\pi} \int_{S^1} g_n \wedge (g_n)_\tau - \frac{1}{2\pi} \int_\omega (u_n \wedge (u_n)_y)\,\varphi_x + ((u_n)_x \wedge u_n)\,\varphi_y. \end{aligned}$$

Hence

$$\deg g_n = \frac{1}{2\pi} \int_\omega (u_n \wedge (u_n)_y)\,\varphi_x + ((u_n)_x \wedge u_n)\,\varphi_y$$

and the conclusion follows (using Theorem 1.11).

Warning: Formula (11) seems to suggest that the degree could be defined for a larger class of maps. Suppose for example that $g \in W^{1-1/p,p}(S^1; S^1)$ with $1 < p < \infty$. Then there exists some $u \in W^{1,p}(\omega; S^1)$ whose trace on $|z| = 1$ is g (see Theorem 1.14). One may be tempted to take as definition of degree the right-hand side integral in (11). However, if $p < 2$ that quantity depends on u, φ and moreover it need not be an integer. (Consider for example $u(z) = (z-a)/|z-a|$ for some a with $1/2 < |a| < 1$, then the right-hand side of (11) equals $1 - \varphi(a)$.)

Remark 1.22. Theorems 1.7 - 1.10 in the previous section hold without any continuity assumption on the boundary traces. In all statements the degree then refers to $H^{1/2}$ degree. This may be seen easily using Corollary 1.3.

Here is one additional property

Theorem 1.12. *With the same notations as at the end of Section 1.3, let*

$$\Omega = G \setminus \bigcup_{i \in I} \omega_i.$$

Let $d_i \in \mathbb{Z}$, $i \in I$ and let w be the associated reference map. Then

$$\mathcal{C} = \{u \in H^1(\Omega; S^1); \deg(u, \partial\omega_i) = d_i\} = \{u = we^{i\psi}; \psi \in H^1(\Omega; \mathbb{R})\}.$$

Moreover the class $\mathcal{C}$ is closed under weak H^1 convergence.

Proof. The inclusion $\supset$ is clear. For the reverse inclusion, let

$$v = uw^{-1}$$

so that $v \in H^1(\Omega; S^1)$ with $\deg(v, \partial\omega_i) = 0$. Applying Corollary 1.3 we fill the holes ω_i by maps in $H^1(\omega_i; S^1)$ and putting these together with v we obtain some map $\tilde{v} \in H^1(G; S^1)$. By Theorem 1.1 there is some φ in $H^1(G; \mathbb{R})$ such that $\tilde{v} = e^{i\varphi}$. The function $\psi = \varphi_{|\Omega}$ provides the desired conclusion.

Next,we prove a slightly stronger statement. Namely, if $u \in H^1(\Omega; S^1)$ and if $(u_n) \in \mathcal{C}$ converges to u weakly in $W^{1,p}, 1 < p \leq 2$, then $u \in \mathcal{C}$. Indeed we may write $u_n = we^{i\psi_n}$ and $\|\nabla\psi_n\|_{L^p} \leq C\|u_n\|_{W^{1,p}} \leq C$. By adding an integer multiple of 2π to ψ_n we may always assume that $\|\psi_n\|_{W^{1,p}} \leq C$. Then, up to a subsequence, $\psi_n \rightharpoonup \psi$ weakly in $W^{1,p}$ and a.e. Hence $u_n \to we^{i\psi}$ a.e. and thus $u = we^{i\psi} \in \mathcal{C}$. An alternative proof of the same conclusion relies on formula (11).

Warning: It would have been natural to prove that $\mathcal{C}$ is closed under weak H^1 convergence by going to traces. If $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$, then $u_{n|\partial\Omega} \rightharpoonup u_{|\partial\Omega}$ weakly in $H^{1/2}(\partial\Omega)$. However this is **not** sufficient to pass to the limit in the degree condition (see Remark 1.20).

We conclude this section with a result essentially due of Rubinstein and Sternberg [1]. Their approach relies on works of White [1] and Bethuel [1]; our method is different.

Consider a smooth, bounded, connected domain ω in $\mathbb{R}^{N-1}$, $N \geq 2$ and consider the "torus"

$$\mathbb{T} = \omega \times S^1.$$

We will study the class of maps $H^1(\mathbb{T}; S^1)$ and show that they have a "degree."

Let $u \in H^1(\mathbb{T}; S^1)$. Clearly, for a.e. $x \in \omega$, the map $t \mapsto u(x,t)$ belongs to $H^1(S^1; S^1)$; so it has a degree.

Theorem 1.13. *Assume $u \in H^1(\omega \times S^1; S^1)$, then the function*

$$x \mapsto \deg u(x, \cdot)$$

is constant a.e. on ω.

The conclusion is rather surprising since u is far from being continuous and even far from being VMO, if $N \geq 3$.

Proof. It is more convenient to consider instead of $\mathbb{T}$ the cylinder

$$\Omega = \omega \times \mathbb{R}$$

and the class

$$H^1_{\text{per}}(\Omega; S^1) = \{u \in H^1_{\text{loc}}(\Omega; S^1); u(x, t + 2\pi) = u(x) \text{ a.e.}\};$$

here, "loc" refers to the t variable only.

We present two different proofs; one using the lifting property of H^1 (Theorem 1.1), the other based on the representation of degree as an integral(formula (1) in Section 1.4).

First approach. Let B be a ball in ω and $v = u_{|B\times\mathbb{R}}$. Since $B \times \mathbb{R}$ is simply connected, we may write $v = e^{i\varphi}$ for some $\varphi \in H^1_{\text{loc}}(B \times \mathbb{R}; \mathbb{R})$.

Clearly,

$$\frac{1}{2\pi}\left[\varphi(x, t + 2\pi) - \varphi(x, t)\right] \in \mathbb{Z}, \quad \text{a.e.}$$

Hence, by Theorem 1.6 (in fact, an easy case of Theorem 1.6) there is a constant $k \in \mathbb{Z}$ such that

$$\varphi(x, t + 2\pi) - \varphi(x, t) = 2k\pi, \text{ a.e.}$$

For a.e. $x \in B$ we have $k = \deg u(x, \cdot)$. It follows that $\deg u(x, \cdot)$ is locally constant, hence constant on ω since ω is connected.

Second approach. For $u \in H^1_{\text{per}}(\Omega; \mathbb{R}^2)$, consider the function

$$v(x) = \fint_0^{2\pi} u(x, t) \wedge D_t u(x, t) dt,$$

which is defined a.e.

Note that the map $u \mapsto v$ is continuous from $H^1_{\text{per}}(\Omega)$ to $L^1(\omega)$.

If $u \in C^\infty_{\rm per}(\overline{\Omega};\mathbb{C})$, then

$$\begin{aligned} D_x v(x) &= \fint_0^{2\pi} D_x u(x,t) \wedge D_t u(x,t)dt + \fint_0^{2\pi} u(x,t) \wedge D_t D_x u(x,t)dt \\ &= 2\fint_0^{2\pi} D_x u(x,t) \wedge D_t u(x,t)dt. \end{aligned}$$

Arguing by density we see that every $u \in H^1_{\rm per}(\Omega;\mathbb{R}^2)$ we have

$$v \in W^{1,1}(\omega;\mathbb{R}) \ \text{ and } \ D_x v(x) = 2\fint_0^{2\pi} D_x u(x,t) \wedge D_t u(x,t)dt.$$

In particular, if $u \in H^1_{\rm per}(\Omega;S^1)$, then for a.e. $x \in \omega$, $v(x) = \deg u(x,\cdot)$, and moreover $D_x u \wedge D_t u = 0$ a.e. on Ω. Therefore, $D_x v = 0$ and hence v is a constant on ω.

Remark 1.23. If ω is simply connected we may write

$$u(x,t) = e^{ikt}e^{i\psi(x,t)}$$

where $\psi(x,t) = \varphi(x,t) - kt$ belongs to $H^1_{\rm per}(\Omega;\mathbb{R})$. (Here φ is defined as above, replacing B by ω.) Going back to the terms, we may write any map $u \in H^1(\mathbb{T};S^1)$ as

$$u(x,z) = z^k e^{i\psi(x,z)}, x \in \omega \quad \text{and } |z| = 1,$$

for some $\psi \in H^1(\mathbb{T};\mathbb{R})$.

Remark 1.24. For every $u \in H^1(\mathbb{T};S^1)$ one has

$$\deg u = \fint_\omega dx \fint_0^{2\pi} (u \wedge u_t)dt = \fint_{\mathbb{T}} (u \wedge u_\theta) dx d\theta. \tag{12}$$

This is clear from the seond approach in the proof of Theorem 1.13. Formula (12) implies that the above degree is stable under weak H^1 convergence, i.e., if (u_j) is a sequence in $H^1(\mathbb{T};S^1)$ such that $u_j \rightharpoonup u$ weakly in H^1, then $\deg u_j \to \deg u$ (same conclusion if $u_j \rightharpoonup u$ weakly in $W^{1,p}$ for any $1 < p < 2$).

Remark 1.25. One might be tempted to use formula (12) as a definition of degree for maps u in larger classes, for example, $u \in W^{1,p}(\mathbb{T};S^1)$ with $1 < p < 2$. However, the integral on the right-hand side of (12) does **not** belong to $\mathbb{Z}$ for a general $u \in W^{1,p}$. For example let $\omega = (1/2,1)$ and consider

$$u(x,t) = \frac{xe^{it} - a}{|xe^{it} - a|}$$

with $1/2 < a < 1$. Then $u \in W^{1,p}$, $\forall p < 2$ and

$$\fint_0^{2\pi} (u \wedge u_t) dt = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

so that $\fint_\omega dx \fint_0^{2\pi} (u \wedge u_t) dt = 2(1-a) \in (0,1)$.

Remark 1.26. When $N = 2$, for every $x \in \omega$, the map $u(x,\cdot)$ belongs to $H^{1/2}(S^1; S^1)$ with continuous dependence on x. Hence every such map has a degree and it does not depend on x. This degree coincides with the degree we have just defined. However, when $N \geq 3$ the existence of a degree is more surprising. Here $u(x,\cdot)$ belongs to $H^{1/2}(S^1; S^1)$ for a.e. x, but the dependence need not be continuous.

Remark 1.27. Many of the above results point towards the direction of showing that, under some appropiate conditions, a degree can be defined and consequently $W^{s,p}(\Omega; S^1)$ admits homotopy classes, i.e., path-connected components. In other cases one expects that a degree cannot be defined and more precisely that $W^{s,p}(\Omega; S^1)$ is path-connected; see OP8 and OP9.

Remark 1.28. Let $\omega \subset \mathbb{R}^{N-1}, N \geq 2$, be a connected domain and set

$$\mathbb{T} = \omega \times S^k, \quad k \geq 1.$$

Maps in $W^{1,k+1}(\mathbb{T}; S^k)$ have a degree; more precisely the function

$$x \mapsto \deg u(x,\cdot)$$

which is well-defined for a.e. $x \in \omega$, is constant on ω. This has been proved by Brezis, Li, Mironescu and Nirenberg [1]. In fact the same conclusion still holds when $W^{1,k+1}$ is replaced by the fractional Sobolev space $W^{s,p}$ with $sp \geq k+1$.

1.5. Traces for $\mathbf{S^1}$-valued maps

A problem which is related to lifting is the question of traces. More precisely, let

$$u \in W^{s,p}(\Omega; S^1)$$

where $\Omega \subset \mathbb{R}^N$ is smooth and bounded. Let $Q = \Omega \times (0,1)$. Can one construct some

$$v \in W^{s+1/p,p}(Q; S^1)$$

whose trace on $\Omega \times \{0\}$ is the given u? This problem has been first studied by Hardt, Kinderlehrer and Lin [1], Hardt and Lin [2] and Bethuel and Demengel [1] for the case

$s = 1 - 1/p$ (and S^1 possibly replaced by S^k or some more general manifold). We will then discuss the case $s \geq 1 - 1/p$ for which there is a complete answer. Finally, the case $s < 1 - 1/p$ is partially open; see Theorem 1.16 and OP10.

We start with the case $s = 1 - 1/p$.

Theorem 1.14. *Let $N \geq 1$ and let*

$$u \in W^{1-1/p,p}(\Omega; S^1).$$

Then u is the trace of some $v \in W^{1,p}(Q; S^1)$ in the following cases:

a) $1 < p < 2$,

b) $p \geq N + 1$.

However, if $N \geq 2$ and

c) $2 \leq p < N + 1$,

the conclusion fails, i.e., there is some u which is not the trace of any v as above.

Proof. We consider separately the three cases:

Case a): This case is due to Hardt, Kinderlehrer and Lin and we sketch their argument. First, construct some

$$w \in W^{1,p}(Q; \mathbb{R}^2)$$

with

$$w_{|\Omega \times \{0\}} = u.$$

This is done using the standard trace theory. For every $a \in B_{1/2} \subset \mathbb{R}^2$ set

$$w_a(x) = \frac{w(x) - a}{|w(x) - a|}.$$

We claim that $w_a \in W^{1,p}$ for **some** suitable a. Indeed, (at least formally)

$$|\nabla w_a| \leq \frac{|\nabla w|}{|w - a|}.$$

Hence

$$\int_{B_{1/2}} da \int_Q |\nabla w_a|^p dx \leq \int_Q |\nabla w(x)|^p \left(\int_{B_{1/2}} \frac{da}{|w(x) - a|^p} \right) dx < \infty$$

since $p < 2$. Fix any a such that $w_a \in W^{1,p}$.

Next consider the map

$$P_a(\xi) = \frac{\xi - a}{|\xi - a|}$$

which is smooth and bijective from S^1 onto itself. Let Π_a be the inverse map. Then

$$v = \Pi_a \circ w_a$$

has all the required properties.

An alternative argument relies on lifting. Here $0 < s < 1$ and $sp = p - 1 < 1$. Thus, by Theorem 1.3 *c*), u may be written as $u = e^{i\psi}$ for some $\psi \in W^{1-1/p,p}(\Omega; \mathbb{R})$. Hence ψ is the trace of some $\varphi \in W^{1,p}(Q; \mathbb{R})$ and $v = e^{i\varphi}$ has the desired properties.

Case b): We may always assume that

$$u \in W^{1-1/p,p}(\Omega'; S^1)$$

for a slightly larger domain $\Omega' \supset \overline{\Omega}$ (this can be achieved by reflexion across $\partial\Omega$). Let $\widetilde{u}$ be any harmonic extension of u to $Q' = \Omega' \times (0, 1)$. Since

$$(1 - 1/p)p = p - 1 \geq N$$

we know that u is continuous or at least VMO in the case of equality. Thus

$$|\widetilde{u}(x, y)| \to 1 \quad \text{uniformly for } x \in \overline{\Omega}, \text{ as } y \to 0,$$

(see e.g. Brezis and Nirenberg [2]). Since $\widetilde{u} \in W^{1,p}(\Omega \times (0, 1/2))$, $v = \widetilde{u}/|\widetilde{u}|$ has all the required properties (after scaling in y).

Case c): Consider any $u \in W^{1-1/p,p}(\Omega; S^1)$ which has no lifting; this is possible by Theorem 1.3 since $1 \leq (1 - 1/p)p = p - 1 < N$. Such a u cannot be the trace of some $v \in W^{1,p}(Q; S^1)$. Otherwise, by Theorem 1.1, we could write $v = e^{i\varphi}$, with $\varphi \in W^{1,p}(Q)$ and then $u = e^{i\psi}$ where $\psi = \varphi_{|\Omega}$ belongs to $W^{1-1/p,p}(\Omega)$.

Example: For $N = 2$, $\Omega =$ the unit disc, $s = 1/2$ and $p = 2$ the maps

$$u(x) = e^{i|x|^{-1/2}}$$

or

$$u(x) = \frac{x}{|x|}$$

both belong to $H^{1/2}(\Omega; S^1)$ but they are not the trace of any $v \in H^1(Q; S^1)$ (since u cannot be lifted in $H^{1/2}$; see the examples in the proof of Theorem 1.3 *b*).

We turn now to the case $s \geq 1 - 1/p$.

Theorem 1.15. *Let $N \geq 1, 0 < s < \infty, 1 < p < \infty$ and assume*

$$(1) \qquad s \geq 1 - 1/p, \quad s \quad \textit{not integer.}$$

Let

$$u \in W^{s,p}(\Omega; S^1).$$

Then u is the trace of some $v \in W^{s+1/p,p}(Q; S^1)$ in the following cases:
a) $sp < 1$
b) $sp \geq N$.

However, if $N \geq 2$ and

c) $1 \leq sp < N$

the conclusion fails, i.e., there is some u which is not the trace of any v as above.

Remark 1.29. The assumption that s is not an integer allows to apply the standard trace theory: any function $u \in W^{s,p}(\Omega; \mathbb{R})$ is the trace of some $v \in W^{s+1/p,p}(Q; \mathbb{R})$ and conversely (see e.g. Adams [1]).

Proof of Theorem 1.15. The case $s = 1 - 1/p$ corresponds precisely to the previous Theorem 1.14. Thus we may assume that

$$(2) \qquad s > 1 - 1/p.$$

We consider separately the three cases:

Case a): sp <1. We use the same ideas as in the previous proof. First construct some

$$w \in W^{s+1/p,p}(Q; \mathbb{R}^2)$$

with

$$w_{|\Omega \times \{0\}} = u.$$

Since $u \in L^\infty(\Omega)$ we may also assume that $w \in L^\infty(Q)$. For every $a \in B_{1/2} \subset \mathbb{R}^2$ set

$$w_a(x) = \frac{w(x) - a}{|w(x) - a|}.$$

We claim that for some $a \in B_{1/2}$ (in fact, for a.e. $a \in B_{1/2}$)

$$(3) \qquad w_a(x) \in W^{s+1/p,p}(\Omega; S^1).$$

Then

$$v = \Pi_a \circ w_a$$

has all the required properties (by Lemma 1.2).

We now turn to the proof of (3). We have

$$1 < \sigma = s + 1/p < 2/p < 2.$$

Recall (see e.g. Triebel [1] or [2]) that a function $f \in L^p$ belongs to $W^{\sigma,p}$ if and only if

$$\int \frac{dh}{|h|^{N+\sigma p}} \int |\delta_h^2 f(x)|^p dx < \infty$$

where $\delta_h f(x) = f(x+h) - f(x)$ and

$$\delta_h^2 f(x) = \delta_h(\delta_h f)(x) = f(x+2h) - 2f(x+h) + f(x).$$

We will prove that $\int da \int \frac{dh}{|h|^{N+\sigma p}} \int |\delta_h^2 w_a(x)|^p dx < \infty$.

Applying Lemma A.1.7 in Appendix A.1.2 with $X = w(x+2h) - a, Y = w(x+h) - a$ and $Z = w(x) - a$, we obtain (for any $\eta \in [1,2]$)

$$|\delta_h^2 w_a(x)| \leq \frac{C}{\rho_{a,h}(x)} |\delta_h^2 w(x)| + \frac{C}{\rho_{a,h}(x)^\eta} \left[|\delta_h w(x+h)|^\eta + |\delta_h w(x)|^\eta \right]$$

where

$$\frac{1}{\rho_{a,h}(x)} = \frac{1}{|w(x+2h) - a|} + \frac{1}{|w(x+h) - a|} + \frac{1}{|w(x) - a|}.$$

Note that

$$p < 2 \quad (\text{ by}(1) \text{ and } a))$$

and

$$\sigma p = sp + 1 < 2 \quad (\text{ by } a)).$$

Fix any $\eta \in [1,2]$ such that

$$\eta > \sigma, \quad \eta p < 2.$$

Then we have

$$\int_{|a| \leq 1/2} \left[\frac{1}{\rho_{a,h}(x)^p} + \frac{1}{\rho_{a,h}(x)^{\eta p}}\right] da \leq C \quad \forall x, \quad \forall h,$$

where C is some absolute constant.

It follows that

$$\int da \int \frac{dh}{|h|^{N+\sigma p}} \int |\delta_h^2 w_a(x)|^p dx \leq C \int \frac{dh}{|h|^{N+\sigma p}} \int |\delta_h^2 w(x)|^p dx$$
$$+ C \int \frac{dh}{|h|^{N+\sigma p}} \int |\delta_h w(x)|^{\eta p} dx.$$

The first integral on the righthand side is finite since $w \in W^{\sigma,p}$. The second integral is finite provided

$$w \in W^{\sigma/\eta,\eta p}.$$

This is a consequence of the fact that $w \in W^{\sigma,p} \cap L^\infty$ and the Gagliardo-Nirenberg inequality.

Case b): sp $\geq$ N. The argument is the same as in the proof of Theorem 1.14.

Case c): 1 $\leq$ sp $<$ N.

Choose (assuming $\Omega = B_1 \subset \mathbb{R}^N$)

$$u(x) = e^{i\varphi(x)} \qquad \text{where} \quad \varphi(x) = 1/|x|^\alpha.$$

We will prove that, for some appropriate α,

$$u \in W^{s,p} \tag{4}$$

and

$$u \text{ is not the trace of any } v \in W^{s+1/p,p}(Q; S^1). \tag{5}$$

Verification of (4). Recall (see the proof of Theorem 1.3*b)*) that (4) holds for any $\alpha > 0$ such that

$$\alpha < \frac{N - sp}{sp}. \tag{6}$$

Verification of (5). We claim that (5) holds with

$$\alpha \geq \frac{N - sp}{sp + 1}. \tag{7}$$

Indeed, suppose by contradiction, that u is the trace of some $v \in W^{s+1/p,p}(Q; S^1)$.

In view of Theorem 1.4$'a$) we may write

$$v = e^{i\psi}$$

for some $\psi \in W^{s+1/p,p} \cap W^{1,sp+1}$. Taking traces we have

$$u = e^{i\varphi} = e^{i\psi_{|\Omega \times \{0\}}}.$$

Since φ is smooth on $\Omega\backslash\{0\}, \psi \in W^{s,p}$ on $\Omega \times \{0\}$(with $sp \geq 1$) and $\eta = \frac{1}{2\pi}(\varphi - \psi)$ takes its values into $\mathbb{Z}$, we conclude (by Theorem 1.6) that η is a constant. But this is impossible since $\psi_{|\Omega\times\{0\}}$ belongs to $W^{1-1/(sp+1),sp+1}$ and φ does not belong to this space (by Lemma A.1.6 in Appendix A.1.2 and (7)).

Finally we consider the case $s < 1 - 1/p$.

Theorem 1.16. *Let $N \geq 1, 1 < p < \infty$ and*

$$s < 1 - 1/p. \tag{8}$$

Let

$$u \in W^{s,p}(\Omega; S^1).$$

Then u is the trace of some $v \in W^{s+1/p,p}(Q; S^1)$ in the following cases:

a) $sp < 1$,

b) $sp \geq N$.

However if $N \geq 2$ and

c) $1 \leq sp < 2$

the conclusion fails, i.e., there is some u which is not the trace of any v as above.

Remark 1.30. Putting together Theorems 1.15 and 1.16 we see that we have a complete answer to the problem of traces for S^1-valued maps when $N = 1$ and when $N = 2$. When $N = 1$ the answer is always positive. When $N = 2$ and s is not an integer the answer is :

a) positive if $sp < 1$,

b) negative if $1 \leq sp < 2$,

c) positive if $sp \geq 2$.

However when $N \geq 3$ there is still a gap; the answer is not known when $s < 1 - 1/p$ and $2 \leq sp < N$ – for example $N = 3, p = 4$ and $1/2 \leq s < 3/4$. See OP10.

Proof of Theorem 1.16.

Case a):sp<1. By Theorem 1.3*c) u* may be written as $u = e^{i\psi}$ for some $\psi \in W^{s,p}(\Omega; \mathbb{R})$. This ψ is the trace of some $\varphi \in W^{s+1/p}(Q; \mathbb{R})$ and then $v = e^{i\varphi}$ has the desired property (since $s + 1/p < 1$, by assumption (8), it is clear that $v \in W^{s+1/p,p}(Q; \mathbb{R})$.

Case b): sp≥N. The argument is the same as in the proof of Theorem 1.14.

Case c): 1≤sp<2. Assume for simplicity that $N = 2$ and that $0 \in \Omega$ (when $N \geq 3$ proceed as in Remark 1.1). Set

$$u(x) = \frac{x}{|x|}.$$

This u belongs to $W^{s,p}$ provided $sp < 2$ (see Appendix A.1.2).

We claim that there is no $v \in W^{s+1/p,p}(Q; S^1)$ whose trace is u. We argue by contradiction as in the proof of Theorem 1.3*b)* (topological obstruction). Let

$$\Sigma_r = \left\{ (x, y) \in \Omega \times (0, 1) \,;\, |x| = r \right\}.$$

For a.e. r the restriction of v to Σ_r belongs to $W^{s+1/p,p}(\Sigma_r; S^1)$. Fix any such r. By standard trace theory $v(\cdot, y)$ belongs to $W^{s,p}(C_r; S^1)$ for every y, where

$$C_r = \left\{ x \in \Omega \,;\, |x| = r \right\},$$

and moreover $v(\cdot, y)$ converges to $u(\cdot)$ in $W^{s,p}(C_r; S^1)$ as $y \to 0$. Since $sp \geq 1$ we know , from the results of Brezis and Nirenberg[1], that maps in $W^{s,p}(S^1, S^1)$ have a degree which is continous under (strong) $W^{s,p}$ convergence. Therefore

$$\deg(v(\cdot, y), C_r) \to 1 \text{ as } y \to 0.$$

On the other hand we claim that

$$\deg(v(\cdot, y), C_r) = 0 \text{ for a.e } y > 0.$$

Indeed for a.e. $y > 0$

$$v(\cdot, y) \in W^{s+1/p,p}(B_r; S^1)$$

where $B_r = \left\{ x \in \Omega;\, |x| < r \right\}$. We may now complete the argument using the following lemma which is an extension of Corollary 1.2.

Lemma 1.12. *Let $\Omega = B_1$ be the unit disc in $\mathbb{R}^2$. Assume $0 < \sigma < \infty, 1 < p < \infty$ satisy*

$$\sigma p \geq 2 \tag{9}$$

and let

$$v \in W^{\sigma,p}(\Omega; S^1).$$

Then

$$\deg(v_{|\partial\Omega}) = 0.$$

Note that $v_{|\partial\Omega}$ belongs to $W^{\sigma-1/p,p}(S^1; S^1)$ and since $(\sigma - 1/p)p = \sigma p - 1 \geq 1, v_{|\partial\Omega}$ belongs to $\mathrm{VMO}(S^1; S^1)$. Therefore it has a degree in the sense of Brezis and Nirenberg [1].

Proof of Lemma 1.12. By Theorem 1.17 below and assumption (9) there is a sequence (v_n) of smooth functions from $\overline{\Omega}$ to S^1 such that $v_n \to v$ in $W^{\sigma,p}$. For each n,

$$\deg(v_{n|\partial\Omega}) = 0$$

since $v_{n|\partial\Omega}$ can be connected to a constant via a smooth homotopy. On the other hand $v_{n|\partial\Omega} \to v_{n|\partial\Omega}$ in $W^{\sigma-1/p,p}$ and this is sufficient to guarantee convergence of degree since $(\sigma - 1/p)p = \sigma p - 1 \geq 1$.

1.6. Density for S^1-valued maps

Another problem which is related to lifting is the question of density. More precisely, let

$$u \in W^{s,p}(\Omega; S^1).$$

Does there exist a sequence of smooth functions (u_n) from $\overline{\Omega}$ into S^1 such that $u_n \to u$ in $W^{s,p}$?

Following earlier work of Schoen and Uhlenbeck [2], this problem has been extensively studied by Bethuel and Zheng [1] and Bethuel [1] when $s = 1$ and by Escobedo [1] for arbitrary s (and some more general target manifolds).

Here is a first partial result due to Bethuel and Zheng [1] when $s = 1$ and to Escobedo [1] in the general case:

Theorem 1.17. *Assume $N \geq 1$, $0 < s < \infty$ and $1 < p < \infty$. Then the answer is positive (i.e., there is density) in the following cases:*

a) $0 < sp < 1$,

b) $sp \geq N$.

However, the answer is negative if $N \geq 2$ and

c) $1 \leq sp < 2$.

Sketch of proof. For the case *a)* we refer to Escobedo [1]. Alternatively, one may use Theorem 1.3 *c)* together with Lemma A.1.5 in Appendix A.1.1. For the case *b)* we use the VMO property as in Lemmas 1.9 and 1.10.

For the case *c)* we use the same construction as in Remark 1.1. To show that u cannot be approximated in $W^{s,p}$ by a sequence (u_n) of smooth maps from $\overline{\Omega}$ to S^1 we rely on degree. Suppose, for simplicity, that $N = 2$ and assume, by contradiction, that such a sequence (u_n) exists. Then, up to a subsequence, $u_n \to u$ in $W^{s,p}(S_r)$ for almost every circle S_r. Since $sp \geq 1$, $W^{s,p}(S_r) \subset$ VMO (S_r) and thus $0 = \deg(u_{n_{|S_r}}) \to \deg(u_{|S_r}) = 1$ by the stability of degree under VMO convergence (see Brezis and Nirenberg [1]).

The case $2 \leq sp < N$ is open when s is not an integer; see OP11. Here is a positive result concerning the case where s is an integer:

Theorem 1.18. *Assume Ω is simply connected, $N \geq 2$, $s \geq 1$ is an integer, $1 \leq p < \infty$ and $sp \geq 2$. Then there is density.*

Proof of Theorem 1.18. Fix $u \in W^{s,p}(\Omega; S^1)$. By Theorem 1.4′$a$) there is some $\varphi \in W^{s,p}(\Omega; \mathbb{R}) \cap W^{1,sp}(\Omega; \mathbb{R})$ such that

$$u = e^{i\varphi}.$$

Hence, there is a sequence of smooth functions (φ_n) such that $\varphi_n \to \varphi$ in $W^{s,p}$ and in $W^{1,sp}$. It is then easy to check that

$$u_n = e^{i\varphi_n} \to u = e^{i\varphi} \quad \text{in} \quad W^{s,p}.$$

Remark 1.30. When Ω is simply connected, the question of lifting for Sobolev maps is closely related to the question of density of smooth maps in Sobolev classes, as was observed in Bethuel and Zheng [1]. For any p, $1 \le p < \infty$, one has

$$\text{(1)} \qquad \overline{C^\infty(\overline{\Omega}; S^1)}^{W^{1,p}} = \{u = e^{i\varphi}; \varphi \in W^{1,p}(\Omega; \mathbb{R})\}.$$

Indeed, if $u = e^{i\varphi}$ for some $\varphi \in W^{1,p}(\Omega; \mathbb{R})$, we may apply the standard density results in Sobolev spaces (see e.g. Adams [1], Chap. III) and assert that there is a sequence (φ_n) in $C^\infty(\overline{\Omega}; \mathbb{R})$ such that $\varphi_n \to \varphi$ in $W^{1,p}$. Then $u_n = e^{i\varphi_n}$ belongs to $C^\infty(\overline{\Omega}; S^1)$ and $u_n \to u$ in $W^{1,p}$.

Conversely, let $u \in W^{1,p}(\Omega; S^1)$ and suppose that there is a sequence $u_n \in C^\infty(\overline{\Omega}; S^1)$ such that $u_n \to u$ in $W^{1,p}$ and a.e. We may write $u_n = e^{i\varphi_n}$ for some sequence $\varphi_n \in C^\infty(\overline{\Omega}; \mathbb{R})$. We have

$$\nabla u_n = ie^{i\varphi_n} \nabla \varphi_n$$

and so

$$\nabla \varphi_n = -i\overline{u}_n \nabla u_n.$$

It follows that

$$\text{(2)} \qquad \nabla \varphi_n \to -i\overline{u}\nabla u \qquad \text{in } L^p.$$

Indeed,

$$\begin{aligned} \|\overline{u}_n \nabla u_n - \overline{u}\nabla u\|_{L^p} &\le \|\overline{u}_n(\nabla u_n - \nabla u)\|_{L^p} + \|(\overline{u}_n - \overline{u})\nabla u\|_{L^p} \\ &\le \|\nabla u_n - \nabla u\|_{L^p} + \|(\overline{u}_n - \overline{u})\nabla u\|_{L^p}. \end{aligned}$$

Note that $\|(\overline{u}_n - \overline{u})\nabla u\|_{L^p} \to 0$, by dominated convergence.

Returning to (2) and applying Poincaré's inequality we see that $(\varphi_n - \fint_\Omega \varphi_n)$ is a Cauchy sequence in $W^{1,p}$ where $\fint$ denotes the average. Recall that Poincaré's inequality asserts that

$$\|\zeta - \fint \zeta\|_{L^p} \le C\|\nabla \zeta\|_{L^p} \qquad \forall \zeta \in W^{1,p}.$$

Set $\alpha_n = \fint_\Omega \varphi_n$ and $\psi_n = \varphi_n - \fint_\Omega \varphi_n$. We have

$$u_n = e^{i\alpha_n} e^{i\psi_n}.$$

Passing to a subsequence we may assume that $e^{i\alpha_n} \to \xi$ with $|\xi| = 1$, so that $\xi = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Since $\psi_n \to \psi$ for some $\psi \in W^{1,p}$ we have

$$u = e^{i(\psi + \alpha)}$$

which is the desired conclusion.

If one replaces $W^{1,p}$ by $W^{s,p}, 0 < s < 1$, equality (1) need not hold anymore. In the next section we will characterize the closure of $C^\infty(\overline{\Omega}; S^1)$ in $H^{1/2}$ when $N = 2$.

Remark 1.31. In the cases where smooth maps are **not** dense in $W^{s,p}$, one may ask whether the class

$$\mathcal{R} = \big\{u \in W^{s,p}(\Omega; S^1); u \text{ is smooth except on a simple set of low dimension}\big\}$$

(where the exceptional set is left free) is dense in $W^{s,p}(\Omega; S^1)$.

This type of question was initially investigated when $s = 1$ in Bethuel and Zheng [1] and Bethuel [2]. When $s \neq 1$ the full picture has not yet been clarified.

Consider, for example, the case $N = 2$. Recall that if $0 < s < \infty, 1 < p < \infty$ then

a) smooth maps are dense when $0 < sp < 1$ or $sp \geq 2$.

b) smooth maps are not dense when $1 \leq sp < 2$.

Set

$$\mathcal{R}_0 = \big\{u \in W^{s,p}(\Omega; S^1); u \text{ is smooth except at a finite number of points}\big\}$$

Is $\mathcal{R}_0$ dense in $W^{s,p}(\Omega; S^1)$ when $1 \leq sp < 2$? The answer is known to be positive in the following cases:

a) $s = 1$ and $1 \leq p < 2$; see Bethuel and Zheng [1]

b) $s = 1 - 1/p$ and $2 < p < 3$; see Bethuel [3]

c) $s = 1/2$ and $p = 2$; see Rivière [2].

See also OP12.

1.7. More about the structure of $\mathbf{H}^{1/2}$

We review here the three main problems (existence of lifting, extension to higher dimension, density of smooth functions) in the special case $s = 1/2, p = 2, N = 2$.

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded simply connected domain (with some slight modifications we could also consider the case where Ω is the boundary of a smooth simply connected domain G in $\mathbb{R}^3$). We are concerned with the structure of $H^{1/2}(\Omega; S^1)$. It will be convenient to assume that $0 \in \Omega$.

Recall that

a) Some u's in $H^{1/2}(\Omega; S^1)$ **cannot be lifted** as $u = e^{i\varphi}$ with $\varphi \in H^{1/2}(\Omega; S^1)$ (see Theorem 1.3*b)*). For example, we may choose

$$u_1(x, y) = e^{i/r^\alpha} \text{ with } r^2 = x^2 + y^2 \text{ and } 1/2 \leq \alpha < 1,$$

or

$$u_2(x, y) = (x, y)/r.$$

b) Some u's in $H^{1/2}(\Omega; S^1)$ are **not the trace** of any $v \in H^1(Q; S^1)$. More precisely, let $Q = \Omega \times (0, 1)$; for some u's in $H^{1/2}(\Omega; S^1)$ there exists **no** $v \in H^1(Q; S^1)$ such that $v_{|\Omega \times \{0\}} = u$. (see Theorem 1.14*c)*). For example we may choose u_1 or u_2 described above.

c) Some u's in $H^{1/2}(\Omega; S^1)$ **cannot be approximated** in the $H^{1/2}$ norm by functions in $C^\infty(\bar{\Omega}; S^1)$ (see Theorem 1.17*c)*). For example we may choose u_2 described above (but not u_1!).

It is therefore natural to introduce the three classes

$$X = \left\{ u \in H^{1/2}(\Omega; S^1); u = e^{i\varphi} \text{ for some } \varphi \in H^{1/2}(\Omega; \mathbb{R}) \right\},$$

$$Y = \left\{ u \in H^{1/2}(\Omega; S^1); u = v_{|\Omega \times \{0\}} \text{ for some } v \in H^1(Q; S^1) \right\}$$

and

$$Z = \overline{C^\infty(\overline{\Omega}; S^1)}^{H^{1/2}}.$$

Lemma 1.13. *We have*

$$X = Y \subsetneqq Z \subsetneqq H^{1/2}(\Omega; S^1).$$

Proof. Let $u \in X$ and choose $\varphi \in H^{1/2}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Let $\psi \in H^1(Q; \mathbb{R})$ be such that $\psi_{|\Omega \times \{0\}} = \varphi$. Then $v = e^{i\psi}$ belongs to $H^1(Q; S^1)$ and $v_{|\Omega \times \{0\}} = u$

Conversely, let $u \in Y$ and choose $v \in H^1(Q; S^1)$ such that $v_{|\Omega \times \{0\}} = u$. By Theorem 1.1 we may write $v = e^{i\psi}$ for some $\psi \in H^1(Q; \mathbb{R})$. Then $u = e^{i\psi}$ where $\varphi = \psi_{|\Omega \times \{0\}}$ in $H^{1/2}(|\Omega; \mathbb{R})$. Thus $u \in X$, and we have proved that $X = Y$.

If $u \in X = Y$ write $u = e^{i\varphi}$ with $\varphi \in H^{1/2}(\Omega;\mathbb{R})$. Let (φ_n) be a sequence in $C^\infty(\overline{\Omega};\mathbb{R})$ such that $\varphi_n \to \varphi$ in $H^{1/2}$. Then $u_n = e^{i\varphi_n}$ converges to $u = e^{i\varphi}$ in $H^{1/2}$ by Lemma A.1.5. Hence $u \in Z$.

To prove that $Z \neq X$ recall that u_1 described above does **not** belong to X. On the other hand, u_1 is the limit in $H^{1/2}$ of $u_\varepsilon = e^{i\psi_\varepsilon}$ with $\psi_\varepsilon = 1/(\varepsilon^2 + r^2)^{\alpha/2}$. (It suffices to adapt the argument showing that $u_1 \in H^{1/2}$ in the proof of Theorem 1.3*b)*). Thus $u_1 \in Z$.

Finally we see that $Z \neq H^{1/2}$ by noting that u_2 lies in $H^{1/2}$ and does not belong to Z.

To every function $u \in H^{1/2}(\Omega;\mathbb{R}^2)$ we associate a distribution

$$T = T(u) \in \mathcal{D}'(\Omega;\mathbb{R}).$$

When $u \in H^{1/2}(\Omega;S^1)$ the distribution T plays an interesting role: it describes the "location" and "topological charge" of the singular set of u.

Let $Q = \Omega \times (0,1)$. Given $u \in H^{1/1}(\Omega;\mathbb{R}^2)$ consider any $v \in H^1(Q;\mathbb{R}^2)$ such that

$$v_{|\Omega\times\{0\}} = u \text{ and } v_{|\Omega\times\{1\}} = 0. \tag{1}$$

Set

$$< T,\zeta >= -2\int_Q (v_y \wedge v_z)\zeta_x - 2\int_Q (v_z \wedge v_x)\zeta_y \tag{2}$$

where $\zeta \in C_0^\infty(\Omega;\mathbb{R})$ (ζ is extended to Q by choosing $\zeta(x,y,z) = \zeta(x,y)$). It is easy to check that this definition of T does not depend on the choice of an orthonormal base (x,y) in $\mathbb{R}^2$. In addition we have

Lemma 1.14. *T is independent of the choice of v.*

Proof. As in the proof of Theorem 1.8 it suffices to verify that

$$I = \int_Q \big[(v_y \wedge w_z) + (w_y \wedge v_z)\big]\zeta_x + \int_Q \big[(v_z \wedge w_x) + (w_z \wedge v_x)\big]\zeta_y = 0$$

$$\forall v \in H^1(Q;\mathbb{R}^2),\ \forall w \in H^1(Q;\mathbb{R}^2) \text{ with } w_{|\Omega\times\{0\}} = w_{|\Omega\times\{1\}} = 0, \text{ and } \forall \zeta \in C_0^\infty(\Omega;\mathbb{R}).$$

By density we may always assume that v and w are smooth with $w_{|\Omega\times\{0\}} = w_{|\Omega\times\{1\}} = 0$.

Next observe that

$$(v_y \wedge w_z) + (w_y \wedge v_z) = (v_y \wedge w)_z + (w \wedge v_z)_y$$

and

$$(v_z \wedge w_x) + (w_z \wedge v_x) = (v_z \wedge w)_x + (w \wedge v_x)_z$$

Hence (since $\zeta_z = 0$),

$$\begin{aligned} I = &- \int_Q (w \wedge v_z)\zeta_{xy} + \int_{\Omega\times\{1\}} (v_y \wedge w)\zeta_x - \int_{\Omega\times\{0\}} (v_y \wedge w)\zeta_x - \int_Q (v_z \wedge w)\zeta_{xy} \\ &+ \int_{\Omega\times\{1\}} (w \wedge v_x)\zeta_y - \int_{\Omega\times\{0\}} (w \wedge v_x)\zeta_y \\ &= 0. \end{aligned}$$

In what follows we will denote the distribution T by $T(u)$ or simply T if there is no ambiguity.

When u has a little more regularity there is also a simpler form for the distribution T:

Lemma 1.15. *If $u \in H^{1/2}(\Omega;\mathbb{R}^2) \cap W^{1,1}(\Omega:\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ then*

$$T(u) = (u \wedge u_y)_x - (u \wedge u_x)_y \ \text{ in } \mathcal{D}'(\Omega;\mathbb{R})$$

Proof. We have to show that

$$(3) \qquad -2\int_Q (v_y \wedge v_z)\zeta_x - 2\int_Q (v_z \wedge v_x)\zeta_y = -\int_Q (u \wedge u_y)\zeta_x + \int_Q (u \wedge u_x)\zeta_y.$$

Let (u_n) be a sequence in $C^\infty(\overline{\Omega};\mathbb{R}^2)$ such that $u_n \to u$ in $H^{1/2}$ and in $W^{1,1}$, and $||u_n||_{L^\infty} \leq C$. We may then choose any sequence (v_n) in $C^\infty(\bar{Q};\mathbb{R}^2)$ such that $v_n \to v$ in $H^1(Q;\mathbb{R}^2)$ and

$$v_{n|\Omega\times\{0\}} = u_n \quad , \quad v_{n|\Omega\times\{1\}} = 0.$$

It suffices to prove that (3) holds for the sequence v_n. For simplicity we drop the subscript n. We have

$$\begin{aligned} 2(v_y \wedge v_z) &= (v \wedge v_z)_y + (v_y \wedge v)_z \\ 2(v_z \wedge v_x) &= (v_z \wedge v)_x + (v \wedge v_x)_z \end{aligned}$$

and therefore

$$\begin{aligned} &- 2\int_Q (v_y \wedge v_z)\zeta_x - 2\int_Q (v_z \wedge v_x)\zeta_y \\ &= 2\int_Q (v \wedge v_z)\zeta_{xy} + \int_{\Omega\times\{0\}} (v_y \wedge v)\zeta_x \\ &+ 2\int_Q (v_z \wedge v)\zeta_x y + \int_{\Omega\times\{0\}} (v \wedge v_x)\zeta_y \\ &= \int_\Omega (u_y \wedge u)\zeta_x + \int_\Omega (u \wedge u_x)\zeta_y, \end{aligned}$$

which is the desired equality.

When u takes its values in S^1 and has only a finite number of singularities there is a very simple expression for the distribution $T(u)$:

Theorem 1.19. *Let (a_j) be k points in Ω. Assume $u \in H^{1/2}(\Omega; S^1) \cap H^1_{loc}(\Omega \setminus \cup_{j=1}^k \{a_j\}; S^1)$.*

Then

$$T(u) = 2\pi \sum_{j=1}^k d_j \delta_{a_j} \quad \text{in } \mathcal{D}'(\Omega)$$

where δ_{a_j} denotes the Dirac mass at a_j and $d_j = deg(u, a_j)$ is the degree of u restricted to any small circle around a_j (in the sense of $H^{1/2}$- degree of Section 1.4)

Proof. We split the argument into 3 steps.

Step 1: supp $T(u) \subset \cup_{j=1}^k \{a_j\}$.

It suffices to verify that if $\widetilde{u} \in H^1(\omega; S^1)$ for some domain ω, then

$$T(\widetilde{u}) = 0 \quad \text{in } \mathcal{D}'(\omega).$$

But, in view of Lemma 1.15,

$$T(\widetilde{u}) = (\widetilde{u} \wedge \widetilde{u}_y)_x - (\widetilde{u} \wedge \widetilde{u}_x)_y \text{ in } \mathcal{D}'(\omega)$$

Let $\widetilde{u}_n$ be a sequence in $C^\infty(\overline{\omega}; S^1)$ such that $\widetilde{u}_n \to \widetilde{u}$ in H^1. We have

$$\begin{aligned} T(\widetilde{u}_n) &= (\widetilde{u}_n \wedge \widetilde{u}_{ny})_x - (\widetilde{u}_n \wedge \widetilde{u}_{nx})_y \\ &= 2\widetilde{u}_{nx} \wedge \widetilde{u}_{ny} = 0 \end{aligned}$$

Moreover $T(\widetilde{u}_n) \to T(\widetilde{u})$ in $\mathcal{D}'(\omega)$ and therefore $T(\widetilde{u}) = 0$.

Step 2: $T(u) = \sum\limits_{j=1}^k c_j \delta_{a_j}$ for some constants c_j.

Proof. By a celebrated result of L. Schwartz [1] we deduce from Step 1 that $T(u)$ may be expressed as a finite sum

$$T(u) = \sum_{j,\alpha} c_{j,\alpha} D^\alpha \delta_{a_j}. \tag{4}$$

Fix j and α with $|\alpha| \geq 1$. We have to prove that $c_{j,\alpha} = 0$. We may assume that $a_j = 0$. Fix a smooth function ζ with support in $B(0,1)$, such that $\zeta(0) = 0$, $D^\alpha \zeta(0) = 1$ and $D^\beta \zeta(0) = 0$ for all $\beta \neq \alpha$. Set

$$\zeta_k(x,y) = \zeta(kx, ky),\ k = 1, 2, ...,$$

so that supp $\zeta_k \subset B(0, 1/k)$. For k sufficiently large, we have by (3),

$$< T(u), \zeta_k >= (-1)^{|\alpha|} k^{|\alpha|} c_{j,\alpha}. \tag{5}$$

On the other hand, from the definition of $T(u)$ (see (2)) we have

$$| < T(u), \zeta_k > | \leq Ck \int_{\{(x,y,z) \in Q;\, r < 1/k\}} |\nabla v|^2 \tag{6}$$

where $r^2 = x^2 + y^2$. Combining (5) and (6) and letting $k \to \infty$ yields $c_{j,\alpha} = 0$, i.e.,

$$< T(u) = \sum_{j=1}^{k} c_j \delta_{a_j}. \tag{7}$$

Step 3: The constant c_j in (7) is given by $c_j = 2\pi$ deg (u, a_j). Assume as above that $a_j = 0$; Fix R such that $R < \min_{k \neq j} |a_k|$ and $R <$ dist $(0, \partial\Omega)$. Consider the annulus

$$\omega = \{(x,y) \in \mathbb{R}^2; R/2 < r < R\}.$$

Choose any smooth function $\zeta(x,y)$ such that

$$\zeta = \begin{cases} 1 & \text{for } r < R/2, \\ 0 & \text{for } r > R. \end{cases}$$

We have , by (7),

$$< T(u), \zeta >= c_j.$$

From the definition of $T(u)$ (see (2)) and since $\nabla\zeta = 0$ outside ω we have

$$< T(u), \zeta >= -2 \int_{\omega \times (0,1)} (v_y \wedge v_z)\zeta_x - 2 \int_{\omega \times (0,1)} (v_z \wedge v_x)\zeta_y$$

Since $u \in H^1(\omega)$ we have, as in the proof of Lemma 1.15,

$$< T(u), \zeta >= \int_\omega (u_y \wedge u)\zeta_x + \int_\omega (u \wedge u_x)\zeta_y$$

Applying formula (11) from Section 1.4 with $\varphi = 1 - \zeta$ yields

$$\begin{aligned} \deg\ (u, a_j) &= -\frac{1}{2\pi} \int_w (u \wedge u_y)\zeta_x - \frac{1}{2\pi} \int (u_x \wedge u)\zeta_y \\ &= \frac{1}{2\pi} < T(u), \zeta >= \frac{1}{2\pi} c_j, \end{aligned}$$

which is the desired conclusion.

Remark 1.32. The concept of a distribution T describing the location and topological charge of the singular set of a map u has been originally introduced by Brezis, Coron and Lieb [1]. There, $u \in H^1(\Omega; S^2)$ where Ω is a domain in $\mathbb{R}^3$. One considers first the vector field

$$D(u) = (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y)$$

which belongs to $L^1(\Omega; \mathbb{R}^3)$. The distribution $T(u)$ is defined by

$$T(u) = \text{ div } D(u) \quad \text{ in } \mathcal{D}'(\Omega; \mathbb{R})$$

When u is smooth except at a finite number of points (a_j) in Ω, then

$$T(u) = 4\pi \sum d_j \delta_{a_j} \quad \text{ in } \mathcal{D}'(\Omega; \mathbb{R})$$

where $d_j =$ deg (u, a_j) denotes the degree of u restricted to a small sphere centered at a_j.

In our current setting, if $u \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, where Ω is a domain in $\mathbb{R}^2$, we consider the vector field

$$H(u) = (u \wedge u_x, u \wedge u_y) \in L^1(\Omega; \mathbb{R}^2)$$

and the distribution

$$T(u) = \text{ curl } H(u) \qquad \text{ in } \mathcal{D}'(\Omega; \mathbb{R}).$$

By Lemma 1.15 it coincides with our distribution $T(u)$ when $u \in H^{1/2}(\Omega; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$. More general situations have been investigated by Jerrard and Soner [1],[2].

By analogy with the results of Brezis, Coron and Lieb [1], and Bethuel, Brezis and Coron [1](concerning maps u from a domain in $\mathbb{R}^3$ with values into S^2) we associate to every map $u \in H^{1/2}(\Omega; \mathbb{R}^2)$ a number $L(u) \geq 0$ defined by

$$L(u) = \frac{1}{2\pi} \operatorname{Sup}_{\substack{\zeta \in C_0^\infty(\Omega; \mathbb{R}) \\ \|\nabla \zeta\|_{L^\infty} \leq 1}} < T(u), \zeta > . \tag{8}$$

It is easy to see from (2) that

$$| < T(u), \zeta > | \leq C \|u\|^2_{H^{1/2}} \|\nabla \zeta\|_{L^\infty} \quad \forall u \in H^{1/2}(\Omega; \mathbb{R}^2), \ \forall \zeta \in C_0^\infty(\Omega) \tag{9}$$

and

(10)

$$| < T(u_1) - T(u_2), \zeta > | \leq C\|u_1 - u_2\|_{H^{1/2}}(\|u_1\|_{H^{1/2}} + \|u_2\|_{H^{1/2}})\|\nabla\zeta\|_{L^\infty} \ \forall u_1, u_2 \in H^{1/2}(\Omega; \mathbb{R}^2), \forall \zeta \in C_0^\infty($$

In particular we deduce that

$$L(u) \leq C||u||^2_{H^{1/2}} \qquad \forall u \in H^{1/2}(\Omega; \mathbb{R}^2) \tag{11}$$

and

$$|L(u_1) - L(u_2)| \leq C||_{u_1 - u_2}||_{H^{1/2}}(||u_1||_{H^{1/2}} + ||u_2||_{H^{1/2}}) \forall u_1, u_2 \in H^{1/2}(\Omega; \mathbb{R}^2). \tag{12}$$

When u takes its values in S^1 and has only a finite number of singularities there is a very simple expression for $L(u)$ in terms of length of a "minimal connection" connecting the singularities of u. In order to define it we introduce a new (semi-)metric on $\mathbb{R}^2$

$$d(x, y) = \text{ min } \{|x - y|, \text{ dist } (x, {}^c\Omega) + \text{ dist } (y, {}^c\Omega)\} \tag{13}$$

where ${}^c|\Omega = \mathbb{R}^2\Omega$. Note that $d(x, y) = 0$ if $x, y \in \Omega$ and $d(x, y) = \text{ dist } (x, \partial\Omega)$ if $x \in \Omega$ and $y \in \partial\Omega$. Given a sequence of "positive" points $(p_j), j = 1, 2, ..., k$ in $\overline{\Omega}$ and a sequence of negative points $(n_j), j = 1, 2, ..., k$ in $\overline{\Omega}$ with an equal number of points, we define

$$L(p, n) = \underset{\sigma}{M}in \sum_{j=1}^{k} d(p_j, n_{\sigma(j)}) \tag{14}$$

where the minimum in (12) is taken over all permutations σ of the integers $1, 2, ..., k$.

Now, given a finite number of points (a_j) in Ω with associated integers (d_j) in $\mathbb{Z}$, we say that a_j is a positive point if $d_j > 0$, respectively a negative point if $d_j < 0$. We list the positive points with each a_j repeated with multiplicity d_j. Likewise, list the negative points, with each a_j repeated $|d_j|$ times. The points a_j with $d_j = 0$ are omitted from these two lists. Write this list as

$$p_1, p_2, ..., p_{k_+}, \quad n_1, n_2, ...n_{k_-}$$

where

$$k_+ = \sum_{d_j > 0} \quad \text{and } k_- = \sum_{d_j < 0} |d_j| = -\sum_{d_j < 0} d_j.$$

If $\sum_j d_j = 0$ we have an equal number of positive and negative points. If $\sum_j d_j > 0$ we have an excess of

$$k_+ - k_- = \sum_j d_j$$

positive points. To balance them we introduce in the list $(k_+ - k_-)$ negative points arbitrarily placed in $\partial\Omega$. Likewise, if $\sum_j d_j < 0$ we introduce in the list $|\sum_j d_j|$ positive points placed in $\partial\Omega$.

In this way, we end up, in all cases, with an equal number of positive and negative points. Write it as

$$p_1, p_2, ...p_k, \; n_1, n_2, ..., n_k.$$

The number L defined by (14) is called the **length of a minimal connection** for the configuration (a_j, d_j) (including connections to the boundary).

Theorem 1.20. *Assume u is as in Theorem 1.19, then the number $L(u)$ defined by (8) coincides with the length L of a minimal connection for the singularities of u (including connections to the boundary).*

Proof. In view of Theorem 1.19 we have

$$T(u) = 2\pi(\sum_{j=1}^{k} \delta_{p_j} - \sum_{j=1}^{k} \delta_{n_j}) \text{ in } \mathcal{D}'(\Omega). \tag{15}$$

(Note that points p_j or n_j on $\partial\Omega$ contribute nothing to the sum in (14). For $\zeta \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \frac{1}{2\pi} < T(u), \zeta > &= \sum_j \zeta(p_j) - \sum_j \zeta(n_j) \\ &= \sum_j \zeta(p_j) - \sum_j \zeta(n_{\sigma(j)}) \end{aligned}$$

for any permutation σ of the integers $1, 2, ..., k$. If we assume in addition that $||\nabla\zeta||_{L^\infty} \leq 1$ it is easy to see that

$$|\zeta(x) - \zeta(y)| \leq d(x, y) \quad \forall x, y \in \overline{\Omega}$$

and thus

$$\frac{1}{2\pi}| < T(u), \zeta > | \leq \sum_j d(p_j, n_{\sigma(j)})$$

for every permutation σ. Consequently

$$L(u) = \frac{1}{2\pi} \operatorname*{Sup}_{\substack{\zeta \in C_0^\infty(\Omega;\mathbb{R}) \\ ||\nabla\zeta||_{L^\infty} \leq 1}} < T(u), \zeta > \leq L$$

where

$$L = \underset{\sigma}{Min} \sum_j d(p_j, n_{\sigma(j)})$$

is the length of a minimal connection for the singularities of u. To prove the reverse inequality we rely on the following elementary, but basic, lemma taken from Brezis, Coron and Lieb [1]:

Lemma 1.16. *Let M be a metric space and let $p_1, p_2, ..., p_k and\, n_1 n_2, ...n_k$ be $2k$ points in M. Then*

$$\underset{\zeta \in Lip_1(M;\mathbb{R})}{Max} \sum_j \zeta(p_j) - \sum_j \zeta(n_j) = L(p,n)$$

where

$$\mathrm{Lip}_1(M;\mathbb{R}) = \{\zeta : M \to \mathbb{R}; |\zeta(x) - \zeta(y)| \leq d(x,y)\ \forall x, y \in M\}$$

and

$$L(p,n) = \underset{\sigma}{Min} \sum_j d(p_j, n_{\sigma(j)}).$$

Sketch of proof. As above, the inequality

$$\underset{\zeta}{Sup} \sum_j \zeta(p_j) - \sum_j \zeta(n_j) \leq L(p,n)$$

is obvious. The reverse inequality is more delicate. It was originally proved by Brezis, Coron and Lieb [1] using a theorem of L.V. Kantorovich (on the transfer of masses) and a theorem of G. Birkhoff (on the extremal points of doubly stochastic matrices). A direct argument is presented in Brezis [2]. The idea is to relabel the points (n_j) so that

$$L(p,n) = \sum_j d(p_j, n_j)$$

and to construct a function ζ defined only on the finite set

$$Q = (\cup_{j=1}^k p_j)(\cup_{j=1}^k n_j)$$

satisfying

$$|\zeta(x) - \zeta(y)| \leq d(x,y) \quad \forall x, y \in Q \tag{16}$$

and

$$\zeta(p_j) - \zeta(n_j) = d(p_j, n_j) \quad \forall j. \tag{17}$$

This is the heart of the matter. It boils down to a **system of linear inequalities** for the unknowns $X_j = \zeta(n_j)$ (since $\zeta(p_j)$ is then given by $X_j = d(p_j, n_j)$). A solution to this system is found using elementary tools from linear programming (sse Brezis [2]).

Once ζ has been defined on the set Q, it may then be extended to all of M by letting

$$\widetilde{\zeta}(x) = \underset{y \in Q}{Inf}\{\zeta(y) + d(x,y)\}, x \in M.$$

It is easy to check that $\widetilde{\zeta} \in$ $\text{Lip}_1(M;\mathbb{R})$ and

$$\sum_j \widetilde{\zeta}(p_j) - \sum_j \widetilde{\zeta}(n_j) = L(p,n).$$

Proof of Theorem 1.20 completed. Using Lemma 1.16 with $M = \mathbb{R}^2$ equipped with the metric $d(x,y)$ defined in (13). We obtain a function $\zeta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{gathered} |\zeta(x) - \zeta(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}^2 \\ \zeta \text{ is constant on } {}^c\Omega \\ \sum_j \zeta(p_j) - \sum_j \zeta(n_j) = L(p,n). \end{gathered}$$

By adding a constant we may always assume that

$$\zeta = 0 \quad \text{ on } {}^c\Omega.$$

Finally, with standard truncation and mollification techniques(see e.g. the proof of Théorème 1X.17 in Brezis [1]) one constructs a sequence (ζ_l) in $C_0^\infty(\Omega;\mathbb{R})$ such that

$$||\nabla\zeta_l||_{L^\infty} \leq 1$$

and

$$\zeta_l \to \zeta \text{ as } l \to \infty, \text{ uniformly on } \overline{\Omega}.$$

Some properties of $T(u)$ and $L(u)$ are easily verified when $u \in H^{1/2}(\Omega;S^1)$ is smooth except at a finite number of points. They can be extended by density to all functions $u \in H^{1/2}(\Omega;S^1)$. Here is an example. The product u_1u_2 of two functions $u_1, u_2 \in H^{1/2}(\Omega;S^1)$ belongs to $H^{1/2}(\Omega;S^1)$ (check)! where u_1u_2 denotes complex multiplication.

Theorem 1.21 *For every $u_1u_2 \in H^{1/2}(\Omega;S^1)$ we have*

(18) $$T(u_1u_2) = T(u_1) + T(u_2) \quad \text{in } \mathcal{D}'(\Omega;\mathbb{R})$$

$$L(u_1 u_2) \leq L(u_1) + L(u_2), \tag{19}$$

$$L(u_1 \bar{u}_2) \leq C||u_1 - u_2||_{H^{1/2}}(||u_1||_{H^{1/2}} + ||u_2||_{H^{1/2}}). \tag{20}$$

The proof relies on the following density result already mentioned at the end of Section 1.6.

Lemma 1.17 (Rivière [2]). Set

$$\mathcal{R}_0 = \{u \in H^{1/2}(\Omega; S^1); u is smooth except at a finite number of points in \Omega\}.$$

Then $\mathcal{R}_0$ *is dense in* $H^{1/2}(\Omega; S^1)$.

Proof of Theorem 1.21. Let (u_{1n}) and (u_{2n}) be sequences in $\mathcal{R}_0$ such that $u_{1n} \to u_1$ in $H^{1/2}$ and $u_{2n} \to u_2$ in $H^{1/2}$. Clearly $u_{1n}u_{2n} \in \mathcal{R}_0$ and moreover $u_{1n}u_{2n} \to u_1u_2$ in $H^{1/2}$ (this is a consequence of Lemma A.1.5 in Appendix A.11 applied to $\Phi(z_1, z_2) = z_1 z_2$ which is globally Lipschitz on $S^1 \times S^1$). It is easy to see, using Theorem 1.19 and the standard properties of degree(see equation(1) in Section 1.3), that

$$T(u_{1n}u_{2n}) = T(u_{1n}) + T(u_{2n}).$$

Passing to the limit (using (10)) yields (18).

Inequality (19) is a direct consequence of (18) and the definition of $L(u)$ (see (8)).

To establish (20) note that, by (18),

$$T(u_1 \bar{u}_2) = T(u_1) - T(u_2)$$

and thus

$$\begin{aligned} L(u_1 \bar{u}_2) &= \frac{1}{2\pi} \operatorname{Sup}_{\substack{\zeta \in C_0^\infty(\Omega;\mathbb{R}) \\ ||\nabla\zeta||_{L^\infty} \leq 1}} < T(u_1) - T(u_2), \zeta > \\ &\leq C||u_1 - u_2||_{H^{1/2}}(||u_1||_{H^{1/2}} + ||u_2||_{H^{1/2}}) \text{ by (10).} \end{aligned}$$

1.7A Superseded

Instead of considering a bounded domain in $\mathbb{R}^2$, we will rather work with the boundary of a smooth bounded $U \subset \mathbb{R}^3$. We assume further that ∂U is simply connected.

Recall that

a) there is some $u \in H^{\frac{1}{2}}(\partial U; S^1)$ which cannot be lifted as $u = e^{i\varphi}$ with $P \in H^{\frac{1}{2}}(\partial U; \mathbb{R})$ (by Theorem 1.5 and the example thereafter). For example, if $0 \in U$, we may choose $u_1(x, y, z) = e^{\frac{i}{|(x,y)|^{1/2}}}$ or $u_2(x, y, z) = \frac{(x,y)}{|(x,y)|}$,

b) there is some $u \in H^{\frac{1}{2}}(\partial U; S^1)$ which cannot written as $u = v_{|\partial U}$ for any $v \in H^1(U; S^1)$ (this is obtained by adapting the proof of Theorem 1.16, c)). As we shall see below, a function has the lifting property if and only if it has the extension property. Hence we may choose u_1 or u_2 as examples,

c) $C^\infty(\partial U; S^1)$ is not dense in $H^{\frac{1}{2}}(\partial U; S^1)$. We know, for example, that $u_2 \notin \overline{C^\infty(\partial U; S^1)}^{H^{\frac{1}{2}}}$ (see sketch of proof of Theorem 1.18; later in this section we will also give a different argument).

It is thus natural to consider the following classes:

$$X = \{u \in H^{\frac{1}{2}}(\partial U; S^1)/\exists \varphi \in H^{\frac{1}{2}}(\partial U; \mathbb{R}) \text{ s.t. } u = e^{i\varphi}\};$$

$$Y = \{u \in H^{\frac{1}{2}}(\partial U; S^1)/\exists v \in H^1(U; S^1) \text{ s.t. } v_{|\partial U} = u\};$$

$$Z = \overline{C^\infty(\partial U; S^1)}^{H^{\frac{1}{2}}}.$$

Lemma 1.9. *We have*

$$X = Y \underset{\neq}{\subset} Z \underset{\neq}{\subset} H^{\frac{1}{2}}(\partial U; S^1).$$

Proof. Let $u \in X$ and $\varphi \in H^{\frac{1}{2}}(\partial U; \mathbb{R})$ be such that $u = e^{i\varphi}$. Let $\psi \in H^1(U; \mathbb{R})$ with $\psi_{|\partial U} = \varphi$ and set $v = e^{i\psi}$. Then $v \in H^1_u(U; S^1)$. Hence $X \subset Y$. Conversely, let $u \in Y$ and $v \in H^1_u(U; S^1)$. By Theorem 1.1, we may write $v = e^{i\psi}$ for some $\psi \in H^1(U; \mathbb{R})$. then $u = e^{i\varphi}$, where $\varphi = \psi_{|\partial U} \in H^{\frac{1}{2}}(\partial U; \mathbb{R})$.

If $u \in X = Y$, let $\varphi \in H^{\frac{1}{2}}(\partial U; \mathbb{R})$ with $u = e^{i\varphi}$. If $(\varphi_n) \subset C^\infty(\partial U; \mathbb{R})$ and $\varphi_n \to \varphi$ in $H^{\frac{1}{2}}$, then $e^{i\varphi_N} \to e^{i\varphi} = u$ in $H^{\frac{1}{2}}$, by Lemma A.1.5. Hence $u \in Z$.

To prove that $Z \neq X$, we assume $0 \in U$. Recall that, by the proof of Theorem 1.5, $u_1 \notin X$. However, $u_1 = \lim_{\varepsilon \to 0^+} u_\varepsilon$ in $H^{\frac{1}{2}}(\partial U; S^1)$, where $u_\varepsilon(x, y, z) = e^{\frac{i}{\varepsilon + |(x,y)|}^{\frac{1}{2}}}$. Since $u_\varepsilon \in H^{\frac{1}{2}} \cap C^0$, by a standard approximation procedure we find that $u_1 \in Z$.

Finally, $Z \neq H^{\frac{1}{2}}(\partial U; S^1)$ by *c*).

There is an interesting distribution associated to every $u \in H^{\frac{1}{2}}(\partial U; \mathbb{R}^2)$. This distribution describes the location and "topological charge" of the singular set of u.

Definition 5. Let $v \in H^1(U;\mathbb{R}^2)$ be any map such that $v_{|\partial U} = u$. Set

$$D = rD(v) = \begin{cases} 2(v_y \wedge v_z, v_z \wedge v_x, v_x \wedge v_y) & \text{in } U \\ 0 & \text{in } \mathbb{R}^3 \backslash U \end{cases}$$

and $T = \text{div } D$ (T makes sense as distribution on $\mathbb{R}^3$ since $D \in L^1(\mathbb{R}^3;\mathbb{R}^3)$).

Remark 1. D depends on the coice of v, but it is easy to see that it does not depend on the choice of a direct orthonormal base in $\mathbb{R}^3$.

Theorem 1.20. *T is independent of the choice of v. Moreover,*

$$\textit{supp } T \subset \partial U.$$

We may thus use the notation $T = T(u)$.

Proof. For the first property, as in the proof of Theorem 1.10, it suffices to check that

$$\int_v \big[f_Y \wedge g_z + f_z \wedge g_y)\xi_x + (f_z \wedge g_x + f_x \wedge g_z)\xi_y + (f_x \wedge g_y + f_y \wedge g_x)\xi_z\big] = 0,$$
$$\forall f \in C_0^\infty(U;\mathbb{R}^2), g \in C^\infty(\overline{U};\mathbb{R}^2), \xi \in C_0^\infty(\mathbb{R}^3;\mathbb{R}).$$

This follows easily by integration by parts.

As for the second property, we clearly have supp $T \subset \overline{U}$. By choosing v smooth in U, we find that $T = 0$ in U. Hence supp $T \subset \partial U$.

Theorem 1.21 *If $u \in H^{\frac{1}{2}}(\partial U;\mathbb{R}^2) \cap W^{1,1}(\partial U;\mathbb{R}^2)$ then*

$$\langle T(u), \xi\rangle = -\int_{\partial U} u \wedge (\nabla_T u \wedge \nabla_T \xi), \xi \in C_0^\infty(\mathbb{R}^3\mathbb{R}). \tag{1}$$

Note that $\nabla_T u(x) \in (T_x^\star(\partial u))^2$ has in intrinsic meaning. The first $\wedge$ denotes the exterior product in $\mathbb{R}^2$ and the second $\wedge$ is the exterior product in $T_x^\star(\partial U)$.

In local coordinates (x, hy) on ∂U, and writing $u = (u_1, u_2)$, we have

$$u_\Lambda(\nabla_T u \wedge \nabla_T \xi) = (u_1 u_{2x} - u_2 u_{1x})\xi_1 - (u_1 u_{2y} - u_2 u_{1y})\xi_x.$$

Proof. We start with $u \in C^\infty(\partial U;\mathbb{R}^2)$ and $v \in C^\infty(\overline{U};\mathbb{R}^2)$. Then

$$\langle T(u), \xi\rangle = -2 \sum_{\text{cyclic}} \int_U (v_y \wedge v_z)\xi_x = 0 \sum_{\text{cyclic}} \int_U [(v_y \wedge v_z)\xi_x + (v_y \wedge v_z)\xi_x]$$
$$= -\sum_{\text{cyclic}} \int_{\partial U} [v \wedge v_z)\xi_x n_y + (v_y \wedge v)\xi_x n_z] ds$$
$$+ \sum_{\text{cyclic}} \int_U [v \wedge v_{yz})\xi_x + (v \wedge v_z)\xi_{xy} + (v_{yz} \wedge v)\xi_x + (v_y \wedge v)\xi_z],$$

so that

$$\begin{aligned}\langle T(u), \xi\rangle &= -\int_{\partial U} v \wedge \det(\nabla v, \nabla \xi, n) = -\int_{\partial U} v \wedge (\nabla_t v \wedge \nabla_t \xi) ds \\ &= -\int_{\partial U} u \wedge (\nabla_T u \wedge \nabla_t \xi).\end{aligned}$$

For a general u as in the theorem, consider a sequence $(u_n) \subset C^\infty(\partial U; \mathbb{R}^2)$ such that $u_n \to u$ in $H^{\frac{1}{2}}$ and in $W^{1,1}$ and $\|u_n\|_\infty \leq C$. By taking for v_n and the harmonic extensions of u_n, u to U, we have $v_n \to v$ in H^1 and hence $D(v_n) \to D(v)$ in L^1. Therefore, $T(u_n) \to T(u)$ in $\mathcal{D}$. On the other hand,

$$\int_{\partial U} u_n \wedge (\nabla_T u_n \wedge \nabla_T \xi) \to \int_{\partial U} u \wedge (\nabla_T u \wedge \nabla_T \xi).$$

The proof is complete.

Remark 1. If u belongs only to $W^{1,1}(\partial U; \mathbb{R}^2)$, we may take as definition of T the right-hand side of (1). In the case of a flat boundary, we have

$$\begin{aligned}\langle T(u), \xi\rangle &= -\int u \wedge (u_x \xi_y - u_x \xi_x) = \langle \frac{\partial}{\partial y}(u \wedge u_x) = \frac{\partial}{\partial x}(u \wedge y), \xi\rangle \\ &= \langle \text{curl } (u \wedge u_x, u \wedge u_y), \xi\rangle - \langle \text{curl } H, \xi\rangle.\end{aligned}$$

. Clearly, $T(u) = 0$ if $u \in C^\infty(\partial U; S^1)$, or, more generally, if $u \in \overline{C^\infty(\partial U; S^1)}^{W^{1,1}}$. The converse (i.e., if $u \in W^{1,1}(\partial U; S^1)$ and $T(u) = 0$, then $u \in \overline{C^\infty(\partial U; S^1)}^{W^{1,1}}$) is also true, see Demengel [1]. A similar result holds for general ∂U.

We now turn to the case of functions which are "smooth" except a finite number of points. More precisely, let $u \in H^{\frac{1}{2}}(\partial U; S^1) \cap H^1_{\text{loc}}(\partial U \backslash \{a_1, -a_k\})$. For $r > 0$ small enough, one may define a geodesic positively oriented circle around a_j. For all such r, $u_{|C_r} \in H^{\frac{1}{2}}$, so it has a degree. As in the proof of theorem 1.15, this degree is independent of r. We all it the degree of u around a_j and denote it $\deg(u, a_j)$.

Theorem 1.22. *If $u \in H^{\frac{1}{2}}(\partial U; S^1) \cap H^1_{\text{loc}}(\partial U \backslash \{a_1 \upharpoonright a_k\})$, then*

$$T(u) = 2\pi \sum_{j=1}^{k} \deg(u, a_j) \delta_{a_j}.$$

Proof. We first prove that supp $T(u) \subset \{a_1, \upharpoonright a_k\}$. Take $a \in \partial U \backslash \{a_1, , \dots, a_k\}$ and a smooth simply connected neighborhood $\sum$ of a in $\partial U \backslash \{a_1 \upharpoonright a_k\}$. In $\sum$ we may write

$u = e^{i\varphi}, \varphi \in H^1(\sum; \mathbb{R})$ and we may extend u to some $v \in H^1(\omega; S^1)$, where ω is a tubular neighborhod of $\sum$ in $U \backslash \{1_1, \restriction a_k\}$. Hence we may tend v to an H^1 function, still denoted v, such that $v \in H^1_u(U; \mathbb{R}^2)$ and $v \in H^1(\omega; S^1)$. If $\xi \in C_0^\infty(\omega)$, then

$$\langle T(u), \xi \rangle = -\int_\omega [(v_y \wedge v_z)\psi_x + (v_z \wedge v_x)\psi_y + (v_x \wedge v_y)\psi_z] = 0,$$

since $\nabla v \perp v$.

Next, we prove that, for some coefficients $k_j \in \mathbb{R}$,

$$T(u) = \sum_{j=1}^{k} k_j \delta_{aj}. \tag{2}$$

By a well-known result of L. Schwartz, we have

$$\text{div } D = T(u) = \sum_{\text{finite}} C_{\alpha_{ij}} \partial^\alpha \delta_{aj}. \tag{3}$$

By taking Fourier transforms in (3), we find that

$$i\zeta \cdot \hat{D}(\zeta) = \sum c_{\alpha,j} (i\zeta)^\alpha e^{-iaj\cdot\zeta}, \qquad \forall \zeta \in \mathbb{R}^3. \tag{4}$$

Now since $D \in L^1$, we have $\lim\limits_{|\zeta|\to\infty} |\hat{D}(\zeta)| = 0$, so that the right-hand side of (3) is $\theta(|\zeta|)$ as $|\zeta| \to \infty$. This implies $c_{\alpha,j} = 0$ if $|\alpha| \geq 1$.

Finally, we identify the coefficients k_j. We reduce the problem to that of a localy flat boundary with the help of

Lemma 1.10. *Let $x : \overline{\omega} \to \overline{U}$ be a positive C^1-diffeo-morphism and $u \in H^{\frac{1}{2}}(\partial U; S^1)$, $\xi \in C_0^\infty(\mathbb{R}^3)$.*

Then

$$\langle T(u), \xi \rangle = \langle T(u_0 \chi), \xi \circ \chi \rangle. \tag{5}$$

We note that $\langle T(U), \xi \rangle$ is well-defined by $\xi_{|\overline{U}}$ and if we merely have $\xi \in C^1$. therefore, equality (5) is meaningful.

Proof of Lemma 1.10. We have, if $v = (v^1, v^2)$,

$$\langle T(u), \xi \rangle = -\int_U D(v) \cdot \nabla \xi = -\int_U \det(\nabla \xi, \nabla v^1, \nabla v^2),$$

and

$$\begin{aligned}\langle T(u\circ\chi),\xi\circ\chi\rangle &= -\int_\omega \det(\nabla(\xi\circ\chi),\nabla(v^1\circ\chi),\nabla(v^2\circ\chi))\\ &= -\int_\omega |\det(\nabla\xi)|[\det(\nabla\xi,\nabla v^1,\nabla v^2)]\circ\chi = -\langle T(u),\xi\rangle.\end{aligned}$$

Proof of Theorem 1.22 completed. Since the coefficients k_j are invariant under a positive diffeomorphism (by Lemma 1.10) and so is the degree of u around aj, it suffices to consider the case of a flat boundary. Assume, e.g., that $u \in H^1_{\mathrm{loc}}(B_1\backslash\{0\};S^1)$ is of degree d around 0, and that locally U is below the $x0y$ plane (hence, positively oriented circles correspond to the usual ones). Let $f \in C^\infty(\mathbb{R}^+,\mathbb{R})$ be such that $f(r)=0$ for $0 \ge r \le \frac{1}{2}$, $f(r)=0$ for $r \ge \frac{2}{3}$. by choosing $\xi \in C_0^\infty(\mathbb{R}^3)$ such that $\xi(x,y,z)=f(|(x,y)|)$ for $|z|\le 1$, $v \in H^1_U(B_1\times\mathbb{R};\mathbb{R}^2)$, it suffices to prove that

$$\langle T(u),\xi\rangle = 2\pi d.$$

We take v such that $v(x,y,z)=u(x,y)f(z)$ for $|(x,y)|\ge\frac{1}{2}$ and $v=0$ for $|z|\ge 1$. Then

$$\begin{aligned}\langle T(u),\xi\rangle &= 2\iiint_{\substack{z\le 0\\ |(x,y)|\ge\frac{1}{2}}} f(z)f'(z)\big[(u_y\wedge u)\xi_x + (u\wedge u_x)\xi_y\big]\\ &= \iint_{|(x,y)|\ge\frac{1}{2}} \big[(u_y\wedge u)\xi_x + (u\wedge u_x)\xi_y\big]\\ &\iint_{|(x,y)|\ge\frac{1}{2}} \big[(u_Y\wedge u)\frac{x}{|(x,y)|} + (u\wedge u_x)\frac{y}{|(x,y)|}\big]f'(|(x,y)|)\\ &= -\iint_{\frac{1}{2}\le r\le 1} (u\wedge u_t)f'(r) = 2\pi d\int_{\frac{1}{2}\le r\le 1} f'(r)dr = 2\pi d.\end{aligned}$$

Theorem 1.23. *Let $u \in H^{\frac{1}{2}}(\partial U;S^1)$. Then $u\in Z \Rightarrow T(u)=0$.*

Proof. If $u \in C^\infty(\partial U;S^1)$, then $u=e^{i\varphi}$ for some $\varphi\in C^\infty(\partial U;\mathbb{R})$, and hence there is some $v\in C_u^\infty(\overline{U};S^1)$. Clearly, $D(v)=0$ in this case. Nof if $u_n\to u$ in $H^{\frac{1}{2}}$, we saw during the proof of Theorem 1 that $T(u_n)\to T(u)$ in $\mathcal{D}'$.

Remark 1. There is an analogue of T for $H^1(U;S^2)$ maps. If $u\in H^1(U;S^2)$, set

$$D(u)=\begin{cases}(u\cdot u_y\wedge u_z,\, u\cdot u_z\wedge u_x,\, u\cdot u_x\wedge u_y) & \text{in } 0\in L^1(\mathbb{R}^3;\mathbb{R}^3)\\ 0 & \text{in } \mathbb{R}^3\backslash U\end{cases}$$

and $T(u) = \text{div } D$.

Then $T(u) = 0$ if $u \in \overline{C^\infty(U;S^2)}^{H^1}$. The converse is also true, that is

$$\overline{C^\infty(U;S^2)}^{H^1} = \{u \in H^1(U;S^2)/T(u) = 0\},$$

see Bethuel [ ].

Remark 1. In connection with the above results, we call attention to some open problems:

a) Is it true that $Z(= \overline{C^\infty(\partial U;S^1)}^{H^{\frac{1}{2}}}) = \{u \in H^1(\partial U;S^1)T(u) = 0\}$ (see OP);
b) Is the set

$$\bigcup_{\substack{k\geq 1\\ \{a_1,\restriction a_k\}\subset\partial U}} H^1_{\text{loc}}(\partial U\backslash\{a_1,\restriction a_k\};S^1) \cap H^{\frac{1}{2}}$$

dense in $H^{\frac{1}{2}}$? (see OP). This could be useful in proving *a*); see OP .

c) Given $u \in H^{\frac{1}{2}}(\partial U;S^1)$ consider the distribution $T(u)$ defined above. Does there exist sequence of points (P_i) and (N_i) on ∂U such that

$$\sum_i |P_i - N_i| < \infty$$

and

$$\langle T(u), \xi\rangle = \sum_i (\xi(P_i) - \xi(N_i)) \quad \forall \xi \in C^\infty(\partial U;\mathbb{R})?$$

see OP .

[The analogous statement for $u \in H^1(U;S^2)$ is true; see XXX.]

e) Is there a single characterization of maps $u \in X = v$; see OP 6.

Remark 1. One may define an analogue of T for functions $u \in W^{1-\frac{1}{k},k}$ $(\partial U;S^{k-1})$, where U is a smooth bounded domain in $\mathbb{R}^{k+1}$.

Since the distribution T belongs to the dual space of $W^{1,\infty}$, it is natural to consider its norm in that space. There are two possibilities, but they yield equivalent norms.

Definition 6. For $u \in H^{\frac{1}{2}}(\partial U;\mathbb{R}^2)$, set

$$L_1(U) = 1 \sup_{\substack{\xi\in Lip(\overline{U};\mathbb{R})\\ |\nabla\xi|\leq 1 \text{ a.e. on } U}} \langle T, \xi\rangle.$$

Definition 7. For $u \in H^{\frac{1}{2}}(\partial U; \mathbb{R}^2)$, set

$$L_2(u) = \frac{1}{2\pi} \sup_{\substack{\xi \in Lip(\partial U;\mathbb{R}) \\ |\nabla_T \xi| \leq 1 \text{ a.e.}}} \quad \text{in } \partial U$$

compute explicitly L_1, L_2 for "nice" singular functions. This is explained below.

Let $u \in H^{\frac{1}{2}}(\partial U; S^1) \cap H^1_{\text{loc}}(\partial U \backslash \{a_1, \rightharpoondown a_k\}; S^1)$. By Theorem 1, we have $T(u) = 2\pi \sum_1^k \deg(u, aj)\delta aj$. Since $\langle T(u), 1\rangle = 0$, it follows that $\sum_1^k \deg(u, aj) = 0$. Call aj a "positive" singularity if $\deg(u, aj) > 0$, a "negative" singularity otherwise. Let $P_1, \dots, P_n$ be the list of positive singularities, each one repeated according to its degree. Let $V_1, \dots, N_n$ (same n) be the list of negative singularities. Applying in Brezis, Coron and Lieb [1] one obtains

Theorem 1. *For $u \in H^{\frac{1}{2}}(\partial U; S^1) \cap H^1_{\text{loc}}(\partial U \backslash \{a_1 \rightharpoondown a_k\}; S^1)$,*

$$L_j(u) = \inf_{\sigma \in S_n} \sum_1^n \; dist_j(P_i, N_{\sigma(i)}), \quad j = 1, 2,$$

where d_1 is the geodesical distance in $\overline{U}$ and d_2 is the geodesical distance on ∂U.

Clearly, $d_1 \leq d_2 \leq K d_1$ for some constant K, so that the same holds for L_1 and L_2.

Remark 1. While L_2 is a more "intrinsic" quantity, L_1 is an important quantity when studying the 3d Ginzburg-Landau equation (see XXX).

Remark 1. Starting from the definition of T as div D, and taking in that definition harmonic extensions, one may easily see that

$$\begin{aligned} &\left|\langle T(u^1), \xi\rangle - \langle T(u^2), \xi\rangle\right| = \left|\langle D(\tilde{u}^1), \nabla\xi\rangle - \langle D(\tilde{u}^2), \nabla\xi\rangle\right| \\ &\leq C \|\nabla(\tilde{u}_1 - \tilde{u}_2)\|_{L^2} (\|\nabla \tilde{u}_1\|_{L^2} + \|\nabla \tilde{u}_2\|_{L^2} \|\nabla\xi\|_{L^\infty} \end{aligned}$$

where $\tilde{u}$ is the harmonic extension of u. It then follows that

$$\left|L_1(u^1) - L_1(u^2)\right| \leq C \|u^1 - u^2\|_{H^{\frac{1}{2}}} (\|u^1\|_{H^{\frac{1}{2}}} + \|u^2\|_{H^{\frac{1}{2}}}). \tag{6}$$

Since L_1 and L_2 are equivalent quantities, we also find that

$$\left|L_2(u^1) = L_2(u^2)\right| \leq C \|u^1 - u^2\|_{H^{\frac{1}{2}}} (\|u^1\|_{H^{\frac{1}{2}}} + \|u^2\|_{H^{\frac{1}{2}}}). \tag{7}$$

Remark 1. In the case of $H^1(U : S^2)$ functions, $L(u)$ plays an important role in computing a relaxed energy; see Bethuel, Brezis and Coron [1] and also Giaquinta, Modica and Soucek

[1] in the framework of Cartesian currents. One ask whether a similar result holds in our setting. More precisely, let $u \in H^{\frac{1}{2}}(\partial U; S^1)$, is it true that

$$\operatorname*{Inf}_{\substack{u_n \in C^\infty(\partial U; S^1) \\ u_n \to u \text{ a.e.}}} \left\{ \liminf \|u_n\|^2_{H^{\frac{1}{2}}(\partial U)} \right\} = \|u\|^2_{H^{\frac{1}{2}}(\partial U)} + 2\pi L_1(u)$$

where $\|u\|^2_{H^{\frac{1}{2}}(\partial U)} = \int_U |\nabla v|^2$ and v is the harmonic extension of u to U? See OP .

$$\lim_{\varepsilon \to 0}$$

1.8. Open problems

OP1. Find a simple proof for Theorem 1.2 when $sp < 1$. More specifically find a simple proof of the existence of a lifting $\varphi \in H^s((0,1); \mathbb{R})$ for every $u \in H^s((0,1); S^1)$ when $0 < s < 1/2$.

OP2. Assume $0 < s < 1$, $1 < p < \infty$ and $sp < 1$. Let $u \in W^{s,p}((0,1); S^1)$. Can one find a lifting $\varphi \in W^{s,p}((0,1); \mathbb{R}) \cap L^\infty((0,1); \mathbb{R})$? [Clearly there is a lifting $\varphi_1 \in L^\infty$ and, by Theorem 1.2, there is another lifting $\varphi_2 \in W^{s,p}$, but they need not coincide]. Recall that the lifting constructed in the proof of Theorem 1.2 belongs to $W^{s,p}$ and also to every $L^q, q < \infty$. This suggests that there may exist a lifting in $W^{s,p} \cap BMO$.

OP3. When $N \geq 3$, the loss of regularity in lifting is not well understood. Consider, for example, the case $N = 3$ and $p = 4$ as in Remark 1.6. If $1/2 \leq s < 3/4$, it is plausible that every $u \in W^{s,4}(\Omega; S^1)$ admits a lifting φ in $W^{\sigma,4}(\Omega; \mathbb{R})$ where

$$\sigma = \frac{7s - 3}{4s}.$$

[**Comment:** This value of σ comes from the following formal computation. If $u = e^{i\varphi}$, then $D\varphi = -i\bar{u}Du$ and hence

$$\int \varphi \operatorname{div}\zeta = -i \int (Du)(\bar{u}\zeta) \quad \forall \zeta. \tag{1}$$

Choose $\zeta = \operatorname{grad}\psi$ where ψ is the solution of

$$\Delta\psi = D^\sigma f \quad , f \in L^{4/3} \text{given}.$$

It follows that

$$\zeta \in W^{1-\sigma,4/3} \subset L^q \quad \text{(by Sobolev)}$$

where

$$\frac{1}{q} = \frac{3}{4} - \frac{1-\sigma}{3}.$$

From (1) we have

$$(2) \qquad \Big| \int (D^\sigma \varphi) f \Big| \leq ||u||_{W^{s,4}} \ \ ||\bar{u}\zeta||_{W^{1-s,4/3}}.$$

On the other hand, by Gagliardo-Nirenberg

$$u \in W^{(1-s),4s/(1-s)}.$$

Since $\sigma < s$ and

$$\frac{1-s}{4s} + \frac{1}{q} = \frac{3}{4}$$

we conclude that $\bar{u}\zeta \in W^{1-s,4/3}$ with

$$\|\bar{u}\zeta\|_{W^{1-s,4/3}} \quad \leq \quad C\|f\|_{L^{4/3}}.$$

Inequality (2) suggests that $D^\sigma \varphi \in L^4$].

One cannot expect a lifting in a better space $W^{\sigma',4}$ with $\sigma' > \sigma$(use $u = e^{i/|x|^\alpha}$ with appropriate α as in the proof of Theorem 1.3 *b)*). The general question of lifting (in "optimal" spaces $W^{\sigma,p}$) for a map $u \in W^{s,p}(\Omega; S^1)$ when $N \geq 3, \ \ 2 \leq sp < N$ is still widely open.

OP4. Let M be a compact Riemannian manifold (without boundary) – for example $M = S^1$. Let $\widetilde{M}$ be a cover space for M. More precisely, assume that $\widetilde{M}$ is a complete Riemannian manifold – for example $\widetilde{M} = \mathbb{R}-$ and that there is a smooth map

$$\pi \quad : \quad \widetilde{M} \to \ M$$

which is onto and such that $\nabla\pi$ is a bijective linear operator with

$$||(\nabla\pi(a))^{-1}|| \quad \leq C \ \forall a \in \widetilde{M}.$$

(for example $\pi(a) = e^{ia}$). Let $\Omega \subset \mathbb{R}^N$ be a smooth simply connected domain and let $u \in W^{1,p}(\Omega; M)$. Under what conditions can one lift u, i.e., find $\varphi \in W^{1,p}(\Omega; \widetilde{M})$ such that

$$u = \pi \circ \varphi \ ?$$

Same question in fractional Sobolev spaces.

OP5. Let $f, g \in C^0(S^1; S^1)$ with Fourier coefficients (a_n) and (b_n). Assume

$$|a_n| = |b_n| \qquad \forall n \in \mathbb{Z}.$$

Can one conclude that

$$\deg(f) = \deg(g)?$$

[The answer is positive if one assumes in addition that $f, g \in H^{1/2}$; see Definition3 and Theorem 1.11 in Section 1.4. See also, in Remark 1.19, the difficulties which may arise when the maps do not belong to $H^{1/2}$].

OP6. Let $f \in C^0(S^1; S^1)$ with Fourier coefficients (a_n) , $n \in \mathbb{Z}$. Assume that

$$\sum_{n=1}^{+\infty} n|a_n|^2 < \infty$$

Can one conclude that $f \in H^{1/2}$?

[Note that the formula

$$\sum_{n=-\infty}^{-1} |n|\ |a_n|^2 \quad = -\deg f + \sum_{n=1}^{+\infty} n|a_n|^2,$$

valid when $f \in H^{1/2}$, is quite suggestive].

OP7. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers satisfying

$$\sum_{n=-\infty}^{+\infty} |n|\ |a_n|^2 < \infty, \tag{3}$$

$$\sum_{n=-\infty}^{+\infty} |a_n|^2 = 1, \tag{4}$$

and

$$\sum_{n=-\infty}^{+\infty} a_n \bar{a}_{n+k} = 0 \quad \forall k \neq 0. \tag{5}$$

Find a direct elementary proof of the fact that

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2 \in \mathbb{Z}. \tag{6}$$

(see Remark 1.19).

OP8. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth bounded domain (not necessarily simply connected). Let $s > 0$ and $p > 1$ with $sp < 2$. Is

$$W^{s,p}(\Omega; S^1) \text{ path - connected?}$$

[**Comment:** The answer is positive in many special cases:

a) If $sp < 1$. One can write any $u \in W^{s,p}(\Omega; S^1)$ as $u = e^{i\varphi}$ with $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ by Theorem 1.3. Then, one may connect u to 1 via the homotopy $e^{it\varphi}$

b) If $s = 1$. This is a special case of Theorem 0.2 in Brezis and Li [1].

c) If $N = 2$. The result is due to P. Mironescu (work in preparation).

d) If $N = 3$ and Ω is a solid torus the result is due to Mironescu (work in preparation).]

Note that the restriction $sp < 2$ is optimal: if $N = 2$ and Ω is an annulus, $W^{s,p}(\Omega; S^1)$ is **not** path-connected when $sp \geq 2$ by a result of Brezis, Li, Mironescu and Nirenberg [1].

More generally, if M is any compact Riemannian manifold, it is plausible that $W^{s,p}(\Omega; M)$ is path-connected for any $s > 0$, and any $p > 1$ with $sp < 2$.

OP9. Let $\Omega \subset \mathbb{R}^N, N \geq 2$, be a smooth, bounded, simply connected domain. Is $W^{s,p}(\Omega; S^1)$ path-connected for any $s > 0$ and any $p > 1$?

[**Comment:** The answer is positive when s is an integer. Indeed:

If s=1 and p<2, we know that $W^{1,p}(\Omega; S^1)$ is path-connected– even for general Ω – by Theorem 0.2 in Brezis and Li [1].

If s=1 and p≥2, we can write $u = e^{i\varphi}$ with $\varphi \in W^{1,p}$ by Theorem 1.1 and then connect u to 1 via the homotopy $e^{it\varphi}$.

If s≥2, then **sp≥2** and we can write $u = e^{i\varphi}$ with $\varphi \in W^{s,p} \cap W^{1,sp}$ by Theorem 1.4'*a)*. We may then connect u to 1 via the homotopy $e^{it\varphi}$.]

OP10. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $N \geq 3$. Assume $1 < p < \infty$ and $s < 1 - 1/p$ with $1 \leq sp < N$. Let

$$u \in W^{s,p}(\Omega; S^1)$$

and

$$Q = \Omega \times (0, 1).$$

Does there exist some

$$v \in W^{s+1/p,p}(Q; S^1)$$

such that

$$v_{|\Omega\times\{0\}} = u \ ?$$

[**Comment:** This is the only problem about traces which is still open. All the other cases have been covered in Section 1.5. We conjecture that the answer is negative and that one may use the same example as in the problem of lifting, (see the proof of Theorem 1.3b)), namely

$$u(x) = e^{i/|x|^\alpha}$$

with some appropriate α, so that $u \in W^{s,p}$. Suppose, by contradiction, that u is the trace of some

$$v \in W^{s+1/p,p}(Q; S^1).$$

Since $1 < (s+1/p), p < N+1 = \dim Q$, this v need **not** have a lifting in $W^{s+1/p,p}$ (by Theorem 1.3). However it is **plausible** (as in OP3) that v admits a lifting ψ in $W^{\sigma,p}$, for **some** σ with $2/p \le \sigma < s + 1/p$. In that case $\psi_{|\Omega\times\{0\}}$ belongs to $W^{\sigma-1/p,p}$. On the other hand $\frac{1}{2\pi}(\psi(x) - \frac{1}{|x|^\alpha}) \in \mathbb{Z}$ and since $(\sigma - 1/p)p \ge 1$ we infer from Theorem 1.6 that

$$\psi(x) = \frac{1}{|x|^\alpha} \ + \ \text{Const.}\ .$$

We could then derive a contradiction if $\frac{1}{|x|^\alpha} \notin W^{\sigma-1/p,p}$, i.e. $(\alpha+\sigma)p \ge N+1$].

OP11. Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded, simply connected domain with $N \ge 3$. Is Theorem 1.18 valid when s is not an integer?

OP12. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Recall that $C^\infty(\overline{\Omega}; S^1)$ is **not** dense in $W^{s,p}(\Omega; S^1)$ in the range $1 \le sp < 2, 0 < s < \infty, 1 < p < \infty$ (see Theorem 1.17). However, it is plausible that

$$\mathcal{R}_0 = \{u \in W^{s,p}(\Omega; S^1); u \text{ is smooth except at a finite number of points}\}$$

is dense in $W^{s,p}(\Omega; S^1)$. Here, the number and location of singular points is left free.

[**Comment:** $\mathcal{R}_0$ is known to be dense in $W^{s,p}$ in the following cases:

a) $s = 1$ and $1 \le p < 2$; see Bethuel and Zheng [1]

b) $s = 1 - 1/p$ and $1 < p < 3$; see Bethuel [3]

c) $s = 1/2$ and $p = 2$; see Riviere [2].]