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Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces

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Dedicated with emotion to the memory of Tosio Kato

I. Introduction

Our main result is the following: let $1 \le s < \infty,\ 1 < p < \infty$, and let

$$m = \begin{cases} s, & \text{if } s \text{ is an integer} \\ [s]+1, & \text{otherwise.} \end{cases}$$

Set

$$R = \{f \in C^m(\mathbb{R})\ ;\ f(0) = 0,\ f, f', \dots, f^{(m)} \in L^\infty(\mathbb{R})\}.$$

THEOREM 1.1. *For every $f \in R$ the map $\psi \mapsto f(\psi)$ is well-defined and continuous from $W^{s,p}(\mathbb{R}^n) \cap W^{1,sp}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$.*

An immediate consequence of Theorem 1.1 is

THEOREM 1.1'. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $f \in C^m$ be such that $f, f', \dots, f^{(m)} \in L^\infty$. Then the map*

$$W^{s,p}(\Omega) \cap W^{1,sp}(\Omega) \ni \psi \mapsto f(\psi) \in W^{s,p}(\Omega)$$

is well-defined and continuous.

Our original motivation in proving Theorem 1.1 comes from the study of properties of the space

$$X = W^{s,p}(\Omega; S^1) = \{u \in W^{s,p}(\Omega; \mathbb{R}^2)\ ;\ |u| = 1 \text{ a.e.}\}.$$

Here, $0 < s < \infty$, $1 < p < \infty$ and Ω is a smooth bounded simply connected domain in $\mathbb{R}^n$. In particular, one may ask whether X is path-connected and whether $C^\infty(\overline{\Omega}; S^1)$ is dense in

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X. Several results concerning the first question were obtained in [10] (and subsequently in [18]) for the spaces $W^{1,p}(M; N)$, where M, N are compact oriented Riemannian manifolds. The second question was studied in [3], [4] and [18] for the spaces $W^{1,p}(M; N)$ and in [16] for the spaces $W^{s,p}(M; S^k)$.

The case where $N = S^1$ is somehow special; one may attempt to answer these questions by lifting the maps $u \in X$. Here is a strategy: given $u \in W^{s,p}(\Omega; S^1)$, one may try to find some $\varphi \in W^{s,p}(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$. Then, hopefully, the path

$$t \in [0, 1] \mapsto e^{it\varphi}$$

will connect continuously $u_0 \equiv 1$ to u.

Moreover, if φ_j are smooth $\mathbb{R}$-valued functions on $\overline{\Omega}$ such that $\varphi_j \to \varphi$ in $W^{s,p}$, then, hopefully, the smooth maps $e^{i\varphi_j}$ converge to u in $W^{s,p}(\Omega; S^1)$.

We are thus naturally led to the study of the mapping

$$W^{s,p}(\Omega) \ni \psi \mapsto f(\psi)$$

for "reasonable" functions f (e.g. $f(x) = e^{ix} - 1$), where Ω is either a smooth bounded domain or $\Omega = \mathbb{R}^n$ and $s \geq 1$.

In a forthcoming paper [12], we will apply Theorem 1.1 to settle the above mentioned questions about $W^{s,p}(\Omega; S^1)$ when $s \geq 1$.

Another motivation for analysing composition and products in fractional Sobolev spaces comes from the study of nonlinear evolution equations (e.g. Schrödinger equation) in H^s spaces; see e.g. T. Kato [20] and the references therein. In fact, the Appendix in [20] contains a result which is a special case of the Runst-Sickel lemma about products: it coincides with Lemma 4.1 below when $q = 2$.

REMARK 1.2. The reader may wonder why we impose the additional condition $\psi \in W^{1,sp}$. At least for the case we are interested in, i.e. $f(x) = e^{ix} - 1$, this condition is also *necessary* in order to conclude that $f(\psi) \in W^{s,p}(\mathbb{R}^n)$.

Indeed, assume that $\psi \in W^{s,p}$ and $(e^{i\psi} - 1) \in W^{s,p}$. Then $(e^{i\psi} - 1) \in W^{s,p} \cap L^\infty \Longrightarrow (e^{i\psi} - 1) \in W^{1,sp}$ (by Gagliardo-Nirenberg, see Corollary 3.2 below). Therefore, $ie^{i\psi} D\psi \in L^{sp}$, so that $D\psi \in L^{sp}$. Thus $\psi \in W^{1,sp}$.

REMARK 1.3. There is a vast literature about composition, starting with the result of Moser [26] asserting that $f(\psi) \in W^{m,p}$ when $\psi \in W^{m,p} \cap L^\infty$, $f \in R$ and m is an integer. (See the historical comments at the end of Section V). Unfortunately, for the application we have in mind, the lifting φ of an arbitrary $u \in W^{s,p}(\Omega; S^1)$ need not belong to L^∞. However, if $s \geq 1$ and if the lifting φ exists in $W^{s,p}(\Omega; \mathbb{R})$, it *must* belong to $W^{1,sp}$, by the above remark.

Surprisingly, Theorem 1.1 is new, but it is closely related and implies two earlier results having a similar flavour; see Adams-Frazier [1] and Runst-Sickel [32], Theorem 1.1, p. 345 and Remark 1, p. 348.

REMARK 1.4. When s is an integer, the proof of Theorem 1.1 is very easy via the standard Gagliardo-Nirenberg inequality [27] (e.g. $W^{k,p} \cap L^\infty \subset W^{\ell,q}$, with $\ell < k$, $\ell q = kp$). When $s > 1$, s is not an integer, our proof is quite involved. The standard form of the Gagliardo-Nirenberg inequality (e.g. $W^{s,p} \cap L^\infty \subset W^{\sigma,q}$, with $\sigma < s$, $\sigma q = sp$) does *not* suffice. We rely on a "microscopic" improvement (due to T. Runst [31]) of the Gagliardo-Nirenberg inequality, in the Triebel-Lizorkin scale, namely $W^{s,p} \cap L^\infty \subset \tilde{F}^{\sigma}_{q,\nu}$ for *every* ν. We present in Section III a more general form of the Gagliardo-Nirenberg inequality due to Oru [28]; see also P. Gérard, Y. Meyer and F. Oru [17] for a special case. We combine this with an important estimate on products of functions in the Triebel-Lizorkin spaces, due to T. Runst and W. Sickel (see [32] and Section IV).

It would be interesting to find a more elementary argument which avoids this excursion into the $\tilde{F}^s_{p,q}$ scale.

The paper is organized as follows. In Section II we recall the definition of the Triebel-Lizorkin spaces $\tilde{F}^s_{p,q}$, their connection with the classical function spaces and some results needed in the proof of Theorem 1.1. In Section III we recall the general form of the Gagliardo-Nirenberg inequality, due to Oru [28]. Section IV deals with the Runst-Sickel lemma. This beautiful result contains all the usual statements about products in fractional Sobolev spaces: e.g. it implies that if $u, v \in W^{s,p} \cap L^\infty$ then $uv \in W^{s,p} \cap L^\infty$, and if $s \geq 1$, then $uDv \in W^{s-1,p}$. More consequences of the Runst-Sickel lemma are presented in Section VI. Theorem 1.1 is proved in Section V.

II. Triebel-Lizorkin spaces and maximal inequalities

We start by recalling the Littlewood-Paley decomposition of temperate distributions. Let $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi_0 \leq 1$, $\psi_0(\xi) = 1$ for $|\xi| \leq 1$, $\psi_0(\xi) = 0$ for $|\xi| \geq 2$. Set $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi)$, $j \geq 1$, and $\varphi_j = F^{-1}(\psi_j)$, $j \geq 0$.

Thus

$$\varphi_j(x) = 2^{nj}\varphi_0(2^j x) - 2^{n(j-1)}\varphi_0(2^{j-1}x), \ \ j \geq 1, \tag{1}$$

and

$$\sum_{k \leq j} \varphi_k(x) = 2^{nj}\varphi_0(2^j x), \ \ j \geq 0. \tag{2}$$

For $f \in S'$, set $f_j = f \star \varphi_j$. We have $f = \sum_{j \geq 0} f_j$ in S'.

DEFINITION. ([34], 2.3.1) For $-\infty < s < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$, set

$$\tilde{F}^s_{p,q} = \{f \in S' \ ; \ \|f\|_{\tilde{F}^s_{p,q}} = \| \|2^{sj} f_j(x)\|_{\ell^q} \|_{L^p(\mathbb{R}^n)} < \infty\}.$$

For $0 < p < \infty$ or $p = q = \infty$, these are the standard Triebel-Lizorkin spaces $F^s_{p,q}$ [34]. We have added the ˜ to avoid confusions in the exceptional cases where they do not coincide. When $0 < p < \infty$, different choices of ψ_0 yield equivalent quasi-norms ([34], 2.3.5). The usual function spaces are special cases of these Triebel-Lizorkin spaces ([34]):

a) $L^p = \tilde{F}^0_{p,2}$, $1 < p < \infty$;
b) $W^{m,p} = \tilde{F}^m_{p,2}$, $m = 1, 2, \dots, 1 < p < \infty$;
c) $W^{s,p} = \tilde{F}^s_{p,p}$, $0 < s < \infty$, s non-integer, $1 \le p < \infty$;
d) $L^{s,p} = \tilde{F}^s_{p,2}$, $s \in \mathbb{R}$, $1 < p < \infty$;
e) $L^\infty \subset \tilde{F}^0_{\infty,\infty}$, i.e.,

$$\sup_{j,x} |f_j(x)| \le C\|f\|_{L^\infty}. \tag{3}$$

In this list, when $1 \le p < \infty$, $0 < s < \infty$, s non-integer, the $W^{s,p}$ are the Sobolev-Slobodeckij spaces. An equivalent norm on these spaces may be obtained as follows: let $s = k + \sigma$, k integer, $0 < \sigma < 1$. Then

$$\|f\|^p_{W^{s,p}} \sim \|f\|^p_{L^p} + \|D^k f\|^p_{L^p} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^k f(x) - D^k f(y)|^p}{|x-y|^{n+\sigma p}} dx\,dy \tag{4}$$

([34], 2.6.1). These spaces also coincide with the Besov spaces $B^s_{p,p}$ (recall that s is not an integer). We warn the reader that, for $p \neq 2$, the spaces $W^{s,p}$ *do not coincide* with the Bessel potential spaces $L^{s,p}$.

We will often use the trivial fact that, for fixed s and p, the space $\tilde{F}^s_{p,q}$ increases with q.

The following result is well-known:

LEMMA 2.1. ([35]) *Let* $0 < s < \infty$, $1 < p < \infty$, $1 < q < \infty$. *For every* $j \ge 0$, *let* $f^j \in S'$ *be such that supp* $F(f^j) \subset B_{2^{j+2}}$. *Then*

$$\left\| \sum_j f^j \right\|_{\tilde{F}^s_{p,q}} \le C\| \|2^{sj} f^j(x)\|_{\ell^q} \|_{L^p(\mathbb{R}^n)}. \tag{5}$$

In the H^s-spaces ($p = q = 2$), this result is proved in [14], p. 21. We postpone the proof of Lemma 2.1 after the discussion of some maximal inequalities. Recall that, for any $f \in L^1_{loc}$, the maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.$$

For $t > 0$, set, for $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\varphi^t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n. \tag{6}$$

We recall some classical inequalities

LEMMA 2.2. *We have:*

a) ([33], p. 13) *for* $1 < p \le \infty$ *and any function* f,

$$\|Mf\|_{L^p} \sim \|f\|_{L^p}; \tag{7}$$

b) ([33], p. 55) *for* $1 < p < \infty$, $1 < q < \infty$, *and any sequence of function* (f^j),

$$\| \|Mf^j(x)\|_{\ell^q} \|_{L^p(\mathbb{R}^n)} \le C \| \|f^j(x)\|_{\ell^q} \|_{L^p(\mathbb{R}^n)}; \tag{8}$$

c) ([33], p. 57) *for any fixed* $\varphi \in S$ *and any function* f,

$$|f \star \varphi^t(x)| \le C\, Mf(x), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^n. \tag{9}$$

By (1), (2) and (9) we obtain the following

COROLLARY 2.3. *For every* $f \in L^1_{loc}$ *we have*

$$|f_j(x)| \le C\, Mf(x), \quad j \ge 0, \quad x \in \mathbb{R}^n, \tag{10}$$

$$\left| \sum_{j \le k} f_j(x) \right| \le C\, Mf(x), \quad k \ge 0, \quad x \in \mathbb{R}^n. \tag{11}$$

We now return to the

Proof of Lemma 2.1. With $f = \sum_j f^j$, we have

$$f_k = \left(\sum_j f^j \right)_k = \left(\sum_{j \ge k-3} f^j \right)_k = \sum_{j \ge k-3} (f^j)_k.$$

Therefore

$$\begin{aligned}\|f\|_{\tilde F^s_{p,q}} &= \left\| \left\| 2^{sk} \sum_{j\ge k-3} (f^j)_k(x) \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} \\ &= \left\| \left(\sum_k 2^{sqk} \left| \sum_{j\ge k-3} (f^j)_k(x) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\le C \left\| \left(\sum_k 2^{sqk} \sum_{j\ge k-3} |(f^j)_k(x)|^q (j-k+4)^{2q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

by the Hölder inequality with exponents q and $q' = \frac{q}{q-1}$ applied to the inner sum. We obtain, using (10), that

$$\begin{aligned}\|f\|_{\tilde F^s_{p,q}} &\le C \left\| \left(\sum_j \sum_{k\le j+3} 2^{sqk} (j-k+4)^{2q} |Mf^j(x)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\le C \left\| \left(\sum_j 2^{sqj} |Mf^j(x)|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &= C \| \|2^{sj} Mf^j(x)\|_{\ell^q} \|_{L^p(\mathbb{R}^n)}. \qquad (12)\end{aligned}$$

The desired conclusion is a consequence of (8) and (12).

III. A "microscopic" improvement of the Gagliardo-Nirenberg inequality

The main result of this section is that, in the Gagliardo-Nirenberg type inequalities for the spaces $\tilde F^s_{p,q}$, there is a gain in the "microscopic" parameter q; this gain is also called sometimes "precised" or "improved" Sobolev inequalities. Let us explain what we mean. In the context of Besov spaces, a typical Gagliardo-Nirenberg inequality asserts that

$$B^s_{p,r} \cap L^\infty \subset B^{s/2}_{2p,2r}, \text{ for } 0 < s < \infty,\ 0 < p < \infty,\ 0 < r \le \infty$$

(see, e.g. [31], Lemma 2.2, p. 331).

Here, the value $2r$ of the microscopic parameter is optimal in general. By contrast, in the scale of $\tilde F$-spaces we have, given $0 < s < \infty,\ 0 < p < \infty,\ 0 < r \le \infty$,

$$\tilde F^s_{p,r} \cap L^\infty \subset \tilde F^{s/2}_{2p,q} \quad \text{for } \textit{every } 0 < q \le \infty$$

([31], Lemma 2.1, p. 329).

A more general version of this phenomenon, due to Oru [28], is the following. Let $-\infty < s_1 < s_2 < \infty$, $0 < q_1, q_2 \le \infty$, $0 < p_1, p_2 \le \infty$, $0 < \theta < 1$, and define

$$s = \theta s_1 + (1-\theta)s_2$$
$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

LEMMA 3.1. *Under the above hypotheses we have, for every* $0 < q \le \infty$*,*

$$\|f\|_{\tilde{F}^s_{p,q}} \le C\|f\|^{\theta}_{\tilde{F}^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{\tilde{F}^{s_2}_{p_2,q_2}}, \tag{13}$$

where C *depends on* s_i, p_i, θ *and* q.

For the convenience of the reader, we reproduce the proof of Oru, since it is not yet published.

Before proving Lemma 3.1, we state some interesting consequences:

COROLLARY 3.2. *We have*

a) *for* $0 \le s_1 < s_2 < \infty$*,* $1 < p_1 < \infty$*,* $1 < p_2 < \infty$*,*

$$s = \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2},$$

$$\|f\|_{W^{s,p}} \le C\|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}; \tag{14}$$

b) ([31], *Lemma* 2.1, p. 329) *for* $0 < s < \infty$*,* $1 < p < \infty$*,* $0 < q \le \infty$*,*

$$\|f\|_{\tilde{F}^{\theta s}_{p/\theta,q}} \le C\|f\|^{\theta}_{W^{s,p}} \|f\|^{1-\theta}_{L^\infty}. \tag{15}$$

In particular, we have

c) *for* $0 < s < \infty$*,* $1 < p < \infty$*,* $0 < \theta < 1$*,*

$$\|f\|_{W^{\theta s,p/\theta}} \le C\|f\|^{\theta}_{W^{s,p}} \|f\|^{1-\theta}_{L^\infty}. \tag{16}$$

REMARK 3.3. Inequality (14) is a special case of (13), with $q = 2$ when s is an integer, $q = p$ otherwise, and similarly for q_1 and q_2. Inequality (15) is a consequence of (13) for $s_1 = 0$, θ replaced by $1-\theta$, $p_1 = q_1 = \infty$, $s_2 = s$, $q_2 = 2$ if s is an integer, $q_2 = p$ otherwise. Here one uses in addition the fact that $\|f\|_{\tilde{F}^0_{\infty,\infty}} \le C\|f\|_{L^\infty}$ (inequality (3) above). Finally, (16) is a special case of (15).

REMARK 3.4. There is something intriguing about inequality (16). It is trivial when $s < 1$ (with $C = 1$) if one takes the usual Gagliardo norm (4). It is also straightforward when both s and θs are integers. We do not know any elementary (i.e. without the Littlewood-Paley machinery) proof when $s = 1$. It would be of interest to establish (16) with control of the constant C, in particular when $s \nearrow 1$. In view of the results in [8], one may expect an inequality of the form

$$\|f\|_{W^{s/2,2p}} \le C(p)(1-s)^{1/2p}\|f\|_{W^{s,p}}^{1/2}\|f\|_{L^\infty}^{1/2} \text{ as } s \nearrow 1,$$

if we take the Gagliardo norms (4).

REMARK 3.5. Inequality (15) may be viewed as an improvement of (16), since for $0 < q < \min\{2, p/\theta\}$ we have $\tilde{F}^{\theta s}_{p/\theta,q} \subset W^{\theta s,p/\theta}$, $\tilde{F}^{\theta s}_{p/\theta,q} \neq W^{\theta s,p/\theta}$. This improvement seems microscopic, however in our situation it is magnified and it plays a central role. A similar (microscopic) improvement of the Sobolev embeddings in the framework of Lorentz spaces which is magnified by the Trudinger inequality is presented in [13], [9].

REMARK 3.6. We call the attention of the reader to the fact that some inequalities à la Gagliardo-Nirenberg are wrong, e.g. $W^{1,1} \cap L^\infty$ is *not contained* in $W^{\theta,1/\theta}$ for $0 < \theta < 1$; see [7], Remark D.1.

We now turn to the proof of Lemma 3.1. It relies on the following inequality:

LEMMA 3.7. *Let* $-\infty < s_1 < s_2 < \infty$, $0 < q < \infty$, $0 < \theta < 1$, *and set* $s = \theta s_1 + (1-\theta)s_2$. *Then for every sequence* (a_j) *we have*

$$\|2^{sj}a_j\|_{\ell^q} \le C\|2^{s_1 j}a_j\|_{\ell^\infty}^{\theta}\|2^{s_2 j}a_j\|_{\ell^\infty}^{1-\theta}. \tag{17}$$

REMARK 3.8. A special case of (17) is implicit in the proof of Theorem 1.1, p. 328, in [31]. For similar inequalities, see also [34], Theorem 2.7.1 or [19].

Proof of Lemma 3.7. Let $C_1 = \sup 2^{s_1 j}|a_j|$, $C_2 = \sup 2^{s_2 j}|a_j|$, so that $C_1 \le C_2$. We may assume $C_1 > 0$. Since $s_1 < s_2$, there is some $j_0 > 0$ such that

$$\min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{s^{s_2 j}}\right\} = \begin{cases} \frac{C_1}{2^{s_1 j}}, & j \le j_0 \\ \frac{C_2}{2^{s_2 j}}, & j > j_0. \end{cases}$$

Since $\dfrac{C_1}{2^{s_1 j_0}} \le \dfrac{C_2}{2^{s_2 j_0}}$ and $\dfrac{C_2}{2^{s_1(j_0+1)}} \le \dfrac{C_1}{2^{s_1(j_0+1)}}$ we find that

$$C_2 \sim C_1 2^{(s_2-s_1)j_0}. \tag{18}$$

Therefore

$$\|2^{s_1 j}a_j\|_{\ell^\infty}^{\theta}\|2^{s_2 j}a_j\|_{\ell^\infty}^{1-\theta} \sim C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{19}$$

On the other hand, we have $a_j \le \min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{s^{s_2 j}}\right\}$, so that

$$a_j \le \frac{C_1}{2^{s_1 j}} \text{ for } 0 \le j \le j_0, \quad a_j \le \frac{C_2}{2^{s_2 j}} \text{ for } j > j_0. \tag{20}$$

It then follows that

$$\begin{aligned}\|2^{sj}a_j\|_{\ell^q} &\le \left(\sum_{j\le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j>j_0} C_2^q 2^{(s-s_2)jq}\right)^{1/q}\\ &\le C\left(\sum_{j\le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j>j_0} C_1^q 2^{-\theta(s_2-s_1)jq+(s_2-s_1)j_0 q}\right)^{1/q},\end{aligned}$$

so that

$$\|2^{sj}a_j\|_{\ell^q} \le C\, C_1 2^{(s_2-s_1)j_0(1-\theta)} \left(\sum_{j\le j_0} 2^{-(1-\theta)(s_2-s_1)(j_0-j)q} + \sum_{j>j_0} 2^{-\theta(s_2-s_1)(j-j_0)q}\right)^{1/q}.$$

Finally, we find that

$$\|2^{sj}a_j\|_{\ell^q} \le C\, C_1 2^{(s_2-s_1)j_0(1-\theta)}, \tag{21}$$

and (17) follows from (19) and (21).

Proof of Lemma 3.1. Since $\|a_j\|_{\ell^\infty} \le \|a_j\|_{\ell^q}$, $0 < q \le \infty$, we find that the r.h.s. of (13) is

$$\ge C\|f\|^{\theta}_{\tilde F^{s_1}_{p_1,\infty}} \|f\|^{1-\theta}_{\tilde F^{s_2}_{p_2,\infty}}.$$

On the other hand, $\|f\|_{\tilde F^s_{p,\infty}} \le \|f\|_{\tilde F^s_{p,q}}$, $0 < q < \infty$. It therefore suffices to prove (13) in the special case $0 < q < \infty$, $q_1 = q_2 = \infty$.

In this case, we have

$$\begin{aligned}\|f\|_{\tilde F^s_{p,q}} &= \|\|2^{sj} f_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)} \le \text{ (by (17))}\\ &\le C\|\|2^{s_1 j} f_j(x)\|^{\theta}_{\ell^\infty}\|2^{s_2 j} f_j(x)\|^{1-\theta}_{\ell^\infty}\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{22}$$

Using the Hölder inequality, (22) yields

$$\begin{aligned}\|f\|_{\tilde F^s_{p,q}} &\le C\|\|2^{s_1 j} f_j(x)\|_{\ell^\infty}\|^{\theta}_{L^{p_1}(\mathbb{R}^n)}\|\|2^{s_2 j} f_j(x)\|_{\ell^\infty}\|^{1-\theta}_{L^{p_2}(\mathbb{R}^n)}\\ &= C\|f\|^{\theta}_{\tilde F^{s_1}_{p_1,\infty}}\|f\|^{1-\theta}_{\tilde F^{s_2}_{p_2,\infty}}.\end{aligned}$$

The proof of Lemma 3.1 is complete.

REMARK 3.9. While talking about microscoping improvements in the $\tilde F$-scale, we call the attention of the reader to the following "improved" Sobolev embedding:

$$W^{s,p} \hookrightarrow \tilde F^{\sigma}_{r,q} \quad \textit{for every } 0 < q \le \infty$$

if $0 \le \sigma < s$ and $\frac{1}{r} = \frac{1}{p} - \frac{s-\sigma}{n} > 0$ (see ([19] or [32], p. 31).

IV. The Runst-Sickel lemma

For the convenience of the reader, we split the statement into two parts; the first one contains the fundamental estimate, the other one deals with the continuity of the product.

Let $0 < s < \infty$, $1 < q < \infty$, $1 < p_1 \le \infty$, $1 < p_2 \le \infty$, $1 < r_1 \le \infty$, $1 < r_2 \le \infty$ be such that

$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1. \tag{23}$$

LEMMA 4.1. ([32], p. 345) *We have, for* $f \in \tilde{F}^s_{p_1,q} \cap L^{r_1}$ *and* $g \in \tilde{F}^s_{p_2,q} \cap L^{r_2}$,

$$\begin{aligned} \|fg\|_{\tilde{F}^s_{p,q}} &\le C(\|\|Mf(x)\|2^{sj} g_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)} \\ &\quad + \|\|Mg(x)\|2^{sj} f_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}) \end{aligned} \tag{24}$$

and

$$\|fg\|_{\tilde{F}^s_{p,q}} \le C(\|f\|_{\tilde{F}^s_{p_1,q}}\|g\|_{L^{r_2}} + \|g\|_{\tilde{F}^s_{p_2,q}}\|f\|_{L^{r_1}}). \tag{25}$$

Proof. We start by noting that (25) follows from (24). Indeed, using the Hölder inequality we find

$$\begin{aligned} &\|\|Mf(x)\|2^{sj} g_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)} + \|\|Mg(x)\|2^{sj} f_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)} \\ &\quad \le \|\|2^{sj} g_j(x)\|_{\ell^q}\|_{L^{p_2}(\mathbb{R}^n)}\|Mf(x)\|_{L^{r_1}(\mathbb{R}^n)} \\ &\qquad + \|\|2^{sj} f_j(x)\|_{\ell^q}\|_{L^{p_1}(\mathbb{R}^n)}\|Mg(x)\|_{L^{r_2}(\mathbb{R}^n)} \\ &\quad \le C(\|f\|_{\tilde{F}^s_{p_1,q}}\|g\|_{L^{r_2}} + \|g\|_{\tilde{F}^s_{p_2,q}}\|f\|_{L^{r_1}}), \end{aligned}$$

by (7).

We turn to the proof of (24). It relies on Lemma 2.1 which is valid since $1 < p < \infty$ and $1 < q < \infty$. We have

$$fg = \sum_k G_k + \sum_j F_j,$$

where $G_k = (\sum_{j \le k} f_j) g_k$, $F_j = (\sum_{k<j} g_k) f_j$. Since supp $\mathcal{F}(F_j) \subset B_{2^{j+2}}$ and supp $\mathcal{F}(G_k) \subset B_{2^{k+2}}$, Lemma 2.1 yields

$$\|fg\|_{\tilde{F}^s_{p,q}} \le C(A + B), \tag{26}$$

with

$$\begin{aligned} A &= \|\|2^{sk} G_k(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}, \\ B &= \|\|2^{sk} F_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

We estimate, e.g. A:

$$A = \left\| \|2^{sk} \left(\sum_{j \le k} f_j(x) \right) g_k(x) \|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} \le \text{ by (11)}$$
$$C \| M f_j(x) \| 2^{sk} g_k(x) \|_{\ell^q} \|_{L^p(\mathbb{R}^n)}. \tag{27}$$

We obtain (24) by combining (26), (27) and the similar estimate for B.

We state the continuity part of this result in the three possible situations:

COROLLARY 4.2. *We have that:*

a) *for* $1 < q < \infty$, $0 < s < \infty$, $1 < p_1 < \infty$, $1 < p_2 < \infty$, $1 < r_1 < \infty$, $1 < r_2 < \infty$, $0 < \dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{r_2} = \dfrac{1}{p_2} + \dfrac{1}{r_1} < 1$, *the map*

$$(\tilde{F}^s_{p_1,q} \cap L^{r_1}) \times (\tilde{F}^s_{p_2,q} \cap L^{r_2}) \ni (f, g) \mapsto fg \in \tilde{F}^s_{p,q}$$

is continuous;

b) *for* $1 < q < \infty$, $0 < s < \infty$, $1 < p < \infty$, *if*

$$\begin{cases} f^\ell \to f \text{ in } \tilde{F}^s_{p,q}, & \|f^\ell\|_{L^\infty} \le C \\ g^\ell \to g \text{ in } \tilde{F}^s_{p,q}, & \|g^\ell\|_{L^\infty} \le C \end{cases}$$

then $f^\ell g^\ell \to fg$ *in* $\tilde{F}^s_{p,q}$*;*

c) *for* $1 < q < \infty$, $0 < s < \infty$, $1 < p_1 < \infty$, $1 < r < \infty$, $1 < p < \infty$ *such that* $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{r}$, *if*

$$\begin{cases} f^\ell \to f \text{ in } \tilde{F}^s_{p_1,q}, & \|f^\ell\|_{L^\infty} \le C \\ g^\ell \to g \text{ in } \tilde{F}^s_{p,q} \cap L^r, & \end{cases}$$

then $f^\ell g^\ell \to fg$ *in* $\tilde{F}^s_{p,q}$.

Proof. a) follows directly from (25).

Some care is needed when one of the r_j's is ∞. We treat, e.g. case c). It clearly suffices to prove the following two assertions:

(i) if $f^\ell \to 0$ in $\tilde{F}^s_{p_1,q}$ and $\|f^\ell\|_{L^\infty} \le C$, then $f^\ell g \to 0$ for each $g \in \tilde{F}^s_{p,q} \cap L^r$.

(ii) if $g^\ell \to 0$ in $\tilde{F}^s_{p,q} \cap L^r$, $\|f^\ell\|_{\tilde{F}^s_{p_1,q}} \le C$, $\|f^\ell\|_{L^\infty} \le C$, then $f^\ell g^\ell \to 0$.

Assertion (ii) is clear from (25). We prove (i) using (24). We have

$$\begin{aligned}\|f^\ell g\|_{\tilde F^s_{p,q}} &\le C(\|f^\ell\|_{\tilde F^s_{p_1,q}}\|g\|_{L^r} + \|Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}) \\ &\le o(1) + C\|Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{28}$$

Set

$$F^\ell(x) = Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}.$$

Then clearly

$$|F^\ell(x)| \le C\|2^{sj}g_j(x)\|_{\ell^q} \in L^p. \tag{29}$$

□

On the other hand, $\tilde F^s_{p_1,q} \hookrightarrow L^{p_1}$ (see, e.g. [34], 2.3.2, or [32], Proposition 2.2.1, p. 29). It follows from the maximal inequality (7) that $Mf^\ell \to 0$ in L^{p_1} and, up to a subsequence, that $Mf^\ell \to 0$ a.e. Then (i) follows from (28) and (29) by dominated convergence.

V. Proof of Theorem 1.1

The conclusion is well-known when s is an integer (this uses the standard Gagliardo-Nirenberg inequalities).

Assume s non integer. Clearly, the map

$$W^{s,p} \cap W^{1,sp} \ni u \mapsto f(u) \in L^p$$

is well-defined and continuous, since $f(0) = 0$, f is Lipschitz and $W^{s,p} \hookrightarrow L^p$.

Thus it suffices to prove that the map

$$W^{s,p} \cap W^{1,sp} \ni u \mapsto D(f(u)) = f'(u)Du \in W^{s-1,p}$$

is well-defined and continuous.

With $m = [s] + 1 \ge 2$, we obtain, using (14), that the inclusion

$$W^{s,p} \cap W^{1,sp} \hookrightarrow W^{m-1,\frac{sp}{m-1}} \cap W^{1,sp} \tag{30}$$

is continuous. Applying Theorem 1.1 to the integer $s = m - 1 \ge 1$, we find that

$$\begin{aligned}&\text{if } u^\ell \to u \text{ in } W^{s,p} \cap W^{1,sp}, \text{ then } f'(u^\ell) \to f'(u) \text{ in } \tilde F^{m-1}_{\frac{sp}{m-1},2} = W^{m-1,\frac{sp}{m-1}} \\ &\quad \text{and } \|f'(u^\ell)\|_{L^\infty} \le C. \end{aligned} \tag{31}$$

On the other hand, we clearly have that

$$\begin{aligned}&\text{if } u^\ell \to u \text{ in } W^{s,p} \cap W^{1,sp}, \text{ then } Du^\ell \to Du \text{ in } W^{s-1,p} \cap L^{sp} \\ &\quad = \tilde F^{s-1}_{p,p} \cap L^{sp}. \end{aligned} \tag{32}$$

Using (31) and the Gagliardo-Nirenberg type inequality (15) (with $q = p$, $s = m-1$, $\theta = \frac{s-1}{m-1}$, $p = \frac{sp}{m-1}$), we obtain

$$\text{if } u^\ell \to u \text{ in } W^{s,p} \cap W^{1,sp}, \text{ then } f'(u^\ell) \to f'(u) \text{ in } \tilde{F}^{s-1}_{\frac{sp}{s-1},p} \text{ and}$$
$$\|f'(u^\ell)\|_{L^\infty} \le C. \tag{33}$$

Finally, by (32), (33), the Runst-Sickel Lemma 4.1 and Corollary 4.2c), we obtain that $f'(u)Du \in \tilde{F}^{s-1}_{p,p} = W^{s-1,p}$ and that

$$\text{if } u^\ell \to u \text{ in } W^{s,p} \cap W^{1,sp}, \text{ then } f'(u^\ell)Du^\ell \to f'(u)Du \text{ in } W^{s-1,p}.$$

REMARK 5.1. The same proof yields the following variant of Theorem 1.1.

THEOREM 5.2. *Assume* $1 < s < \infty$, *s non integer,* $1 < p < \infty$, $1 < q < \infty$. *Then, for every* $f \in R$, *the map*

$$\tilde{F}^s_{p,q} \cap W^{1,sp} \ni \psi \mapsto f(\psi) \in \tilde{F}^s_{p,q}$$

is well-defined and continuous.

REMARK 5.3. There is a natural strategy for proving Theorem 1.1: assume, e.g. that $1 < s < 2$ and try to prove that $f'(u)Du \in W^{s-1,p}$. Set $s = 1 + \sigma$. On the one hand, we have $Du \in W^{\sigma,p} \cap L^{(1+\sigma)p}$. On the other hand, since $u \in W^{1,(1+\sigma)p}$, we find that $f'(u) \in W^{1,(1+\sigma)p} \cap L^\infty$. By the "standard" Gagliardo-Nirenberg inequality, we obtain $f'(u) \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^\infty$. The conclusion of Theorem 1.1 would follow if we can prove that

$$\left.\begin{aligned} U &\in W^{\sigma,p} \cap L^{(1+\sigma)p} \\ V &\in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^\infty \end{aligned}\right\} \Longrightarrow UV \in W^{\sigma,p}. \tag{34}$$

Using the Gagliardo norm (4), we have to estimate

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x+h)V(x+h) - U(x)V(x)|^p}{|h|^{n+\sigma p}} dx\, dh \\ &\le C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|V(x)|^p|U(x+h) - U(x)|^p}{|h|^{n+\sigma p}} dx\, dh \right.\\ &\qquad \left. + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^p|V(x+h) - V(x)|^p}{|h|^{n+\sigma p}} dx\, dh\right) \\ &\le C\left(\|V\|^p_{L^\infty}\|U\|^p_{W^{\sigma,p}} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^p|V(x+h) - V(x)|^p}{|h|^{n+\sigma p}} dx\, dh\right). \end{aligned} \tag{35}$$

It is natural to estimate the last integral in (34) using the Hölder inequality with exponents $1+\sigma$ and $\frac{1+\sigma}{\sigma}$. We find

$$\|UV\|^p_{W^{\sigma,p}} \le C\Bigg(\|V\|^p_{L^\infty}\|U\|^p_{W^{\sigma,p}} + \|V\|^p_{W^{\sigma,\frac{1+\sigma}{\sigma}p}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^{(1+\sigma)p}}{|h|^n} dx\, dh\right)^{\frac{1}{1+\sigma}} \Bigg).$$

Unfortunately, the last integral diverges, but we are "close" to convergence. In fact, we suspect that (34) is wrong.

It is here that the microscopic improvement of the Gagliardo-Nirenberg inequality Lemma 3.1, combined with the Runst-Sickel Lemma 4.1, magically saves the proof. We make use, in an essential way, of the additional information that $V = f'(u) \in F^{\sigma}_{\frac{1+\sigma}{\sigma}p,p}$.

We conclude this section with a brief survey of earlier results dealing with composition.

a) if $0 < s \le 1$, $1 < p < \infty$, $f(0) = 0$, f Lipschitz, then

$u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$ (trivial for $s < 1$; see [21] and [22] for $s = 1$);

b) if $s = n/p$, $1 < p < \infty$, $f \in R$, where $m = \begin{cases} s, & \text{if } s \text{ is an integer} \\ [s]+1, & \text{otherwise} \end{cases}$,
then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$.

This result is explicitely stated in [11]; G. Bourdaud has pointed out that it may also be derived from a result of T. Runst and W. Sickel, see p. 345 in [32], combined with a result in [19] which asserts that, when $s = n/p$, $W^{s,p} \hookrightarrow \tilde{F}^{\theta s}_{p/\theta,q}$ for $0 < \theta < 1$ and *every* $0 < q < \infty$ (see Remark 3.9 above);

c) if $s > n/p$, $1 < p < \infty$, $f(0) = 0$ and $f \in C^m$, then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$; see [25] for $p = 2$ and [29] for the general case;

d) if $1 < s < n/p$, we have to impose additional restrictions on u. Indeed, if $1 + 1/p < s < n/p$, the only C^2 f's that act on $W^{s,p}$ are of the form $f(t) = ct$; see [15] for s integer and [31], Theorem 3.2, p. 319, for a general s. For $1 < s < n/p$, it follows from Remark 1.2 in the Introduction that R does not act on $W^{s,p}$, since $W^{s,p} \not\subset W^{1,sp}$. A standard additional condition on u is $u \in L^\infty$: if $f(0) = 0$ and $f \in C^m$, then $u \in W^{s,p} \cap L^\infty \Longrightarrow f(u) \in W^{s,p}$; see [29], [16];

e) an improvement is that, for f as above and $0 < \sigma < 1$ we have $u \in W^{s,p} \cap W^{\sigma,sp/\sigma} \Longrightarrow f(u) \in W^{s,p}$; see [11]. This result implies the previous one, since $W^{s,p} \cap L^\infty \hookrightarrow W^{\sigma,sp/\sigma}$ (by Corollary 3.2);

f) a finer result asserts that, for f as above, we have $u \in W^{s,p} \cap \tilde{F}^1_{sp,q}$ (with $q \le 2$ sufficiently small depending on s and p) $\Longrightarrow f(u) \in W^{s,p}$; see [32], Theorem 1.1, p. 345. This hypothesis on u is weaker than the previous one, since $W^{s,p} \cap W^{\sigma,sp/\sigma} \hookrightarrow \tilde{F}^1_{sp,q}$ for all $q > 0$, by Lemma 3.1. This result is contained in Theorem 1.1, since $\tilde{F}^1_{sp,q} \hookrightarrow W^{1,sp} = \tilde{F}^1_{sp,2}$ as soon as $q \le 2$ (recall that $\tilde{F}^s_{p,q}$ increases with q). However, when $p \le 2$ or $1 < s < 2$, Runst and Sickel point out in Remark 1, p. 348 that the above smallness condition on q is precisely $q \le 2$. This means that Runst and Sickel had established Theorem 1.1 when $p \le 2$ or $1 < s < 2$.

g) in the framework of Bessel potential spaces

$$L^{s,p} = \{f = G_s \star g \ ; \ g \in L^p, \hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}\} = \tilde{F}^s_{p,2},$$

there are various similar results about composition, starting with [23], [24] when $s > n/p$, [30], [2] and [14] for $H^s \cap L^\infty$ when $s \ge 1$. The ultimate result for $s \ge 1$ was obtained by Adams-Frazier in [1]: if $1 \le s < \infty$, $1 < p < \infty$, $f \in R$, then $u \in L^{s,p} \cap L^{1,sp} \Longrightarrow f(u) \in L^{s,p}$. This is a special case ($q = 2$) of Theorem 5.2 since $L^{1,sp} = W^{1,sp}$.

h) Other questions concerning composition in Sobolev spaces have been investigated e.g. in [5], [6], [32].

VI. More about products

In this last section, we state some natural results about products which may be derived from the Runst-Sickel lemma.

Let $1 < p < \infty$, $0 < s < \infty$, $1 < r < \infty$, $0 < \theta < 1$, $1 < t < \infty$ be such that

$$\frac{1}{r} + \frac{\theta}{t} = \frac{1}{p}.$$

LEMMA 6.1. *For $f \in W^{s,t} \cap L^\infty$, $g \in W^{\theta s,p} \cap L^r$, we have $fg \in W^{\theta s,p}$ and*

$$\|fg\|_{W^{\theta s,p}} \le C(\|f\|_{L^\infty}\|g\|_{W^{\theta s,p}} + \|g\|_{L^r}\|f\|^{\theta}_{W^{s,t}}\|f\|^{1-\theta}_{L^\infty}). \tag{36}$$

In the special case $s > 1$, $\theta = \frac{s-1}{s}$, we have $r = sp$ and we obtain the following

COROLLARY 6.2. *If $1 < s < \infty$, $1 < p < \infty$ and $f \in W^{s,p} \cap L^\infty$, $g \in W^{s-1,p} \cap L^{sp}$, then $fg \in W^{s-1,p}$ and*

$$\|fg\|_{W^{s-1,p}} \le C(\|f\|_{L^\infty}\|g\|_{W^{s-1,p}} + \|g\|_{L^{sp}}\|f\|^{1-1/s}_{W^{s,p}}\|f\|^{1/s}_{L^\infty}). \tag{37}$$

In particular, if $f, g \in W^{s,p} \cap L^\infty$, then $Dg \in W^{s-1,p} \cap L^{sp}$, so that Corollary 6.2 contains as a special case the following result

COROLLARY 6.3. ([7], Lemma 2.2) *If* $1 < s < \infty$, $1 < p < \infty$ *and* $f, g \in W^{s,p} \cap L^\infty$, *then* $f\,Dg \in W^{s-1,p}$.

REMARK 6.4. Clearly, Corollary 6.3 implies the well-known assertion that $W^{s,p} \cap L^\infty$ is an algebra.

Proof of Lemma 6.1. Let $q = 2$ if θs is an integer, $q = p$ otherwise. By (15), we find that $f \in \tilde{F}^{\theta s}_{t/\theta,q}$ and

$$\|f\|_{\tilde{F}^{\theta s}_{t/\theta,q}} \le C\|f\|^{\theta}_{W^{s,t}}\|f\|^{1-\theta}_{L^\infty}. \tag{38}$$

From the Runst-Sickel lemma, we deduce that $fg \in \tilde{F}^{\theta s}_{p,q}$ and

$$\begin{aligned}\|fg\|_{W^{\theta s,p}} &= \|fg\|_{\tilde{F}^{\theta s}_{p},q} \le C(\|f\|_{L^\infty}\|g\|_{\tilde{F}^{\theta s}_{p,q}} + \|g\|_{L^r}\|f\|_{\tilde{F}^{\theta s}_{t/\theta,q}})\\ &\le C(\|f\|_{L^\infty}\|g\|_{W^{\theta s,p}} + \|g\|_{L^r}\|f\|^{\theta}_{W^{s,t}}\|f\|^{1-\theta}_{L^\infty}).\end{aligned}$$

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