C. R. Acad. Sci. Paris, t. 331, Série I, p. 365–370, 2000
Équations aux dérivées partielles/*Partial Differential Equations*
(Topologie/*Topology*)

# Topology and Sobolev spaces

**Haïm BREZIS** [a], **YanYan LI** [b]

[a] Analyse numérique, Université Pierre-et-Marie-Curie, B.C. 187, 4, place Jussieu, 75252 Paris cedex 05, France
E-mail: brezis@ccr.jussieu.fr

[b] Department of Mathematics, Rutgers University, Hill Center, Busch Campus, 110 Frelinghuysen Rd, Piscataway, NJ 08854-8019, USA
E-mail: yyli@math.rutgers.edu

(Reçu et accepté le 19 juin 2000)

**Abstract.** Let $M$ and $N$ be compact manifolds and consider the Sobolev space $W^{1,p}(M,N)$. Our main concern is to determine whether or not $W^{1,p}(M,N)$ is path-connected and, if not, what can be said about its path-connected components, i.e., its $W^{1,p}$-homotopy classes. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

***Topologie et espaces de Sobolev***

**Résumé.** *Étant donné deux variétés compactes $M$ et $N$ on considère l'espace de Sobolev $W^{1,p}(M,N)$. Notre objectif est de déterminer si $W^{1,p}(M,N)$ est connexe par arc et, sinon, d'analyser ses composantes, c'est-à-dire les classes d'homotopie relatives à $W^{1,p}$.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

Soient $M$ et $N$ deux variétés. Les classes d'homotopie usuelles correspondent aux composantes connexes par arc de l'espace $C^0(M,N)$. Si l'on remplace l'espace des fonctions continues par l'espace de Sobolev $W^{1,p}$ il est essentiel de comprendre comment les composantes connexes par arc de $W^{1,p}(M,N)$ dépendent de $p$. Curieusement, cette question à l'interface entre l'Analyse et la Topologie reste en grande partie à défricher. Voici quelques résultats et conjectures frappants :

THÉORÈME 1. – *Si $p \geqslant \dim M$, alors $W^{1,p}(M,N)$ possède la même topologie que $C^0(M,N)$.*

THÉORÈME 2. – *Si $p < 2$ et $\dim M \geqslant 2$, alors $W^{1,p}(M,N)$ est connexe par arc ($\forall M, \forall N$).*

THÉORÈME 3. – *Si $p \geqslant 2$ et $N = S^1$, alors $W^{1,p}(M,N)$ et $C^0(M,N)$ possèdent la même topologie.*

CONJECTURE 1. – *Dans toute composante connexe par arc de $W^{1,p}(M,N)$ il existe au moins une fonction régulière ($\forall p, \forall M, \forall N$).*

CONJECTURE 2. – *Soient $u, v \in W^{1,p}(M,N)$ ; si $u \sim v$ dans $W^{1,[p]}$, alors $u \sim v$ dans $W^{1,p}$.*

Note présentée par Haïm BREZIS.

S0764-4442(00)01656-6/FLA

La conjecture 2 exprime que les changements de structure topologique de $W^{1,p}(M,N)$ ont lieu seulement pour des valeurs entières de $p$. Au vu de ces résultats il est extrêmement intéressant d'analyser par quel mécanisme certaines classes d'homotopie « fusionnent » alors que d'autres persistent, lorsque $p$ décroît depuis $p = \dim M$ jusqu'à $p = 2$ et de trouver les valeurs de $p$ où un changement de topologie apparaît.

---

Let $M$ and $N$ be compact connected oriented smooth Riemannian manifolds with or without boundary. Throughout this note we assume that $\dim M \geqslant 2$ but $\dim N$ could possibly be one, for example $N = S^1$ is of interest. Our functional framework is the Sobolev space $W^{1,p}(M,N)$ which is defined by considering $N$ as smoothly embedded in some Euclidean space $\mathbb{R}^K$ and then

$$W^{1,p}(M,N) = \{u \in W^{1,p}(M, \mathbb{R}^K);\ u(x) \in N \text{ a.e.}\},$$

with $1 \leqslant p < \infty$. $W^{1,p}(M,N)$ is equipped with the standard metric $d(u,v) = \|u - v\|_{W^{1,p}}$. Our main concern is to determine whether or not $W^{1,p}(M,N)$ is path-connected and if not what can be said about its path-connected components, i.e., its $W^{1,p}$-homotopy classes. We say that $u$ and $v$ are $W^{1,p}$-homotopic if there is a path $u^t \in C([0,1], W^{1,p}(M,N))$ such that $u^0 = u$ and $u^1 = v$. We denote by $\sim_p$ the corresponding equivalence relation. Let $\sim$ denote the equivalence relation on $C^0(M,N)$, i.e., $u \sim v$ if there is a path $u^t \in C([0,1], C^0(M,N))$ such that $u^0 = u$ and $u^1 = v$.

First an easy result:

THEOREM 1. – *Assume* $p \geqslant \dim M$, *then* $W^{1,p}(M,N)$ *is path-connected if and only if* $C^0(M,N)$ *is path-connected.*

Theorem 1 is basically known (and relies on an idea introduced by Schoen and Uhlenbeck [7] when $p = \dim M$; *see* also Brezis and Nirenberg [5]).

Since, in general, $C^0(M,N)$ is not path-connected, this means that $W^{1,p}(M,N)$ is not path-connected when $p$ is "large". On the other hand if $p$ is "small", we expect $W^{1,p}(M,N)$ to be path-connected for all $M$ and $N$. Indeed we have:

THEOREM 2. – *Let* $1 \leqslant p < 2$ (*and recall that* $\dim M \geqslant 2$). *Then* $W^{1,p}(M,N)$ *is path-connected.*

Our proof of Theorem 2 is surprisingly involved and requires a number of technical tools (*see* [3]).

*Remark* 1. – Assumption $1 \leqslant p < 2$ in Theorem 2 is sharp (for general $M$ and $N$). For example if $\Lambda$ is any open connected set (or a connected Riemannian manifold) of dimension $\geqslant 1$, then $W^{1,2}(S^1 \times \Lambda, S^1)$ is not path-connected. This may be seen using the results of B. White [9] or Rubinstein–Sternberg [6]. This is also a consequence of the result in [4] which we recall for the convenience of the reader. Let $\Lambda$ be a connected open set (or Riemannian manifold) of dimension $\geqslant 1$ and let $u \in W^{1,p}(S^n \times \Lambda, S^n)$ with $p \geqslant n+1$ $(n \geqslant 1)$. Then for a.e. $\lambda \in \Lambda$ the map $u(\cdot,\lambda) : S^n \to S^n$ belongs to $W^{1,p}$ and thus it is continuous. So $\deg(u(\cdot,\lambda))$ is well-defined. In this setting, the result of [4] asserts that this degree is independent of $\lambda$ (a.e.) and that it is stable under $W^{1,n}$ convergence. Clearly, this implies that $W^{1,p}(S^n \times \Lambda, S^n)$ is not path-connected for $p \geqslant n+1$.

Our next result, proved in [3], is a generalization of Theorem 2.

THEOREM 3. – *Let* $1 \leqslant p < \dim M$, *and assume that* $N$ *is* $[p-1]$*-connected, i.e.,*

$$\pi_0(N) = \cdots = \pi_{[p-1]}(N) = 0.$$

*Then* $W^{1,p}(M,N)$ *is path-connected.*

An immediate consequence of Theorem 3 is:

COROLLARY 1. – *For* $1 \leqslant p < n$, $W^{1,p}(S^n, S^n)$ *is path-connected.*

*Remark* 2. – If $1 \leqslant p < 2$ (i.e., the setting of Theorem 2) then the hypothesis on $N$ in Theorem 3 reads $\pi_0(N) = 0$, i.e., $N$ is connected (which is always assumed), and thus Theorem 3 implies Theorem 2. Assumption $p < \dim M$ is sharp. Just take $M = N = S^n$ and $p = n$, and recall (*see*, e.g., [5]) that $W^{1,n}(S^n, S^n)$ is not path-connected since a degree is well-defined.

Corollary 1 may also be derived from the following general result.

PROPOSITION 1. – *For any* $1 \leqslant p < n$ *and any* $N$, $W^{1,p}(S^n, N)$ *is path-connected.*

In the same spirit we also have:

PROPOSITION 2. – *For any* $m \geqslant 1$, *any* $1 \leqslant p < n+1$ *and any* $N$, $W^{1,p}(S^n \times B_1^m, N)$ *is path-connected.*

Here $B_1^m$ is the unit ball in $\mathbb{R}^m$.

*Remark* 3. – As in Remark 1, assumption $p < n + 1$ is optimal since $W^{1,p}(S^n \times B_1^m, N)$ is not path-connected when $p \geqslant n + 1$ and $\pi_n(N) \neq 0$. This is again a consequence of a result in [4] (Section 2, Theorem 2′).

An interesting problem which we have not settled is the following:

CONJECTURE 1. – Given $u \in W^{1,p}(M, N)$ (any $1 \leqslant p < \infty$, any $M$, any $N$), there exists a $v \in C^\infty(M, N)$ and a path $u^t \in C([0,1], W^{1,p}(M, N))$ such that $u^0 = u$ and $u^1 = v$.

We have strong evidence that the above conjecture is true. First, we know that if $p \geqslant \dim M$, Conjecture 1 holds. Next, it is a consequence of Theorem 2 that the conjecture holds when $\dim M = 2$. Indeed if $p < 2$, any $u$ may be connected to a constant map; if $p \geqslant 2 = \dim M$ we are again in the situation just mentioned above. Conjecture 1 also holds when $M = S^n$ (any $p$ and any $N$); this is a consequence of Proposition 1 when $p < n$.

Here are two additional results, proved in [3], in support of Conjecture 1.

THEOREM 4. – *If* $\dim M = 3$ *and* $\partial M \neq \varnothing$ (*any* $N$ *and any* $p$), *Conjecture* 1 *holds.*

THEOREM 5. – *If* $N = S^1$ (*any* $M$ *and any* $p$), *Conjecture* 1 *holds.*

Next, we analyze how the topology of $W^{1,p}(M, N)$ "deteriorates" as $p$ decreases from infinity to 1. We denote by $[u]$ and $[u]_p$ the equivalence classes associated with $\sim$ and $\sim_p$. It is not difficult to see that if $u$, $v \in W^{1,p}(M, N) \cap C^0(M, N)$, $1 \leqslant p < \infty$, with $u \sim v$, then $u \sim_p v$. As a consequence we have a well-defined map

$$i_p : [u] \longmapsto [u]_p$$

going from $C^1(M, N)/\sim$ to $W^{1,p}(M, N)/\sim_p$.

The following definition is natural:

DEFINITION 1. – If $i_p$ is bijective, we say that $W^{1,p}(M, N)$ and $C^0(M, N)$ have the same topology (or more precisely the same homotopy classes).

We know that:

PROPOSITION 3. – *For* $p \geqslant \dim M$, $W^{1,p}(M, N)$ *and* $C^0(M, N)$ *have the same topology.*

Another, much more delicate, case where $W^{1,p}(M, N)$ and $C^0(M, N)$ have the same topology is:

THEOREM 6. – *For any* $p \geqslant 2$ *and any* $M$, $W^{1,p}(M, S^1)$ *and* $C^0(M, S^1)$ *have the same topology.*

*Remark* 4. – On the other hand, $W^{1,p}(M,S^1)$ and $C^0(M,S^1)$ do not have the same topology for $p<2$ if $C^0(M,S^1)$ is not path-connected; this is a consequence of Theorem 2.

For $q \geqslant p$ we also have a well-defined map

$$i_{p,q} : W^{1,q}(M,N)/\sim_q \longrightarrow W^{1,p}(M,N)/\sim_p .$$

It is then natural to introduce the following:

DEFINITION 2. – Let $1<p<\infty$. We say that a change of topology occurs at $p$ if for every $0<\varepsilon<p-1$, $i_{p-\varepsilon,p+\varepsilon}$ is not bijective. Otherwise, we say that there is no change of topology at $p$. We denote by $CT(M,N)$ the set of $p$'s where a change of topology occurs.

Note that if $p>1$ is not in CT, then there exists $0<\bar{\varepsilon}<p-1$ such that $i_{p_1,p_2}$ is bijective for all $p-\bar{\varepsilon}<p_1<p_2<p+\bar{\varepsilon}$. Consequently, CT is closed. In fact we have the following property of $CT(M,N)$ which relies on Theorem 2.

PROPOSITION 4. – *$CT(M,N)$ is a compact subset of $[2,\dim M]$.*

*Remark* 5. – Assuming that Conjecture 1 holds, then $i_{p,q}$ is always surjective. As a consequence, a change of topology occurs at $p$ if for every $0<\varepsilon<p-1$, $i_{p-\varepsilon,p+\varepsilon}$ is not injective, i.e., for every $0<\varepsilon<p-1$, there exist $u_\varepsilon$ and $v_\varepsilon$ in $C^1$ such that $[u_\varepsilon]_{p-\varepsilon}=[v_\varepsilon]_{p-\varepsilon}$ while $[u_\varepsilon]_{p+\varepsilon}\neq[v_\varepsilon]_{p+\varepsilon}$.

Another consequence of Theorem 2 is:

PROPOSITION 5. – *If $CT(M,N)=\varnothing$, then $C^0(M,N)$ and $W^{1,p}(M,N)$ are path-connected for all $p\geqslant 1$.*

*Remark* 6. – Assuming that Conjecture 1 holds, then the following statements are equivalent:
a) $CT(M,N)=\varnothing$;
b) $C^0(M,N)$ is path-connected;
c) $W^{1,p}(M,N)$ is path-connected for all $p\geqslant 1$.

Here is another very interesting conjecture:

CONJECTURE 2. –

$$CT(M,N)\subset\{2,3,\ldots,\dim M\}.$$

A stronger form of Conjecture 2 is:
CONJECTURE 2′. – *For every integer $j\geqslant 1$ and any $p$, $q$ with $j\leqslant p\leqslant q<j+1$, $i_{p,q}$ is bijective.*

*Remark* 7. – If Conjecture 1 holds, then Conjecture 2′ can be stated as follows: assume $u$, $v\in W^{1,p}(M,N)$ (any $p$, any $M$, and any $N$) are homotopic in $W^{1,[p]}(M,N)$, then they are homotopic in $W^{1,p}(M,N)$.

In connection with Conjecture 2 we may also raise the following:
OPEN PROBLEM. – *Is it true that for any $n\geqslant 2$ and any $\Gamma\subset\{2,3,\ldots,n\}$, there exist $M$ and $N$ such that* $\dim M=n$ *and*

$$CT(M,N)=\Gamma?$$

We list some more properties of $CT(M,N)$ discussed in [3]:
1) for all $N$,

$$CT(B_1^n,N)=\varnothing;$$

2) for all $N$,

$$\mathrm{CT}(\mathrm{S}^n, N) = \begin{cases} \{n\} & \text{if } \pi_n(N) \neq 0, \\ \varnothing & \text{if } \pi_n(N) = 0. \end{cases}$$

In particular,

$$\mathrm{CT}(\mathrm{S}^n, \mathrm{S}^n) = \{n\};$$

3) for all $M$,

$$\mathrm{CT}(M, \mathrm{S}^1) = \begin{cases} \{2\} & \text{if } \mathrm{C}^0(M, \mathrm{S}^1) \text{ is not path-connected}, \\ \varnothing & \text{if } \mathrm{C}^0(M, \mathrm{S}^1) \text{ is path-connected}; \end{cases}$$

4) if $\mathrm{CT}(M, N)$ is non-empty and $\pi_0(N) = \cdots = \pi_k(N) = 0$ for some $k \geqslant 0$, then

$$\min\{p;\ p \in \mathrm{CT}(M, N)\} \geqslant \min\{k+2, \dim M\};$$

5) if $\Lambda$ is compact and connected with $\dim \Lambda \geqslant 1$, then

$$\min\{p;\ p \in \mathrm{CT}(\mathrm{S}^n \times \Lambda, \mathrm{S}^n)\} = n+1, \quad n \geqslant 1.$$

It would be interesting to determine $\mathrm{CT}(M, N)$ in some concrete cases, e.g., $M$ and $N$ are products of spheres. We plan to return to this question in the future.

In [3] we have investigated the structure of the path-connected components of $\mathrm{W}^{1,p}(M, N)$, i.e.,

$$\pi_0(\mathrm{W}^{1,p}(M, N)).$$

It would be interesting to analyze $\pi_k(\mathrm{W}^{1,p}(M, N))$ for $k \geqslant 1$, starting from $\pi_1(\mathrm{W}^{1,p}(M, N))$. Of course it is natural to consider first the case where $1 \leqslant p < 2$ since we already know that $\mathrm{W}^{1,p}$ is path-connected.

**Warning**

People have also considered several other spaces of maps closely related to $\mathrm{W}^{1,p}(M, N)$, for example

$$Z^{1,p}(M, N) = \text{the closure in } \mathrm{W}^{1,p} \text{ of } \mathrm{C}^\infty(M, N),$$

or the weak sequential closure in $\mathrm{W}^{1,p}$ of $\mathrm{C}^\infty(M, N)$ (*see*, e.g., White [8] and [9]). $Z^{1,p}(M, N)$ is a subset of $\mathrm{W}^{1,p}(M, N)$ and in general a strict subset (*see* Bethuel [1]). One may ask the same questions as above (i.e., path-connectedness, etc.) for $Z^{1,p}(M, N)$. We warn the reader that the properties of $Z^{1,p}(M, N)$ may be quite different from the properties of $\mathrm{W}^{1,p}(M, N)$. For example, if $1 \leqslant p < 2$, then $\mathrm{W}^{1,p}(\mathrm{S}^1 \times \Lambda, \mathrm{S}^1)$ ($\Lambda$ connected, $\dim \Lambda \geqslant 1$) is path-connected by Theorem 2. On the other hand $Z^{1,p}(\mathrm{S}^1 \times \Lambda, \mathrm{S}^1)$ is not path-connected. Indeed, note that if $u \in \mathrm{C}^\infty(\mathrm{S}^1 \times \Lambda, \mathrm{S}^1)$ then

$$\psi(u) := \fint_\Lambda \fint_{\mathrm{S}^1} (u \times u_\theta)\, \mathrm{d}\theta\, \mathrm{d}\lambda \in \mathbb{Z}$$

(and $\psi(u)$ represents the degree of the map $u(\cdot, \lambda)$ for any $\lambda \in \Lambda$). By density $\psi(u) \in \mathbb{Z}$ for all $u \in Z^{1,p}(\mathrm{S}^1 \times \Lambda, \mathrm{S}^1)$ and since $\psi$ can take any integer value it follows that $Z^{1,p}$ is not path-connected.

F. Bethuel [1] (*see* also [2]) has been mostly concerned with the question of density of smooth maps in $W^{1,p}(M, N)$. B. White [9] deals with the question of how much the topological properties are preserved by $W^{1,p}$ (or $Z^{1,p}$, etc.). We have tried to analyze how much of the topology "deteriorates" when passing to $W^{1,p}$, i.e., whether two smooth maps $u, v \in C^\infty(M, N)$ in different homotopy classes (in the usual sense) can nevertheless be connected in $W^{1,p}$ for appropriate $p$'s. Roughly speaking our concerns complement those of B. White as well as those in [4]. However, some of our techniques resemble those of B. White and F. Bethuel.

## References

[1] Bethuel F., The approximation problem for Sobolev maps between two manifolds, Acta Math. 167 (1991) 153–206.
[2] Bethuel F., Zheng X., Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (1988) 60–75.
[3] Brezis H., Li Y.Y., Topology and Sobolev spaces, Preprint.
[4] Brezis H., Li Y.Y., Mironescu P., Nirenberg L., Degree and Sobolev spaces, Topol. Methods in Nonlin. Anal. 13 (1999) 181–190.
[5] Brezis H., Nirenberg L., Degree theory and BMO, Part I: Compact manifolds without boundaries, Selecta Math. 1 (1995) 197–263; Part II: Compact manifolds with boundaries, Selecta Math. 2 (1996) 309–368.
[6] Rubinstein J., Sternberg P., Homotopy classification of minimizers of the Ginzburg–Landau energy and the existence of permanent currents, Commun. Math. Phys. 179 (1996) 257–263.
[7] Schoen R., Uhlenbeck K., Boundary regularity and the Dirichlet problem for harmonic maps, J. Differ. Geom. 18 (1983) 253–268.
[8] White B., Infima of energy functionals in homotopy classes of mappings, J. Differ. Geom. 23 (1986) 127–142
[9] White B., Homotopy classes in Sobolev spaces and the existence of energy minimizing maps, Acta Math. 160 (1988) 1–17.