93

J. Math. pures et appl.,
61, 1982, p. 245 à 259

SOME PROPERTIES OF HIGHER ORDER SOBOLEV SPACES

By Haim BREZIS and Felix E. BROWDER

Introduction

Assume Ω is an open set in $\mathbb{R}^n$, $m \geqq 1$ is an integer and $1 < p < \infty$. Let u be a function in the Sobolev space $W_0^{m,p}(\Omega)$ and let T be a distribution in $L^1_{loc}(\Omega) \cap W^{-m,p'}(\Omega)$. In the present paper we are concerned with the following question: Under what conditions is the function $T(x)u(x)$ integrable on Ω (for the Lebesgue measure) and if so does $\int_\Omega T(x)u(x)\,dx$ equal $\langle T, u \rangle$ (where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the duality of $W^{-m,p'}(\Omega)$ with $W_0^{m,p}(\Omega)$)?

In earlier work ([3], [4]) the authors have considered the case of first order Sobolev spaces (i. e. $m=1$). Such results have had applications to the study of singular elliptic equations, singular either because or a strong nonlinearity or because of singularities in the coefficients as for example in Schrödinger operators with singular potentials (*see* [2], [5]).

The techniques we use in the present paper differ considerably from the technique we used in our previous work. The main reason is that there is no obvious truncation operation within the space $W_0^{m,p}$. Instead we must rely on the delicate truncation procedure introduced by L. Hedberg [11] for $W^{m,p}$ spaces. The usefulness of Hedberg's technique in the study of strongly nonlinear elliptic equations has been originally pointed out by J. R. L. Webb [15].

The plan of our paper is the following: In section 1 we discuss Hedberg's truncation method and we prove:

THEOREM 1. — *Assume* $T \in L^1_{loc}(\mathbb{R}^N) \cap W^{-m,p'}(\mathbb{R}^N)$ *and* $u \in W^{m,p}(\mathbb{R}^N)$. *Suppose* $T(x)u(x) \geqq f(x)$ *a. e. on* $\mathbb{R}^N$ *for some* $f \in L^1(\mathbb{R}^N)$. *Then* $Tu \in L^1(\mathbb{R}^N)$ *and:*

$$\int T(x)u(x)\,dx = \langle T, u \rangle.$$

In section 2 we deal with various extensions of Theorem 1, in particular with the case where $\Omega \subset \mathbb{R}^N$ and the case where T is a measure.

In section 3 we present an application to the solvability of a strongly nonlinear elliptic equation.

In section 4 we consider the connection between our results and earlier work of J. Deny [8] dealing with the case where $T \geqq 0$. We thank F. Murat and A. Ancona for stimulating discussions concerning paragraph 4.

1. Hedberg's truncation method for $\Omega = \mathbb{R}^N$

We start with a self contained exposition of some of Hedberg's devices [10], [11] which are relevant for our study. We follow essentially the presentation given in [15]. Throughout section 1 we shall deal only with the case where $\Omega = \mathbb{R}^N$. For simplicity we write $W^{m,p}$ instead of $W^{m,p}(\mathbb{R}^N)$, etc. and we denote by C various constants depending only on m, p and N.

The following result plays a central role.

THEOREM 2. — *Given u in* $W^{m,p}$ *there exists a sequence* $\{u_n\}$ *such that:*

(1) $$u_n \in W^{m,p} \cap L^\infty, \quad \operatorname{supp} u_n \text{ is compact;}$$

(2) $$|u_n(x)| \leqq |u(x)| \quad \text{and} \quad u_n(x)u(x) \geqq 0 \quad \text{a.e. on } \mathbb{R}^N;$$

(3) $$u_n \to u \quad \text{in } W^{m,p} \quad \text{as } n \to \infty.$$

We first deduce Theorem 1 as a simple consequence of Theorem 2.

Proof of Theorem 1. — Let $\{u_n\}$ be the sequence defined in Theorem 2. It follows easily from (1) (using convolution with mollifiers) that:

(4) $$\int T(x)\,u_n(x)\,dx = \langle T, u_n \rangle.$$

By Theorem 2 the right hand side in (4) converges as $n \to \infty$ to $\langle T, u \rangle$. On the other hand, we have $Tu_n \geqq -|f|$ a.e. We deduce from Fatou's Lemma that $Tu \in L^1$. We conclude by dominated convergence that $\int Tu_n\,dx \to \int Tu\,dx$ and thus $\int Tu\,dx = \langle T, u \rangle$.

Proof of Theorem 2. — We distinguish two cases:

(i) The case $mp > N$;

(ii) The case $mp \leqq N$.

(i) *The case* $mp > N$.

We fix a function $\zeta_1 \in C_0^\infty$ such that $0 \leqq \zeta_1 \leqq 1$ and $\zeta_1(x) = 1$ near $x = 0$. Let $\zeta_n(x) = \zeta_1(x/n)$. Then $u_n(x) = \zeta_n(x)\,u(x)$ satisfies all the required properties (since $u \in L^\infty$ by Sobolev's Theorem).

(ii) *The case* $mp \leqq N$.

We may always assume that u has compact support (otherwise we apply the following construction to $\zeta_n u$ instead of u). Since $u \in W^{m, p_1}$ for every $1 \leqq p_1 \leqq p$ we may represent u as a Riesz potential:

$$u = I_m \star \varphi \tag{5}$$

(see e. g. [14], Chap. V) where $I_m(x) = |x|^{m-N}$ and $\varphi \in L^{p_1}$ for every $1 \leqq p_1 \leqq p$ with:

$$\| \varphi \|_{L^p} \leqq C \| u \|_{W^{m,p}}. \tag{6}$$

We recall that by the Hardy-Littlewood-Sobolev Theorem (see e. g. [14], Theorem 1, p. 119) the convolution $I_m \star \varphi$ is well defined provided $\varphi \in L^{p_1}$ for some p_1 such that $1 \leqq p_1 < N/m$ (and then $I_m \star \varphi \in L^{q_1}$ if $p_1 > 1$ and $1/q_1 = (1/p_1) - (m/N)$).

Let:

$$v = I_m \star |\varphi|. \tag{7}$$

Let $H \in C^\infty(\mathbb{R})$ be a function such that $0 \leqq H \leqq 1$ and:

$$H(t) = \begin{cases} 1 & \text{when} \quad |t| \leqq \dfrac{1}{2}, \\ 0 & \text{when} \quad |t| \geqq 1. \end{cases}$$

Set:

$$u_n(x) = H\left(\frac{1}{n} v(x)\right) u(x). \tag{8}$$

We shall now verify that u_n satisfies all the required properties in Theorem 2. The argument relies heavily on the following Lemma due to Hedberg [10] which we prove in the Appendix for the convenience of the reader.

LEMMA 1. — *Let $\psi \in L^{p_1}$ with $p_1 < N/m$. Let $M\psi$ be the maximal function of ψ (1). Then there is a constant C depending only on m and N such that:*

$$|D^\beta (I_m \star \psi)| \leqq C (M\psi)^{|\beta|/m} (I_m \star |\psi|)^{1-|\beta|/m} \qquad a.e. \quad in \ \mathbb{R}^N,$$

for every multi-integer β with $0 \leqq |\beta| \leqq m-1$.

Proof of Theorem 2 continued. — It is clear that $|u(x)| \leqq v(x)$, $|u_n(x)| \leqq n$, $u_n \to u$ in L^p. In order to prove (3) it suffices to check that $D^\alpha u_n \to D^\alpha u$ in L^p for every α with $|\alpha| = m$. An easy computation based on Lemma 1 shows that:

$$\left| D^\beta H\left(\frac{1}{n} v\right) \right| \leqq C n^{-|\beta|/m} |M\varphi|^{|\beta|/m} \qquad \text{a.e.} \quad \text{on } \mathbb{R}^N, \tag{9}$$

(1) $(M\psi)(x) = \sup_{r>0} r^{-N} \int_{|y-x| \leqq r} |\psi(y)| \, dy.$

for every β with $0 \leqq |\beta| \leqq m-1$ and:

$$(10) \qquad \left| D^\alpha H\left(\frac{1}{n} v\right) \right| \leqq C n^{-1} (M\varphi + |D^\alpha v|)$$

for every α with $|\alpha| = m$.

On the other hand we have by the Leibnitz rule:

$$(11) \qquad \left| D^\alpha u_n - \left[D^\alpha H\left(\frac{1}{n} v\right)\right] u - H\left(\frac{1}{n} v\right) D^\alpha u \right| \leqq C \sum_{\substack{|\beta|+|\gamma|=m \\ |\beta|\geqq 1, |\gamma|\geqq 1}} \left| D^\beta H\left(\frac{1}{n} v\right) \right| |D^\gamma u|.$$

Combining (9), (10), (11) and Lemma 1 we find a. e.:

$$(12) \qquad \left| D^\alpha u_n - H\left(\frac{1}{n} v\right) D^\alpha u \right| \leqq C n^{-1} (M\varphi + |D^\alpha v|) v + C n^{-1/m} v^{1/m} M\varphi.$$

It follows that $D^\alpha u_n \to D^\alpha u$ a. e.

Also we have:

$$(13) \qquad |D^\alpha u_n| \leqq C (M\varphi + |D^\alpha v| + |D^\alpha u|).$$

Recall that by the theory of singular integrals $D^\alpha v \in L^p$ (see e. g. [14], Chap. II). Recall also (see e. g. [14], Chap. I) that $M\varphi \in L^p$. We conclude that $D^\alpha u_n \to D^\alpha u$ in L^p.

Remark 1. — The same proof shows that if we define:

$$\tilde{u}_n(x) = H\left(\frac{1}{n} \tilde{v}(x)\right) u(x) \qquad \text{where} \quad \tilde{v}(x) = I_m \star \psi$$

and ψ is *any* function such that $\psi \in L^p \cap L^{p_1}$ ($p_1 < N/m$) and $|\varphi| \leqq \psi$ a. e., then $\tilde{u}_n$ satisfies (1), (2) and (3).

Remark 2. — The proof of Theorem 2 shows that the sequence u_n defined by (8) has the additional property that $\| u_n \|_{W^{m,p}} \leqq C \| u \|_{W^{m,p}}$ where C depends only on m, p and $\mathbb{N}$.

2. Various extensions of Theorem 1

We shall first be concerned with the extension of Theorem 1 to domains $\Omega \subset \mathbb{R}^N$. More precisely the question we shall investigate is the following:

Assume $\Omega \subset \mathbb{R}^N$ is an *arbitrary* open set. Suppose:

$$T \in L^1_{loc}(\Omega) \cap W^{-m,p'}(\Omega), \quad u \in W_0^{m,p}(\Omega)$$

and suppose:

$$T(x) u(x) \geqq f(x) \qquad \text{a. e. on } \Omega \text{ for some } f \in L^1(\Omega).$$

Question (Q_1). — Can one conclude that $Tu \in L^1(\Omega)$ and that:

$$\int_\Omega Tu\, dx = \langle T, u \rangle ?$$

The answer to Q_1 is positive when $m=1$ (see [4]); however the answer seems to be unknown in general.

We shall prove that the answer to Q_1 is positive under various kinds of additional assumptions.

In section 2.1 we prove that if in addition $\int_{\Omega\cap B_R}|T(x)|\,dx<\infty$ for every $R<\infty$ where $B_R=\{x\in\mathbb{R}^N;\ |x|<R\}$, then the answer to Q_1 is positive.

In section 2.2, we prove that if $\partial\Omega$ is regular (locally) then the answer to Q_1 is positive. In fact using a result of Hedberg [11] it suffices to assume very little regularity for $\partial\Omega$.

In section 2.3 we consider the case where T is a measure (instead of a function in L^1_{loc}) and also the case of multiple T_i and u_i such that $\sum_i T_i u_i \geqq f$ for some $f\in L^1$.

2.1. The case where Ω is arbitrary and:

$$\int_{\Omega\cap B_R}|T(x)|\,dx<\infty \qquad \textit{for every} \quad R<\infty.$$

Throughout section 2.1 we assume that Ω is arbitrary. We shall prove the following.

Theorem 3. — *Let* $T\in L^1_{loc}(\Omega)\cap W^{-m,p'}(\Omega)$ *be such that:*

$$\int_{\Omega\cap B_R}|T(x)|\,dx<\infty \qquad \textit{for every} \quad R<\infty. \tag{14}$$

Assume $u\in W^{m,p}_0(\Omega)$ *and* $T(x)u(x)\geqq f(x)$ *a.e. on* Ω *for some* $f\in L^1(\Omega)$.
Then $Tu\in L^1(\Omega)$ *and:*

$$\int_\Omega Tu\,dx=\langle T, u\rangle.$$

Proof. — The proof is straightforward when $mp>N$; therefore we may assume that $mp\leqq N$. Using $\zeta_n u$ in place of u we may always reduce to the case where supp u is bounded.

Set:

$$\bar{u}(x)=\begin{cases} u(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\in\mathbb{R}^N\setminus\Omega.\end{cases}$$

Then $\bar{u}$ lies in $W^{m,p}(\mathbb{R}^N)$ and also $\bar{u}\in W^{m,p_1}(\mathbb{R}^N)$ for every $1\leqq p_1\leqq p$. Thus we may write:

$$\bar{u}=I_m\star\varphi$$

for some $\varphi\in L^p(\mathbb{R}^N)$ and $\varphi\in L^{p_1}$ for $1\leqq p_1\leqq p$.

As in the proof of Theorem 2 set:

$$v=I_m\star|\varphi|,$$
$$u_n=H\left(\frac{1}{n}v\right)u.$$

Since $u \in W_0^{m,p}(\Omega)$, there is a sequence $u^j \in C_0^\infty(\Omega)$ such that $u^j \to u$ in $W^{m,p}(\Omega)$ (and a. e.). For each j we perform the above construction and we set:

$$\bar{u}^j = I_m \star \varphi^j,$$
$$v^j = I_m \star |\varphi^j|,$$
$$u_n^j = H\left(\frac{1}{n} v^j\right) u^j.$$

Fix $\zeta \in C_0^\infty(\mathbb{R}^N)$. We clearly have:

$$\int_\Omega T \zeta u_n^j \, dx = \langle T, \zeta u_n^j \rangle. \tag{15}$$

As we keep n fixed and let $j \to \infty$ we see that:

$$\int_\Omega T \zeta u_n^j \, dx \to \int_\Omega T \zeta u_n$$

by dominated convergence [and assumption (14)].

On the other hand, by Remark 2 we have:

$$\| u_n^j \|_{W^{m,p}(\Omega)} \leqq C \| \bar{u}^j \|_{W^{m,p}(\mathbb{R}^N)} \leqq C$$

(where C is independent of j and n).

Therefore u_n^j converges weakly in $W_0^{m,p}(\Omega)$ to u_n as $j \to \infty$; and thus $\zeta u_n^j \rightharpoonup \zeta u_n$ as $j \to \infty$. Passing to the limit in (15) as $j \to \infty$ we find:

$$\int_\Omega T \zeta u_n = \langle T, \zeta u_n \rangle.$$

We conclude easily (by the same argument as in the proof of Theorem 1) that $Tu \in L^1(\Omega)$ and:

$$\int_\Omega Tu \, dx = \langle T, u \rangle.$$

2.2. THE CASE WHERE $\partial\Omega$ IS LOCALLY SMOOTH. — Throughout section 2.2 we assume that $\partial\Omega$ is locally smooth — but Ω is not necessarily bounded. We shall prove the following.

THEOREM 4. — *Assume* $T \in L^1_{loc}(\Omega) \cap W^{-m,p'}(\Omega)$ *and* $u \in W_0^{m,p}(\Omega)$ *are such that:*

$$T(x) u(x) \geqq f(x) \quad a.e. \quad on \ \mathbb{R}^N$$

for some $f \in L^1(\Omega)$.

Then $Tu \in L^1(\Omega)$ *and:*

$$\int_\Omega Tu \, dx = \langle T, u \rangle.$$

Proof. — We may always assume that $\operatorname{supp}(u)$ is bounded. Using the smoothness of $\partial\Omega \cap \operatorname{supp} u$ and a standard technique (see e. g. [12], Theorem 11.8 in Chapter 1) one constructs a sequence $\zeta_n \in C_0^\infty(\Omega)$ such that $0 \leqq \zeta_n \leqq 1$ and $\zeta_n u \to u$ in $W_0^{m,p}(\Omega)$. By Theorem 1 or 3 we know that $T\zeta_n u \in L^1(\Omega)$ and:

$$\int T\,\zeta_n u\, dx = \langle \zeta_n T, u \rangle.$$

The conclusion follows easily as $n \to \infty$.

2.3. T IS A MEASURE-MULTIPLE T AND u. — In order to simplify matters we start again with the case where $\Omega = \mathbb{R}^N$; similar results hold when $\Omega \subset \mathbb{R}^N$ (under additional assumptions of the kind introduced in Theorem 3 and 4; see e. g. Corollary 6). We recall some basic notions about capacities (see e. g. [13]). For a compact set $K \subset \mathbb{R}^N$ we define:

$$\operatorname{cap} K = \operatorname{Inf}\{ \|\alpha\|_{W^{m,p}}^p;\ \alpha \in C_0^\infty(\mathbb{R}^N),\ \alpha \geqq 1 \text{ on } K \}.$$

For an arbitrary set $A \subset \mathbb{R}^N$ we define:

$$\operatorname{cap}_* A = \operatorname{Sup}\{ \operatorname{cap} K;\ K \text{ compact},\ K \subset A \}$$

and:

$$\operatorname{cap}^* A = \operatorname{Inf}\{ \operatorname{cap}_* G\text{: } G \text{ open},\ A \subset G \}.$$

We recall that if $u_n \in C_0^\infty(\mathbb{R}^N)$ is a Cauchy sequence in $W^{m,p}$ then there is a subsequence u_{n_k} which converges everywhere to a function u except for a set A with $\operatorname{cap}^* A = 0$. Besides if we pick another subsequence u'_{n_k} of the original sequence in a similar manner we would obtain a pointwise limit u' which equals u except on a set A' with $\operatorname{cap}^* A' = 0$. In this way a function u in $W^{m,p}$ can be defined pointwise except possibly on a set A with $\operatorname{cap}^* A = 0$. Let M denote the space of all regular Borel measures on $\mathbb{R}^N$ (not necessarily bounded measures or non-negative measures); M^+ consists of all non-negative measures. Let $\mu \in M^+$ be a measure satisfying:

(16) For every set $A \subset \mathbb{R}^N$ with $\operatorname{cap}^* A = 0$ then $|\mu|(A) = 0$ (2).

Let $g_1, g_2, \ldots, g_k \in L^1_{\text{loc}}(\mathbb{R}^N; d\mu)$ and consider the measures:

$$T_i = g_i \mu \qquad \text{for} \quad 1 \leqq i \leqq k.$$

We assume that:

$$T_i \in W^{-m,p'} \qquad \text{for all} \quad 1 \leqq i \leqq k.$$

Let $u_1, u_2, \ldots, u_k \in W^{m,p}$.

We shall prove the following.

(2) $|\mu|(A) = \int_A d|\mu|$.

THEOREM 5. — *Assume:*

$$g.u \equiv \sum_{i=1}^{k} g_i u_i \geqq f \ \mu\text{-}a.e. \quad on\ \mathbb{R}^N \tag{17}$$

for some $f \in L^1(\mathbb{R}^N; d\mu)$ (*note that each* u_i *is defined* μ *a. e.*).

Then $g.u \in L^1(\mathbb{R}^N; d\mu)$ *and:*

$$\int g.u\, d\mu = \langle T, u \rangle \equiv \sum_{i=1}^{k} \langle T_i, u_i \rangle.$$

Remark 3. — Choosing μ to be the Lebesgue N-measure and $k=1$ we recover Theorem 1.

Remark 4. — Assume $T_1, T_2, \ldots, T_k$ are *given* elements of $M \cap W^{-m, p'}$ and set $\mu = \sum_{i=1}^{k} |T_i|$. Then μ satisfies (16). This is a consequence of the following.

LEMMA 2. — *Assume* $T \in M \cap W^{-m, p'}$. *Let* $A \subset \mathbb{R}^N$ *be a set such that* $\text{cap}^* A = 0$. *Then* A *is* T-*measurable and* $|T|(A) = 0$.

When $m=1$, Lemma 2 is proved in [9]; the argument given in [9] extends readily to the case $m \geqq 1$.

Since T_i is absolutely continuous with respect to μ we can write $T_i = g_i \mu$ be some $g_i \in L^1_{\text{loc}}(\mathbb{R}^N, d\mu)$. Therefore Theorem 5 applies provided (17) holds.

Proof of Theorem 5. — Set $\vec{u} = (u_1, u_2, \ldots, u_k)$.

We may always assume that supp $\vec{u}$ is bounded and that $mp \leqq N$. We write $\vec{u} = I_m \star \vec{\varphi}$ for some $\vec{\varphi} \in (L^p \cap L^{p_1})^k (1 \leqq p_1 \leqq p)$. Set:

$$\varphi(x) = |\vec{\varphi}(x)|$$

and:

$$v(x) = (I_m \star \varphi)(x),$$

$$\vec{u}_n(x) = H\left(\frac{1}{n} v(x)\right) \vec{u}(x)$$

where $|\xi|$ denotes the euclidean norm of ξ.

By Theorem 2 (see Remark 1) we know that $\vec{u}_n \to \vec{u}$ in $(W^{m, p})^k$ as $n \to \infty$. In addition $|\vec{u}_n(x)| \leqq n$ a. e. on $\mathbb{R}^N$. Let $\zeta \in C_0^\infty(\mathbb{R}^N)$ with $0 \leqq \zeta \leqq 1$; then the function $\zeta \vec{g}.\vec{u}_n \in L^1(\mathbb{R}^N; d\mu)$. In addition:

$$\int \zeta \vec{g}.\vec{u}_n\, d\mu = \langle \vec{T}, \zeta \vec{u}_n \rangle. \tag{18}$$

[Indeed let ρ_ε be a sequence of mollifiers;

$$\rho_\varepsilon \star (\zeta \vec{u}_n) \to \zeta \vec{u}_n \text{ in } [W^{m, p}]^k \text{ as } \varepsilon \to 0$$

and thus – extracting an infinite subsequence – we may assume that $\rho_\varepsilon \star (\zeta \vec{u}_n) \to \zeta \vec{u}_n$ as $\varepsilon \to 0$ everywhere except on a set A with $\text{cap}^* A = 0$. We have:

(19)
$$\int \vec{g} . [\rho_\varepsilon \star (\zeta \vec{u}_n)]\, d\mu = \langle \vec{T}, \rho_\varepsilon \star (\zeta u_n) \rangle.$$

Passing to the limit in (19) as $\varepsilon \to 0$ using dominated convergence we obtain (18)]. Next we pass to the limit in (18) as $n \to \infty$ and as $\zeta \to 1$ using Fatou and dominated convergence.

We now indicate briefly how Theorem 5 extends to the case where $\Omega \subset \mathbb{R}^N$, for example under the additional assumption that $\partial\Omega$ is locally smooth [Ω not necessarily bounded].

Let $\mu \in M^+(\Omega)$ be a non-negative measure on Ω – possibly unbounded – such that:

(16′) for every set $A \subset \Omega$ such that $\text{cap}^* A = 0$, then $|\mu|(A) = 0$. Let $g_1, g_2, \ldots, g_k \in L^1_{\text{loc}}(\Omega; d\,\mu)$ and consider the measures:

$$T_i = g_i \mu, \qquad 1 \leqq i \leqq k.$$

We assume that $T_i \in W^{-m, p'}(\Omega)$ for $1 \leqq i \leqq k$. Let $u_1, u_2, \ldots, u_k$ lie in $W_0^{m, p}(\Omega)$.

Corollary 6. – *Assume:*

(17′)
$$g.u \geqq f, \qquad \mu - a.e. \text{ on } \Omega,$$

for some $f \in L^1(\Omega; d\mu)$.

Then $g.u \in L^1(\Omega, d\mu)$ *and:*

$$\int_\Omega g.u\, d\mu = \langle T, u \rangle.$$

Proof. – We may always reduce to the case where supp u is bounded. Since $\partial\Omega$ is locally smooth there is a sequence $\zeta_n \in C_0^\infty(\Omega)$ such that $0 \leqq \zeta_n \leqq 1$, $\zeta_n u \to u$ in $W_0^{m, p}$ as $n \to \infty$. By Theorem 5 we know that $\int_\Omega g.u \zeta_n d\mu = \langle \zeta_n T, u \rangle$ and we pass easily to the limit as $n \to \infty$.

3. An application to a strongly nonlinear elliptic equation

Let Ω be an arbitrary open subset of $\mathbb{R}^N$. Let $g(x, u): \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that:

(20)
$$\text{for each } s > 0, \qquad \operatorname{Sup}_{|u| \leqq s} |g(x, u)| \leqq h_s(x) \in L^1(\Omega).$$

Assume:

(21)
$$g(x, u).u \geqq 0 \qquad \text{a.e.} \quad x \in \Omega, \quad \forall u \in \mathbb{R}.$$

Assume A: $W_0^{m,p}(\Omega) \to W^{-m,p'}(\Omega)$ is a pseudomonotone operator which maps bounded sets into bounded sets and which is coercive i. e.:

$$\lim_{\|u\|_{m,p} \to +\infty} \frac{\langle Au, u \rangle}{\|u\|_{W^{m,p}}} = +\infty$$

(For general examples of such nonlinear elliptic operators see e. g. [6].)

THEOREM 7. — *For every $f \in W^{-m,p'}(\Omega)$ there exists a $u \in W_0^{m,p}(\Omega)$ such that:*

$$(22) \quad \left\{ \begin{array}{l} g(x, u) \in L^1(\Omega), \quad g(x, u)u \in L^1(\Omega) \\ \textit{and:} \\ \langle Au, u \rangle + \displaystyle\int g(x, u)v\,dx = \langle f, v \rangle, \quad \forall v \in W_0^{m,p} \cap L^\infty \quad \textit{and for} \quad v = u. \end{array} \right.$$

Furthermore if g is nondecreasing in u and u_1, u_2 are two solutions corresponding to f_1 and f_2 then:

$$(23) \quad \langle Au_1 - Au_2, u_1 - u_2 \rangle + \int [g(x, u_1) - g(x, u_2)](u_1 - u_2)\,dx = \langle f_1 - f_2, u_1 - u_2 \rangle$$

Remark 5. — The existence part in Theorem 7 is due to J. L. Webb [15] under some (mild) additional regularity assumptions on $\partial\Omega$.

Remark 6. — If $\partial\Omega$ is (locally) smooth, (20) may be weakened. Instead of (20) we assume

(20′) for $s > 0$, $\operatorname{Sup}_{|u| \leq s} |g(x, u)| \leq h_s(x) \in L^1_{loc}(\Omega)$.

Then for every $f \in W^{-m,p'}(\Omega)$, there exists a $u \in W_0^{m,p}(\Omega)$ such that:

$$(22') \quad \left\{ \begin{array}{l} g(x, u) \in L^1_{loc}(\Omega), \quad g(x, u)u \in L^1(\Omega) \\ \text{and:} \\ \langle Au, u \rangle + \displaystyle\int_\Omega g(x, u)v\,dx = \langle f, v \rangle, \quad \forall v \in C_0^\infty(\Omega) \quad \text{and for} \quad v = u. \end{array} \right.$$

Furthermore (23) holds.

The proof is similar to the proof of Theorem 7 except that we use Theorem 4 in place of Theorem 3.

Proof of Theorem 7. — Set $g_n(x, u) = \zeta_1(x/n) P_n g(x, u)$ where $\zeta_1 \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \zeta_1 \leq 1$, $\zeta_1(x) = 1$ near $x = 0$ and:

$$P_n \xi = \left\{ \begin{array}{ll} \xi & \text{if} \quad |\xi| \leq n, \\ \dfrac{n\xi}{|\xi|} & \text{if} \quad |\xi| > n. \end{array} \right.$$

It follows easily from the theory of pseudo-monotone operators that there exists $u_n \in W_0^{m,p}(\Omega)$ such that:

$$Au_n + g_n(x, u_n) = f.$$

In addition:

$$\| u_n \|_{W^{m,p}} \leqq C \qquad \text{and} \qquad \int_\Omega g_n(x, u_n) u_n \, dx \leqq C.$$

Without loss of generality we may assume that u_n converges weakly in $W_0^{m,p}(\Omega)$ and also a. e. to some u, and Au_n converges weakly to some χ in $W^{-m,p'}(\Omega)$. A standard measure theoretic argument shows that $g_n(x, u_n) \to g(x, u)$ in $L^1(\Omega)$ and that (22) holds.

On the other hand we have by Fatou's Lemma that:

$$\overline{\lim} \, \langle Au_n, u_n \rangle \leqq \langle f, u \rangle - \int_\Omega g(x, u) u \, dx.$$

Set $T = g(x, u) = f - \chi$.

We have $T \in L^1(\Omega) \cap W^{-m,p'}(\Omega)$, and by Theorem 3 we conclude that:

$$\int_\Omega g(x, u) u \, dx = \langle f - \chi, u \rangle.$$

Therefore $\overline{\lim} \, \langle Au_n, u_n \rangle \leqq \langle \chi, u \rangle$ and consequently $Au = \chi$. The conclusion follows readily. Property (22) is again a direct consequence of Theorem 3.

4. Connection with a Theorem of J. Deny

We assume first that $\Omega = \mathbb{R}^N$. When $p = 2$, the following result is a special case of a theorem of J. Deny [8] (Th. 1, p. 138).

THEOREM 8. — *Assume* $T \in M^+ \cap W^{-m,p'}$ *and* $u \in W^{m,p}$.
Then:

$$u \in L^1(\mathbb{R}^N; dT) \qquad \textit{and} \qquad \int u \, dT = \langle T, u \rangle.$$

Proof. — We may write u as $u = u_1 - u_2$ with $u_1, u_2 \in W^{m,p}$, $u_1 \geqq 0$ and $u_2 \geqq 0$. Indeed we may represent u as $u = G_m \star \varphi$ where G_m is a Bessel potential and $\varphi \in L^p$ (see e. g. [14], Chap. V); we set $u_1 = G_m \star \varphi^+$ and $u_2 = G_m \star \varphi^-$. Next we apply Theorem 5 with $\mu = T$, $g = 1$, $u = u_1$ and $u = u_2$. The conclusion follows readily.

Remark 7. — One can give a direct and elementary proof of Theorem 8 which does not make use of Hedberg's approximation technique. Indeed set:

$$\varphi = G_m \star T.$$

Note that $\varphi(x)$ makes sense as a measurable function — which possibly equals $+\infty$.

We shall prove that:

$$\varphi \in L^{p'}(\mathbb{R}^N), \tag{24}$$

$$G_m \star f \in L^1(\mathbb{R}^N; dT) \quad \text{for every} \quad f \in L^p(\mathbb{R}^N), \tag{25}$$

$$\int \varphi(x) f(x)\,dx = \int (G_m \star f)\,dT = \langle T, G_m \star f \rangle \quad \text{for every} \quad f \in L^p(\mathbb{R}^N). \tag{26}$$

The conclusion of Theorem 8 follows readily since every function $u \in W^{m,p}$ can be represented as $u = G_m \star f$ for some $f \in L^p$. Let $\zeta \in \mathcal{D}_+$ and let $f \in \mathcal{D}_+$: we clearly have [since $G_m \in L^1(\mathbb{R}^N)$]:

$$\int (G_m \star \zeta T) f\,dx = \int (G_m \star f) \zeta\,dT = \langle T, \zeta (G_m \star f) \rangle. \tag{27}$$

Choose $\zeta(x) = \zeta_n(x) = \zeta_1(x/n)$ where $\zeta_1 \in C_0^\infty(\mathbb{R}^N)$ is fixed with $0 \leqq \zeta_1 \leqq 1$ and $\zeta_1(x) = 1$ near $x = 0$.

Since $\| \zeta_n (G_m \star f) \|_{W^{m,p}} \leqq C \| G_m \star f \|_{W^{m,p}}$ we conclude from (27) and the monotone convergence Theorem that $(G_m \star T) f \in L^1(\mathbb{R}^N)$, that $(G_m \star f) \in L^1(\mathbb{R}^N; dT)$ and:

$$\int (G_m \star T) f\,dx = \int (G_m \star f)\,dT = \langle T, G_m \star f \rangle.$$

Since:

$$| \langle T, G_m \star f \rangle | \leqq C \| G_m \star f \|_{W^{m,p}} \leqq C \| f \|_{L^p}.$$

We also conclude that $\varphi = G_m \star T \in L^{p'}(\mathbb{R}^N)$. Properties (25) and (26) follow immediately — first for a function $f \in L^p$ with $f \geqq 0$ and then for a general $f \in L^p$.

We turn now to the case of a domain $\Omega \subset \mathbb{R}^N$.

THEOREM 9. — *Assume $\Omega \subset \mathbb{R}^N$ is bounded and smooth. Suppose $T \in W^{-m,p'}(\Omega)$ and $T \geqq 0$ (so that T is a measure). Let $u \in W_0^{m,p}(\Omega)$.*

Then $u \in L^1(\Omega; dT)$ and $\int_\Omega u\,dT = \langle T, u \rangle$.

The proof of Theorem 9 relies on the following Lemma due to Ancona [1].

LEMMA 3. — *Assume $\Omega \subset \mathbb{R}^N$ is bounded and smooth. Given any function $u \in W_0^{m,p}(\Omega)$ there exist two functions $u_1, u_2 \in W_0^{m,p}(\Omega)$ such that $u_1 \geqq 0$, $u_2 \geqq 0$ and $u = u_1 - u_2$.*

Proof of Theorem 9. — We apply Corollary 6 with $k = 1$, $g = 1$ respectively to u_1 and u_2. Note that (16') holds by Lemma 2 and (17') holds with $f = 0$. We deduce that

$$u_1 \in L^1(\Omega; dT), \quad u_2 \in L^1(\Omega; dT)$$

and:

$$\int_\Omega u_1\,dT = \langle T, u_1 \rangle, \qquad \int_\Omega u_2\,dT = \langle T, u_2 \rangle.$$

Remark 8. — Assume $\Omega \subset \mathbb{R}^N$ is an *arbitrary* open set. Let $T \in W^{-m,p'}(\Omega)$ with $T \geqq 0$ and let $u \in W_0^{m,p}(\Omega)$. Then $u \in L^1_{\text{loc}}(\Omega; dT)$ — this a direct consequence of Theorem 8.

When $m=1$, we may even conclude that u lies in $L^1(\Omega; dT)$ — this follows from the main result in [4] and the fact that every function u in $W_0^{1,p}(\Omega)$ can be written as $u=u^+-u^-$ with $u^+, u^- \in W_0^{1,p}(\Omega)$.

However when $m \geqq 2$ and Ω is not smooth, u does *not* necessarily lie in $L^1(\Omega; dT)$. Here is an example suggested to us by Ancona and which is based on a construction due to Coffman-Grover [7]. Let $\Omega=\{x\in\mathbb{R}^2; 0<|x|<1\}$. Let $u(x)=x_1\zeta(x)$ where ζ denotes any smooth function with support in $\{x\in\mathbb{R}^2; |x|<1\}$ and such that $\zeta(x)=1$ near $x=0$. Let $T=\Delta f$ where $f(x)=|x|^{-1}(1-\log|x|)^{-1}$. It is easy to check that $u\in H_0^2(\Omega)$:

$$T\in H^{-2}(\Omega)\cap L^1_{\text{loc}}(\Omega),\quad T\geqq 0 \qquad \text{and} \qquad \int_\Omega |u|\,T=\infty$$

Note that such a function u can not be written as a difference of two nonnegative functions in $H_0^2(\Omega)$.

APPENDIX

Proof of Lemma 1. — We have $|D^\beta(I_m\star\psi)|\leqq C(I_{m-|\beta|}\star|\psi|)$.

Let δ be a positive real number, to be chosen later. We write:

$$I_{m-|\beta|}\star|\psi|=E_1+E_2,$$

where:

$$E_1=\int_{|y-x|<\delta} I_{m-|\beta|}(x-y)\,|\psi(y)|\,dy;$$

$$E_2=\int_{|y-x|\geqq\delta} I_{m-|\beta|}(x-y)\,|\psi(y)|\,dy.$$

Next, we consider the following estimates:

$$E_1=\sum_{n=0}^{\infty}\int_{2^{-n-1}\delta<|y-x|\leqq 2^{-n}\delta} I_{m-|\beta|}(x-y)\,|\psi(y)|\,dy$$
$$\leqq C\,M\psi(x)\,\delta^{m-|\beta|}\sum_{n=0}^{\infty}2^{-n(m-|\beta|)}\leqq C\,M\psi(x)\,\delta^{m-|\beta|}$$
$$E_2\leqq\delta^{-|\beta|}(I_m\star|\psi|)(x).$$

We conclude by choosing δ such that:

$$\delta^m=(I_m\star|\psi|)(x)(M\psi(x))^{-1}.$$

REFERENCES

[1] A. ANCONA, *Une propriété des espaces de Sobolev* (*C. R. Acad. Sc.*, Paris, Vol. 292, 1981, pp. 477-480).
[2] H. BREZIS, *Localized self-adjointness of Schrödinger Operators*, (*J. Operator Th.*, Vol. 1, 1979, p. 287-290).
[3] H. BREZIS et F. BROWDER, *Sur une propriété des espaces de Sobolev* (*C.R. Acad. Sc.*, Paris, T. 287, 1978, pp. 113-115).

[4] H. BREZIS et F. BROWDER, *A Property of Sobolev Spaces* (*Comm. in P.D.E.*, Vol. 4 (9), 1979, pp. 1077-1083).
[5] H. BREZIS et T. KATO, *Remarks on the Schrödinger Operators with Singular Complex Potentials* (*J. Math. Pures et Appl.*, Vol. 58, 1979, pp. 137-151).
[6] F. BROWDER, *Pseudo-Monotone Operators and Nonlinear Elliptic Boundary Value Problems on Unbounded Domains* (*Proc. Nat. Acad. Sc.*, Vol. 74, 1977, pp. 2659-2661).
[7] C. V. COFFMAN et C. L. GROVER, *Obtuse Cones in Hilbert Spaces and Applications to Partial Differential Equations* (*J. Funct. Anal.*, Vol. 35, 1980, pp. 369-396).
[8] J. DENY, *Les potentiels d'énergie finie* (*Acta Math.*, Vol. 82, 1950, pp. 107-183).
[9] M. GRUN-REHOMME, *Caractérisation du sous-différentiel d'intégrandes convexes dans les espaces de Sobolev* (*J. Math. Pures et Appl.*, Vol. 56, 1977, pp. 149-156).
[10] L. HEDBERG, *On Certain Convolution Inequalities* (*Proc. Amer. Math. Soc.*, Vol. 36, 1972, pp. 505-510).
[11] L. HEDBERG, *Two Approximation Properties in Function Spaces* (*Ark. Mat.*, Vol. 16, 1978, pp. 51-81).
[12] J. L. LIONS et E. MAGENES, *Problèmes aux limites non homogènes*, Dunod, Paris, Vol. 1, 1968.
[13] V. G. MAZ'YA et V. P. KHAVIN, *Non-linear Potential Theory* (*Uspehi Mat. Nauk*, Vol. 27, 1972, pp. 67-138 [*Russian Math. Surveys*, Vol. 27, 1972, pp. 71-148]).
[14] E. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
[15] J. R. L. WEBB, *Boundary Value Problems for Strongly Nonlinear Elliptic Equations* (*J. London Math. Soc.*, Vol. 21, 1980, pp. 123-132).

H. BREZIS,
Département de Mathématiques,
Université Paris-VI,
4, place Jussieu,
75230 Paris Cedex 05;

F. BROWDER,
Department of Mathematics,
University of Chicago,
Chicago, Ill. 60637.

(Manuscrit reçu en décembre 1980.)

ADDENDUM 1

(Dec. 5, 1980) One of the authors (F. B.) has been informed by his colleague Peter Jones at the University of Chicago that L. I. Hedberg has just sent the latter a statement and outline proof of a new result concerning properties of the space $W_0^{m,p}(\Omega)$ for arbitrary open sets in $\mathbb{R}^N$ which answers in large part the question (Q_1) posed in Section 2 above. As in [11], Hedberg treats the characterization of elements of the class $W_0^{m,p}(\Omega)$ in $W^{m,p}(\mathbb{R}^N)$ by their trace properties on the boundary of Ω, i. e. each derivative $D^\beta u$ for $|\beta|<m$ must vanish on the boundary of Ω, in the sense of $(m-|\beta|, p)$-capacity, but with the sole restriction that $p>2-1/N$. The techniques of proof are relatively complex but involve the construction of a sequence $\{\zeta_j\}$ of functions with compact support in Ω and with $0\leqq\zeta_j\leqq 1$ such that $\zeta_j u$ converges to u in $W^{m,p}(R^N)$. If we apply Hedberg's procedure to a given u which we already know to lie in $W_0^{m,p}(\Omega)$, we can conclude that the conclusion of Theorem 1 holds for any T in $L^1_{loc}(\Omega)\cap W^{-m,p'}(\Omega)$ and any u in $W_0^{m,p}(\Omega)$ under the sole restriction that $p>2-1/N$.

ADDENDUM 2

Hedberg's result mentioned in the previous Addendum has appeared in L. I. Hedberg, "Spectral synthesis in Sobolev spaces and uniqueness of solutions of the Dirichlet problem", *Acta Math.*, Vol. 147, 1981, pp. 237-264. The technical problems restricting the result to the case $p > 2 - (1/N)$ have been removed by some new results in nonlinear potential theory obtained by L. I. Hedberg and T. H. Wolff, "Thin sets in nonlinear potential theory" (preprint).