

Semilinear Equations in \mathbb{R}^N Without Condition at Infinity

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Abstract. In this paper we establish that some nonlinear elliptic (and parabolic) problems are well posed in all of \mathbb{R}^N without prescribing the behavior at infinity. A typical example is the following: Let $1 < p < \infty$. For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there is a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{on } \mathbb{R}^N.$$

1. Introduction

The purpose of this paper is to point out that some nonlinear elliptic (and parabolic) problems are well-posed in all of \mathbb{R}^N without conditions at infinity. A typical example is the following:

Theorem 1. *Let $1 < p < \infty$. For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (1)$$

Moreover, if $f \geq 0$ a.e. then $u \geq 0$ a.e.

Remark 1. It was previously known that for every $f \in L^1(\mathbb{R}^N)$ there exists a unique $u \in L^p(\mathbb{R}^N)$ satisfying (1) (see [3], Theorem 5.11). However, we emphasize that in Theorem 1 there is *no limitation on the growth at infinity of the data f and the solution u is unique without prescribing its behavior at infinity.*

2. Proof of Theorem 1

Existence

Let $B_R = \{x \in \mathbb{R}^N; |x| < R\}$. We start with some local estimate:

Lemma 1. *Let $R < R'$ and assume $u \in L^p_{\text{loc}}(B_{R'})$ satisfies*

$$-\Delta u + |u|^{p-1}u = f(x) \quad \text{in } \mathcal{D}'(B_{R'}) \quad (2)$$

with $f \in L^1(B_{R'})$. Then

$$\int_{B_R} |u|^p \leq C \left(1 + \int_{B_{R'}} |f| \right) \quad (3)$$

where C depends only on p , R and R' .

Remark 2. The conclusion of Lemma 1 is a rather unusual *localization property*. Indeed, let Ω and Ω' be bounded open sets in \mathbb{R}^N such that $\overline{\Omega} \cap \overline{\Omega'} = \emptyset$ and let u be the solution of (1). On the one hand the values of f in Ω' "affect" the solution u in Ω : for example, if $f > 0$ in Ω' and $f \equiv 0$ outside Ω' it follows from the strong maximum principle that $u > 0$ in Ω . On the other hand the values of f in Ω' affect only "mildly" u in Ω : in view of (3) $u|_{\Omega}$ may be estimated independently of $f|_{\Omega'}$; even if $f \rightarrow \infty$ on Ω' , $\int_{\Omega} |u|^p$ still remains bounded.

Proof of Lemma 1. We use a device introduced by P. Baras and M. Pierre [2]. By Kato's inequality (see [10]) and (2) we have

$$-\Delta |u| + |u|^p \leq |f| \quad \text{in } \mathcal{D}'(B_{R'}). \quad (4)$$

Let $\zeta \in \mathcal{D}(B_{R'})$ be such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on B_R . Multiplying (4) through by ζ^α where α is an integer, and integrating we find

$$\int |u|^p \zeta^\alpha \leq \int |f| \zeta^\alpha + C \int |u| \zeta^{\alpha-2} \leq \int |f| \zeta^\alpha + C \int |u| \zeta^{\alpha/p}, \quad (5)$$

provided $\alpha - 2 \geq \alpha/p$, i.e., $\alpha \geq 2p/(p-1)$ and we fix any such α . The conclusion of Lemma 1 follows easily from (5).

Proof of Theorem 1. Existence

Let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n. \end{cases}$$

Let $u_n \in L^p(\mathbb{R}^N)$ be the unique solution of

$$-\Delta u_n + |u_n|^{p-1}u_n = f_n \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \quad (6)$$

(see [3], Theorem 5.11).

We deduce from Lemma 1 that there is a constant C such that

$$\|u_n\|_{L^p(B_R)} \leq C$$

where C depends only on p , R and f , and thus we also have

$$\|\Delta u_n\|_{L^1(B_R)} \leq C.$$

It follows that (for some subsequence still denoted by u_n) we have

$$u_n \rightarrow u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N)$$

$$u_n \rightarrow u \quad \text{a.e. on } \mathbb{R}^N.$$

We claim that

$$|u_n|^{p-1}u_n \rightarrow |u|^{p-1}u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N).$$

It suffices to verify that $|u_n|^{p-1}u_n$ is a Cauchy sequence in $L^1(B_R)$ for any R . By Kato's inequality and (6) we have

$$-\Delta |u_n - u_m| + ||u_n|^{p-1}u_n - |u_m|^{p-1}u_m| \leq |f_n - f_m|.$$

Let $\zeta \in \mathcal{D}(\mathbb{R}^N)$ be such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ on B_R . We have

$$\int ||u_n|^{p-1}u_n - |u_m|^{p-1}u_m| \zeta \leq \int |f_n - f_m| \zeta + \int |u_n - u_m| \Delta \zeta$$

and the *RHS* tend to zero as $m, n \rightarrow \infty$. Passing to the limit in (6) we obtain (1).

Uniqueness

We shall need the following:

Lemma 2. Assume $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfies

$$-\Delta u + |u|^{p-1}u \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (7)$$

Then $u \leq 0$ a.e. on \mathbb{R}^N .

Remark 3. Lemma 2 is closely related to the results of J. Keller [11] and R. Osserman [13] (see also the earlier works quoted in these papers).

Proof. We use a comparison function of the same type as in Osserman [13] (see also C. Loewner and L. Nirenberg [12]). Set

$$U(x) = \frac{CR^\alpha}{(R^2 - |x|^2)^\alpha} \quad \text{in } B_R$$

where $R > 0$, $\alpha = 2/(p-1)$ and $C^{p-1} = 2\alpha \max\{N, \alpha + 1\}$. A direct computation shows that

$$-\Delta U + U^p \geq 0 \quad \text{in } B_R \quad (8)$$

and thus

$$-\Delta(u - U) + |u|^{p-1}u - U^p \leq 0 \quad \text{in } \mathcal{D}'(B_R). \quad (9)$$

Using a variant of Kato's inequality (see Lemma A.1 in the Appendix) we deduce from (9) that

$$-\Delta(u - U)^+ + (|u|^{p-1}u - U^p) \text{sign}^+(u - U) \leq 0 \quad \text{in } \mathcal{D}'(B_R) \quad (10)$$

and therefore

$$-\Delta(u - U)^+ \leq 0 \quad \text{in } \mathcal{D}'(B_R). \quad (11)$$

From Lemma A.1 and (7) we deduce that

$$-\Delta u^+ + (u^+)^p \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

and therefore

$$-\Delta u^+ \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

i.e., u^+ is subharmonic and in particular $u^+ \in L_{\text{loc}}^\infty(\mathbb{R}^N)$. It follows that for some $\delta > 0$ we have

$$(u - U)^+ = 0 \quad \text{for } R - \delta < |x| < R \quad (12)$$

(since $U(x) \rightarrow +\infty$ as $|x| \rightarrow R$, $x \in B_R$). Combining (11) and (12) we obtain that $(u - U)^+ = 0$ a.e. on B_R , i.e., $u \leq U$ a.e. on B_R .

Keeping x fixed and letting $R \rightarrow \infty$ we see that $u \leq 0$ a.e. on \mathbb{R}^N .

Proof of Theorem 1. Uniqueness

Let u_1 and u_2 be two solutions of (1) and let $u = u_1 - u_2$. By Kato's inequality we have

$$-\Delta|u| + ||u_1|^{p-1}u_1 - |u_2|^{p-1}u_2| \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (13)$$

On the other hand, there is a constant $\delta > 0$ —depending only on p —such that

$$||a|^{p-1}a - |b|^{p-1}b| \geq \delta|a - b|^p \quad \forall a, b \in \mathbb{R}. \quad (14)$$

From (13) and (14) we deduce that

$$-\Delta|u| + \delta|u|^p \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Using Lemma 2 we conclude that $u = 0$.

3. Miscellaneous Remarks and Generalizations*A. Monotone Nonlinearities*

The proof of Theorem 1 extends easily to the case where $|u|^{p-1}u$ is replaced by a more general function $g(u)$. Assume $g: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function such that

$$g'(u) \geq a|u|^{p-1} \quad \forall u \in \mathbb{R},$$

for some constants $a > 0$ and $1 < p < \infty$ (for example, $g(u) = \sinh u$, etc. ...).

Theorem 1'. *For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$-\Delta u + g(u) = f(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (16)$$

B. Nonmonotone g 's

Let $g(x, u): \mathbb{R}^N \times \mathbb{R}$ be measurable in x and continuous in u . We assume that:

$$g(x, u) \text{ sign } u \geq a|u|^p - \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^N, \quad \text{for all } u \in \mathbb{R} \quad (17)$$

where $\omega \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $a > 0$, $1 < p < \infty$ and also

$$h_M(x) = \sup_{|u| \leq M} |g(x, u)| \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } M > 0. \quad (18)$$

Theorem 2. *There exists $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ such that $g(\cdot, u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$-\Delta u + g(x, u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (19)$$

Sketch of the Proof. First we consider the case of a smooth *bounded* domain $\Omega \subset \mathbb{R}^N$.

Claim. There exists $u \in W_0^{1,1}(\Omega)$ such that $g(\cdot, u) \in L^1(\Omega)$, satisfying

$$-\Delta u + g(x, u) = 0 \quad \text{on } \Omega. \quad (20)$$

This type of result is closely related—but not quite contained in [6]. Here it suffices to assume (17) with $a = 0$.

For $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we set

$$\tau_n r = \begin{cases} r & \text{if } |r| \leq n \\ n & \text{if } r \geq n. \\ -n & \text{if } r \leq -n. \end{cases}$$

By the Schauder fixed point theorem there exists $u_n \in W_0^{1,1}(\Omega)$ satisfying

$$-\Delta u_n + g(x, \tau_n u_n) = 0 \quad \text{on } \Omega. \quad (21)$$

Using the fact that $-\int_{\Omega} \Delta u_n \operatorname{sign} u_n \geq 0$ we find

$$\int_{\Omega} |g(x, \tau_n u_n)| \leq 2 \int_{\Omega} |\omega|.$$

Therefore

$$\int_{\Omega} |\Delta u_n| \leq 2 \int_{\Omega} |\omega|. \quad (22)$$

After extracting a subsequence we may assume that

$$u_n \rightarrow u \quad \text{in } W_0^{1,1}(\Omega)$$

$$u_n \rightarrow u \quad \text{a.e.}$$

$$g(x, \tau_n u_n) \rightarrow g(x, u) \quad \text{a.e.}$$

To show that $g(x, \tau_n u_n) \rightarrow g(x, u)$ in $L^1(\Omega)$ we use a new device introduced in [8] by Th. Gallouet and J. M. Morel (with an observation of L. Boccardo).

Set

$$p_M(r) = \begin{cases} 1 & \text{if } r > M \\ 0 & \text{if } -M \leq r \leq M \\ -1 & \text{if } r < -M \end{cases}$$

where $r \in \mathbb{R}$ and $M > 0$. It is well known that

$$-\int_{\Omega} \Delta u \cdot p_M(u) \geq 0 \quad \forall u \in W_0^{1,1}(\Omega), \Delta u \in L^1(\Omega).$$

Therefore we have

$$\int_{\Omega} g(x, \tau_n u_n) p_M(u_n) \leq 0,$$

That is,

$$\int_{|u_n| \geq M} g(x, \tau_n u_n) \operatorname{sign}(u_n) \leq 0$$

and hence

$$\int_{|u_n| > M} |g(x, \tau_n u_n)| \leq 2 \int_{|u_n| > M} |\omega|. \quad (23)$$

From (22) we see that $\|u_n\|_{L^1} \leq C$ and thus $M \operatorname{meas} [|u_n| > M] \leq C$.

Given $\varepsilon > 0$ we may fix M large enough so that $2 \int_{|u_n| > M} |\omega| < \varepsilon$.
Next, for any measurable $A \subset \Omega$, we have

$$\begin{aligned} \int_A |g(x, \tau_n u_n)| &\leq \int_{\substack{A \\ |u_n| \leq M}} |g(x, \tau_n u_n)| + \int_{|u_n| > M} |g(x, \tau_n u_n)| \\ &\leq \int_A h_M(x) + \varepsilon \leq 2\varepsilon \end{aligned}$$

provided $\operatorname{meas} A < \delta$ and δ is small enough. In other words, we have established that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \int_A |g(x, \tau_n u_n)| < 2\varepsilon \quad \text{when} \quad \operatorname{meas} A < \delta.$$

We conclude that $g(x, \tau_n u_n) \rightarrow g(x, u)$ in $L^1(\Omega)$.

We turn now to problem (19). For each n let

$$\Omega_n = \{x \in \mathbb{R}^N; |x| < n\}.$$

By the previous step there exists $u_n \in W_0^{1,1}(\Omega_n)$ such that $g(\cdot, u_n) \in L^1(\Omega_n)$ satisfying

$$-\Delta u_n + g(x, u_n) = 0 \quad \text{on } \Omega_n. \quad (24)$$

From Kato's inequality and (24) we obtain

$$-\Delta |u_n| + g(x, u_n) \operatorname{sign} u_n \leq 0 \quad \text{in } \mathcal{D}'(\Omega_n).$$

And therefore we also have

$$-\Delta|u_n| + a|u_n|^p \leq \omega \quad \text{in } \mathcal{D}'(\Omega_n) \quad (25)$$

$$-\Delta|u_n| + |g(x, u_n)| \leq 2|\omega| \quad \text{in } \mathcal{D}'(\Omega_n). \quad (26)$$

Using the same device as in the proof of Theorem 1 we deduce from (25) that

$$u_n \text{ is bounded in } L^p_{\text{loc}}(\mathbb{R}^N).$$

It follows from (26) that

$$g(\cdot, u_n) \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N)$$

and thus

$$\Delta u_n \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Hence we may assume that

$$u_n \rightarrow u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N)$$

$$u_n \rightarrow u \quad \text{a.e. on } \mathbb{R}^N$$

$$g(x, u_n) \rightarrow g(x, u) \quad \text{a.e. on } \mathbb{R}^N.$$

Finally we prove that $g(x, u_n) \rightarrow g(x, u)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. By a variant of Kato's inequality (see Lemma A.2) we have

$$-\Delta P_M(u) + g(x, u_n) p_M(u_n) \leq 0 \quad \text{in } \mathcal{D}'(\Omega_n)$$

where

$$P_M(t) = \int_0^t p_M(s) ds.$$

Therefore we have

$$\int_{|u_n| > M} |g(x, u_n)| \zeta \leq 2 \int_{|u_n| > M} |\omega| \zeta + \int_{|u_n| > M} |u_n| \Delta \zeta \quad \forall \zeta \in \mathcal{D}_+(\Omega_n).$$

It follows easily that $g(x, u_n)$ is equi-integrable on bounded sets of \mathbb{R}^N and thus $g(x, u_n) \rightarrow g(x, u)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$.

C. Measures or More General Distributions as Right Hand Side Data

Let T be a distribution of the form $T = f + \Delta\phi$ where $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\phi \in L^p_{\text{loc}}(\mathbb{R}^N)$. Then the problem

$$-\Delta u + |u|^{p-1}u = T \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \quad (27)$$

has a unique solution $u \in L^p_{\text{loc}}(\mathbb{R}^N)$.

Indeed it suffices to consider the new unknown $v = u + \phi$ and to apply the result of Section B to v (see also [8] for similar questions on bounded domains).

Suppose now that T is a *measure* on \mathbb{R}^N (not necessarily a bounded measure). Suppose $1 < p < N/(N-2)$ (no restriction when $N=1,2$). Then there exists a unique solution $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ for (27). Related questions for bounded measures are considered in [5] and [8].

D. Nonlinearities with Growth Close to Linear

Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $g(u)$ behaves like $u|\log u|^k$ as $|u| \rightarrow \infty$ with $k > 2$. Then for every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + g(u) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

As before, we use Kato's inequality to find

$$-\Delta|u| + g(u)\text{sign } u \leq |f|.$$

We multiply (28) through by $\eta = e^{-1/s^\beta}$ where $\beta > 2/(k-2)$. Then we estimate $\int |u| |\Delta \eta|$ with the help of Young's inequality. Recently Gallouet and Morel have proved the following result. Suppose $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, nondecreasing, odd, convex, and $\int_1^\infty [G(x)]^{-1/2} dx < \infty$ where G is a primitive of g . Then for every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists a unique function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying $-\Delta u + g(u) = f$ in $\mathcal{D}'(\mathbb{R}^N)$.

E. Unbounded Domains

Let $\Omega \subset \mathbb{R}^N$ be any domain (bounded or unbounded) with smooth boundary. Using the same principles as in the proof of Theorem 1 one can show that for every $f \in L^1_{\text{loc}}(\bar{\Omega})$ and $\phi \in L^1_{\text{loc}}(\partial\Omega)$ there exists a unique $u \in L^p_{\text{loc}}(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

where $1 < p < \infty$ and the boundary condition is understood in some appropriate sense.

F. Local Regularity

Let $\Omega \subset \mathbb{R}^N$ be any domain. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and nondecreasing function.

Theorem 3. Suppose $u \in L^1_{\text{loc}}(\Omega)$ is such that $g(u) \in L^1_{\text{loc}}(\Omega)$ and satisfies

$$-\Delta u + g(u) = f(x) \quad \text{in } \mathcal{D}'(\Omega) \tag{29}$$

where $f \in L^q_{\text{loc}}(\Omega)$ and $1 < q < \infty$.

Then $u \in W^{2,q}_{\text{loc}}(\Omega)$.

Proof. We may assume that $g(0) = 0$. We have

$$-\Delta|u| + g(u)\text{sign } u \leq |f| \quad \text{in } \mathcal{D}'(\Omega)$$

and thus

$$-\Delta|u| \leq |f| \quad \text{in } \mathcal{D}'(\Omega).$$

It follows that $u \in L^q_{\text{loc}}(\Omega)$. Set

$$g_n(r) = g(\tau_n r) \quad \text{and} \quad P_n(r) = \text{sign } r \int_0^r |g_n(s)|^{q-1} ds$$

so that

$$|P_n(r)| \leq |r| |g_n(r)|^{q-1} \quad \forall r \in \mathbb{R}.$$

By Lemma A.2 and (29) we have

$$\Delta P_n(u) \geq |g_n(u)|^{q-1} \text{sign } u [g(u) - f] \geq |g_n(u)|^q - |f| |g_n(u)|^{q-1}. \quad (30)$$

Let $\zeta \in \mathcal{D}(\Omega)$ with $0 \leq \zeta \leq 1$; from (30) we see that

$$\begin{aligned} \int |g_n(u)|^q \zeta^\alpha &\leq C \int |P_n(u)| \zeta^{\alpha-2} + \int |f| |g_n(u)|^{q-1} \zeta^\alpha \\ &\leq C \int |u| |g_n(u)|^{q-1} \zeta^{\alpha-2} + \int |f| |g_n(u)|^{q-1} \zeta^\alpha \end{aligned}$$

where C is independent of u .

Fix any integer $\alpha \geq 2q$; by Hölder's inequality we have

$$\int |g_n(u)|^q \zeta^\alpha \leq C \int_{\text{supp } \zeta} (|u|^q + |f|^q).$$

As $n \rightarrow \infty$ we see that $g(u) \in L^q_{\text{loc}}(\Omega)$.

G. Parabolic Equations

Consider the problem

$$\begin{cases} u_t - \Delta u + |u|^{p-1}u = 0 & \text{on } \mathbb{R}^N \times (0, +\infty) \text{ with } 1 < p < \infty \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N. \end{cases} \quad (31)$$

Using the same principles as in the proof of Theorem 1 one can show that for every $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ there is a unique function $u \in C^2(\mathbb{R}^N \times (0, +\infty)) \cap C([0, +\infty); L^1_{\text{loc}}(\mathbb{R}^N))$ satisfying (31).

Results of the same nature for the problem

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = 0 & \text{on } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N \end{cases}$$

have been obtained by M. Herrero–M. Pierre [9] when $0 < m < 1$. When $m > 1$ the situation is totally different; see [1, 4, 7].

Appendix: Some Variants of Kato's Inequality

Let $\Omega \subset \mathbb{R}^N$ be any open set.

Lemma A.1. *Let $u \in L^1_{\text{loc}}(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$\Delta u^+ \geq f \operatorname{sign}^+ u \quad \text{in } \mathcal{D}'(\Omega).$$

Lemma A.2. *Let $p: \mathbb{R} \rightarrow \mathbb{R}$ be a monotone, nondecreasing function such that p is continuous except at a finite number of jumps and $p(\mathbb{R})$ is bounded.*

Let $P(r) = \int_0^r p(s) ds$ and let $u \in L^1_{\text{loc}}(\Omega)$ with $\Delta u \in L^1_{\text{loc}}(\Omega)$.

Then

$$\Delta P(u) \geq (\Delta u)p(u) \quad \text{in } \mathcal{D}'(\Omega).$$

The proofs are easy modifications of Kato's original argument in [10], and we shall omit them.

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References

1. Aronson D, Caffarelli L (in press) The initial trace of a solution of the porous medium equation. Trans. Am Math Soc
2. Baras P, Pierre M (1982) Singularités éliminables d'équations elliptiques semilinéaires, C.R.A.S. Paris 295:519–522; see also Ann Inst Fourier to appear
3. Bénéilan Ph, Brezis H, and Crandall M (1975) A semilinear equation in $L^1(\mathbb{R}^N)$, Ann Sc Norm Sup Pisa 2:523–555
4. Bénéilan Ph, Crandall M, and Pierre M Solutions of the porous medium equation under optimal conditions on initial values. Indiana Univ Math J 33:51–87

5. Brezis H (1980) Some variational problems of the Thomas-Fermi type. In: Cottle, Giannessi, and Lions (eds) *Variational Inequalities and Complementarity Problems*. Wiley, New York: 53–73
6. Brezis H, Strauss W (1973) Semilinear second-order elliptic equations in L^1 . *J Math Soc Japan* 25:565–590
7. Dahlberg B, Kenig C (in press) Non-negative solutions of the porous medium equation
8. Gallouet Th, Morel JM (in press) Resolution of a semilinear equation in L^1
9. Herrero M, Pierre M (in press) Some results for the Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$
10. Kato T (1972) Schrödinger operators with singular potentials. *Israel J Math* 13:135–148
11. Keller JB (1957) On solutions of $\Delta u = f(u)$. *Comm Pure Appl Math* 10:503–510
12. Loewner C, Nirenberg L (1974) Partial differential equations invariant under conformal or projective transformations. In: *Contributions to Analysis*. Acad. Press, New York: 245–272
13. Osserman R (1957) On the inequality $\Delta u \geq f(u)$. *Pacific J Math* 7:1641–1647

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