Some characterizations of Sobolev spaces

Quelques caractérisations des espaces Sobolev

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Abstract

We establish a new characterization of Sobolev spaces $W^{1,p}(\mathbb{R}^N)$, $1 < p < \infty$ and extend a result in [1].

Résumé

Nous présentons une nouvelle caractérisation des espaces Sobolev ainsi que généralisons un résultat dans [1].

1. Version française abrégée

Nous présentons une caractérisation des espaces Sobolev. Elle est motivée par les travaux de J. Bourgain, H. Brezis, P. Mironescu [1] (voir aussi [3]). Notre résultat principal est le Théorème 1.

2. Introduction

In this paper we will give some characterizations of Sobolev spaces. It is motivated by works of J. Bourgain, H. Brezis, P. Mironescu [1] (see also [3]).

Our main result is as follows :

Theorem 1 *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < \infty$. We have*

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1 If $g \in W^{1,p}(\mathbb{R}^N)$, then there exists a constant $C_{N,p}$ depending only on N and p such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx, \quad \forall\,\delta>0.$$

2 If there exists a constant $C>0$ independent of δ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C, \quad \forall\,0<\delta<1.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$.

Moreover for any $g \in W^{1,p}(\mathbb{R}^N)$ there exists a positive constant $K_{N,p}$ depending only on N and p such that

$$\lim_{\delta\longrightarrow 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\limits_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx. \tag{1}$$

3. Proof of Theorem 1

The following lemma is useful.

Lemma 1 *Let $g \in C^\infty(\mathbb{R}^N)$ such that*

$$C(g) := \sup_{0<\alpha<1}(1-\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|g(x+h)-g(x)|^{p+1-\alpha}}{|x-y|^{N+p}} < \infty.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx \le C(g).$$

Proof of Lemma 1.

Let B_R be the ball with center at 0 and radius $R>0$.
From the change of variables formula, one has

$$\sup_{0<\alpha<1}(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{|g(x+h\cdot\sigma)-g(x)|^{p+1-\alpha}}{h^{p+1}}\,dh\,dx\,d\sigma = C(g).$$

Thus

$$\sup_{0<\alpha<1}(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^1 \frac{|g(x+h\cdot\sigma)-g(x)|^{p+1-\alpha}}{h^{p+1}} \le C(g).$$

On the other hand $\frac{g(x+h\cdot\sigma)-g(x)}{h}$ converges uniformly to $\nabla g(x)\cdot\sigma$ on every compact subset of $\mathbb{R}^N$ when h converges to 0.
Thus for all $\varepsilon>0$ there exists a constant $\tau>0$, independent of σ, such that

$$\int_{B_R}\Big|\Big|\frac{g(x+h\cdot\sigma)-g(x)}{h^{p+1-\alpha}}\Big|^{p+1-\alpha} - |\nabla g(x)\cdot\sigma|^{p+1-\alpha}\Big|\,dx \le \varepsilon, \quad \forall\,0<h<\tau,\ \forall\,0<\alpha<1.$$

Hence

$$\Big|(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^{\tau}\frac{|g(x+h\cdot\sigma)-g(x)|^{p+1-\alpha}}{h^{p+1}}\,dh\,dx\,d\sigma - (1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^{\tau}\frac{|\nabla g(x)\cdot\sigma|^{p+1-\alpha}}{h^{\alpha}}\,dh\,dx\,d\sigma\Big| \le C\varepsilon. \tag{2}$$

Hereafter C denotes a positive constant independent of g and α.
On the other hand

$$\lim_{\alpha\longrightarrow 1}(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_{\tau}^{1}\frac{|g(x+h\cdot\sigma)-g(x)|^{p+1-\alpha}}{h^{p+1}}\,dh\,dx\,d\sigma \le \lim_{\alpha\longrightarrow 1}C_N R^N(||\nabla g||_{L^\infty(B_{R+1})}+1)^{p+1}(1-\alpha)\int_{\tau}^{1}\frac{1}{h^{\alpha}}\,dh = 0, \tag{3}$$

and

$$\lim_{\alpha\longrightarrow 1}(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_{\tau}^{1}|\nabla g(x)\cdot\sigma|^{p+1-\alpha}\,dh\,dx\,d\sigma \le \lim_{\alpha\longrightarrow 1}C_N R^N(||\nabla g||_{L^\infty(B_R)}+1)^{p+1}(1-\alpha)\int_{\tau}^{1}\frac{1}{h^{\alpha}}\,dh = 0. \tag{4}$$

Furthermore

$$\lim_{\alpha\longrightarrow 1}\int_{\mathbb{S}^{N-1}}\int_{B_R}|\nabla g(x)\cdot\sigma|^{p+1-\alpha}\,dx\,d\sigma = \int_{\mathbb{S}^{N-1}}\int_{B_R}|\nabla g(x)\cdot\sigma|^{p}\,dx\,d\sigma. \tag{5}$$

Therefore combine (2), (3), (4) and (5), after noticing that $(1-\alpha)\int_0^1 \frac{1}{h^{\alpha}} = 1$, yields

$$\lim_{\alpha\longrightarrow 1}(1-\alpha)\int_{\mathbb{S}^{N-1}}\int_{B_R}\int_0^{1}\frac{|g(x+h\cdot\sigma)-g(x)|^{p+1-\alpha}}{h^{p+1}}\,dh\,dx\,d\sigma = K_{N,p}\int_{B_R}|\nabla g(x)|^p\,dx.$$

Hence $g\in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx \le C(g).$$

The proof is finished □

Sketch of the proof of Theorem 1.

We give here only the proof : (2) ⇒ (1). The details of the proof of Theorem 1 can be found in [4].
Let $g\in L^p(\mathbb{R}^N)$ such that

$$C(g) := \sup_{0<\delta<1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>\delta}\frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < \infty. \tag{6}$$

Our goal is to show that $g\in W^{1,p}(\mathbb{R}^N)$. We split the proof in 2 cases.

Case 1 : $g\in L^\infty(\mathbb{R}^N)$.
Set

$$\xi(\delta) = \begin{cases} C(g) & \text{if } 0<\delta<1,\\ 2^p\cdot C(g)\cdot ||g||^p_{L^\infty(\mathbb{R}^N)} & \text{otherwise.}\end{cases}$$

From (6) one has, for all $0 \leq \delta \leq 2||g||_{L^\infty(\mathbb{R}^N)}$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy < \xi(\delta). \tag{7}$$

Let $0 < \alpha < 1$, α will tend to 1 later on.
We multiply the inequality (7) by $\dfrac{1-\alpha}{\delta^\alpha}$. Integrating the expression obtained with respect to δ from 0 to $2||g||_{L^\infty(\mathbb{R}^N)}$, one gets

$$(1-\alpha)\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^{p+1-\alpha}}{|x-y|^{N+p}}\, dx\, dy \leq C(g) + 2^p(1-\alpha)C(g)||g||^p_{L^\infty(\mathbb{R}^N)}. \tag{8}$$

Let (γ_ε) be an any sequence of smooth mollifiers.
Set

$$g_\varepsilon = g * \gamma_\varepsilon.$$

By the convexity of the function $t^{p+1-\alpha}$ on the interval $[0,\infty)$, one has

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g_\varepsilon(x)-g_\varepsilon(y)|^{p+1-\alpha}}{|x-y|^{N+p}}\, dx\, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^{p+1-\alpha}}{|x-y|^{N+p}}\, dx\, dy.$$

Thus from (8) one gets

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g_\varepsilon(x)-g_\varepsilon(y)|^{p+1-\alpha}}{|x-y|^{N+p}}\, dx\, dy \leq C(g) + C_p(1-\alpha)C(g)||g||^p_{L^\infty(\mathbb{R}^N)}. \tag{9}$$

Applying Lemma 1 for g_ε, one obtains $g_\varepsilon \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N} |\nabla g_\varepsilon(x)|^p\, dx \leq \lim_{\alpha \longrightarrow 1}(1-\alpha)\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g_\varepsilon(x)-g_\varepsilon(y)|^{p+1-\alpha}}{|x-y|^{p+1}}\, dx\, dy, \tag{10}$$

where

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\, d\sigma. \tag{11}$$

Combining (9) and (10), then let α go to 1 in (9), one gets

$$K_{N,p}\int_{\mathbb{R}^N} |\nabla g_\varepsilon(x)|^p\, dx \leq C(g).$$

Now let ε tend to 0 in the above inequality we deduce that $g \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx \leq \sup_{0<\delta<1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy.$$

Case 2 : General case.
Set, for any $A > 0$,

$$g_A(x) = \begin{cases} g(x) & \text{if } |g(x)| < A, \\ A\dfrac{g(x)}{|g(x)|} & \text{otherwise.} \end{cases}$$

One has

$$|g_A(x) - g_A(y)| \leq |g(x) - g(y)|, \quad \forall\, x,\, y \in \mathbb{R}^N.$$

Thus

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g_A(x)-g_A(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy < \infty.$$

Applying the previous case, one obtains $g_A \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N} |\nabla g_A(x)|^p\,dx \le \sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy.$$

Since A is arbitrary, one has $g \in W^{1,p}(\mathbb{R}^N)$ and

$$K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx \le \sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy.$$

□

4. Another characterization

Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of functions such that

$$\rho_n \ge 0,\ \rho_n(x) = \rho_n(|x|) \tag{12}$$

and

$$\lim_{n\longrightarrow\infty} \int_\tau^\infty \rho_n(r) r^{N-1}\,dr = 0 \qquad \text{for all } \tau > 0. \tag{13}$$

We recall a result of characterization of Sobolev space which was due to Bourgain, H. Brezis and P. Mironescu, see [1].

Proposition 1 *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of functions satisfying (12), (13) and*

$$\lim_{n\longrightarrow\infty} \int_0^\tau h^{N-1}\rho_n(h)\,dh = 1 \textit{ for all } \tau > 0. \tag{14}$$

Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if there exists a constant C, independent of n, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy \le C, \qquad \forall\, n \in \mathbb{N}.$$

Moreover there exists a constant $K_{N,p}$, defined by (11), depending only N, p such that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx.$$

A lighter generalization of this proposition is as follows :

Theorem 2 *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < \infty$ and $(\rho_n)_{n\in N}$ be a sequence of functions satisfying the conditions (12), (13) and (14). Then the two following conditions are equivalent*

(i) $g \in W^{1,p}(\mathbb{R}^N)$.

(ii) There exists a constant C independent of n such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(|g(x)-g(y)|)}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy \le C \text{ for all } n \in \mathbb{N}, \tag{15}$$

where $F : [0,\infty) \longrightarrow [0,\infty)$ is a function verifying, for some $\delta > 0$ and $C > 0$,

$$\begin{cases} (i) \quad F(t) = t^p \text{ for all } t \in [0,\delta), \\ \\ (ii) \quad F(t) \le Ct^p \text{ for all } t \ge 0, \\ \\ (iii) \quad F \text{ is non-decrease.} \end{cases} \tag{16}$$

Moreover there exists a constant $K_{N,p}$, defined by (11), depending only N and p such that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(|g(x)-g(y)|)}{|x-y|^p}\rho_n(|x-y|)\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

Remark 1 *The convexity of F is not required in Theorem 2.*

Proof of Theorem 2 : See in [4].

Remark 2 *A generalized version of Theorem 2 will be found in [4].*

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