COMPLEMENTS TO THE PAPER "LIFTING, DEGREE AND THE DISTRIBUTIONAL JACOBIAN REVISITED"

JEAN BOURGAIN[(1)], HAIM BREZIS[(2),(3)] AND PETRU MIRONESCU[(4)]

1. Existence of a degree and optimal estimates.

Let $0 < s < \infty, 1 \le p < \infty$ and set $X = W^{s,p}(S^N; S^N)$. We say that there is a (topological) degree in X if

a) $C^\infty(S^N; S^N)$ is dense in X;

b) the mapping $g \mapsto \deg g$, defined on $C^\infty(S^N; S^N)$, extends by continuity to X.

We recall the following result, which is part of the folklore:

Lemma 1.1. *There is a degree in X if and only if $sp \ge N$.*

Proof. Property a) holds for each s and p. When s is not an integer and $sp < N$, this was proved in [15]. When $s = 1$ and $p < N$, this assertion can be found in [4]; the same argument holds when $s \ge 2$ is an integer and $sp < N$.

When $sp > N$, property a) follows immediately from the embedding $W^{s,p} \hookrightarrow C^0$. Finally, property a) when $sp = N$ is essentially established in [13].

We next turn to property b). When $sp > N$, it is easy to see that the usual Brouwer degree of the (continuous) maps in $W^{s,p}$ has the required properties. When $sp = N$, we have $W^{s,p} \hookrightarrow VMO$; in this case, the degree of VMO maps (studied in [13]) is the desired extension. Finally, we prove that b) does not hold when $sp < N$.

We fix a map $g \in C^\infty(\mathbb{R}^N; S^N)$ such that $g(x) \equiv P$ when $|x| \ge 1$ and $\deg g = 1$; here, P is the North pole of S^N. Let $\pi : S^N \to \mathbb{R}^N$ be the stereographic projection and set $g_k(x) = g(k\pi(x)),\ x \in S^N$. Then $\deg g_k = 1, \forall k$. However, it is easy to see that $g_k \to P$ strongly in $W^{s,p}$ and, therefore, the degree is not preserved in the strong limit.

In view of Lemma 1.1, it is natural to ask whether, for $sp \ge N$, there is a control of the form

$$|\deg g| \le F(|g|_{s,p}), \forall g \in W^{s,p}(S^N; S^N). \tag{1.1}$$

It follows from Theorem 0.6 and the Sobolev embeddings that the answer is yes. Indeed, if $sp \ge N$ and $(s,p) \ne (1,1)$, then there is some $q > N$ such that $W^{s,p}(S^N; S^N) \hookrightarrow W^{N/q,q}(S^N; S^N)$. On the other hand, if $s = p = N = 1$, we have the estimate

$$|\deg g| = \left| \frac{1}{2i\pi} \int_{S^1} \frac{\dot g}{g} \right| \le \frac{1}{2\pi} |g|_{1,1}.$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

We next examine the optimality of the estimates in Theorem 0.6, which we restate in a slightly more general form.

Theorem 1. *Let* $1 \le p < \infty$ *and* $g \in W^{N/p,p}(S^N; S^N)$*. Then*

$$|\deg g| \le C_{p,N} |g|^p_{N/p,p}. \tag{1.2}$$

Proof. When $p > N$, this is the content of Theorem 0.6. When $p = N$, estimate (1.2) is an immediate consequence of the Kronecker formula

$$\deg g = \frac{1}{|S^N|} \int_{S^N} \det(\nabla g). \tag{1.3}$$

Finally, when $1 \le p < N$, (1.2) follows from (1.3) and the Gagliardo-Nirenberg type inequality

$$|g|_{1,N} \le |g|^{p/N}_{N/p,p} \|g\|^{1-p/N}_{L^\infty}. \tag{1.4}$$

Estimate (1.2) is optimal in the following sense:

Lemma 1.2. *For* $1 \le p < \infty$*, there is a sequence* $(g_k) \subset W^{N/p,p}(S^N; S^N)$ *such that* $|g_k|_{N/p,p} \xrightarrow{k} \infty$ *and* $\deg g_k \ge C'_{p,N} |g_k|^p_{N/p,p}$.

Proof. Let $h : \mathbb{R}^N \to S^N$ be such that $h(x) \equiv P$ for $|x| \ge 1$. Then, clearly, $|h \circ \pi|_{N/p,p(S^N)} \sim |h|_{N/p,p(\mathbb{R}^N)}$. In view of this remark, it suffices to construct a sequence $(h_k) \subset \dot{W}^{N/p,p}(\mathbb{R}^N; S^N)$ such that $|h_k|_{N/p,p} \xrightarrow{k} \infty, h_k \equiv P$ for $|x| \ge 1, \deg h_k \ge C'_{p,N} |h_k|^p_{N/p,p}$. Fix a map $g \in C^\infty(\mathbb{R}^N; S^N)$ such that $\deg g = 1$ and $g(x) \equiv P$ for $|x| \ge 1$. For $k \ge 1$, we fix k distinct points $a_1, \dots, a_k \in B_1$. Let

$$g_{\lambda,k}(x) = P + \sum_{j=1}^{k} (g - P)\left(\frac{x - a_j}{\lambda}\right), \lambda > 0.$$

It is easy to see that

$$|g_{\lambda,k}|^p_{N/p,p} \xrightarrow{\lambda \to 0} k|g|^p_{N/p,p}.$$

In addition, for sufficiently small λ, the map $g_{\lambda,k}$ is S^N-valued, has degree k, and equals P for $|x| \ge 1$. If we set, for sufficiently large λ_k, $h_k = g_{\lambda_k,k}$, then $|h_k|^p_{N/p,p} \sim k$ and $\deg h_k = k$.

2. Existence of a distributional Jacobian.

As in the previous section, we discuss whether, given $0 < s < \infty, 1 \le p < \infty$, there is a notion of a distributional Jacobian in $W^{s,p}(S^{N+1}; S^N)$. As noted in the discussion before Theorem 0.8, the answer is yes in $W^{N/p,p}(S^{N+1}; S^N)$, and therefore also in $W^{s,p}(S^{N+1}; S^N)$ if $sp \ge N$ (via the Sobolev embeddings). On the other hand, there is **no** natural notion of distribution Jacobian if $sp < N$. Indeed, in this case $C^\infty(S^{N+1}; S^N)$ is dense in $W^{s,p}(S^{N+1}; S^N)$ (this follows from [4] and [5]). Let $g : S^{N+1} \to S^N, g(x', x_{N+1}) = \dfrac{x'}{|x'|}$, for which $\mathrm{Det}(\nabla g) \ne 0$. Consider a sequence $(g_k) \subset C^\infty(S^{N+1}; S^N)$ such that $g_k \to g$ in $W^{s,p}$. If a natural $\mathrm{Det}(\nabla)$ would exist, this would yield

$$0 = \lim_k \text{ Det } (\nabla g_k) = \text{ Det } (\nabla g) \ne 0, \text{ impossible }.$$

However, the answer given by Theorem 0.8 is not completely satisfactory. Indeed, the perfect analogs of a), b) in Section 1 are, for $0 < s < \infty, 1 \le p < \infty$ such that $N \le sp < N + 1$:

a$'$) that the class $\mathcal{R}_{s,p} = \{g \in W^{s,p}(S^{N+1}; S^N);\ g \in C^\infty$ except a finite set, $g \in W^{1,N}\}$ is dense in $W^{s,p}(S^{N+1}; S^N)$;

b$'$) that $\mathrm{Det}(\nabla)$ extends by continuity from $\mathcal{R}_{s,p}$ to $W^{s,p}$.

The proof of Theorem 0.8 combined with the Sobolev embeddings shows that b$'$) holds, provided a$'$) holds. However, we established a$'$) only for $0 < s < 1$; when $s = 1$, a$'$) holds also, see [4]. It is plausible that a$'$) holds for any s.

Concerning the estimate

$$\| \text{ Det } (\nabla g)\|_{(W^{1,\infty})^*} \le C|g|^p_{N/p,p}, \quad g \in W^{N/p,p}(S^{N+1}; S^N),\ p > N, \tag{2.1}$$

Theorem 2.4 implies its optimality.

3. The closure of $C^\infty(S^{N+1}; S^N)$.

As we have already noted, $C^\infty(S^{N+1}; S^N)$ in dense in $W^{s,p}(S^{N+1}; S^N)$ if $sp < N$ or $sp \ge N + 1$. It is easy to see that this is **not** true if $N \le sp < N + 1$.

We mention the following straightforward generalization of a result due to Bethuel [19] when $p = N$.

Theorem 2. *Let $N < p < \infty$. For $g \in W^{N/p,p}(S^{N+1}; S^N)$, the following are equivalent:*

a) $g \in \overline{C^\infty(S^{N+1}; S^N)}^{W^{N/p,p}}$;

b) $\text{Det } (\nabla g) = 0$.

A proof is presented in [20].

It is plausible that the assumptions $s < 1, sp = N$ are irrelevant here.

Question. *Let $0 < s < \infty, 1 \le p < \infty$ be such that $N \le sp < N+1$. It is true that*

$$g \in \overline{C^\infty(S^{N+1}; S^N)}^{W^{N/p,p}} \Leftrightarrow \text{Det }(\nabla g) = 0?$$

4. An Alternative proof of Theorem 0.1.

In this section, we present another argument that yields the estimate

$$|\varphi|_{BMO(I)} \le C_p(|e^{i\varphi}|^p_{1/p,p(I)} + |e^{i\varphi}|_{1/p,p(I)}), \quad 1 < p < \infty, \ \varphi \in W^{1/p,p}(I). \tag{4.1}$$

We start with some preliminary results.

4.1. Basic estimates.

If $\varphi \in L^1(I)$, with $I \subset \mathbb{R}$ interval, we set $\varphi_I = \dfrac{1}{|I|}\displaystyle\int_I \varphi$.

Lemma 4.1. *Let $\varphi \in C^0((-\rho,\rho))$. Assume that*

$$|e^{i\varphi}|_{1/p,p(-\rho,\rho)} \le C_1 \tag{4.2}$$

and

$$|\varphi|_{BMO(-\rho,0)} + |\varphi|_{BMO(0,\rho)} \le C_2. \tag{4.3}$$

Then

$$|\varphi_{(-\rho,0)} - \varphi_{(0,\rho)}| \le C(1 + C_1^p)(1 + C_2). \tag{4.4}$$

Proof. We start by introducing some notations. For $0 < l_1 < l_2 \le \rho$, set

$$f(l_1) = \varphi_{(0,l_1)} - \varphi_{(-l_1,0)},$$
$$h(l_1,l_2) = \varphi_{(l_1,l_2)} - \varphi_{(-l_2,-l_1)}.$$

Let $C_3 = 2 + 2C_2$. If $|f(\rho)| \le 10^3 C_3$, there is nothing to prove. Otherwise, assume, e.g., $f(\rho) > 0$ and set $t = \dfrac{f(\rho)}{C_3} > 10^3$. Let $J = \left[\dfrac{t}{2}\right] - 1$. For $j = 1, \dots, J$, we will construct inductively $0 < \rho_1 < \dots < \rho_J < \rho$ such that, for $j = 1, \dots, J-1$,

$$f(\rho_{j+1}) - f(\rho_j) \in [C_3, 2C_3]; \tag{4.5}$$

$$\rho_{j+1} \ge 2\rho_j; \tag{4.6}$$

$$\text{dist}\,(h(\rho_j,\rho_{j+1}), 2\pi\mathbb{Z}) > \frac{1}{2}. \tag{4.7}$$

Assume the ρ_j' s constructed, for the moment. By Corollary A.5, it follows that

$$|e^{i\varphi}|^p_{1/p,p((\rho_j,\rho_{j+1})\cup(-\rho_{j+1},-\rho_j))} \geq C, \; j = 1, \ldots, J-1,$$

and thus

$$|e^{i\varphi}|^p_{1/p,p(-\rho,\rho)} \geq C(J-1) \geq C' f(\rho),$$

from which the conclusion of the lemma follows.

It remains to construct the ρ_j's. Let ρ_1 be the first $l > 0$ such that $f(l) = C_3$. Assuming $\rho_1, \ldots, \rho_j$ constructed ($j < J$), let a be the largest $l > 0$ such that $f(l) = f(\rho_j) + C_3$ and let b be the smallest $l > a$ such that $f(l) = f(\rho_j) + 2C_3$.

We claim that

(4.8) $$a \geq 4\rho_j$$

(4.9) $$\text{for at least one } l \in [a,b], \text{ it holds that } \operatorname{dist}(h(\rho_j, l), 2\pi\mathbb{Z}) > \frac{1}{2}.$$

Properties (4.5) - (4.7) follow immediately from (4.8) - (4.9); it suffices to take $\rho_{j+1} = l$, where $l \in [a,b]$ is such that (4.9) holds. It is also clear from our construction that the ρ_j's exist up to $j = J$.

Proof of (4.8).

By Lemma A.1, we have

$$C_3 = |f(a) - f(\rho_j)| \leq |\varphi_{(0,a)} - \varphi_{(0,\rho_j)}| + |\varphi_{(-a,0)} - \varphi_{(-\rho_j,0)}| \leq \frac{\rho_j}{a}\left(|\varphi|_{\mathrm{BMO}(-\rho,0)} + |\varphi|_{\mathrm{BMO}(0,\rho)}\right) \leq \frac{\rho_j}{a} C_2,$$

and (4.8) follows from our choice of C_3.

Proof of (4.9). Argue by contradiction and assume that

$$\operatorname{dist}(h(\rho_j, l), 2\pi\mathbb{Z}) \leq \frac{1}{2}, \quad \forall\, l \in [a,b].$$

Since $l \mapsto h(\rho_j, l)$ is continuous, there is some fixed $d \in \mathbb{Z}$ such that

$$|h(\rho_j, l) - 2\pi d| \leq \frac{1}{2}, \forall\, l \in [a,b].$$

In particular, $|h(\rho_j, a) - h(\rho_j, b)| \leq 1$. By Lemma A.2, we have

$$h(\rho_j, a) = \frac{a}{a-\rho_j} f(a) - \frac{\rho_j}{a-\rho_j} f(\rho_j) = f(\rho_j) + C_3 \frac{a}{a-\rho_j},$$

and similarly

$$h(\rho_j, b) = f(\rho_j) + 2C_3 \frac{b}{b-\rho_j}.$$

Thus

$$C_3 \left| 2\frac{b}{b-\rho_j} - \frac{a}{a-\rho_j} \right| \leq 1.$$

Since $2\frac{b}{b-\rho_j} - \frac{a}{a-\rho_j} \geq \frac{1}{2}\frac{a}{a-\rho_j}$, we find that $C_3 \frac{a}{a-\rho_j} \leq 2$, which is impossible, since $C_3 \geq 2$.

Corollary 4.2. *Let J, K be two adjacent intervals and $\varphi \in C^0(J \cup K)$. Assume that*

$$|e^{i\varphi}|_{1/p,p(J\cup K)} \le C_1 \tag{4.10}$$

and

$$|\varphi|_{\mathrm{BMO}(J)} + |\varphi|_{\mathrm{BMO}(K)} \le C_2. \tag{4.11}$$

Then

$$|\varphi|_{\mathrm{BMO}(J\cup K)} \le C(1 + C_1^p)(1 + C_2). \tag{4.12}$$

Proof. Let $L \subset J \cup K$ be an interval. We have to prove that

$$\frac{1}{|L|}\int_L |\varphi - \varphi_L| \le C(1 + C_1^p)(1 + C_2).$$

If $L \subset J$ or $L \subset K$, this is clear. Otherwise, assume, e. g., $L = (-a, b)$, with $(-a, 0) \subset J$ and $(0, b) \subset K$. By Lemma A.3, we have

$$\frac{1}{|L|}\int_L |\varphi - \varphi_L| \le 3(|\varphi|_{\ \mathrm{BMO}(-a,0)} + |\varphi|_{\mathrm{BMO}(0,b)} + |\varphi_{(-a,0)} - \varphi_{(0,a)}|),$$

and the conclusion follows from Lemma 4.1.

We will also need the following variant of Lemma 4.1

Lemma 4.3. *Let $0 < \rho' \le \frac{1}{4}\rho$ and $\varphi \in C^0((\rho', \rho) \cup (-\rho, -\rho'))$.*

Assume that

$$|e^{i\varphi}|_{1/p,p((4\rho',\rho)\cup(-\rho,-4\rho'))} \le C_1 \tag{4.13}$$

and

$$|\varphi|_{\mathrm{BMO}((\rho',\rho)\cup(-\rho,-\rho'))} \le C. \tag{4.14}$$

Then

$$\left|(\varphi_{(\rho',\rho)} - \varphi_{(-\rho,-\rho')}) - (\varphi_{(\rho',4\rho')} - \varphi_{(-4\rho',-\rho')})\right| \le C(1 + C_1^p)(1 + C_2). \tag{4.15}$$

Proof. Let $C_3 = 2 + 3C_2$. We may assume, e. g., that

$$tC_3 = \big(\varphi_{(\rho',\rho)} - \varphi_{(-\rho.-\rho')}\big) - \big(\varphi_{(\rho',4\rho')} - \varphi_{(-4\rho',-\rho')}\big) \ge 10^3 C_3.$$

Let $J = \left[\frac{t}{4}\right] - 1$. We construct inductively $\rho_j, j = 1, \dots, J$, as follows: set $\rho_1 = 4\rho'$. Assume $\rho_1, \dots, \rho_j$ already constructed such that

$$\left(\varphi_{(\rho',\rho_k)} - \rho_{(-\rho_k,-\rho')}\right) - \left(\varphi_{(\rho',\rho_{k-1})} - \varphi_{(-\rho_{k-1},-\rho')}\right) \in [C_3, 2C_3]; \tag{4.16}$$

$$\rho_k \geq 2\rho_{k-1}; \tag{4.17}$$

$$\text{dist}\left(\varphi_{(\rho_{k-1},\rho_k)} - \varphi_{(-\rho_k,-\rho_{k-1})}, 2\pi\mathbb{Z}\right) > \frac{1}{2}, \tag{4.18}$$

$k = 1, \dots, j$.

Let a be the largest $l > \rho_j$ such that

$$\left(\varphi_{(\rho',l)} - \varphi_{(-l,-\rho')}\right) - \left(\varphi_{(\rho',\rho_j)} - \varphi_{(-\rho_j,-\rho')}\right) = C_3,$$

and let b be the smallest $l > a$ such that

$$\left(\varphi_{(\rho',l)} - \varphi_{(-l,-\rho')}\right) - \left(\varphi_{(\rho',\rho_j)} - \varphi_{(-\rho,-\rho')}\right) = 2C_3.$$

As in the proof of Lemma 4.1, we claim that

$$a \geq 2\rho_j; \tag{4.19}$$

$$\text{there is some } l \in [a,b] \text{ such that dist}\left(\varphi_{(\rho_j,l)} - \varphi_{(-l,\rho_j)}, 2\pi\mathbb{Z}\right) > \frac{1}{2}. \tag{4.20}$$

Proof of (4.19). We have, by Lemma A.1,

$$\begin{aligned} C_3 &= \left|(\varphi_{(\rho',a)} - \varphi_{(\rho',\rho_j)}) - (\varphi_{(-a,-\rho')} - \varphi_{(-\rho_j,-\rho')})\right| \\ &\leq \left|\varphi_{(\rho',a)} - \varphi_{(\rho',\rho_j)}\right| + \left|\varphi_{(-a,-\rho')} - \varphi_{(-\rho_j,\rho')}\right| \\ &\leq \frac{a-\rho'}{\rho_j - \rho'}|\varphi|_{\text{BMO}(\rho',\rho)} + \frac{a-\rho'}{\rho_j - \rho'}|\varphi|_{\text{BMO}(-\rho,-\rho')} \leq \frac{a-\rho'}{\rho_j-\rho}C_2, \end{aligned}$$

so that

$$a \geq \frac{C_3}{C_2}\rho_j - \frac{C_3 + C_2}{C_2}\rho' \geq 2\rho_j;$$

the last inequality follows from the inequalities $C_3 \geq 3C_2$ and $\rho_j \geq 4\rho'$.

Proof of (4.20). Argue by contradiction. As in the proof of Lemma 4.1, it follows that

$$\left|\left(\varphi_{(\rho_j,a)} - \varphi_{(-a,-\rho_j)}\right) - \left(\varphi_{(\rho_j,b)} - \varphi_{(-b,-\rho_j)}\right)\right| \leq 1.$$

Starting from the identity

$$\varphi_{(\rho_j,a)} = \frac{a-\rho'}{a-\rho_j}\varphi_{(\rho',a)} - \frac{\rho_j-\rho'}{a-\rho_j}\varphi_{(\rho',\rho_j)},$$

we obtain, as in the proof of Lemma 4.1, that

$$C_3\left|2\frac{b-\rho'}{b-\rho_j} - \frac{a-\rho'}{a-\rho_j}\right| \leq 1.$$

As in the proof of Lemma 4.1, this implies that $C_3\dfrac{a-\rho'}{a-\rho_j} \leq 2$, which is impossible, since $C_3 \geq 2$.

The remaining part of the proof of Lemma 4.3 is identical to the one of Lemma 4.1 and will be omitted.

Proof of Theorem 0.1. We may assume that $\varphi \in C^0 \cap W^{1/p,p}$. As explained in the main paper, when $|e^{i\varphi}|_{1/p,p}$ is sufficiently small, (4.1) follows from the inequality $|g|_{\mathrm{BMO}(I)} \leq C|g|_{1/p,p(I)}$ combined with

Lemma 4.4 ([14]). *Let $\varphi \in$VMO(I). There are constants $C>0, \delta>0$ such that*

$$|\varphi|_{\mathrm{BMO}(I)} \leq C|e^{i\varphi}|_{\mathrm{BMO}(I)} \quad \textit{if } |e^{i\varphi}|_{\mathrm{BMO}(I)} \leq \delta. \tag{4.21}$$

Let $\gamma = \min(\delta^{1/p}, \delta)$. It suffices to establish (4.1) when $|e^{i\varphi}|^p_{1/p,p(I)} \geq \delta$. Let N be the smallest integer $\geq \dfrac{|g|^p_{1/p,p}}{\gamma}$. We consider a partition of I with N successive intervals $I_1,\ldots,I_N$ chosen such that $|e^{i\varphi}|^p_{1/p(I_1\cup\cdots\cup I_j)} = j\gamma, j=1,\ldots,N-1$. Thus $|e^{i\varphi}|^p_{1/p,p(I_j)} \leq \gamma, \forall\, j$.

It suffices to establish the estimate

$$\frac{1}{|I|}\int_I |\varphi - \varphi_I| \leq C|e^{i\varphi}|^p_{1/p,p(I)}. \tag{4.22}$$

In view of lemmas A.6 and 4.4, we have

$$\frac{1}{|I|}\int_I |\varphi-\varphi_I| \leq C\gamma + \frac{1}{|I|^2}\sum_{j,k}|I_j||I_k||\varphi_{I_j}-\varphi_{I_k}|,$$

so that (4.22) bounds to proving

$$\frac{1}{|I|^2}\sum_{j,k}|I_j||I_k||\varphi_{I_j}-\varphi_{I_k}| \leq C|e^{i\varphi}|^p_{1/p,p(I)}. \tag{4.23}$$

The remaining part of the proof is devoted to estimating the differences $|\varphi_{I_j}-\varphi_{I_k}|$. Without any loss of generality, we will assume $j=1$.

Lemma 4.5. *Assume that* $|I_1| = |I_k| \geq \frac{1}{2}\max |I_l|$. *Let* l_0 *be such that* $|I_{l_0}| = \max\{|I_l|; 2 \leq l \leq k-1\}$ *and set* $J = I_2 \cup \cdots \cup I_{k-1}$. *Assume that* $|I_1| \leq 4|J|$, *and consider the following intervals*

picture
(with $|\bar{I}_1| = |\bar{I}_{l_0}| = |\tilde{I}_{l_0}| = |\tilde{I}_k| = \frac{1}{2}|I_{l_0}|$*). Then*

$$|\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\tilde{I}_{l_0}} - \varphi_{\tilde{I}_k}| + C\left(1 + \log\frac{|I|}{|I_{l_0}|}.\right) \tag{4.24}$$

Proof. We have

$$|\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I_1} - \varphi_{\bar{I}_1}| + |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\bar{I}_{l_0}} - \varphi_{\tilde{I}_{l_0}}| + |\varphi_{\tilde{I}_{l_0}} - \varphi_{\tilde{I}_k}| + |\varphi_{\tilde{I}_k} - \varphi_{I_k}|. \tag{4.25}$$

By Lemma A.7, we have

$$|\varphi_{I_1} - \varphi_{\bar{I}_1}| + |\varphi_{\bar{I}_{l_0}} - \varphi_{\tilde{I}_{l_0}}| + |\varphi_{\tilde{I}_k} - \varphi_{I_k}| \leq C\left(1 + \log\frac{|I|}{|I_{l_0}|}\right), \tag{4.26}$$

and the conclusion follows.

Lemma 4.6. *Same hypotheses as above, except that we assume* $|I_1| > 4|J|$. *Let* I_1^*, I_k^* *be as below.*

picture
(with $|I_1^*| = |I_k^*| = 4|J|$*). Then*

$$|\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\tilde{I}_{l_0}} - \varphi_{\tilde{I}_k}| + C\left(1 + \log\frac{4|J|}{|I_{l_0}|} + |e^{i\varphi}|^p_{1/p,p((I_1\setminus I_1^*)\cup(I_k\setminus I_k^*))}\right). \tag{4.27}$$

Proof. We have

$$|\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{I_1^*} - \varphi_{I_k^*}| + \left|(\varphi_{I_1} - \varphi_{I_1^*}) - (\varphi_{I_k} - \varphi_{I_k^*})\right|. \tag{4.28}$$

By Lemma 4.5, we have

$$|\varphi_{I_1^*} - \varphi_{I_k^*}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\tilde{I}_{l_0}} - \varphi_{\tilde{I}_k}| + C\left(1 + \log\frac{4|J|}{|I_{l_0}|}\right). \tag{4.29}$$

On the other hand, Lemma 4.3 (with $(-\rho', \rho')$ replaced by J and $(-\rho, \rho)$ replaced by $I_1 \cup J \cup I_k$) yields

$$|(\varphi_{I_1} - \varphi_{I_1^*}) - (\varphi_{I_k} - \varphi_{I_k^*})| \leq C\left(1 + |e^{i\varphi}|^p_{1/p,p((I_1\setminus I_1^*)\cup(I_k\setminus I_k^*))}\right), \tag{4.30}$$

and the conclusion follows.

Corollary 4.7. *If $|I_1| = |I_k| \geq \frac{1}{2} \max_{2\leq l\leq k-1} |J_l|$, then (with l_0 as above)*

$$
|\varphi_{I_1} - \varphi_{I_k}| \leq |\varphi_{\bar{I}_1} - \varphi_{\bar{I}_{l_0}}| + |\varphi_{\tilde{I}_{l_0}} - \varphi_{\tilde{I}_k}| + C\left(1 + \log\frac{\min\{4|J|, |I_1|\}}{|I_{l_0}|} + |e^{i\varphi}|^p_{1/p,p((I_1\setminus I_1^*)\cup(I_k\setminus I_k^*))}\right). \tag{4.31}
$$

(Here, $I_1 \setminus I_1^*$ and $I_k \setminus I_k^*$ could be empty).

Lemma 4.8. *Assume that $|I_1| = |I_k| \geq \frac{1}{2} \max_{2\leq l\leq k-1} |J_l|$. Then*

$$
|\varphi_{I_1} - \varphi_{I_k}| \leq C(k + |e^{i\varphi}|^p_{1/p,p(I)}). \tag{4.32}
$$

Proof. We start by applying Corollary 4.7. We note that, by construction, we may apply again Corollary 4.7 to the consecutive intervals $\bar{I}_1, I_2, \ldots, I_{l_0-1}, \bar{I}_{l_0}$, respectively to $\tilde{I}_{l_0}, I_{l_0+1}, \ldots I_{k-1}, \tilde{I}_k$; next we iterate this procedure.

We find that

$$
|\varphi_{I_1} - \varphi_{I_k}| \leq \sum_{l=1}^{2(k-1)} |\varphi_{J_l} - \varphi_{K_l}| + 2C(k-1) + \Sigma_1 + \Sigma_2. \tag{4.33}
$$

Here, J_l, K_l are adjacent intervals of equal length, each one contained into one of the original I_j's; Σ_1 is the sum of the logarithmic terms, while Σ_2 is the sum of the $|\ |^p_{1/p,p}$ terms. Lemma 4.1 implies that

$$
\sum_{l=1}^{2(k-1)} |\varphi_{J_l} - \varphi_{K_l}| \leq C(k-1) \leq Ck. \tag{4.34}
$$

On the other hand, the $|\ |^p_{1/p,p}$ terms we consider appear on disjoint intervals, and thus

$$
\Sigma_2 \leq C|e^{i\varphi}|^p_{1/p,p(I)}. \tag{4.35}
$$

Therefore,

$$
|\varphi_{I_1} - \varphi_{I_k}| \leq Ck + C|e^{i\varphi}|^p_{1/p,p(I)} + \Sigma_1. \tag{4.36}
$$

Claim We have $\Sigma_1 \leq Ck$. In order to prove the claim, we give a formal description of how Σ_1 is computed.

Let $Z_\phi = \{2, \ldots, k-1\}$ and let $s_\phi \in Z_\phi$ be such that $|I_{s_\phi}| = \max |I_s|$ (with the notations used up to now, we have $s_\emptyset = l_0$). Let $Z_{(0)} = \{2, \ldots, s_\phi - 1\}, Z_{(1)} = \{s_\phi + 1, \ldots, k-1\}$ if s_ϕ

is closer to 2 than to $k-1$; otherwise, let $Z_{(0)} = \{s_\phi+1, \dots, k-1\}, Z_{(1)} = \{2, \dots, s_\phi - 1\}$. We have

$$|Z_\phi| = 1 + |Z_{(0)}| + |Z_{(1)}|, |Z_{(0)}| \le |Z_{(1)}|.$$

Assuming Z_c constructed, we proceed to constructing $Z_{(c,0)}$ and $Z_{(c,1)}$ as above. More specifically, if $Z_c \neq \phi$, we pick $s_c \in Z_c$ such that $|I_{s_c}| = \max_{s\in Z_c} |I_s|$. We next write $Z_c \setminus \{s_c\} = Z_{(c,0)} \sqcup Z_{(c,1)}$, with $Z_{(c,0)}, Z_{(c,1)}$ intervals of integers and $|Z_{(c,0)}| \le |Z_{(c,1)}|$. If $Z_c = \emptyset$, we stop.

If $Z_c = \{m, m+1, \dots, n\}$, then the corresponding term in Σ_1 is of the form

$$\log \frac{\min\{|K|, 4\sum_{s\in Z_c} |I_s|\}}{|I_{s_c}|}; \tag{4.37}$$

here, K is an interval contained in I_{m-1} and of length $\le \min\{|I_{m-1}, |I_{n+1}|\}$.

Assume $c \neq \emptyset$. If $\hat{c}$ is the predecessor of c, we have either $m-1 \in Z_{\hat{c}}$, or $n+1 \in Z_{\hat{c}}$, and thus $|K| \le |I_{s_{\hat{c}}}|$.

In conclusion,

$$\Sigma_1 \le C \log \frac{\min\{4|J|, |I_1|\}}{|I_{l_0}|} + C \sum_{c\neq\emptyset} \log 4 \underbrace{\frac{\min\{|I_{s_{\hat{c}}}|, \sum_{s\in Z_c} |I_s|\}}{|I_{s_c}|}}_{R_c}. \tag{4.38}$$

Setting $R_\emptyset = \dfrac{\min\{|J|, |I_1|\}}{|I_{l_0}|}$, the claim amounts to proving that

$$\sum_c \log(4R_c) \le Ck. \tag{4.39}$$

This is an immediate consequence of the two following

Lemma 4.9. *We have*

$$\sum_c \log R_c \le Ck + \sum_c \log |Z_{(c,0)}|. \tag{4.40}$$

Lemma 4.10. *We have*

$$\sum_c \log |Z_{(c,0)}| \le Ck. \tag{4.41}$$

Proof of Lemma 4.9. Let t be the largest integer such that $Z_{(\underbrace{1,1,\dots,1}_{t \text{ times}},0)} \neq \phi$. Set $a_0 = Z_{(0)}, a_1 = Z_{(1,0)}, \dots, a_t = Z_{(\underbrace{1,1,\dots,1}_{t \text{ times}},0)}, x_0 = R_\phi, x_1 = R_{(1)}, \dots, x_t = R_{(\underbrace{1,\dots,1}_{t \text{ times}})}$.

Writing

$$Z_\phi = (Z_{(0)} \cup \{s_\phi\}) \cup (Z_{(1,0)} \cup \{s_{(1)}\}) \cup (Z_{(1,1,0)} \cup \{s_{(1,1)}\}) \cup \ldots,$$

we find that

$$R_\phi \leq \frac{\sum_{s \in Z_\phi} |I_s|}{|I_{s_\phi}|} = \sum_{s \in Z_{(0)} \cup \{s_\phi\}} \frac{|I_s|}{|I_{s_\phi}|} + \sum_{s \in Z_{(1,0)} \cup \{s_1\}} \frac{|I_s|}{|I_{s_\phi}|} + \sum_{s \in Z_{(1,1,0)} \cup \{s_{(1,1,0)}\}} \frac{|I_s|}{|I_{s_\phi}|} + \ldots,$$

so that

$$R_\phi \leq 2|Z_{(0)}| + 2|Z_{(1,0)}| \frac{|I_{s_{(1)}}|}{|I_{s_\phi}|} + 2|Z_{(1,1,0)}| \frac{|I_{s_{(1,1)}}|}{|I_{s_\phi}|} + \ldots.$$

Since

$$\frac{|I_{s_{(1)}}|}{|I_{s_\phi}|} \leq \frac{1}{R_{(1)}}, \frac{|I_{s_{(1,1)}}|}{|I_{s_\phi}|} = \frac{|I_{s_{(1,1)}}|}{|I_{s_{(1)}}|} \cdot \frac{|I_{s_{(1)}}|}{|I_{s_\phi}|} \leq \frac{1}{R_{(1)} R_{(1,1)}}, \ldots,$$

we obtain

$$\frac{1}{2} x_0 \leq a_0 + \frac{a_1}{x_1} + \frac{a_2}{x_1 x_2} + \cdots + \frac{a_t}{x_1 x_2 \ldots x_t} \tag{4.42}$$

and similarly

$$\begin{cases} \frac{1}{2} x_1 \leq a_1 + \frac{a_2}{x_2} + \cdots + \frac{a_t}{x_2 \ldots x_t} \\ \vdots \\ \frac{1}{2} x_{t-1} \leq a_{t-1} + \frac{a_t}{x_t}. \end{cases} \tag{4.43}$$

Noting that $x_t = 1$, we find from (4.42) - (4.43) by backward induction on j that

$$x_j \ldots x_t \leq \sum_{m=1}^{t-j} 2^m \sum_{\substack{J \subset \{j,\ldots,t\} \\ |J|=m}} \prod_{l \in J} a_l.$$

In particular, since $a_j \geq 1, \forall j$, we obtain

$$x_0 \ldots x_t \leq \sum_{m=1}^{t} 2^m \sum_{J \subset \{0,\ldots,t\} |J|=m} \prod_{l \in J} a_l \leq \sum_{m=1}^{t} 2^m \binom{t+1}{m} \prod_0^t a_l \leq 3^{t+1} \prod_0^t a_l. \tag{4.44}$$

Similarly, for any fixed $\bar{c}$ we have

$$\prod_{\substack{c \text{ contains} \\ \text{only 1's}}} R_{(\bar{c},c)} \leq \prod_{\substack{c \text{ contains} \\ \text{only 1's}}} (3|Z_{(\bar{c},c,0)}|). \tag{4.45}$$

Since each c can be uniquely written as $c = (\bar{c}, 0, \bar{\bar{c}})$ where $\bar{\bar{c}}$ contains only 1's, by multiplying the inequalities of type (4.45) we find that

$$\prod_c R_c \leq 3^{2k} \prod_c |Z_{(c,0)}|,$$

from which the conclusion of the lemma follows.

Proof of Lemma 4.10. Let, for $l \geq 0$, $S_l = \{c \; ; \; |Z_{(c,0)}| \in [2^l, 2^{l+1})\}$. We claim that

$$[c \neq c', c, c' \in S_l] \Rightarrow Z_{(c,0)} \cap Z_{(c',0)} = \phi.$$

Argue by contradiction and assume that $Z_{(c,0)} \cap Z_{(c',0)} \neq \phi$.

Then, for example, we have $Z_{(c,0)} \subsetneq Z_{(c',0)}$, so that $Z_c \subset Z_{(c',0)}$, by construction. Thus $|Z_{(c,0)}| \leq \dfrac{1}{2}|Z_c| \leq \dfrac{1}{2}|Z_{(c',0)}|$, which is impossible if $c, c' \in S_l$. Therefore,

$$\prod |Z_{(c,0)}| = \prod_{l=1}^{[\log_2 k]+1} \prod_{c \in S_l} |Z_{(c,0)}| \leq \prod_{l=1}^{[\log_2 k]+1} 2^{l|S_l|} \leq \prod_{l \geq 1} 2^{lk/2^l} = 2^{Ak},$$

where $A = \sum_{l \geq 1} l 2^{-l}$.

Lemma 4.11. *Assume that* $|I_1| \geq |I_l|, l = 2, \ldots, k$. *Then*

$$|\varphi_{I_1} - \varphi_{I_k}| \leq C \left(k + |e^{i\varphi}|^p_{1/p, p(I_1 \cup \cdots \cup I_k)} + \log \frac{|I_1|}{|I_k|} \right). \tag{4.46}$$

Proof. Let $l_0 = 1$ and define inductively l_j such that $|I_{lj}| = \max_{l_{j-1} < l \leq k} |I_l|$. Then

$$|\varphi_{I_1} - \varphi_{I_k}| \leq \sum_{j \geq 1} |\varphi_{I_{s_{j-1}}} - \varphi_{I_{s_j}}|.$$

Let $\bar{I}_{s_{j-1}}$ be as follows:

picture

(such that $|\bar{I}_{s_{j-1}}| = |I_{s_j}|$). We may apply Lemma 4.7 to the sequence of intervals $\bar{I}_{s_{j-1}}, \ldots, I_{s_j}$, and find that

$$|\varphi_{\bar{I}_{s_{j-1}}} - \varphi_{I_{s_j}}| \leq C(s_j - s_{j-1} + |e^{i\varphi}|^p_{1/p, p(I_{s_{j-1}} \cup \cdots \cup I_{s_j})}). \tag{4.47}$$

On the other hand, Lemma A.7 yields

$$|\varphi_{I_{s_{j-1}}} - \varphi_{\bar{I}_{s_{j-1}}}| \leq C \left(1 + \log \frac{|I_{s_{j-1}}|}{|I_{s_j}|} \right). \tag{4.48}$$

By summing up all the inequalities of type (4.47)-(4.48), we find that (4.46) holds.

Lemma 4.12. *For each j, k we have*

$$|\varphi_{I_j} - \varphi_{I_k}| \leq C\left(|k-j| + |e^{i\varphi}|^p_{1/p,p(I)} + \log\frac{|I|^2}{|I_j||I_k|}\right). \tag{4.49}$$

Proof. Assume $j = 1$. If I_1 (or I_k) is the largest among the intervals $I_1, \ldots, I_k$, the conclusion follows from Lemma 4.11. Otherwise, let $l \in \{2, \ldots, k-1\}$ be such that $|I_l| \geq |I_t|, t = 1, \ldots, k$. By Lemma 4.11, we have

$$|\varphi_{I_1} - \varphi_{I_l}| \leq C\left((l-1) + |e^{i\varphi}|^p_{1/p,p(I_1\cup\cdots\cup I_l)} + \log\frac{|I_l|}{|I_1|}\right) \tag{4.50}$$

and

$$|\varphi_{I_k} - \varphi_{I_k}| \leq C\left((k-l) + |e^{i\varphi}|^p_{1/p,p(I_l\cup\cdots\cup I_k)} + \log\frac{|I_l|}{|I_k|}\right), \tag{4.51}$$

from which the lemma follows.

Corollary 4.13. *We have*

$$|\varphi_{I_j} - \varphi_{I_k}| \leq C\left(N + |e^{i\varphi}|^p_{1/p,p(I)} + \frac{|I|}{\sqrt{|I_j||J_k|}}\right). \tag{4.51}$$

Proof of Theorem 0.1. We have to estimate the r.h.s. of (4.23). In view of Corollary 4.13, we have

$$\begin{aligned}\frac{1}{|I|^2}\sum_{j,k}|I_j||I_k||\varphi_{I_j} - \varphi_{I_k}| \leq & C\left(N + |e^{i\varphi}|^p_{1/p,p(I)}\right) + \frac{C}{|I|}\sum_{j,k}|I_j|^{1/2}|I_k|^{1/2} \\ \leq & C\left(N + |e^{i\varphi}|^p_{1/p,p(I)}\right) \leq C|e^{i\varphi}|^p_{1/p,p(I)},\end{aligned}$$

since $N \leq C|e^{i\varphi}|^p_{1/p,p(I)}$.

5. An improvement of Theorem 0.1 and the answer to OP2 when $N = 1$.

If $I \subset \mathbb{R}$ is an interval and $g : I \to \mathbb{C}$, we set, for $\delta > 0$,

$$J(g, \delta, I) = \iint_{\{(x,y)\in I^2; |g(x)-g(y)|\geq\delta\}} \frac{1}{|x-y|^2}.$$

In this section, we prove the following generalization of Theorem 0.1.

Theorem 3. *For sufficiently small $\delta > 0$, we have*

$$|\varphi|_{BMO(I)} \leq C(\delta + J(e^{i\varphi}, \delta, I)), \quad \forall\, \varphi \in C^0(I; \mathbb{R}). \tag{5.1}$$

An immediate consequence is the following

Theorem 4. *Let $g \in C^0(S^1; S^1)$. Then, for sufficiently small $\delta > 0$, we have*

$$|\deg g| \leq CJ(g, \delta, S^1). \tag{5.2}$$

This answer OP 2 when $N = 1$.

Proof of Theorem 4. By Lemma A.8, we have $|g|_{\text{BMO}(S^1)} \leq \delta + 2J(g, \delta, S^1)$. Recall that $\deg g = 0$ provided $|g|_{\text{BMO}(S^1)}$ is sufficiently small (see [13]). Thus (5.2) holds (for small $\delta > 0$) provided $J(g, \delta, S^1)$ is sufficiently small.

When $J(g, \delta, S^1)$ is not to small, estimate (5.2) is obtained from (5.1) in the same way (0.6) follows from Theorem 0.1.

Proof of Theorem 3. The proof is the same as the one of Theorem 0.1, except that $|\ \ |^p_{1/p,p}$ has to be replaced by $J(g, \delta, I)$. The only two places where $|\ \ |^p_{1/p,p}$ comes into the picture are the inequality

$$|g|_{\text{BMO}(I)} \leq |g|_{1/p,p(I)} \tag{5.3}$$

and Lemma A.4 (together with Corollary A.5). The substitute of (5.3) is Lemma A.8. The analog of Lemma A.4/Corollary A.5 are Lemma A.9/Corollary A.10 presented into the appendix.

Appendix. Elementary properties of averages.

Lemma A.1. *Let $J \subset K$. Then* $|\varphi_J - \varphi_K| \leq \dfrac{|K|}{|J|}|\varphi|_{\text{BMO}(K)}$.

Proof. We have

$$|\varphi_J - \varphi_K| = \frac{1}{|J|}\left|\int_J (\varphi - \varphi_K)\right| \leq \frac{1}{|J|}\int_K |\varphi - \varphi_K| = \frac{|K|}{|J|}\frac{1}{|K|}\int_K |\varphi - \varphi_K| \leq \frac{|K|}{|J|}|\varphi|_{\text{BMO}(K)}.$$

The following identities are trivial:

Lemma A.2. *Let J, K be two adjacent intervals. Then*

$$\varphi_J - \varphi_{J\cup K} = \frac{|K|}{|J| + |K|}(\varphi_J - \varphi_K), \varphi_J = \frac{|J| + |K|}{|J|}\varphi_{J\cup K} - \frac{|K|}{|J|}\varphi_K \tag{A.1}$$

and

$$\varphi_{J\cup K} = \frac{|K|}{|J| + |K|}\varphi_K + \frac{|J|}{|J| + |K|}\varphi_J. \tag{A.2}$$

Lemma A.3. *Let $0 < a \le b$. Then*

$$\frac{1}{a+b}\int_{-a}^{b} |\varphi - \varphi_{(-a,b)}| \le 3\left(|\varphi|_{\mathrm{BMO}(-a,0)} + |\varphi|_{\mathrm{BMO}(0,b)}\right) + |\varphi_{(-a,0)} - \varphi_{(0,a)}|.$$

Proof. We have

$$\begin{aligned} M = \int_{-a}^{b} |\varphi - \varphi_{(-a,b)}| \le & \int_{-a}^{0} |\varphi - \varphi_{(-a,0)}| + \int_{-a}^{0} |\varphi_{(-a,0)} - \varphi_{(-a,b)}| + \\ & \int_{0}^{b} |\varphi - \varphi_{(0,b)}| + \int_{0}^{b} |\varphi_{(0,b)} - \varphi_{(-a,b)}| \\ \le & a|\varphi|_{\mathrm{BMO}(-a,0)} + b|\varphi|_{\mathrm{BMO}(0,b)} + a|\varphi_{(-a,0)} - \varphi_{(-a,b)}| + b|\varphi_{(0,b)} - \varphi_{(-a,b)}|. \end{aligned}$$

By Lemma A.2, we further obtain

$$\begin{aligned} M &\le \underbrace{(a+b)\left(|\varphi|_{\mathrm{BMO}(-a,0)} + |\varphi|_{\mathrm{BMO}(0,b)}\right)}_{N} + \frac{2ab}{a+b}|\varphi_{(-a,0)} - \varphi_{(0,b)}| \\ &\le N + \frac{2ab}{a+b}|\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2ab}{a+b}|\varphi_{(0,a)} - \varphi_{(0,b)}|, \end{aligned}$$

and Lemma A.1 implies that

$$M \le N + \frac{2ab}{a+b}|\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2b^2}{a+b}|\varphi|_{\mathrm{BMO}(0,b)}. \tag{A.3}$$

Dividing (A.3) by $a+b$, we find that

$$\begin{aligned} \frac{1}{a+b}\int_{-a}^{b} |\varphi - \varphi_{(-a,b)}| \le & |\varphi|_{\mathrm{BMO}(-a,0)} + |\varphi|_{\mathrm{BMO}(0,b)} + \\ & \frac{2ab}{(a+b)^2}|\varphi_{(-a,0)} - \varphi_{(0,a)}| + \frac{2b^2}{(a+b)^2}|\varphi|_{\mathrm{BMO}(0,b)} \\ \le & 3\left(|\varphi|_{\mathrm{BMO}(-a,0)} + |\varphi|_{\mathrm{BMO}(0,b)}\right) + |\varphi_{(-a,0)} - \varphi_{(0,a)}|. \end{aligned}$$

Lemma A.4. *Let $L \ge 2l > 0$ and $\varphi \in C^0\left((l,L) \cup (-L,-l)\right)$. There is some $\gamma > 0$ such that*

$$\left[\operatorname{dist}\left(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}\right) > \frac{1}{2} \text{ and } |\varphi|_{\mathrm{BMO}(l,L)} + |\varphi|_{\mathrm{BMO}(-L,-l)} < \gamma\right] \Rightarrow$$

$$\int_{l}^{L}\int_{-L}^{-l} \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2} \ge C;$$

here, γ and C depend only on l, L, φ.

Proof. We have, with $I = (l, L), J = (-L, -l)$,

$$\frac{1}{L-l}\int_I |\varphi - \varphi_I|^p \leq C|\varphi|^p_{\mathrm{BMO}(I)},$$

with C independent of I (this is the scale invariant form of the John-Nirenberg inequality). Thus

$$\frac{1}{L}\int_l^L |e^{i\varphi} - e^{i\varphi_I}|^p \leq \frac{1}{L-l}\int_I |\varphi - \varphi_I|^p \leq C|\varphi|^p_{\mathrm{BMO}(I)}; \tag{A.4}$$

a similar inequality holds for J.

Since, for $x \in I, y \in J$, we have

$$|e^{i\varphi_I} - e^{i\varphi_J}|^p \leq 3^p \left(|e^{i\varphi(x)} - e^{i\varphi_I}|^p + |e^{i\varphi(x)} - e^{i\varphi(y)}|^p + |e^{i\varphi(y)} - e^{i\varphi_J}|^p\right), \tag{A.5}$$

we find that

$$\int_I\int_J |e^{i\varphi_I} - e^{i\varphi_J}|^p \leq 3^p \int_I\int_J \left(|e^{i\varphi(x)} - e^{i\varphi_I}|^p + |e^{i\varphi(x)} - e^{i\varphi(y)}|^p + |e^{i\varphi(y)} - e^{i\varphi_J}|^p\right),$$

so that

$$\begin{aligned} |e^{i\varphi_I} - e^{i\varphi_J}|^p \leq & \frac{C}{L}\left(\int_I |e^{i\varphi(x)} - e^{i\varphi_I}|^p + \int_J |e^{i\varphi(y)} - e^{i\varphi_J}|^p\right) + \\ & \frac{C}{L^2}\int_I\int_J |e^{i\varphi(x)} - e^{i\varphi(y)}|^p \\ \leq & C'\left(|\varphi|_{\mathrm{BMO}(I)} + |\varphi|^p_{\mathrm{BMO}(J)}\right) + C''\int_I\int_J \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2}. \end{aligned} \tag{A.6}$$

Thus

$$\int_I\int_J \frac{|e^{i\varphi(x)} - e^{i\varphi(y)}|^p}{(x-y)^2} \geq C'''\left(|e^{i\varphi_I} - e^{i\varphi_J}|^p - |\varphi|^p_{\mathrm{BMO}(I)} - |\varphi|^p_{\mathrm{BMO}(J)}\right), \tag{A.7}$$

from which the lemma follows immediately.

Corollary A.5. *Let $L \geq 2l > 0$ and $\varphi \in C^0((l, L) \cup (-L, -l))$.*

Then

$$\mathrm{dist}\left(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}\right) > \frac{1}{2} \Rightarrow |e^{i\varphi}|_{1/p,p((l,L)\cup(-L,-l))} \geq C,$$

for some C independent of l, L, φ.

Proof. If $|\varphi|_{\mathrm{BMO}(l,L)} + |\varphi|_{\mathrm{BMO}(-L,-l)} < \gamma$, the conclusion follows from Lemma A.4. Otherwise, Lemma 4.4 combined with the embedding $W^{1/p,p} \subset$VMO implies that $|e^{i\varphi}|_{1/p,p(l,L)} + |e^{i\varphi}|_{1/p,p(-L,-l)} \geq C$ for some C depending only on γ, and the conclusion follows again.

Lemma A.6. *Let $\varphi \in BMO(I)$ and consider a partition $I = \bigcup_j I_j$ of I with intervals. If $|\varphi|_{\mathrm{BMO}(I_j)} \le C_2, \forall j,$ then*

$$\frac{1}{|I|}\int_I |\varphi - \varphi_I| \le C_2 + \frac{1}{|I|^2}\sum_{j,k} |I_j||I_k||\varphi_{I_j} - \varphi_{I_k}|.$$

Proof. We have

$$\begin{aligned}\frac{1}{|I|}\int_I |\varphi - \varphi_I| =& \frac{1}{|I|}\sum_j \int_{I_j} |\varphi - \varphi_I| \le \frac{1}{|I|}\sum_j \left(\int_{I_j} |\varphi - \varphi_{I_j}| + \int_{I_j} |\varphi_I - \varphi_{I_j}|\right)\\ =& \frac{1}{|I|}\sum_j |I_j| \frac{1}{|I_j|}\int_{I_j} |\varphi - \varphi_{I_j}| + \frac{1}{|I|}\sum_j \int_{I_j} \left|\sum_k \frac{|I_k|}{|I|}\varphi_{I_k} - \varphi_{I_j}\right|\\ \le& \frac{C_2}{|I|}\sum_j |I_j| + \frac{1}{|I|^2}\sum_{j,k}\int_{I_j} |I_k||\varphi_{I_k} - \varphi_{I_j}| \le C_2 + \frac{1}{|I|^2}\sum_{j,k} |I_j|\,|I_k|\,|\varphi_{I_j} - \varphi_{I_k}|.\end{aligned}$$

Lemma A.7. *Let $J \subset K$ be intervals. Then*

$$|\varphi_J - \varphi_K| \le C\left(1 + \log\frac{|K|}{|J|}\right)|\varphi|_{\mathrm{BMO}(K)}.$$

Proof. If $|J| \ge \frac{1}{2}|K|$, the conclusion follows from Lemma A.1. Otherwise, let $l \in \mathbb{N}$ be such that $\dfrac{|J|}{|K|} \in [2^{-l-1}, 2^{-l})$ and consider a sequence of intervals $J_1, \ldots, J_{l+2}$, such that $J_1 = J, J_k \subset J_{k+1}, J_{l+2} = K, |J_k| = 2^{k-l-2}, k = 2, \ldots, l+1$. Then

$$\begin{aligned}|\varphi_J - \varphi_K| \le& \sum_{j=1}^{l+1} |\varphi_{J_{j+1}} - \varphi_{J_j}| \le |\varphi|_{\mathrm{BMO}(K)} \sum_{j=1}^{l+1} \frac{|J_{j+1}|}{|J_j|}\\ \le& 2(l+1)|\varphi|_{\mathrm{BMO}(K)} \le C\left(1 + \log\frac{|K|}{|J|}\right)|\varphi|_{\mathrm{BMO}(K)};\end{aligned}$$

here, we use again Lemma A.1.

Lemma A.8. *We have, for $g \in C^0(I; S^1)$, $|g|_{\mathrm{BMO}(I)} \le \delta + 2J(g, \delta, I)$.*

Proof. Let $K \subset I$ be an interval. Then

$$\begin{aligned}\frac{1}{|K|^2}\int_K\int_K |g(x) - g(y)| \le& \frac{\delta}{|K|^2}\iint_{\{(x,y)\in K^2; |g(x)-g(y)|<\delta\}} dxdy +\\ & \frac{2}{|K|^2}\iint_{\{(x,y)\in K^2; |g(x)-g(y)|\ge\delta\}} dxdy\\ \le& \delta + 2J(g, \delta, K) \le \delta + 2J(g, \delta, I).\end{aligned}$$

Lemma A.9. *Let $L \geq 2l > 0$ and $\varphi \in C^0((l,L) \cup (-L,-l))$. There is some $\gamma > 0$ such that*

$$\left[\operatorname{dist}\left(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}\right) > \frac{1}{2} \text{ and } |\varphi|_{\mathrm{BMO}(l,L)} + |\varphi|_{\mathrm{BMO}(-L,-l)} < \gamma\right] \Rightarrow$$

$$\iint_{\{x\in(l,L),y\in(-L,-l);|g(x)-g(y)|\geq\delta\}} \frac{1}{(x-y)^2} \geq C;$$

here, γ and C depend only on l, L, φ and δ is small.

Proof. We start from (A.6). With $g = e^{i\varphi}$, we have

$$\begin{aligned} C_1 \leq |e^{i\varphi_I} - e^{i\varphi_J}|^p \leq & C'(|\varphi|^p_{\mathrm{BMO}(I)} + |\varphi|^p_{\mathrm{BMO}(J)}) + \frac{C}{L^2}\int_I\int_J |e^{i\varphi(x)} - e^{i\varphi(y)}|^p \\ \leq & C'(|\varphi|^p_{\mathrm{BMO}(I)} + |\varphi|^p_{\mathrm{BMO}(J)}) + \\ & \frac{C\delta}{L^2}\iint_{\{|g(x)-g(y)|\leq\delta\}} dxdy + \frac{2C}{L^2}\iint_{\{|g(x)-g(y)|>\delta\}} dxdy, \end{aligned}$$

so that

$$C_1 \leq C'(|\varphi|^p_{\mathrm{BMO}(I)} + |\varphi|^p_{\mathrm{BMO}(J)}) + C\delta + C''\iint_{\{|g(x)-g(y)|>\delta\}} \frac{1}{(x-y)^2}dxdy, \tag{A.8}$$

and the lemma follows.

Corollary A.10. *Let $\delta > 0$ be sufficiently small, $L \geq 2l > 0$ and $\varphi \in C^0((l,L) \cup (-L,-l))$. Then*

$$\operatorname{dist}\left(\varphi_{(l,L)} - \varphi_{(-L,-l)}, 2\pi\mathbb{Z}\right) > \frac{1}{2} \Rightarrow \iint_{\{l\leq|x|,|y|\leq L;|g(x)-g(y)|>\delta\}} \frac{1}{(x-y)^2}dxdy \geq C,$$

for some C independent of l, L, φ.

Proof. If $|\varphi|_{\mathrm{BMO}(l,L)} + |\varphi|_{\mathrm{BMO}(-L,-l)} < \gamma$, the conclusion follows from Lemma A.9. Otherwise, Lemma A.8 combined with Lemma 4.4 imply that

$$J(e^{i\varphi}, \delta, (l,L)) + J(e^{i\varphi}, \delta, (-L,-l)) \geq C$$

for C independent of l, L, φ, and the lemma follows.

Additional References

19. Bethuel, F., *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **7** (1990), 269–286.

20. Brezis, H., Mironescu, P. and Ponce, A. C.,, *Complements to the paper "$W^{1,1}$-maps with values into S^1"*, http://www.ann.jussieu.fr/publications/2004.php3 (2004), 387–404.

(1) INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540
E-mail address: bourgain@math.ias.edu

(2) LABORATOIRE J. -L. LIONS
UNIVERSITÉ P. ET M. CURIE, B.C. 187
4 PL. JUSSIEU
75252 PARIS CEDEX 05
E-mail address: brezis@ccr.jussieu.fr

(3) RUTGERS UNIVERSITY
DEPT. OF MATH., HILL CENTER, BUSCH CAMPUS
110 FRELINGHUYSEN RD, PISCATAWAY, NJ 08854
E-mail address: brezis@math.rutgers.edu

(4) DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS-SUD
91405 ORSAY
E-mail address: Petru.Mironescu@math.u-psud.fr