Lifting, Degree, and Distributional Jacobian Revisited

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0 Introduction

Let $g : I = (0,1) \to \mathbb{S}^1$. If $g \in \mathrm{VMO}$, we may write $g = e^{i\varphi}$ for some $\varphi \in \mathrm{VMO}$; this φ is unique modulo 2π (see [13] and the earlier work [14]). There is no control of $|\varphi|_{\mathrm{BMO}}$ in terms of $|g|_{\mathrm{BMO}}$, since we always have $|g|_{\mathrm{BMO}} \le 2$ and $|\varphi|_{\mathrm{BMO}}$ can be arbitrarily large; recall, however, that, when $|g|_{\mathrm{BMO}}$ is sufficiently small, there is a linear estimate $|\varphi|_{\mathrm{BMO}} \le C|g|_{\mathrm{BMO}}$ (see [13, theorem 4], [14], and Remark 0.2 below).

We are going to establish that a norm slightly stronger than $|g|_{\mathrm{BMO}}$ *does* control $|\varphi|_{\mathrm{BMO}}$. Consider, for $1 < p < \infty$, $0 < s < 1$, the fractional Sobolev space $W^{s,p}(I)$, equipped with its standard seminorm

$$|g|_{s,p} = \left(\int_I \int_I \frac{|g(x) - g(y)|^p}{|x-y|^{1+sp}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Set

$$W^{s,p}(I;\mathbb{S}^1) = \{ g \in W^{s,p}(I;\mathbb{R}^2);\ |g| = 1 \text{ a.e.} \}.$$

Recall (see [6]) that, if $g \in W^{1/p,p}(I;\mathbb{S}^1)$, then $g = e^{i\varphi}$ for some $\varphi \in W^{1/p,p}(I;\mathbb{R})$; this φ is unique modulo 2π. Again, there is no estimate of $|\varphi|_{1/p,p}$ in terms of $|g|_{1/p,p}$. The canonical example (see [6]) is the following: let

$$\varphi_n(x) = \begin{cases} 0 & \text{if } \ 0 < x < \frac{1}{2} \\ 2n\pi(x - \frac{1}{2}) & \text{if } \ \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 2\pi & \text{if } \ x > \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Then $|\varphi_n|_{1/p,p} \to \infty$, while $|e^{i\varphi_n}|_{1/p,p} \le C$.

In view of the injection

$$W^{\frac{1}{p},p}(I) \hookrightarrow \mathrm{VMO}(I), \quad 1 < p < \infty,$$

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(see, e.g., [13, 18]), it is natural to ask whether a control of $|g|_{1/p,p}$ yields a control of $|\varphi|_{\mathrm{BMO}}$. This is indeed true:

THEOREM 0.1 *Let* $1 < p < \infty$. *Let* $\varphi \in W^{1/p,p}(I;\mathbb{R})$ *and* $g = e^{i\varphi}$. *Then*

$$|\varphi|_{\mathrm{BMO}} \le C_p\big(|g|^p_{1/p,p} + |g|_{1/p,p}\big)\,. \tag{0.1}$$

Remark 0.2. The p^{th} power growth in (0.1) is optimal when $|g|_{1/p,p}$ is large. This is easily seen by choosing $\varphi_n(x) = nx$. When $|g|_{1/p,p}$ is small, the linear growth in (0.1) is a special case of a result of [14], namely,

$$|\varphi|_{\mathrm{BMO}} \le C|g|_{\mathrm{BMO}} \quad \text{if } |g|_{\mathrm{BMO}} \le \delta\,, \tag{0.2}$$

where δ is a sufficiently small constant.

Remark 0.3. When $p = 2$, estimate (0.1) can be derived from [9, theorem 3] (announced in [7]; see also [5]), which asserts that, if $g \in H^{1/2}(I;\mathbb{S}^1)$, then we may write $g = e^{i(\varphi_1+\varphi_2)}$, with

$$|\varphi_1|_{1/2,2} \le C|g|_{1/2,2} \tag{0.3}$$

and

$$|\varphi_2|_{W^{1,1}} \le C|g|^2_{1/2,2}. \tag{0.4}$$

Since

$$|\varphi_1+\varphi_2|_{\mathrm{BMO}} \le C(|\varphi_1|_{1/2,2} + |\varphi_2|_{W^{1,1}})\,,$$

estimate (0.1) for $p = 2$ follows from (0.3)–(0.4).

Note that if Theorem 0.1 holds for some p, it also holds for every $q \in (1, p)$; this follows from (0.1) and (0.2). Hence Theorem 0.1 for $1 < p \le 2$ is a consequence of (0.3) - (0.4). The main novelty concerns the case $p > 2$; our argument relies on a completely different approach. In fact, we do not know whether (0.3)–(0.4) still hold when 2 is replaced by p:

OPEN PROBLEM 1 Let $\varphi \in C^\infty(\bar{I};\mathbb{R})$, $g = e^{i\varphi}$, and $p > 2$. Does there exist a decomposition $\varphi = \varphi_1 + \varphi_2$, with

$$|\varphi_1|_{1/p,p} \le C|g|_{1/p,p} \tag{0.3$'$}$$

and

$$|\varphi_2|_{W^{1,1}} \le C|g|^p_{1/p,p}\,? \tag{0.4$'$}$$

We are also interested in the same question when I is replaced by $(0,1)^N$.

An immediate consequence of Theorem 0.1 is the following:

COROLLARY 0.4 *Set* $Q = (0,1)^N$. *Let* $N < p < \infty$, $\varphi \in W^{N/p,p}(Q;\mathbb{R})$, *and* $g = e^{i\varphi}$. *Then*

$$|\varphi|_{\mathrm{BMO}} \le C_{p,N}\big(|g|^p_{N/p,p} + |g|_{N/p,p}\big)\,. \tag{0.5}$$

We now turn to similar questions for the degree. If $g \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$, then g has a well-defined degree; see [13]. Clearly, there is no estimate of the degree in terms of $|g|_{\mathrm{BMO}}$; however, $\deg g = 0$ provided $|g|_{\mathrm{BMO}}$ is sufficiently small; see [13]. An easy consequence of Theorem 0.1 asserts that $\deg g$ can be controlled in terms of $|g|_{1/p,p}$:

COROLLARY 0.5 *Let* $1 < p < \infty$ *and* $g \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. *Then*

$$|\deg g| \le C_p |g|^p_{1/p,p}\,. \tag{0.6}$$

When $p = 2$, estimate (0.6) was well-known: it may be easily deduced from the degree formula

$$\deg g = \frac{1}{2i\pi}\int_{\mathbb{S}^1} \frac{\dot g}{g} = \frac{1}{2i\pi}\langle \bar g, \dot g\rangle_{H^{1/2},H^{-1/2}}\,, \tag{0.7}$$

which implies that

$$|\deg g| \le C|g|^2_{1/2,2}\,.$$

Estimate (0.6) can be obtained from Theorem 0.1 as follows: set $h(t) = g(e^{it})$, $t \in \mathbb{R}$, and write $h = e^{i\varphi}$. Note that

$$|\deg g| = \frac{1}{4\pi^2}\int_0^{2\pi} |\varphi(t+2\pi) - \varphi(t)|dt \le C|\varphi|_{\mathrm{BMO}(0,4\pi)} \tag{0.8}$$

and apply Theorem 0.1 on $(0, 4\pi)$.

Corollary 0.5 extends to higher dimensions:

THEOREM 0.6 *Let* $p > N$ *and* $g \in W^{N/p,p}(\mathbb{S}^N;\mathbb{S}^N)$. *Then*

$$|\deg g| \le C_{p,N}|g|^p_{N/p,p}\,. \tag{0.9}$$

Although the conclusions of Theorems 0.1 and 0.6 are different in nature, the proofs we present below bear some similarities.

Remark 0.7. For $g \in W^{1,N}(\mathbb{S}^N;\mathbb{S}^N)$, the estimate

$$|\deg g| \le C_N \int_{\mathbb{S}^N} |\nabla g|^N \tag{0.9'}$$

is well-known and follows from Kronecker's formula

$$\deg g = \fint_{\mathbb{S}^N} \det(\nabla g) = \fint_{\mathbb{S}^N} \det(\nabla g, g) \tag{0.10}$$

(in the first integral, g is regarded as a map from $\mathbb{S}^N$ into itself and "det" denotes the determinant of an $N \times N$ matrix; in the second integral, g is considered as an $\mathbb{R}^{N+1}$-valued map, and "det" denotes the determinant of an $(N+1)\times(N+1)$ matrix).

In fact, we will use (0.10) in the proof of Theorem 0.6. It is presumably possible to rederive (0.9′) as a limiting case of (0.9) via a careful analysis of $C_{p,N}$ as $p \searrow N$, in the spirit of [8].

Estimate (0.9), which asserts that for every $p > N$,

$$|\deg g| \le C_{p,N} \int_{\mathbb{S}^N}\int_{\mathbb{S}^N} \frac{|g(x) - g(y)|^p}{|x - y|^{2N}}\, dx\, dy$$

suggests the following stronger estimate:

OPEN PROBLEM 2 Is it true that, for every $g \in C^0(\mathbb{S}^N; \mathbb{S}^N)$,

$$|\deg g| \le C_N \iint_{\{(x,y)\in\mathbb{S}^N\times\mathbb{S}^N;\, |g(x)-g(y)|>\frac{1}{10}\}} |x - y|^{-2N}\, dx\, dy\,?$$

The answer to Open Problem 2 is positive when $N = 1$; the proof is given in [10], where we also present an improvement of Theorem 0.1 in the same spirit.

We next discuss the distributional Jacobian of maps $g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$. Recall that if g is a smooth map from $\mathbb{S}^{N+1}$ into $\mathbb{R}^{N+1}$, its distributional Jacobian is defined through its action on smooth functions $\zeta \in C^\infty(\mathbb{S}^{N+1}; \mathbb{R})$ by the formula

$$\langle \mathrm{Det}(\nabla g), \zeta\rangle = - \frac{1}{N+1} \sum_{j=1}^{N+1} \int_{\mathbb{S}^{N+1}} \zeta_{x_j} \det(g_{x_1}, \dots, g_{x_{j-1}}, g, g_{x_{j+1}}, \dots, g_{x_{N+1}})\,; \tag{0.11}$$

here, the derivatives are computed pointwise in an orthonormal frame such that $(x_1, \dots, x_{N+1}, n)$ is direct, where n is the outward normal to $\mathbb{S}^{N+1}$ (this integrand is frame invariant).

Note that formula (0.11) still makes sense when

$$g \in W^{1,N}(\mathbb{S}^{N+1}; \mathbb{R}^{N+1}) \cap L^\infty$$

and $\zeta \in W^{1,\infty}(\mathbb{S}^{N+1}; \mathbb{R})$. Observe also that if $g \in C^1(\mathbb{S}^{N+1}; \mathbb{S}^N)$, then its Jacobian determinant vanishes pointwise. By density, it follows that $\mathrm{Det}(\nabla g) = 0$ for every $g \in W^{1,N+1}(\mathbb{S}^{N+1}; \mathbb{S}^N)$. On the other hand, it is standard to construct maps in $W^{1,N}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ (and even in $W^{1,q}$, $\forall q < N + 1$), e.g., with point singularities, such that $\mathrm{Det}(\nabla g) \ne 0$; see, e.g., [11].

One of the main goals of this paper is to give a meaning to the distribution $\mathrm{Det}(\nabla g)$ for maps $g : \mathbb{S}^{N+1} \to \mathbb{S}^N$ that do not necessarily belong to $W^{1,N}$. It has been observed in [7] (see also [9]) that it is possible to define $\mathrm{Det}(\nabla g)$ for $g \in H^{1/2}(\mathbb{S}^2; \mathbb{S}^1)$. The construction there was painless (using the fact that $H^{1/2}$ is the trace space of H^1). The same technique allows us to define $\mathrm{Det}(\nabla g)$ for $g \in W^{N/(N+1),N+1}(\mathbb{S}^{N+1}; \mathbb{S}^N)$. Consequently, $\mathrm{Det}(\nabla g)$ makes sense for $g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$, $N \le p \le N + 1$. In this paper, we are able to define $\mathrm{Det}(\nabla g)$ for $g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ in the more delicate case where $N + 1 < p < \infty$. The new idea involves an adaptation of the method (and the estimates) introduced in the proof of Theorem 0.6.

Our main result is the following:

THEOREM 0.8 *Let $N < p < \infty$. There exists a (unique) strongly continuous map*

$$T : W^{\frac{N}{p},p}(\mathbb{S}^{N+1};\mathbb{S}^N) \to (W^{1,\infty}(\mathbb{S}^{N+1}))^*$$

such that, for every $\zeta \in W^{1,\infty}(\mathbb{S}^{N+1};\mathbb{R})$,

$$|\langle T(g),\zeta\rangle| \le C_{p,N}|g|^p_{N/p,p}\|\nabla\zeta\|_{L^\infty} \quad \forall g \in W^{\frac{N}{p},p} \tag{0.12}$$

and

$$\langle T(g),\zeta\rangle = \langle \mathrm{Det}(\nabla g),\zeta\rangle \quad \forall g \in W^{1,N}\cap W^{\frac{N}{p},p}\,. \tag{0.13}$$

For each $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$, there are sequences $(P_i), (N_i) \subset \mathbb{S}^{N+1}$ such that

$$\sum_i |P_i - N_i| \le C_p|g|^p_{N/p,p} \tag{0.14}$$

and

$$\langle T(g),\zeta\rangle = \omega_{N+1}\sum(\zeta(P_i)-\zeta(N_i)) \quad \forall\zeta \in W^{1,\infty}(\mathbb{S}^{N+1};\mathbb{R})\,. \tag{0.15}$$

If $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)\cap C^0(\mathbb{S}^{N+1}\setminus A)$, where A is a finite set, then we may choose $P_i, N_i \in A$.

Moreover, we have

$$\langle T(g),\zeta\rangle = \omega_{N+1}\int_{\mathbb{R}} \deg(g;\Gamma_\lambda)d\lambda \quad \forall\zeta \in C^\infty(\mathbb{S}^{N+1};\mathbb{R})\,. \tag{0.16}$$

Here, ω_{N+1} is the volume of the unit ball in $\mathbb{R}^{N+1}$ and, for each regular value λ of ζ, Γ_λ is the level set $\Gamma_\lambda = \{x : \zeta(x) = \lambda\}$, positively oriented with respect to the outward normal of the open set $\{x \in \mathbb{S}^{N+1} : \zeta(x) > \lambda\}$.

Note that, for a.e. λ, $g|_{\Gamma_\lambda} \in W^{N/p,p}(\Gamma_\lambda;\mathbb{S}^N)\subset \mathrm{VMO}(\Gamma_\lambda;\mathbb{S}^N)$ so that $\deg(g;\Gamma_\lambda)$ makes sense (by [13]).

Remark 0.9.

(i) Since $W^{1,N}(\mathbb{S}^{N+1};\mathbb{S}^N)\cap W^{N/p,p}$ is dense in $W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$, $N < p < \infty$ (see the appendix), it follows that T is the unique extension of the distributional Jacobian restricted to $W^{1,N}(\mathbb{S}^{N+1};\mathbb{S}^N)\cap W^{N/p,p}$.

(ii) If $N \ge 2$, we have $W^{1,N}\cap L^\infty \subset W^{N/p,p}$, $N < p < \infty$ (see, e.g., [17]), and thus $W^{1,N}(\mathbb{S}^{N+1};\mathbb{S}^N)\cap W^{N/p,p} = W^{1,N}(\mathbb{S}^{N+1};\mathbb{S}^N)$. However, this conclusion fails when $N = 1$.

(iii) We will establish in Section 2 that $T(g)$ is "intrinsic"; more precisely, if $g \in W^{N/p,p}$, then $g \in W^{N/q,q}$ for every $q > p$, and the two definitions of $T(g)$ (relative to p and to q) coincide.

(iv) We have reached here the "largest" Sobolev classes to which one can extend the distributional Jacobian; when $sp < N$, there is no good definition of the distributional Jacobian in the class $W^{s,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$; see [10].

(v) Formula (0.15) has its source in [11] for special maps (having a finite number of singularities); the general case (0.15) is an extension of theorem 1 in [9].

1 Proofs of Theorems 0.1 and 0.6

Let $g \in \mathrm{VMO}(\mathbb{S}^N;\mathbb{S}^N)$ and let u be its harmonic extension to B^{N+1} (with values into B^{N+1}). Let $v(x,\varepsilon) = u((1-\varepsilon)x)$, $x \in \mathbb{S}^N$, $0 < \varepsilon \le 1$. We have

$$|v(x,\varepsilon)| \to 1 \quad \text{uniformly in } x \text{ as } \varepsilon \to 0\,, \tag{1.1}$$

$$|\nabla v(x,\varepsilon)| \le \frac{C}{\varepsilon} \quad \forall x \in \mathbb{S}^N \quad \text{where } C \text{ is an absolute constant} \tag{1.2}$$

(for the proof of (1.1), see [13]).

Set, for every $x \in \mathbb{S}^N$,

$$d(x) = \begin{cases} \frac{1}{2} & \text{if } |v(x,\varepsilon)| > \frac{1}{2} \text{ for every } \varepsilon \in (0,\frac{1}{2}] \\ \mathrm{Min}\{\varepsilon \in (0,\frac{1}{2}] : |v(x,\varepsilon)| \le \frac{1}{2}\} & \text{otherwise.} \end{cases}$$

In other words, $d(x) = \min(\ell(x), \frac{1}{2})$, where $\ell(x)$ is the length of the largest radial interval coming from $x \in \mathbb{S}^N$ on which $|u| \ge \frac{1}{2}$.

Clearly,

$$G = \{y \in B^{N+1} : |u(y)| \le \tfrac{1}{2}\} \subset \bigcup_{x \in \mathbb{S}^N} [0, (1-d(x))x]\,. \tag{1.3}$$

We start with the following ingredient, which is of interest in itself:

THEOREM 1.1 *For $g \in C^1(\mathbb{S}^N;\mathbb{S}^N)$, we have*

$$|\deg g| \le C I(g) \quad \textit{where } I(g) = \int_{\mathbb{S}^N} \frac{1}{(d(x))^N}\,.$$

The proof of Theorem 1.1 relies on the following:

LEMMA 1.2 *We have*

$$\int_G |\nabla u|^{N+1} \le C I(g)\,. \tag{1.4}$$

PROOF: By (1.2) and (1.3), we have

$$\begin{aligned}\int_G |\nabla u|^{N+1}\,dy &\le C \int_{\mathbb{S}^N} \left(\int_0^{1-d(x)} \frac{r^N}{(1-r)^{N+1}}\,dr\right)dx \\ &\le C \int_{\mathbb{S}^N} \left(\int_0^{1-d(x)} \frac{1}{(1-r)^{N+1}}\,dr\right)dx = C' I(g)\,.\end{aligned}$$

□

PROOF OF THEOREM 1.1: Set, for $y \in B^{N+1}$,

$$\tilde{u}(y) = \begin{cases} \frac{u(y)}{|u(y)|} & \text{if } |u(y)| > \frac{1}{2} \\ 2u(y) & \text{if } |u(y)| \le \frac{1}{2}. \end{cases}$$

Note that $\tilde{u} = g$ on $\mathbb{S}^N$ and thus, by Kronecker's formula (0.10), we have

$$\deg g = \fint_{\mathbb{S}^N} \det(\nabla g) = \fint_{B^{N+1}} \det(\nabla \tilde{u}) \,.$$

(To prove the last equality, consider the vector field

$$D = (D_1, \ldots, D_{N+1})$$

where

$$D_j = \det(\tilde{u}_{x_1}, \ldots, \tilde{u}_{x_{j-1}}, \tilde{u}, \tilde{u}_{x_{j+1}}, \ldots, \tilde{u}_{x_{N+1}}) \,.$$

Clearly, we have

$$\operatorname{div} D = (N+1) \det(\nabla \tilde{u})$$

and thus

$$\fint_{B^{N+1}} \det(\nabla \tilde{u}) = \frac{(N+1)^{-1}}{|B_{N+1}|} \int_{\mathbb{S}^N} D \cdot \nu \,,$$

where ν is the outward normal to $\mathbb{S}^N$. On the other hand, it is easy to see that $D \cdot \nu = \det(\nabla g)$, where the $N \times N$ Jacobian determinant $\det(\nabla g)$ is computed with respect to any orthonormal frame in the tangent space to $\mathbb{S}^N$ at x and in the tangent space to $\mathbb{S}^N$ at $g(x)$.)

Since $|\tilde{u}(y)| = 1$ on $B^{N+1} \setminus G$ we have $\det(\nabla \tilde{u}) = 0$ on $B^{N+1} \setminus G$ and thus

$$\deg g = \frac{1}{|B^{N+1}|} \int_G \det(\nabla \tilde{u}) = \frac{2^{N+1}}{|B^{N+1}|} \int_G \det(\nabla u) \,.$$

Hence

$$|\deg g| \le C \int_G |\nabla u|^{N+1} \le C' I(g) \quad \text{by Lemma 1.2.}$$

(There is an alternative proof of the first inequality above using differential forms. As is well-known

$$\deg g = \deg(u, B^{N+1}, 0) \,.$$

The latter can be given as the integral of the pullback, under the map u, of any smooth $(N+1)$–form μ, with compact support in the open ball B^{N+1}, and whose integral is 1. Take $\mu = h(z)dz$, where h is any smooth function with support in $\{z \in \mathbb{R}^{N+1} : |z| < \frac{1}{2}\}$ and whose integral is 1. Then we find

$$\deg g = \deg(u, B^{N+1}, 0) = \int_{B^{N+1}} h(u(y)) \det(\nabla u(y)) dy,$$

which yields the desired estimate.) □

In the proof of Theorem 0.6 we will also use the following:

LEMMA 1.3 *Let $p > N$, $g \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$. Then*

$$\int_{\mathbb{S}^N} \frac{1}{(d(x))^N} \leq C\big(|g|^p_{N/p,p} + 1\big) . \tag{1.5}$$

PROOF: It suffices to consider only the x's such that $d(x) < \frac{1}{2}$. For any such x, we have

$$\begin{aligned}\frac{1}{2} \leq |u((1-d(x))x) - g(x)| &\leq d(x)^{\frac{N}{p}} |v|_{C^{0,N/p}(\{x\}\times(0,\frac{1}{2}))} \\ &\leq C\, d(x)^{\frac{N}{p}} |v|_{\frac{N+1}{p},p(\{x\}\times(0,\frac{1}{2}))} ,\end{aligned}$$

by the embedding $W^{s,p}(0,1) \subset C^{0,\alpha}(0,1)$ where $sp > 1$ and $\alpha = s - \frac{1}{p}$. Thus

$$\frac{1}{d(x)^N} \leq C|v|^p_{(N+1)/p,p(\{x\}\times(0,1/2))} . \tag{1.6}$$

Let, for f defined on B^{N+1} and $x \in \mathbb{S}^N$, $f^x(r) = f(rx)$, $\frac{1}{2} \leq r \leq 1$. Recall the Besov-type inequality (see, e.g., [1, pp. 208–214])

$$\int_{\mathbb{S}^N} |f^x|^p_{s,p(1/2,1)}\, dx \leq C|f|^p_{s,p(B^{N+1})} \quad \forall f \in W^{s,p}(B^{N+1}) . \tag{1.7}$$

Inequality (1.5) follows by combining (1.6) and (1.7) with the standard estimate $|v|_{(N+1)/p,p(\mathbb{S}^N\times(0,1/2))} \leq C|u|_{(N+1)/p,p} \leq C|g|_{N/p,p}$. □

PROOF OF THEOREM 0.6: We want to show that for every $g \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$

$$|\deg g| \leq C|g|^p_{N/p,p} . \tag{1.8}$$

By density of $C^1(\mathbb{S}^N; \mathbb{S}^N)$ in $W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$ and continuity of the degree under VMO convergence, it suffices to prove (1.8) for $g \in C^1(\mathbb{S}^N; \mathbb{S}^N)$. When $|g|_{N/p,p}$ is sufficiently small, we have $\deg g = 0$, once more by continuity of the degree under VMO convergence, and thus (1.8) holds. Otherwise, (1.8) follows from Theorem 1.1 and Lemma 1.3. □

PROOF OF THEOREM 0.1: We will prove that

$$|\varphi|_{\mathrm{BMO}(I)} \leq C\big(|g|^p_{1/p,p} + |g|_{1/p,p}\big) . \tag{1.9}$$

As above, we may assume that g is smooth. When $|g|_{1/p,p}$ is sufficiently small, (1.9) follows from the estimate

$$|\varphi|_{\mathrm{BMO}(I)} \leq C|g|_{\mathrm{BMO}(I)} \quad \text{if } |g|_{\mathrm{BMO}(I)} \leq \delta$$

(δ a small constant) of Coifman and Meyer [14]. In view of this and scale invariance, it suffices to establish the following weaker form of (1.9)

$$\fint_I \fint_I |\varphi(x) - \varphi(y)| \le C\big(|g|^p_{1/p,p} + 1\big)\,. \tag{1.10}$$

Extending g by symmetry, we may always assume g and φ are periodic and thus defined on a circle (with g of degree 0). We will prove that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} |\varphi(x_1) - \varphi(x_2)|dx_1\,dx_2 \le C\big(|g|^p_{1/p,p} + 1\big)\,, \tag{1.11}$$

where $\varphi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$ and $g = e^{i\varphi}$. As in the proof of Lemma 1.2, and by Lemma 1.3, we have

$$\int_{\{y=rx:r\le 1-d(x)\}} |\nabla u|^2\,dy \le C\big(|g|^p_{1/p,p} + 1\big)\,. \tag{1.12}$$

By the co-area formula, (1.3), and (1.12), we have

$$\begin{aligned}\int_{\frac13}^{\frac12}\Bigg(\int_{\{y\in B^2:|u(y)|=t\}} |\nabla u|\Bigg)dt &= \int_{\{y\in B^2:\frac13<|u(y)|<\frac12\}} |\nabla u|\,|\nabla|u||\\ &\le \int_G |\nabla u|^2 \le C\big(|g|^p_{1/p,p} + 1\big)\,.\end{aligned}$$

Thus we may find some regular value $t \in (\frac13, \frac12)$ of $|u|$ such that

$$\int_\Gamma |\nabla u| \le C\big(|g|^p_{1/p,p} + 1\big)\,, \tag{1.13}$$

where $\Gamma = \{y : |u(y)| = t\}$. Let $\gamma_1, \gamma_2, \dots,$ be the connected components of Γ. By (1.13), we have

$$\sum_j |\deg(u,\gamma_j)| \le \frac{1}{2\pi t}\sum_j \int_{\gamma_j} |\nabla u| \le C\big(|g|^p_{1/p,p} + 1\big)\,. \tag{1.14}$$

On the other hand, if $j \ne k$, then the domains enclosed by γ_j and γ_k have disjoint interiors, by the maximum principle.

Let now $x, y \in \mathbb{S}^1$ and consider the domains

$$U = \{z : |u(z)| > t\}\,, \qquad V \text{ as in Figure 1.1 and } \tilde W = U \cap V\,.$$

Let W be the connected component of $\tilde W$ whose boundary contains x and y. Since ∂U is a finite union of analytic curves, ∂W will generically be a finite union of segments and curves contained in Γ:

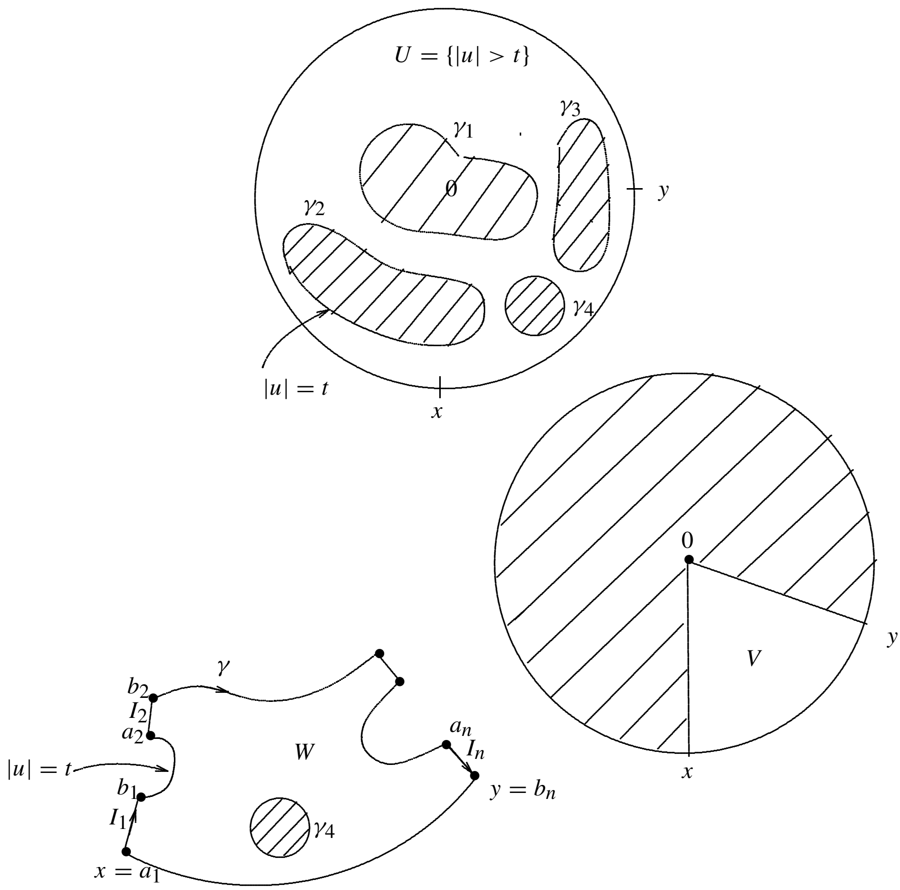

FIGURE 1.1

Let γ be the arc from x to y as in Figure 1.1. Let

$$h : U \to \mathbb{S}^1, \quad h(z) = \frac{u(z)}{|u(z)|}.$$

Since $u \in W^{2/p,p}$, we have $h \in W^{2/p,p}$. Next we note that, since $g \in W^{1/p,p} \cap L^\infty$, it suffices to establish (1.10) for $p \geq 2$. Assuming $p \geq 2$, we have $|h|_{2/p,p} \leq C|u|_{2/p,p} \leq C|g|_{1/p,p}$. Clearly, it suffices to prove that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} |\varphi(x) - \varphi(y)| dx\, dy \leq C\left(|v|^p_{2/p,p(\mathbb{S}^N \times (0,1/2))} + |g|^p_{1/p,p} + 1\right). \tag{1.15}$$

Let ψ be the lifting of h on γ such that $\psi(x) = \varphi(x)$. Then

$$\varphi(y) - \varphi(x) = \psi(y) - \psi(x) \pm 2\pi \sum \deg(u, \gamma_j),$$

where the above summation is done over the j's such that $\gamma_j \subset W$. By (1.14), we have

$$|\varphi(y) - \varphi(x)| \leq |\psi(y) - \psi(x)| + C\big(|g|^p_{1/p,p} + 1\big)\,. \tag{1.16}$$

We next note that if $\tilde{\gamma}$ is an arc on $\gamma \cap \Gamma$ with endpoints a and b, then

$$|\psi(b) - \psi(a)| \leq \frac{1}{t} \int_{\tilde{\gamma}} |\nabla u|\,. \tag{1.17}$$

We write

$$\gamma = I_1 \cup \tilde{\gamma}_1 \cup I_2 \cup \cdots \cup I_n\,,$$

where $I_1, \ldots, I_n$ are line segments, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n-1}$ are on $\gamma \cap \Gamma$, I_1 has endpoints $a_1 = x$ and b_1, $\tilde{\gamma}_1$ has endpoints b_1 and a_2, etc. By (1.13), (1.16), and (1.17), we find that

$$|\psi(y) - \psi(x)| \leq C\big(|g|^p_{1/p,p} + 1\big) + \sum_1^n |\psi(b_j) - \psi(a_j)|\,. \tag{1.18}$$

We estimate the terms $|\psi(b_1) - \psi(a_1)|$ and $|\psi(b_n) - \psi(a_n)|$ in (1.18) with the help of the following lemma:

LEMMA 1.4 *Let* $\psi \in C^{0,\alpha}((0,l);\mathbb{R})$ *with* $0 < \alpha \leq 1$ *and set* $h = e^{i\psi}$. *Then*

$$|\psi(l) - \psi(0)| \leq 4\big(l|h|^{1/\alpha}_{C^{0,\alpha}} + l^\alpha |h|_{C^{0,\alpha}}\big)\,. \tag{1.19}$$

PROOF OF LEMMA 1.4: After scaling, we may always take $l = 1$. Suppose first that $|h|_{C^{0,\alpha}} \leq 1$. Then, clearly,

$$|\psi(1) - \psi(0)| \leq 2|h(1) - h(0)| \leq 2|h|_{C^{0,\alpha}}$$

and the desired conclusion follows.

When $|h|_{C^{0,\alpha}} > 1$, let n be the integer part of $|h|^{1/\alpha}_{C^{0,\alpha}} + 1$. For $j = 0, \ldots, n$, set $a_j = \frac{j}{n}$. Since

$$|h(a_{j+1}) - h(a_j)| \leq |h|_{C^{0,\alpha}} \left(\frac{1}{n}\right)^\alpha \leq 1\,,$$

we deduce as above that

$$\begin{aligned}|\psi(a_{j+1}) - \psi(a_j)| &\leq 2|h(a_{j+1}) - h(a_j)| \\ &\leq 2|h|_{C^{0,\alpha}} \left(\frac{1}{n}\right)^\alpha, \quad j = 0, \ldots, n-1.\end{aligned}$$

Summing these inequalities for $j = 0, \ldots, n-1$, we find

$$|\psi(1) - \psi(0)| \leq 2|h|_{C^{0,\alpha}} n^{1-\alpha} \leq 4|h|^{1/\alpha}_{C^{0,\alpha}}\,,$$

since $n \leq |h|^{1/\alpha}_{C^{0,\alpha}} + 1 \leq 2|h|^{1/\alpha}_{C^{0,\alpha}}$; this is again the desired conclusion. □

Now, using Lemma 1.4, the one-dimensional embedding $W^{2/p,p} \hookrightarrow C^{0,1/p}$, and the inequality

$$|\nabla u(y)| \le C \quad \text{if } |y| \le \frac{1}{2}\,, \tag{1.20}$$

we find that

$$\begin{aligned}|\psi(b_1) - \psi(a_1)| + |\psi(b_n) - \psi(a_n)| \le \\ C(|v|^p_{2/p,p(\{x\}\times(0,1/2))} + |v|^p_{2/p,p(\{y\}\times(0,1/2))} + 1)\,.\end{aligned} \tag{1.21}$$

The ingredient for estimating the terms $|\psi(b_j) - \psi(a_j)|$, $j = 2, \dots, n-1$, is the inequality

$$|\psi(b_j) - \psi(a_j)| = \left| \int_{[a_j,b_j]} \bar{h}\frac{\partial h}{\partial \tau} \right| \le C \int_{[a_j,b_j]} |\nabla u|\,. \tag{1.22}$$

Estimate (1.22), used in conjunction with (1.20), yields

$$\sum_2^{n-1} |\psi(b_j) - \psi(a_j)| \le C\Bigg(\int_{\{rx:\frac{1}{2}\le r\le 1-d(x)\}} |\nabla u| + \int_{\{ry:\frac{1}{2}\le r\le 1-d(y)\}} |\nabla u| + 1\Bigg)\,. \tag{1.23}$$

By (1.18), (1.21), and (1.23), we find that

$$\begin{aligned}|\varphi(x) - \varphi(y)| \le C\Bigg(&\int_{\{rx:\frac{1}{2}\le r\le 1-d(x)\}} |\nabla u| \\ &+ \int_{\{ry:\frac{1}{2}\le r\le 1-d(y)\}} |\nabla u| + |g|^p_{1/p,p} + |v|^p_{2/p,p(\{x\}\times(0,1/2))} \\ &+ |v|^p_{2/p,p(\{y\}\times(0,1/2))} + 1\Bigg)\,.\end{aligned} \tag{1.24}$$

The conclusion follows, with the help of (1.7) and (1.12), by integrating (1.24). □

PROOF OF COROLLARY 0.4: Recall that we want to obtain the estimate

$$|\varphi|_{\mathrm{BMO}} \le C\big(|g|^p_{N/p,p} + |g|_{N/p,p}\big)\,. \tag{1.25}$$

When $|g|_{N/p,p}$ is small, the conclusion follows from [13, theorem 4]. Otherwise, assume, e.g., $N = 2$. It suffices (after scaling) to prove that

$$J = \iint_{(0,1)^2\times(0,1)^2} |\varphi(x) - \varphi(y)| \le C\big(|g|^p_{2/p,p} + 1\big)\,. \tag{1.26}$$

This follows from

$$|\varphi(x) - \varphi(y)| \le |\varphi(x_1, x_2) - \varphi(x_1, y_2)| + |\varphi(x_1, y_2) - \varphi(y_1, y_2)|\,, \tag{1.27}$$

which, combined with Theorem 0.8, yields

$$J \le C\left(1+\int |g|^p_{1/p,p(\{s\}\times[0,1])}\,ds+\int |g|^p_{1/p,p([0,1]\times\{t\})}\,dt\right) \le C\left(|g|^p_{2/p,p}+1\right). \tag{1.28}$$

□

2 Proof of Theorem 0.8

We want to prove that the distribution $\mathrm{Det}(\nabla g)$, initially defined in (0.11) for $g \in W^{1,N}(\mathbb{S}^{N+1};\mathbb{S}^N)$, makes sense for $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$, $N < p < \infty$, and satisfies (0.12)–(0.16). The strategy of the proof is the following:

(i) we define $\langle T(g),\zeta\rangle$ for a general $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$ via an integral formula;

(ii) with T defined in (i), we prove that (0.12) holds and that the map $g \mapsto T(g)$ is strongly continuous from $W^{N/p,p}$ into $(W^{1,\infty})^*$;

(iii) we establish (0.13);

(iv) we note that (0.14)–(0.16) hold for some special g's; for a general $g \in W^{N/p,p}$, (0.14)–(0.16) will be obtained by density.

2.1 Step 1: Definition of $T(g)$, Continuity of $T(g)$, and Proof of (0.12)

The definition of $T(g)$ relies on a formula that is in the same spirit as the one presented in [9] for maps in $H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$. Let us start with a smooth map $g : \mathbb{S}^{N+1} \to \mathbb{R}^{N+1}$ and a Lipschitz function $\zeta : \mathbb{S}^{N+1} \to \mathbb{R}$. Let F be any smooth extension of g to B^{N+2} (with values into $\mathbb{R}^{N+1}$), and let ξ be any Lipschitz extension of ζ to B^{N+2}. Set

$$X(F,\xi) = \sum_{j=1}^{N+2}\int_{B^{N+2}} H_j \xi_{x_j}\,, \tag{2.1}$$

where $H = (H_1,\dots,H_{N+2})$ and

$$H_j = (-1)^{N+j} F_{x_1}\wedge\cdots\wedge F_{x_{j-1}}\wedge F_{x_{j+1}}\wedge\cdots\wedge F_{x_{N+2}}\,. \tag{2.2}$$

It is easy to see that $\operatorname{div} H = 0$, that X depends only on g and ζ, and (after a number of integration by parts) that

$$X(F,\xi) = \langle \mathrm{Det}(\nabla g),\zeta\rangle\,. \tag{2.3}$$

In the case $N = 1$ and $g \in H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$, we took in [9] an *arbitrary* extension $F \in H^1(B^3;\mathbb{R}^2)$ of g; then the corresponding H given by (2.2) belongs to L^1. Consequently, formula (2.3) allows to define $\mathrm{Det}(\nabla g) \in (W^{1,\infty})^*$ for every $g \in H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$. We may still use the same technique when $g \in W^{N/(N+1),N+1}(\mathbb{S}^{N+1};\mathbb{S}^N)$. However, this method does not seem to work when $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$ and $p > N+1$. In this case, we are going to choose a *special* extension F of g such that:

(i) $F \in C^\infty(B^{N+2};\mathbb{R}^{N+1})$,

(ii) $F \in W^{(N+1)/p,p}(B^{N+2})$, and

(iii) H (defined by (2.2)) belongs to L^1.

For every $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$, let u be the harmonic extension of g to B^{N+2} (with values into B^{N+1}).

(*Warning*: Here, g need not be VMO, in contrast with the situation we encountered in the proofs of Theorems 0.1 and 0.6. In general, $|u(y)|$ does *not* tend to 1 as $|y| \to 1$ and the set $\{y \in \overline{B^{N+2}} : |u(y)| \le \frac{1}{2}\}$ is *not* a compact subset of the open ball B^{N+2}. This will become particularly transparent later on at the points of $\mathbb{S}^{N+1}$ where g has topological singularities.)

Fix any map $\Phi \in C^\infty(\mathbb{R}^{N+1};\mathbb{R}^{N+1})$ such that $\Phi(X) = X/|X|$ if $|X| \ge \frac{1}{2}$. The *special* F we will use is defined by

$$F(y) = \Phi(u(y)) \quad \forall y \in B^{N+2}. \tag{2.4}$$

Note that $F \in C^\infty(B^{N+2}; B^{N+1})$ and that $F(y) \in \mathbb{S}^N$ when $|u(y)| \ge \frac{1}{2}$. Consider the vector field H defined by (2.2) for this F and observe that $H = 0$ in the open set $\{y \in B^{N+2} : |u(y)| > \frac{1}{2}\}$.

For every $\xi \in W^{1,\infty}(B^{N+2};\mathbb{R})$, define

$$Y(\xi) = X(F,\xi) \tag{2.5}$$

as in (2.1)–(2.2). This requires a justification, since it is not clear that $H \in L^1$. A key ingredient in the proof of Theorem 0.8 is the following:

LEMMA 2.1 *For each $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$, we have $H \in L^1(B^{N+2};\mathbb{R}^{N+2})$, so that the quantity $Y(\xi)$ is well-defined. Moreover*:

(i) *$Y(\xi_1) = Y(\xi_2)$ when $\xi_1 = \xi_2$ on $\mathbb{S}^{N+1}$.*

(ii) *Set $\langle T(g),\zeta\rangle = Y(\xi)$, where ξ is any Lipschitz extension of a given $\zeta \in W^{1,\infty}(\mathbb{S}^{N+1};\mathbb{R})$. Then*

$$|\langle T(g),\zeta\rangle| \le C|g|^p_{N/p,p}\|\nabla\zeta\|_{L^\infty} \quad \forall \zeta \in W^{1,\infty}(\mathbb{S}^{N+1};\mathbb{R}). \tag{2.6}$$

(iii) *The map $g \mapsto T(g)$ is strongly continuous from $W^{N/p,p}$ into $(W^{1,\infty}(\mathbb{S}^{N+1}))^*$.*

PROOF: We start by proving that $H \in L^1$. Assume first that $N < p \le N+1$. Then u (the harmonic extension of g) belongs to $W^{(N+1)/p,p}(B^{N+2}) \cap L^\infty$, and thus to $W^{1,N+1}$. Therefore, with our choice of F, we have $H \in L^1$. Moreover, in this case, the map $g \mapsto H \in L^1$ is clearly continuous, so that (iii) follows (provided we establish (i)).

Assume next that $p > N+1$. In the open set $\{y \in B^{N+2} : |u(y)| > \frac{1}{2}\}$, F is $\mathbb{S}^N$-valued, and thus $H = 0$ pointwise. Therefore,

$$\int_{B^{N+2}} |H| = \int_{\{y:|u(y)|\le\frac{1}{2}\}} |H|.$$

Clearly, $|\nabla F| \leq C|\nabla u|$ and therefore $|H| \leq C|\nabla u|^{N+1}$. By the proof of Lemma 1.2, we have

$$\int_{\{y:|u(y)|\leq\frac{1}{2}\}} |H| \leq C \int_{\{y:|u(y)|\leq\frac{1}{2}\}} |\nabla u|^{N+1} \leq C \int_{\mathbb{S}^{N+1}} \frac{1}{(d(x))^N},$$

where $d(x)$ is defined as in Section 1. □

By the proof of Lemma 1.3, we further obtain that

$$\int_{\mathbb{S}^{N+1}} \frac{1}{(d(x))^N} \leq C\big(|g|^p_{N/p,p} + 1\big),$$

and thus

$$\int_{\{y:|u(y)|\leq\frac{1}{2}\}} |H| \leq C\big(|g|^p_{N/p,p} + 1\big).$$

Hence $H \in L^1$ and consequently $Y(\xi)$ is well-defined.

We now turn to the proof of (i). Let $\xi_1, \xi_2 \in W^{1,\infty}(B^{N+2};\mathbb{R})$ be such that $\xi_1 = \xi_2$ on $\mathbb{S}^{N+1}$ and set $\eta = \xi_1 - \xi_2 \in W_0^{1,\infty}(B^{N+2})$. Consider a sequence $(\eta_j) \subset C_c^\infty(B^{N+2})$ such that $\nabla\eta_j \to \nabla\eta$ a.e. and $\|\nabla\eta_j\|_{L^\infty} \leq C$. Since $\operatorname{div} H = 0$, we clearly have $\int_{B^{N+2}} H \cdot \nabla\eta_j = 0\ \forall j$, and thus $\int_{B^{N+2}} H \cdot \nabla\eta = 0$.

We next establish (ii). It suffices to estimate $\langle T(g), \zeta\rangle$ when

$$\int_{\mathbb{S}^{N+1}} \zeta = 0. \tag{2.7}$$

In view of (2.7), we may find an extension ξ of ζ to B^{N+2} such that

$$\|\nabla\xi\|_{L^\infty} \leq C\|\nabla\zeta\|_{L^\infty} \tag{2.8}$$

and

$$\operatorname{Supp}\xi \subset \left\{y \in \overline{B^{N+2}} : |y| \geq \frac{1}{2}\right\}. \tag{2.9}$$

For such a ξ, we have

$$|\langle T(g), \zeta\rangle| \leq \int_{B^{N+2}} |H|\,|\nabla\xi| \leq C\|\nabla\zeta\|_{L^\infty} \int_{\{y:|y|\geq\frac{1}{2}\text{ and }|u(y)|\leq\frac{1}{2}\}} |\nabla u|^{N+1}. \tag{2.10}$$

Going back to the proofs of Lemmas 1.2 and 1.3, we see that

$$\int_{\{y:|y|\geq\frac{1}{2}\text{ and }|u(y)|\leq\frac{1}{2}\}} |\nabla u|^{N+1} \leq C|g|^p_{N/p,p}, \tag{2.11}$$

so that (ii) is a consequence of (2.10) and (2.11).

Finally, we prove (iii). As we already observed, it suffices to consider the case $p > N+1$. Let $g_n, g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ be such that $g_n \to g$ in $W^{N/p,p}$ and let H_n and H be the corresponding vector fields. We claim that

$$\int_{B^{N+2}} |H_n - H| \to 0\,. \tag{2.12}$$

By the uniqueness of the limit, it suffices to establish (2.12) for a subsequence. With u_n and u the corresponding harmonic extensions, we have $u_n \to u$ in $C^\infty(B^{N+2})$ and in $W^{(N+1)/p,p}$. For $x \in \mathbb{S}^{N+1}$ and $t \in I = (0, \frac{1}{2})$, set

$$v_n(x,t) = u_n((1-t)x) \quad \text{and} \quad v(x,t) = u((1-t)x)\,.$$

In view of (1.7), we know that

$$v_n \to v \quad \text{in } L^p(\mathbb{S}^{N+1}; W^{s,p}(I))$$

where $s = (N+1)/p$. Passing to a subsequence (still denoted by v_n) we obtain a function $K \in L^1(\mathbb{S}^{N+1})$ such that

$$|v_n(x,\cdot)|^p_{s,p(I)} \le K(x) \quad \forall n \text{ and a.e. } x \in \mathbb{S}^{N+1}\,. \tag{2.13}$$

As in the proof of Lemma 1.3 we find, using (2.13),

$$\frac{1}{d_n(x)^N} \le CK(x) \quad \forall n \text{ and a.e. } x \in \mathbb{S}^{N+1}\,, \tag{2.14}$$

(where d_n, corresponding to g_n, is defined as in Section 1). Next we have (using (1.2) and (1.3))

$$|H_n(rx)| \le \begin{cases} 0 & \text{if } 1 - d_n(x) < r < 1 \\ \dfrac{C}{(1-r)^{N+1}} & \text{if } 0 \le r < 1. \end{cases} \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$|H_n(y)| \le M(y) \quad \forall y \in B^{N+2} \tag{2.16}$$

for some $M \in L^1$. Since clearly $H_n \to H$ in $C^\infty(B^{N+2})$, (2.12) follows from inequality (2.16).

2.2 Step 2: Proof of (0.13)

As we already observed, we may still define $T(g)$ if $g \in W^{N/(N+1),N+1}(\mathbb{S}^{N+1}; \mathbb{R}^{N+1})$ (note that here g need not be $\mathbb{S}^N$-valued). Indeed, for such a g, we have $u \in W^{1,N+1}(B^{N+2}; \mathbb{R}^{N+1})$ and thus $H \in L^1$. Similarly, the definition (0.11) of $\mathrm{Det}(\nabla g)$ still makes sense for $g \in W^{1,N}(\mathbb{S}^{N+1}; \mathbb{R}^{N+1}) \cap L^\infty$. An easy adaptation of the proof of lemma 1 in [9] yields, in $(W^{1,\infty})^*$, the equality

$$\mathrm{Det}(\nabla g) = T(g) \quad \forall g \in W^{1,N}(\mathbb{S}^{N+1}; \mathbb{R}^{N+1}) \cap W^{\frac{N}{N+1},N+1} \cap L^\infty\,. \tag{2.17}$$

This completes the proof of (0.13) when $N \ge 2$. Indeed, if $N \ge 2$ we have $W^{1,N}(\mathbb{S}^{N+1}; \mathbb{S}^N) \subset W^{N/p,p}$, $\forall p > N$, so that (0.13) is a special case of (2.17).

We now turn to the proof of (0.13) when $N = 1$, i.e.,

$$\operatorname{Det}(\nabla g) = T(g) \quad \forall p > 1,\ \forall g \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1) \cap W^{\frac{1}{p},p}\,. \tag{2.18}$$

It is useful to introduce the class

$$\mathcal{R} = \big\{ g \in W^{1,q}(\mathbb{S}^{N+1}; \mathbb{S}^N) \text{ for every } 1 \le q < N+1; \\ g \in C^\infty(\mathbb{S}^{N+1} \setminus A) \text{ for some finite set } A \big\}\,.$$

Note that every $g \in \mathcal{R}$ belongs to $W^{1,N}$ and also to $W^{N/(N+1),N+1}$. Thus (2.17) holds for every $g \in \mathcal{R}$.

Equality (2.18) follows from

- Lemma 2.2 below,
- (2.17) applied to $g \in \mathcal{R}$,
- the continuity of $g \mapsto T(g)$ from $W^{1/p,p}(\mathbb{S}^2; \mathbb{S}^1)$ into $(W^{1,\infty})^*$, and
- the continuity of $g \mapsto \operatorname{Det}(\nabla g)$ from $W^{1,1}(\mathbb{S}^2; \mathbb{S}^1)$ into $(W^{1,\infty})^*$ (which is obvious from (0.11)).

LEMMA 2.2 *Let $p > 1$. For every $g \in W^{1,1}(\mathbb{S}^2; \mathbb{S}^1) \cap W^{1/p,p}$, there is a sequence $(g_n) \subset \mathcal{R}$ such that $g_n \to g$ in $W^{1,1}$ and in $W^{1/p,p}$.*

The proof of Lemma 2.2 is given in the appendix.

2.3 Step 3: Proof of (0.14)–(0.16)

The proof of (0.14)–(0.15) is a straightforward adaptation—left to the reader—of the proof of theorem 1 in [9]. It relies on four facts:

- $\mathcal{R}$ is dense in $W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ (see the appendix).
- $g \mapsto T(g)$ is continuous from $W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ into $(W^{1,\infty})^*$.
- The following equality holds:

$$\operatorname{Det}(\nabla g) = T(g) = \omega_{N+1} \sum_{\text{finite}} d_a \delta_a \quad \forall g \in \mathcal{R}\,, \tag{2.19}$$

  where ω_{N+1} is the volume of the unit ball in $\mathbb{R}^{N+1}$ and d_a denotes the degree of g restricted to a small sphere around a in $\mathbb{S}^{N+1}$ (with appropriate orientation). Equality (2.19) is proven as in [9, lemma 2].
- If $g, h \in \mathcal{R}$ and we write

$$\operatorname{Det}(\nabla g) - \operatorname{Det}(\nabla h) = \omega_{N+1} \sum_{a \in A} d_a \delta_a\,, \tag{2.20}$$

  then (see [11])

$$\|\operatorname{Det}(\nabla g) - \operatorname{Det}(\nabla h)\|_{(W^{1,\infty})^*} = \omega_{N+1} L\,, \tag{2.21}$$

  where

$$L = \operatorname*{Min}_{\sigma \in S_k} \sum_{i=1}^{k} d(P_i, N_{\sigma(i)})\,; \tag{2.22}$$

here P_i and N_i are the points $a \in A$ repeated according to their multiplicity and d is the geodesic distance on $\mathbb{S}^{N+1}$.

The proof of (0.16) relies on the following variant of [12, theorem 4]:

LEMMA 2.3 *Let $g, h \in \mathcal{R}$. Then, for $\zeta \in C^\infty(\mathbb{S}^{N+1}; \mathbb{R})$, we have*

$$\int |\deg(g;\Gamma_\lambda) - \deg(h;\Gamma_\lambda)| d\lambda \le \frac{1}{\omega_{N+1}} \|\nabla\zeta\|_{L^\infty} \|\mathrm{Det}(\nabla g) - \mathrm{Det}(\nabla h)\|_{(W^{1,\infty})^*} .$$

PROOF: Let $g, h \in \mathcal{R}$. Assume that

$$T(g) = \omega_{N+1} \sum_{i=1}^{I} (\delta_{P_i} - \delta_{N_i}) , \quad T(h) = \omega_{N+1} \sum_{j=1}^{J} (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) .$$

If λ is a regular value of ζ such that $\zeta(P_i) \neq \lambda$, $\zeta(N_i) \neq \lambda$, $\zeta(\tilde{P}_j) \neq \lambda$, and $\zeta(\tilde{N}_j) \neq \lambda$, for every i and j, then

$$\deg(g;\Gamma_\lambda) = \mathrm{card}\{1 \le i \le I : \zeta(P_i) > \lambda\} - \mathrm{card}\{1 \le i \le I : \zeta(N_i) > \lambda\} ,$$

so that, clearly,

$$\deg(g;\Gamma_\lambda) = \frac{1}{2} \sum_{i=1}^{I} \big(\mathrm{sgn}(\zeta(P_i) - \lambda) - \mathrm{sgn}(\zeta(N_i) - \lambda)\big) . \tag{2.23}$$

It follows from (2.23) that

$$\deg(g;\Gamma_\lambda) - \deg(h;\Gamma_\lambda) = \frac{1}{2} \sum_{k=1}^{I+J} \big(\mathrm{sgn}(\zeta(P_k^*) - \lambda) - \mathrm{sgn}(\zeta(N_k^*) - \lambda)\big) , \tag{2.24}$$

where the sets $\{P_i\} \cup \{\tilde{N}_j\}$ and $\{N_i\} \cup \{\tilde{P}_j\}$ are now labeled as $\{P_k^*\}$ and $\{N_k^*\}$, respectively. Assume, e.g., that the length of the minimal connection in (2.22) is given by $L = \sum_{k=1}^{I+J} d(P_k^*, N_k^*)$, and let γ_k be a geodesic from P_k^* to N_k^* $\forall k$. Since clearly

$$\frac{1}{2} \big| (\mathrm{sgn}(\zeta(P_k^*) - \lambda) - \mathrm{sgn}(\zeta(N_k^*) - \lambda) \big| \le \mathrm{card}\{x \in \gamma_k : \zeta(x) = \lambda\} ,$$

we find, using the area formula and (2.22), that

$$
\begin{aligned}
\int |\deg(g;\Gamma_\lambda) - \deg(h;\Gamma_\lambda)|d\lambda
&\le \sum_k \int \operatorname{card}\{x \in \gamma_k : \zeta(x) = \lambda\} d\lambda \\
&= \sum_k \int_{\gamma_k} \left|\frac{\partial \zeta}{\partial \tau}\right| \\
&\le L \|\nabla \zeta\|_{L^\infty} \\
&= \frac{1}{\omega_{N+1}} \|\nabla \zeta\|_{L^\infty} \|\operatorname{Det}(\nabla g) - \operatorname{Det}(\nabla h)\|_{(W^{1,\infty})^*} .
\end{aligned}
\tag{2.25}
$$

□

PROOF OF (0.16): As in [9], we have

$$
\langle \operatorname{Det}(\nabla g), \zeta \rangle = \int_{\mathbb{R}} \deg(g;\Gamma_\lambda) d\lambda \quad \forall \zeta \in C^\infty(\mathbb{S}^{N+1};\mathbb{R}),\ \forall g \in \mathcal{R} . \tag{2.26}
$$

Let $g \in W^{N/p,p}(\mathbb{S}^{N+1};\mathbb{S}^N)$ and let $(g_n) \subset \mathcal{R}$ be such that $g_n \to g$ in $W^{N/p,p}$ and

$$
\sum_n \|\operatorname{Det}(\nabla g_{n+1}) - \operatorname{Det}(\nabla g_n)\|_{(W^{1,\infty})^*} < \infty .
$$

By Lemma 2.3 we have, for a fixed $\zeta \in C^\infty(\mathbb{S}^{N+1};\mathbb{R})$,

$$
\sum_n \int_{\mathbb{R}} |\deg(g_{n+1};\Gamma_\lambda) - \deg(g_n;\Gamma_\lambda)| d\lambda < \infty . \tag{2.27}
$$

On the other hand, passing to a subsequence, we have, for a.e. λ, $g_{n|\Gamma_\lambda} \to g_{|\Gamma_\lambda}$ in $W^{N/p,p}$ and thus in VMO. Therefore,

$$
\deg(g_n;\Gamma_\lambda) \to \deg(g;\Gamma_\lambda) \quad \text{for a.e. } \lambda . \tag{2.28}
$$

From (2.27) and (2.28) we obtain

$$
\deg(g_n;\Gamma_\lambda) \to \deg(g;\Gamma_\lambda) \quad \text{in } L^1(\mathbb{R}) . \tag{2.29}
$$

Property (0.16) follows by combining (2.26), (2.29), and the continuity of T. □

We conclude this section by showing, in the spirit of [9, 11, 12], that, given points (P_i) and (N_i) in $\mathbb{S}^{N+1}$, the minimal "energy" (in the $W^{N/p,p}$ sense) required to produce topological singularities at the P_i's and N_i's is of the same order as the length of a minimal connection connecting the P_i's to the N_i's.

Let $\mathcal{P} = (P_i)$ and $\mathcal{N} = (N_i) \subset \mathbb{S}^{N+1}$ be such that $\sum_i |P_i - N_i| < \infty$. We define the length of a minimal connection to be

$$
L(\mathcal{P},\mathcal{N}) = \operatorname{Inf}\left\{ \sum d(\tilde{P}_j, \tilde{N}_j) : \sum (\delta_{P_i} - \delta_{N_i}) = \sum (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) \right\} .
$$

As observed in [9], if

$$T = \omega_{N+1} \sum (\delta_{P_i} - \delta_{N_i}) ,$$

then

$$\|T\|_{(W^{1,\infty})^*} = \omega_{N+1} L(\mathcal{P}, \mathcal{N}) . \tag{2.30}$$

THEOREM 2.4 *Given $\mathcal{P}$ and $\mathcal{N}$, we have, for $N < p < \infty$,*

$$L(\mathcal{P}, \mathcal{N}) \sim \operatorname{Inf}\left\{ |g|^p_{N/p,p} : g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N),\ T(g) = \omega_{N+1} \sum (\delta_{P_i} - \delta_{N_i}) \right\} . \tag{2.31}$$

(The equivalence in (2.31) is up to constants depending on p and N.)

PROOF: In view of (0.12) and (2.30), it suffices to find, for $\mathcal{P}$ and $\mathcal{N}$ as above, a map $g \in W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ such that $T(g) = \omega_{N+1} \sum (\delta_{P_i} - \delta_{N_i})$ and $|g|^p_{N/p,p} \leq CL(\mathcal{P}, \mathcal{N})$. We rely on [2, theorem 5.6], which asserts that, given $\mathcal{P} = (P_i)$ and $\mathcal{N} = (N_i) \subset \mathbb{S}^{N+1}$ such that $\sum |P_i - N_i| < \infty$, there is some $g \in W^{1,N}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ such that

$$\operatorname{Det}(\nabla g) = \omega_{N+1} \sum (\delta_{P_i} - \delta_{N_i}) \tag{2.32}$$

and

$$\|\nabla g\|^N_{L^N} \leq CL(\mathcal{P}, \mathcal{N}) . \tag{2.33}$$

If $N \geq 2$, we have the inclusion $W^{1,N}(\mathbb{S}^{N+1}; \mathbb{S}^N) \hookrightarrow W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$, $N < p < \infty$, and Theorem 2.4 follows from the inequality

$$|g|^p_{N/p,p} \leq C \|\nabla g\|^N_{L^N} \leq CL(\mathcal{P}, \mathcal{N}) . \tag{2.34}$$

The above inclusion is false when $N = 1$. However, in this case we rely on the proof of lemma 16 in [9]. More specifically, given $1 < p < \infty$, and given points (P_i), $(N_i) \subset \mathbb{S}^2$ such that $\sum |P_i - N_i| < \infty$, we constructed in [9] a map $g \in W^{1/p,p}(\mathbb{S}^2; \mathbb{S}^1) \cap W^{1,1}$ such that $\operatorname{Det}(\nabla g) = \pi \sum (\delta_{P_i} - \delta_{N_i})$ and (2.34) holds. Estimate (2.34) is established in [9] only for $p = 2$, but the argument there can be easily adapted to every p, $1 < p < \infty$. For this purpose, one needs to generalize lemma 17 in [9] with the help of the obvious inequality

$$\left| |a+b|^p - |a|^p - |b|^p \right| \leq C_p (|a|^{p-1}|b| + |a||b|^{p-1}) \quad \forall a, b \in \mathbb{C},\ \forall p > 1 .$$

The proof of Theorem 2.4 is complete. □

Appendix: Density of the Class $\mathcal{R}$

The appendix is devoted to density results for classes of $\mathbb{S}^N$-valued maps. Recall that, if $0 < s < 1$, $1 < p < \infty$, and $sp \geq N + 1$, then $C^\infty(\mathbb{S}^{N+1}; \mathbb{S}^N)$ is dense in $W^{s,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ (see, e.g., [3] or [13, lemma A.12]). We now turn to the remaining case: $sp < N + 1$.

LEMMA A.1 *Assume* $0 < s < 1$, $1 < p < \infty$, *and* $sp < N + 1$. *Then the class*

$$\mathcal{R} = \big\{ g \in W^{1,q}(\mathbb{S}^{N+1}; \mathbb{S}^N) \text{ for every } 1 \le q < N+1 \quad g \in C^\infty(\mathbb{S}^{N+1} \setminus A) \text{ for some finite set } A \big\}$$

is dense in $W^{s,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$.

For $N = 1$, $s = \frac{1}{2}$, and $p = 2$, the above result is due to T. Rivière [16] (following earlier works of F. Bethuel [3], F. Bethuel and X. Zheng [4], and M. Escobedo [15]). A different proof is presented in [9, lemma 23]. We explain below how to adapt the proof of [9] to the general case.

Let $g \in W^{s,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$ and let g_ε be an ε-smoothing of g. Then g_ε satisfies

$$\|g_\varepsilon - g\|_{L^p} \le C\varepsilon^s \,, \tag{A.1}$$

$$|g_\varepsilon|_{s,p} \le C \,, \tag{A.2}$$

$$\|\nabla g_\varepsilon\|_{L^p} \le C\varepsilon^{s-1} \,. \tag{A.3}$$

Given a point $a \in \mathbb{R}^{N+1}$ with $|a| \le \frac{1}{10}$, let $\pi_a\colon \mathbb{R}^{N+1} \setminus \{a\} \to \mathbb{S}^N$ be the radial projection onto $\mathbb{S}^N$ with vertex a. Using (A.1)–(A.3), we find, with exactly the same proof as in [9, lemma 23], that there is a family (a_ε) such that $|a_\varepsilon| \le \frac{1}{10}$ and $h_\varepsilon = \pi_{a_\varepsilon}(g_\varepsilon) \to g$ in $W^{s,p}$. Moreover, as explained in [9], we may choose a_ε to be a regular value of g_ε, and for such a choice we have $h_\varepsilon \in \mathcal{R}\ \forall n$.

COROLLARY A.2 *For* $N < p < \infty$, *the class* $W^{1,N}(\mathbb{S}^{N+1}; \mathbb{S}^N) \cap W^{N/p,p}$ *is dense in* $W^{N/p,p}(\mathbb{S}^{N+1}; \mathbb{S}^N)$.

PROOF OF LEMMA 2.2: Let g_ε be as above. Then g_ε satisfies (A.1)–(A.3) (with $s = \frac{1}{p}$) and, in addition,

$$\|\nabla g_\varepsilon\|_{L^1} \le C \,. \tag{A.4}$$

On the other hand, we have

$$\int_{\{a:|a|\le\frac{1}{10}\}} \|\nabla(\pi_a \circ g_\varepsilon)\|_{L^1(\mathbb{S}^2)}\, da \le C\|\nabla g_\varepsilon\|_{L^1(\mathbb{S}^2)} \tag{A.5}$$

(this is inequality (5.34) in [9]). By combining (A.1)–(A.5), we find, exactly as in [9], that there is a family (a_ε) such that $|a_\varepsilon| \le \frac{1}{10}$ and $h_\varepsilon = \pi_{a_\varepsilon}(g_\varepsilon) \to g$ in $W^{1/p,p}$ and $\|\nabla h_\varepsilon\|_{L^1} \le C$. In order to prove that, in addition, $h_\varepsilon \to g$ in $W^{1,1}$, one may adapt the argument in [9]. Convergence in $W^{1/p,p}$ is obtained there with the help of the property (5.43). To establish convergence in $W^{1,1}$, it suffices to note that the analogue of (5.43) also holds in $W^{1,1}$; this is easily obtained by dominated convergence. □

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