LIMITING EMBEDDING THEOREMS FOR $W^{s,p}$ WHEN $s \uparrow 1$ AND APPLICATIONS

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Dedicated to the memory of T. Wolff

1. Introduction.

This is a follow-up of our paper [3] where we establish that

$$\lim_{s\uparrow 1}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \sim \|\nabla f\|_{L^p(\Omega)}, \tag{1}$$

for any $p \in [1,\infty)$, where Ω is a smooth bounded domain in $\mathbb{R}^d, d \geq 1$.

On the other hand, if $0 < s < 1$, $p > 1$ and $sp < d$, the Sobolev inequality for fractional Sobolev spaces (see e.g. [1], Theorem 7.57 or [6], Section 3.3) asserts that

$$\|f\|^p_{W^{s,p}(\Omega)} \geq C(s,p,d)\|f - ⨍f\|_{L^q(\Omega)} \tag{2}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d}. \tag{3}$$

Here we use the standard semi-norm on $W^{s,p}$

$$\|f\|^p_{W^{s,p}(\Omega)} = \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy. \tag{4}$$

When $s=1$ the analog of (2) is the classical Sobolev inequality

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(p,d)\|f - ⨍f\|^p_{L^{p^*}(\Omega)} \tag{5}$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \text{ and } 1 \leq p < d.$$

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The behaviour of the best constant $C(p,d)$ in (5) as $p \uparrow d$ is known (see e.g. [5], Section 7.7 and also Remark 1 below); more precisely one has

$$\|\nabla f\|^p_{L^p(\Omega)} \geq C(d)(d-p)^{p-1}\|f - \textstyle\fint f\|_{L^{p^*}(\Omega)}. \tag{6}$$

Putting together (1), (4) and (6) suggests that (2) holds with

$$C(s,p,d) = C(d)(d-sp)^{p-1}/(1-s), \tag{7}$$

for all $s < 1$, s close to 1 and $sp < d$.

This is indeed our main result. For simplicity we work with $\Omega =$ the unit cube Q in $\mathbb{R}^d$.

Theorem 1. *Assume $d \geq 1, p \geq 1, 1/2 \leq s < 1$ and $sp < 1$. Then*

$$\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}dxdy \geq C(d)\frac{(d-sp)^{p-1}}{1-s}\|f - \textstyle\fint f\|^p_{L^q(Q)} \tag{8}$$

where q is given by (3) and $C(d)$ depends only on d.

As can be seen from (8) there are two phenomena that govern the behaviour of the constant in (8). As $s \uparrow 1$ the constant gets bigger, while as $s \uparrow d/p$ the constant deteriorates. This explains why the we consider several cases in the proof.

As an application of Theorem 1 with $p = 1$ and $f = \chi_A$, the characteristic function of a measurable set $A \subset Q$ we easily obtain

Corollary 1. *For all $0 < \varepsilon \leq 1/2$,*

$$|A|\,|{}^cA| \leq \left(C(d)\varepsilon \int_A\int_{{}^cA} \frac{dxdy}{|x-y|^{d+1-\varepsilon}}\right)^{d/(d-1+\varepsilon)}. \tag{9}$$

Note that in the special case $d = 1$, (9) takes the simple form

$$|A|\,|{}^cA| \leq \left(C^*\varepsilon \int_A\int_{{}^cA} \frac{dxdy}{|x-y|^{2-\varepsilon}}\right)^{1/\varepsilon} \tag{10}$$

for some absolute constant C^*. Estimate (10) is sharp as can be easily seen when A is an interval.

The conclusion of Corollary 1 is related to a result stated in [3] (Remark 4). There is however an important difference. In [3] the set A was *fixed* (independent of ε) and the statement there provides a bound for $|A|\,|{}^cA|$ in terms of the limit, as $\varepsilon \to 0$, of the RHS in (9). The improved version - which requires a more delicate argument- is used in Section 7; we apply Corollary 1 (with $d = 1$) to give a proof of a result announced in [2] (Remark E.1). Namely, on $\Omega = (-1, +1)$ consider the function

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for} \quad -1 < x < 0, \\ 2\pi x/\delta & \text{for} \quad 0 < x < \delta, \\ 2\pi & \text{for} \quad \delta < x < 1, \end{cases}$$

where $\delta = e^{-1/\varepsilon}, \varepsilon > 0$ small.

Set $u_\varepsilon = e^{i\varphi_\varepsilon}$. It is easy to check (by scaling) that

$$\|u_\varepsilon\|_{H^{1/2}} = \|u_\varepsilon - 1\|_{H^{1/2}} \leq C$$

as $\varepsilon \to 0$ and consequently $\|u_\varepsilon\|_{H^{(1-\varepsilon)/2}} \leq C$ as $\varepsilon \to 0$. On the other hand, a straightforward computation shows that $\|\varphi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \sim \varepsilon^{-1/2}$.

The result announced in [2] asserts that *any* lifting φ_ε of u_ε blows up in $H^{(1-\varepsilon)/2}$ (at least) in the same rate as φ_ε:

Theorem 2. *Let $\psi_\varepsilon : \Omega \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$. Then*

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}} \geq c\varepsilon^{-1/2}, \forall \varepsilon \in (0, 1/2),$$

for some absolute constant $c > 0$.

Remark 1. There are various versions of the Sobolev inequality (5). All these forms hold with equivalent constants:

Form 1: $\|\nabla f\|_{L^p(Q)} \geq A_1 \|f - \fint_Q f\|_{L^q(Q)} \quad \forall f \in W^{1,p}(Q)$.

Form 2: $\|\nabla f\|_{L^p(Q)} \geq A_2 \|f - \fint_Q f\|_{L^q(Q)}$ for all Q-periodic functions $f \in W^{1,p}_{loc}(\mathbb{R}^d)$.

Form 3: $\|\nabla f\|_{L^p(\mathbb{R}^d)} \geq A_3 \|f\|_{L^q(\mathbb{R}^d)} \quad \forall f \in C_0^\infty(\mathbb{R}^d)$.

Form 1 ⇒ Form 2. Obvious with $A_2 = A_1$.

Form 2 ⇒ Form 1. Given any function $f \in W^{1,p}(Q)$, it can be extended by reflections to a periodic function on a larger cube $\widetilde{Q}$ so that Form 2 implies Form 1 with $A_1 \geq CA_2$, and C depends only on d.

Form 1 ⇒ Form 3. By scale invariance, Form 1 holds with the same constant A_1 on the cube Q_R of side R. Fix a function $f \in C_0^\infty(\mathbb{R}^d)$ and let $R >$ diam (Supp f). We have

$$\|\nabla f\|_{L^p(Q_R)} \geq A_1 \|f - \fint_{Q_R} f\|_{L^q(Q_R)}.$$

As $R \to \infty$ we obtain Form 3 with $A_3 = A_1$.

Form 3 ⇒ Form 2. Given a smooth periodic function f on $\mathbb{R}^d$, let ρ be a smooth cut-off function with $\rho = 1$ on Q and $\rho = 0$ outside $2Q$. Then

$$\|\nabla(\rho f)\|_{L^p(\mathbb{R}^d)} \geq A_3 \|\rho f\|_{L^q(\mathbb{R}^d)}$$

and thus

$$A_3\|f\|_{L^q(Q)} \le C(\|\nabla f\|_{L^p(Q)} + \|f\|_{L^p(Q)})$$

where C depends only on d. Replacing f by $(f - \fint_Q f)$ and applying Poincaré's inequality (see e.g. [5], Section 7.8) yields

$$A_3\|f - \fint f\|_{L^q(Q)} \le C\|\nabla f\|_{L^q(Q)}.$$

The reader will check easily that the same considerations hold for the fractional Sobolev norms such as in (8). The proof of the last implication (Form 3 $\Rightarrow$ Form 2) involves a Poincaré-type inequality. What we use here is the following

Fact: Let $1 \le p < \infty$, $1/2 \le s < 1$, then

$$(1-s)\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \ge c(d)\|f - \fint_Q f\|^p_{L^p(Q)}.$$

The proof of this fact is left to the reader. (It is an adaptation of the argument in the beginning of Section 5. In (3) of Section 5 one uses an obvious lower bound:

$$(3) \ge c\Big(\sum_r \|f_r\|_{L^p}\Big)^p \ge c\|f - \fint f\|^p_{L^p}.)$$

For the convenience of the reader we have divided the proof of Theorem 1 into several cases. The plan of the paper is the following:

1. Introduction.
2. Proof of Theorem 1 when $p = 1$ and $d = 1$.
3. Proof of Theorem 1 when $p = 1$ and $d \ge 2$.
4. Square function inequalities.
5. Proof of Theorem 1 when $1 < p < 2$.
6. Proof of Theorem 1 when $p \ge 2$.
7. Proof of Theorem 2.

Appendix: Proof of square function inequality.

2. Proof of Theorem 1 when $p = 1$ and $d = 1$.

For simplicity, we work with periodic functions of period 2π (for non-periodic functions see Remark 1 in the Introduction). All integrals, L^p norms, etc...., are understood on the interval $(0, 2\pi)$. We must prove that, (with $\varepsilon = 1 - s$), for all $\varepsilon \in (0, 1/2]$,

$$C\varepsilon \iint \frac{|f(x)-f(y)|}{|x-y|^{2-\varepsilon}}dxdy \ge \|f - \fint f\|_{L^{1/\varepsilon}}. \tag{1}$$

Write the left side as

$$\varepsilon \int \frac{1}{|h|^{2-\varepsilon}} \|f - f_h\|_1 dh \sim$$
$$\varepsilon \sum_{k\geq 0} 2^{k(2-\varepsilon)} \int_{|h|\sim 2^{-k}} \|f - f_h\|_1 dh. \tag{2}$$

For $|h| \sim 2^{-k}$

$$\|f - f_h\|_1 \geq$$
$$\|(f - f_h) * F_{N_k}\|_1 = \left(N_k = 2^{k-100}, F_N(x) = \sum_{|n|\leq N} \frac{N - |n|}{N} e^{inx} = \text{ Féjer kernel} \right)$$
$$\left\| \sum_{|n|<N_k} \frac{N_k - |n|}{N_k} \hat{f}(n)(e^{inh} - 1)e^{inx} \right\|_1 \sim$$
$$2^{-k} \left\| \sum_{|n|<N_k} \frac{N_k - |n|}{N_k} n\hat{f}(n)e^{inx} \right\|_1 \qquad \text{(by the choice of } N_k\text{)}.$$

This last equivalence is justified via a smooth truncation as in the following

Lemma 1. $\left\|\sum_{|n|<N} \hat{f}(n)(e^{inh} - 1)e^{inx}\right\|_1 \gtrsim \frac{1}{N} \left\|\sum_{|n|<N} n\hat{f}(n)e^{inx}\right\|_1$ *for* $|h| < \frac{1}{100N}$.

Proof. Write

$$\left\| \sum_{|n|<N} n\hat{f}(n)e^{inx} \right\|_1 \leq \left\| \sum_{|n|<N} \hat{f}(n)(e^{inh} - 1)e^{inx} \right\|_1 . \left\| \sum \varphi\left(\frac{n}{N}\right) \frac{n}{e^{inh} - 1} e^{inx} \right\|_1$$

where $0 \leq \varphi \leq 1$ is a smooth function with

$$\varphi(t) = \begin{cases} 1 & \text{for} \quad |t| \leq 1 \\ 0 & \text{for} \quad |t| \geq 2 \end{cases}$$

We have from assumption

$$\left\| \sum \varphi\left(\frac{n}{N}\right) \frac{n}{e^{inh} - 1} e^{inx} \right\|_1 \sim N \left\| \sum \varphi\left(\frac{n}{N}\right) \frac{nh}{e^{inh} - 1} e^{inx} \right\|_1$$

and the second factor remains uniformly bounded. This may be seen by expanding

$$\frac{y}{e^{iy} - 1} \sim \frac{1}{i} + 0(y)$$

for $|y| < \frac{1}{50}$ and using standard multiplier bounds.

We now return to the proof of Theorem 1 ($p = 1, d = 1$).

Substitution in (2) gives thus

$$\varepsilon \sum_{k\geq 0} 2^{-\varepsilon k} \Big\| \sum_{|n|<N_k} \frac{N_k - |n|}{N_k} n\hat{f}(n)e^{inx} \Big\|_1 . \tag{3}$$

Define

$$k_0 = \frac{10}{\varepsilon}.$$

For $k_0 < k < 2k_0$, minorate (using Lemma 1)

$$\Big\| \sum_{|n|<N_k} \frac{N_k - |n|}{N_k} n\hat{f}(n)e^{inx} \Big\|_1 \gtrsim \Big\| \sum_{|n|<N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} n\hat{f}(n)e^{inx} \Big\|_1$$

and therefore

$$(3) \gtrsim \Big\| \sum_{|n|<N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} n\hat{f}(n)e^{inx} \Big\|_1 =$$

$$\Big\| \sum_{|n|<N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} \hat{f}(n)e^{inx} \Big\|_{W^{1,1}} \geq$$

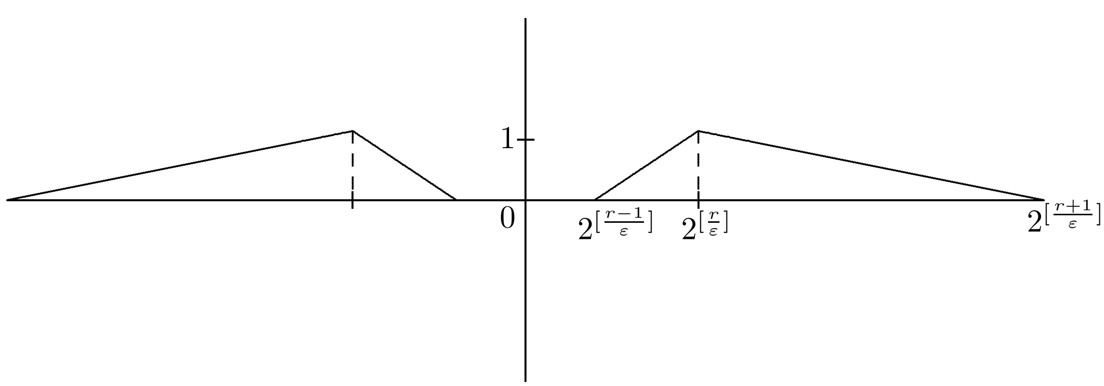
$$\Big\| \sum_{0<|n|<N_{k_0}} \frac{N_{k_0} - |n|}{N_{k_0}} \hat{f}(n)e^{inx} \Big\|_\infty . \tag{4}$$

Next write also

$$(3) \gtrsim \varepsilon \sum_{r\geq 1} 2^{-r} \sum_{[\frac{r+2}{\varepsilon}]\leq k<[\frac{r+3}{\varepsilon}]} \Big\| \sum_{|n|<N_k} \frac{N_k - |n|}{N_k} n\hat{f}(n)e^{inx} \Big\|_1$$

$$\gtrsim \sum_{r\geq 1} 2^{-r} \Big\| \sum_{|n|<2^{[\frac{r+1}{\varepsilon}]}} \frac{2^{[\frac{r+1}{\varepsilon}]} - |n|}{2^{[\frac{r+1}{\varepsilon}]}} e^{inx} \Big\|_1 . \tag{5}$$

Denote for each r by $\lambda_r = \{\lambda_r(n) | n \in \mathbb{Z}\}$ the following multiplier



Thus

$$\lambda_r(n) = \lambda_r(-n)$$

$$\Big\| \sum \lambda_r(n) e^{inx} \Big\|_1 < C.$$

(This multiplier may be reconstructed from Féjer-kernels F_N with $N = 2^{[\frac{r+1}{\varepsilon}]}, 2^{[\frac{r}{\varepsilon}]}, 2^{[\frac{r-1}{\varepsilon}]}$).

Also

$$\Big\| \sum_{|n|<2^{[\frac{r+1}{\varepsilon}]}} \frac{2^{[\frac{r+1}{\varepsilon}]} - |n|}{2^{[\frac{r+1}{\varepsilon}]}} n\hat{f}(n) e^{inx} \Big\|_1 \gtrsim$$

$$\Big\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)\, n\hat{f}(n) e^{inx} \Big\|_1 \tag{6}$$

and

$$(5) \gtrsim \sum_{r\geq 1} 2^{-r} \Big\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)(\text{sign } n)|n|\ \hat{f}(n) e^{inx} \Big\|_1. \tag{7}$$

We claim that for $q > 2$

$$\Big\| \sum_{N_1<|n|<N_2} \hat{g}(n) e^{inx} \Big\|_q \leq C N_1^{-\frac{1}{q}} \Big\| \sum_{N_1<|n|<N_2} |n|(\text{sign } n)\hat{g}(n) e^{inx} \Big\|_1 \tag{8}$$

with the constant C independent of q.

Applying (8) with

$$q = \frac{1}{\varepsilon}, \quad \hat{g}(n) = \lambda_r(n)\hat{f}(n), \quad N_1 = 2^{[\frac{r-1}{\varepsilon}]},\ N_2 = 2^{[\frac{r+1}{\varepsilon}]}$$

we obtain the minoration

$$(7) \gtrsim \sum_{r\geq 1} \Big\| \sum_{2^{[\frac{r-1}{\varepsilon}]}<|n|<2^{[\frac{r+1}{\varepsilon}]}} \lambda_r(n)\hat{f}(n) e^{inx} \Big\|_q. \tag{9}$$

By construction

$$\sum_{r\geq 1} \lambda_r(n) = 1 \text{ for } |n| > 2^{[\frac{1}{\varepsilon}]}.$$

Using also minoration (4) together with the triangle-inequality yields

$$\text{LHS in (1)} \gtrsim (3) + (8) \gtrsim \Big\| \sum_{n\neq 0} \hat{f}(n) e^{inx} \Big\|_q$$

which proves the inequality.

Proof of (8).

Estimate

$$\Big\|\sum_{N_1<|n|<N_2} \hat{g}(n)e^{inx}\Big\|_q \le \Big\|\sum_{N_1<|n|<N_2} |n|^{-1}(\text{sign } n)e^{inx}\Big\|_q \Big\|\sum_{N_1<|n|<N_2} |n|(\text{sign } n)\hat{g}(n)e^{inx}\Big\|_1$$

where the first factor equals

$$\Big\|\sum_{N_1<n<N_2} \frac{1}{n}\sin\ nx\Big\|_q \lesssim$$

$$\begin{aligned}
&\Big\|\sum_{\log N_1<k<\log N_2} \Big|\sum_{n\sim 2^k} \frac{1}{n}\sin\ nx\Big|\ \Big\|_q \qquad \text{(assume } N_1, N_2 \text{ powers of 2)}\\
&\lesssim \Big\|\sum_{\log N_1<k<\log N_2} \min(2^k|x|, 2^{-k}|x|^{-1})\Big\|_q\\
&\lesssim \Big\|\frac{1}{1+N_1|x|}\Big\|_q \lesssim N_1^{-1/q}. \qquad (10)
\end{aligned}$$

This proves (8) and completes the proof of Theorem 1 when $p=1$ and $d=1$.

3. Proof of Theorem 1 when $p=1$ and $d\ge 2$.

We have to prove that

$$\iint \frac{|f(x)-f(y)|}{|x-y|^{d+s}}dxdy \ge \frac{C(d)}{1-s}\|f - {\textstyle\int} f\|_q \qquad (1)$$

where $q = d/(d-s)$. We assume $d=2$. The case $d>2$ is similar. Write

$$\begin{aligned}
\iint \frac{|f(x)-f(y)|}{|x-y|^{d+s}}dxdy &\sim \sum_{0\le k} 2^{k(d+s)} \int_{|h|\sim 2^{-k-10}} \|f(x+h)-f(x-h)\|_1 dh\\
&\ge \sum 2^{k(d+s)} \int_{\substack{|h_1|\sim 2^{-k-10}\\ |h_2|\sim 2^{-k-10}}} \Big\|\sum_{n\in\mathbb{Z}^d} \hat{f}(n)(\sin n.h)e^{in.x}\Big\|_1 dh_1dh_2 \qquad (2)
\end{aligned}$$

Let φ be a smooth function on $\mathbb{R}$ s.t. $0\le\varphi\le 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{for } |t|\le 1\\ 0 & \text{for } |t|\ge 2\end{cases}$$

As for $d=1$, consider (radial) multipliers λ_0 and $\lambda_r, r\geq 1$

$$\begin{aligned}\lambda_0(n) &= \varphi(2^{-\frac{1}{\varepsilon}}|n|)\\ \lambda_r(n) &= \varphi(2^{-\frac{r+1}{\varepsilon}}|n|) - \varphi(2^{-\frac{r}{\varepsilon}}|n|)\end{aligned} \tag{3}$$

where $\varepsilon = 1-s$ and $\varepsilon \in (0,1/2)$.

Hence

$$\sum \lambda_r(n) = 1$$

$$\|\lambda_r\|_{M(L^1,L^1)} \leq C \qquad \text{(multiplier norm)} \tag{4}$$

$$\text{supp } \lambda_0 \subset B(0, 2^{\frac{1}{\varepsilon}+1}) \tag{5}$$

$$\text{supp } \lambda_r \subset B(0, 2^{\frac{r+1}{\varepsilon}+1})\backslash B(0, 2^{\frac{r}{\varepsilon}}). \tag{6}$$

Write

$$(2) = \sum_{\frac{1}{\varepsilon}<k<\frac{2}{\varepsilon}} + \sum_{r\geq 1}\ \sum_{\frac{r+1}{\varepsilon}<k<\frac{r+2}{\varepsilon}} . \tag{7}$$

For $\frac{2}{\varepsilon} > k > \frac{1}{\varepsilon}$ and $|h| < 2^{-k-10}$, (4), (5) permit us to write

$$\begin{aligned}\left\| \sum_n \hat{f}(n) e^{in.x} \sin n.h \right\|_1 &\gtrsim \left\| \sum_n \lambda_0(n) \hat{f}(n) e^{in.x} \sin n.h \right\|_1\\ &\sim \left\| \sum_n \lambda_0(n)(n.h) \hat{f}(n) e^{in.x} \right\|_1\end{aligned}$$

and thus

$$\begin{aligned}&2^{k(d+1-\varepsilon)} \int\limits_{|h_1|,|h_2|\sim 2^{-k-10}} \left\| \sum \hat{f}(n)(\sin n.h) e^{in.x}\right\|_1 dh_1 dh_2 \gtrsim\\ &2^{k(3-\varepsilon)} 8^{-k} \left(\left\| \sum \lambda_0(n) n_1 \hat{f}(n) e^{in.x} \right\|_1 + \left\| \sum \lambda_0(n) n_2 \hat{f}(n) e^{in.x}\right\|_1 \right)\\ &= 2^{-k\varepsilon} \left(\left\| \partial_{x_1} \left(\sum \lambda_0(n) \hat{f}(n) e^{in.x}\right)\right\|_1 + \|\partial_{x_2}(\cdots)\| \right) \sim\\ &\left\| \sum \lambda_0(n) \hat{f}(n) e^{in.x} \right\|_{W^{1,1}}.\end{aligned} \tag{8}$$

Similarly, for

$$\frac{r+1}{\varepsilon} < k < \frac{r+2}{\varepsilon}$$

we have

$$2^{k(d+1-\varepsilon)} \int\limits_{|h_1|,|h_2|\sim 2^{-k-10}} \left\| \sum \hat{f}(n)(\sin nh) e^{in.x} \right\|_1 \gtrsim 2^{-r} \left\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \right\|_{W^{1,1}}. \tag{9}$$

Since in the summation (7), each of the terms (8), (9) appear at least $\frac{1}{\varepsilon}$ times, we have

$$\varepsilon.(2) \gtrsim \Big\| \sum \lambda_0(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}} + \sum_r 2^{-r} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \tag{10}$$

Write

$$\frac{2-s}{2} = 1 - s + \frac{s}{2}$$

and by Hölder's inequality

$$\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{\frac{2}{2-s}} \leq \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_2^s \, \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_1^{1-s}. \tag{11}$$

By the Sobolev embedding theorem ($d = 2$)

$$\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_2 \leq C \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \tag{12}$$

We estimate the last factor in (11).

Recalling (6),

$$2^{\frac{r+1}{\varepsilon}+1} > \max(|n_1|, |n_2|) > 2^{\frac{r}{\varepsilon}-1}$$

if $\lambda_r(n) \neq 0, r \geq 1$.

Hence, with φ as above

$$\lambda_r(n) = \lambda_r(n).(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) + \lambda_r(n).\varphi(2^{-\frac{r-1}{\varepsilon}} n_1).(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2)$$

and thus

$$\begin{aligned}
&\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_1 \leq \\
&\Big\| \sum \lambda_r(n) n_1 \hat{f}(n) e^{in.x} \Big\|_1 \, \Big\| \sum \frac{1}{n_1} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) e^{in.x} \Big\|_1 \\
&+ \Big\| \sum \lambda_r(n) n_2 \hat{f}(n) e^{in.x} \Big\|_1 \, \Big\| \sum \frac{1}{n_2} \varphi(2^{-\frac{r-1}{\varepsilon}} n_1)(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2) e^{in.x} \Big\|_1 \leq \\
&\Big(\Big\| \sum_{n_1} \frac{1}{n_1} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) e^{in_1 x_1} \Big\|_{L^1_{x_1}} + \Big\| \sum_{n_2} \frac{1}{n_2} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2) e^{in_2 x_2} \Big\|_{L^1_{x_2}} \Big). \\
&\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}.
\end{aligned} \tag{13}$$

Since $(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) = 0$ for $|n_1| \leq 2^{\frac{r-1}{\varepsilon}}$, one easily checks that

$$\Big\| \sum_{n_1} \frac{1}{n_1} (1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_1) \varepsilon^{in_1 x_1} \Big\|_{L^1_{x_1}} \lesssim \sum_{\ell \geq \frac{r-1}{\varepsilon}} 2^{-\ell} < 2^{\frac{r-2}{\varepsilon}}.$$

Similarly

$$\Big\| \sum_{n_2} \frac{1}{n_2}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}} n_2) e^{in_2x_2} \Big\|_{L^1_{x_2}} \leq 2^{-\frac{r-2}{\varepsilon}}.$$

Thus (13) implies that

$$\Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_1 \leq 2^{-\frac{r-2}{\varepsilon}} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \tag{14}$$

Substitution of (12), (14) in (11) gives

$$\begin{aligned} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{\frac{2}{2-s}} &\lesssim 2^{-\frac{r-2}{\varepsilon}(1-s)} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}} \\ &\sim 2^{-r} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{W^{1,1}}. \end{aligned} \tag{15}$$

By (12), (15)

$$\begin{aligned} \varepsilon.(2) &\geq \Big\| \sum \lambda_0(n) \hat{f}(n) e^{in.x} \Big\|_2 + \sum_{r\geq 1} \Big\| \sum \lambda_r(n) \hat{f}(n) e^{in.x} \Big\|_{\frac{2}{2-s}} \\ &\geq \Big\| f - \textstyle\fint f \Big\|_{\frac{2}{2-s}} \end{aligned}$$

by (3).

This proves (1) and completes the proof of Theorem 1 when $p = 1$.

4. Square function inequalities.

We present here some known inequalities used in the proof of Theorem 1 when $p > 1$. Let $\{\Delta_j f\}_{j=1,2,\ldots}$ be a Littlewood-Paley decomposition with $\Delta_j f$ obtained from a Fourier multiplier of the form $\varphi(2^{-j}|n|) - \varphi(2^{-j+1}|n|)$ with $0 \leq \varphi \leq 1$ a smooth function satisfying $\varphi(t) = 1$ for $|t| \leq 1$ and $\varphi(t) = 0$ for $|t| > 2$.

Recall the square-function inequality for $1 < q < \infty$

$$\frac{1}{C(q)} \Big\| \Big(\sum |\Delta_j f|^2 \Big)^{1/2} \Big\|_q \leq \|f\|_q \leq C(q) \Big\| \Big(\sum |\Delta_j f|^2 \Big)^{1/2} \Big\|_q. \tag{1}$$

We will also consider square-functions wrt a martingale filtration. Denote thus $\{\mathbb{E}_j\}$ the expectation operators wrt a dyadic partition of $[0,1]^d$ and

$$\widetilde{\Delta}_j f = (\mathbb{E}_j - \mathbb{E}_{j-1}) f \tag{2}$$

the martingale differences.

We will use the square-function inequality

$$\|f\|_q \leq C\sqrt{q} \Big\| \Big(\sum |\widetilde{\Delta}_j f|^2 \Big)^{1/2} \Big\|_q \qquad \text{for } \infty > q \geq 2 \tag{3}$$

which is precise in terms of the behaviour of the constant for $q \to \infty$ $\big($see [4] and also the Appendix for a proof of (3)$\big)$.

Remark 2. *One should expect (3) also to hold if $\widetilde{\Delta}_j$ is replaced by Δ_j above but we will not need this fact.*

We do use later on the following inequality.

Let

$$p < q \text{ and } s = d\left(\frac{1}{p} - \frac{1}{q}\right) \geq \frac{1}{2}.$$

Then, for $q \geq 2$

$$\|f\|_q \leq C\sqrt{q}\left[\sum_k (2^{ks}\|\Delta_k f\|_p)^2\right]^{1/2}. \tag{4}$$

Proof of (4)

It follows from (3) that since $q \geq 2$

$$\|f\|_q \leq C\sqrt{q}\left(\sum_j \|\widetilde{\Delta}_j f\|_q^2\right)^{1/2}. \tag{5}$$

Write

$$\begin{aligned}
\widetilde{\Delta}_j f &= \sum_{k \leq j} \widetilde{\Delta}_j \Delta_k f + \sum_{k > j} \widetilde{\Delta}_j \Delta_k f \\
\|\widetilde{\Delta}_j f\|_q &\lesssim \sum_{k \leq j} 2^{k-j}\|\Delta_k f\|_q + \sum_{k > j} 2^{js}\|\Delta_k f\|_p \\
&\lesssim \sum_{k \leq j} 2^{k-j}(2^{ks}\|\Delta_k f\|_p) + \sum_{k > j} 2^{(j-k)s}(2^{ks}\|\Delta_k f\|_p). \qquad (6)
\end{aligned}$$

Substitution of (6) in (5) gives

$$\begin{aligned}
\|f\|_q &\leq C\sqrt{q}\left\{\left(\sum_{k \leq j}(j-k)^2 4^{k-j}(2^{ks}\|\Delta_k f\|_p)^2\right)^{1/2} + \left(\sum_{k > j}(k-j)^2 4^{(j-k)s}(2^{ks}\|\Delta_k f\|_p)^2\right)^{1/2}\right\} \\
&\leq C\sqrt{q}\left(\sum_k (2^{ks}\|\Delta_k f\|_p)^2\right)^{1/2}. \qquad (7)
\end{aligned}$$

5. Proof of Theorem 1 when $1 < p < 2$.

Write

$$\begin{aligned}
\iint \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} dx dy &\sim \sum_{k \geq 0} 2^{k(d+ps)} \int_{|h| \sim 2^{-k-10}} \|f(x+h) - f(x-h)\|_p^p dh \\
&\geq \sum_{k \geq 0} 2^{k(d+ps)} \int_{|h| \sim 2^{-k-10}} \left\| \sum \hat{f}(n)(\sin n.h) e^{in.x} \right\|_p^p dh. \qquad (1)
\end{aligned}$$

Following the argument in Section 3 (formula (10)), we get again for

$$s = d\Big(\frac{1}{p} - \frac{1}{q}\Big), 1 - s = \varepsilon \tag{2}$$

$$\varepsilon.(1) \gtrsim \sum_r \Bigg(2^{-r}\Big\|\sum_n \lambda_r(n)\hat{f}(n)e^{in.x}\Big\|_{W^{1,p}}\Bigg)^p \tag{3}$$

where the multipliers λ_r are defined as before.

Case $\underline{d=1}$

Define

$$f_r = \sum_n \lambda_r(n)\hat{f}(n)e^{in.x}.$$

We will make 2 estimates.

First write

$$f_r = \Big(\sum n\lambda_r(n)\hat{f}(n)e^{in.x}\Big) * \Bigg(\sum_{2^{\frac{r}{\varepsilon}}<|n|<2^{\frac{r+1}{\varepsilon}}} \frac{1}{n}e^{in.x}\Bigg)$$

implying

$$\|f_r\|_q \le \|f_r\|_{W^{1,p}}\Bigg\|\sum_{2^{\frac{r}{\varepsilon}}<n<2^{\frac{r+1}{\varepsilon}}} \frac{1}{n}\sin nx\Bigg\|_{(\frac{1}{p'}+\frac{1}{q})^{-1}} \tag{4}$$

and by estimate (10) in Section 2,

$$\|f_r\|_q \lesssim 2^{-\frac{r}{\varepsilon}(\frac{1}{p'}+\frac{1}{q})}\|f_r\|_{W^{1,p}} = 2^{-\frac{r}{\varepsilon}(1-s)}\|f_r\|_{W^{1,p}} = 2^{-r}\|f_r\|_{W^{1,p}}. \tag{5}$$

Estimate then

$$\|f\|_q \le \sum_r \|f_r\|_q \le C\sum_r (2^{-r}\|f_r\|_{W^{1,p}}). \tag{6}$$

Next apply inequality (4) of Section 4. Observe that

$$|\Delta_k f| \le \sum_r |\Delta_k f_r|$$

where, by construction, there are, for fixed k, at most 2 nonvanishing terms.

Thus

$$\|\Delta_k f\|_p^2 \lesssim \sum_r \|\Delta_k f_r\|_p^2. \tag{7}$$

Also, for fixed r

$$\sum_k (2^{ks}\|\Delta_k f_r\|_p)^2 = \sum_r 4^{-k\varepsilon}\|\Delta_k f_r\|_{W^{1,p}}^2 \lesssim \frac{1}{\varepsilon}4^{-r}\|f_r\|_{W^{1,p}}^2. \tag{8}$$

Substituting (7), (8) in (4) of Section 4 gives

$$\|f\|_q \lesssim C\sqrt{q}\Big[\sum_k\sum_r (2^{ks}\|\Delta_k f_r\|_p)^2\Big]^{1/2} \leq [C\sqrt{\frac{q}{\varepsilon}})\Big[\sum_r (2^{-r}\|f_r\|_{W^{1,p}})^2\Big]^{1/2} \tag{9}$$

which is the second estimate.

Interpolation between (6) and (9) implies thus

$$\|f\|_q \leq C\bigg(\sqrt{\frac{q}{\varepsilon}}\bigg)^{2(1-\frac{1}{p})}\Big[\sum_r (2^{-r}\|f_r\|_{W^{1,p}})^p\Big]^{1/p}. \tag{10}$$

Recalling (3) and also (2) (which implies that $1-\varepsilon = \frac{1}{p} - \frac{1}{q} < \frac{1}{p}$, hence $\varepsilon > 1 - \frac{1}{p}$) it follows that

$$\varepsilon.(1) \gtrsim \bigg(\frac{1}{q}\bigg)^{p-1}\|f\|_q^p \tag{11}$$

which gives the required inequality.

Case $\underline{d > 1}$

We will distinguish the further 2 cases

Case A: $0 < \frac{1}{p} - \frac{1}{d}$ is not near 0

Case B: $\frac{1}{p} - \frac{1}{d}$ is near 0

Observe that case B may only happen for $d = 2$ and p near 2 (we assumed $1 < p < 2$).

Case A.

Define q_1 by

$$1 = d\bigg(\frac{1}{p} - \frac{1}{q_1}\bigg) \tag{12}$$

so that $q < q_1$ and q_1 is bounded from above by assumption.

Thus we have the Sobolev inequality

$$\|g\|_{q_1} \leq C\|g\|_{W^{1,p}}. \tag{13}$$

Next, we make the obvious adjustment of the argument in Section 3, (11)-(15).

Thus Hölder's inequality gives

$$\|f_r\|_q \leq \|f_r\|_{q_1}^{1-\theta}\|f_r\|_q^{\theta} \tag{14}$$

with

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{p}, \text{ hence } \theta = 1 - s = \varepsilon \text{ by (2), (12).}$$

Hence, by (13)

$$\|f_r\|_q \le C\|f_r\|_{W^{1,p}}^{1-\varepsilon}\|f_r\|_p^{\varepsilon}. \tag{15}$$

To estimate $\|f_r\|_p$, proceed as in (13) of Section 3. Thus

$$\begin{aligned}\|f_r\|_p &\lesssim \left\| \sum \frac{1}{n}(1-\varphi)(2^{-\frac{r-1}{\varepsilon}}n)e^{inx}\right\|_{L^1_x(\mathbb{T})} \|f_r\|_{W^{1,p}}\\ &\lesssim 2^{-\frac{r-1}{\varepsilon}}\|f_r\|_{W^{1,p}}. \end{aligned}\tag{16}$$

Substitution of (16) in (15) gives

$$\|f_r\|_q \lesssim 2^{-r}\|f_r\|_{W^{1,p}}. \tag{17}$$

Substitution of (17) in (3) gives (since q is bounded by case A hypothesis)

$$\begin{aligned}\varepsilon.(1) \gtrsim \sum_r \|f_r\|_q^p &\sim \sum_r \left\| \Big(\sum_j |\Delta_j f_r|^2\Big)^{1/2}\right\|_q^p\\ &\gtrsim \left\| \Big(\sum_{r,j} |\Delta_j f_r|^2\Big)^{1/2}\right\|_q^p\\ &\gtrsim \left\| \Big(\sum_j |\Delta_j f|^2\Big)^{1/2}\right\|_q^p \sim \|f\|_q^p \end{aligned}\tag{18}$$

(the second inequality requires distinction of the cases $q \ge 2$ and $p < q \le 2$).

(18) gives the required inequality.

Case B.

Thus $d = 2$ and p is near 2.

Going back to (3) and applying (1), (4) of Section 4, we obtain

$$\begin{aligned}\varepsilon.(1) &\gtrsim \sum_r (2^{-r}\|f_r\|_{W^{1,p}})^p\\ &\gtrsim \Big(\sum_r 4^{-r}\sum_j \|\Delta_j f_r\|_p^2 4^j\Big)^{\frac{p}{2}}\\ &\gtrsim \Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^2\Big)^{\frac{p}{2}}\\ &\gtrsim q^{-\frac{p}{2}}\|f\|_q^p \end{aligned}\tag{19}$$

where

$$q^{-\frac{p}{2}} = \left(\frac{1}{p} - \frac{s}{2}\right)^{\frac{p}{2}} \sim (2-ps)^{p-1} \tag{20}$$

which again gives the required inequality.

6. Proof of Theorem 1 when $p \geq 2$.

¿From (3) in Section 5, we get now the minoration

$$\varepsilon.(1) \gtrsim \sum_j (2^{sj} \|\Delta_j f\|_p)^p \tag{1}$$

which we use to majorize $\|f\|_q$.

We have already inequality (4) in Section 5, thus

$$\|f\|_q \leq C\sqrt{q}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^2\Big)^{1/2}. \tag{2}$$

Our aim is to prove that

$$\|f\|_q \leq Cq^{1-\frac{1}{p}}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^p\Big)^{\frac{1}{p}} \tag{3}$$

which will give the required inequality together with (1).

Using interpolation for $2 \leq p < \frac{d}{s}$, it clearly suffices to establish (3) for large values of q. To prove (3), we assume $2 \leq p \leq 4$ (other cases may be treated by adaption of the argument presented below). Assume further (taking previous comment into account)

$$q \geq 2p. \tag{4}$$

Again by interpolation, (3) will follow from (2) and the inequality

$$\|f\|_q \leq Cq^{\frac{3}{4}}\Big(\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big)^{1/4}. \tag{5}$$

We use the notation from Section 4 and start from the martingale square function inequality (3) in Section 4; thus

$$\|f\|_q \leq C\sqrt{q}\Big\|\Big(\sum |\widetilde{\Delta}_j f|^2\Big)^{1/2}\Big\|_q. \tag{6}$$

Write

$$|\widetilde{\Delta}_j f| \leq \sum_k |\widetilde{\Delta}_j \Delta_k f| = \sum_{m\in\mathbb{Z}} |\widetilde{\Delta}_j \Delta_{j+m} f|$$

(putting $\Delta_k = 0$ for $k < 0$).

Writing

$$\Big\|\Big(\sum_j |\widetilde{\Delta}_j f|^2)^{1/2}\Big\|_q \leq \sum_{m\in\mathbb{Z}} \Big\|\Big(\sum_j |\widetilde{\Delta}_j \Delta_{j+m} f|^2\Big)^{1/2}\Big\|_q \tag{7}$$

we estimate each summand.

Fix m. Write

$$\Big\|\Big(\sum_j |\widetilde{\Delta}_j\Delta_{j+m}f|^2\Big)^{1/2}\Big\|_q^4 = \Big\|\Big(\sum_j |\widetilde{\Delta}_j\Delta_{j+m}f|^2\Big)^2\Big\|_{\frac{q}{4}}$$

$$\leq 2\sum_{j_1\leq j_2} \|\, |\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f|^2\, |\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^2\|_{\frac{q}{4}} \tag{8}$$

and

$$\|\, |\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f|^2\, |\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^2\|_{\frac{q}{4}} = \Big[\int |\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f|^{\frac{q}{2}}.\mathbb{E}_{j_1}\big[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^{\frac{q}{2}}\big]\Big]^{\frac{4}{q}}$$

$$\leq \|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2\ \big\|\big(\mathbb{E}_{j_1}[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^{\frac{q}{2}}]\big)^{\frac{2}{q}}\big\|_q^2$$

$$\leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})}\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2\ \big\|\big(\mathbb{E}_{j_1}[|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f|^p]\big)^{1/p}\|_q^2$$

$$\leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})}\ 4^{dj_1(\frac{1}{p}-\frac{1}{q})}\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q^2\ \|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f\|_p^2. \tag{9}$$

Assume $\underline{m\leq 0}$

Estimate

$$\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q \lesssim 2^m\|\Delta_{j_1+m}f\|_q \leq 2^m 2^{d(j_1+m)(\frac{1}{p}-\frac{1}{q})}\|\Delta_{j_1+m}f\|_p \tag{10}$$

$$\|\widetilde{\Delta}_{j_2}\Delta_{j_2+m}f\|_p \lesssim 2^m\|\Delta_{j_2+m}f\|_p. \tag{11}$$

Substitution of (10), (11) in (9) gives

$$4^{(1-d(\frac{1}{p}-\frac{1}{q}))m+m}\ 4^{-\frac{d}{q}(j_2-j_1)}[2^{d(\frac{1}{p}-\frac{1}{q})(j_1+m)}\|\Delta_{j_1+m}f\|_p]^2\ [2^{d(\frac{1}{p}-\frac{1}{q})(j_2+m)}\|\Delta_{j_2+m}f\|_p]^2 \tag{12}$$

where

$$d\Big(\frac{1}{p}-\frac{1}{q}\Big) = s.$$

Summing (12) for $j_1<j_2$ and applying Cauchy-Schwartz implies for $m<0$

$$(8) < 4^{(2-s)m}\Big(\sum_{\ell\geq 0} 4^{-\frac{d}{q}\ell}\Big)\Big[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big]$$

$$\lesssim 4^{(2-s)m} q\Big[\sum_j (2^{sj}\|\Delta_j f\|_p)^4\Big]. \tag{13}$$

Assume next $\underline{m>0}$.

Estimate

$$\|\widetilde{\Delta}_{j_1}\Delta_{j_1+m}f\|_q \lesssim 2^{d_{j_1}(\frac{1}{p}-\frac{1}{q})}\|\Delta_{j_1+m}f\|_p$$

and

$$\begin{aligned}(9) &\leq 4^{d(j_2-j_1)(\frac{1}{p}-\frac{2}{q})}\ 16^{dj_1(\frac{1}{p}-\frac{1}{q})}\|\Delta_{j_1+m}f\|_p^2\ \|\Delta_{j_2+m}f\|_p^2 \\ &\leq 16^{-ms}4^{-(j_2-j_1)\frac{d}{q}}\|2^{s(j_1+m)}\Delta_{j_1+m}f\|_p^2\ \|2^{s(j_2+m)}\Delta_{j_2+m}f\|_p^2. \end{aligned} \tag{14}$$

Summing over $j_1 < j_2$ implies that for $m > 0$

$$(8) \lesssim 16^{-ms}q\Big[\sum_j(2^{sj}\|\Delta_j f\|_p)^4\Big]. \tag{15}$$

Summing (13), (15) in m implies that

$$\begin{aligned}(7) &\leq \Big(\sum_{m\leq 0}2^{(1-\frac{s}{2})m}+\sum_{m>0}2^{-sm}\Big)q^{1/4}\Big[\sum_j(2^{sj}\|\Delta_j f\|_p)^4\Big]^{1/4} \\ &\leq q^{1/4}\Big[\sum_j(2^{sj}\|\Delta_j f\|_p)^4\Big]^{1/4}. \end{aligned} \tag{16}$$

To bound $\|f\|_q$, apply (6) which introduces an additional $q^{1/2}$-factor. This establishes (5) and completes the argument and the proof of Theorem 1.

7. Proof of Theorem 2.

We will make use of the following two lemmas

Lemma 2. *Let $I \subset \mathbb{R}$ be an interval and let $\psi : I \to \mathbb{Z}$ be any measurable function. Then, there is some $k \in \mathbb{Z}$ such that*

$$|\{x \in I; \psi(x) \neq k\}| \leq 2\Big(C^*\varepsilon\int_I\int_I\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2-\varepsilon}}dxdy\Big)^{1/\varepsilon},$$

for all $\varepsilon \in (0, 1/2]$. where C^* is the absolute constant in Corollary 1 (inequality (10) in Section 1).

Proof of Lemma 2. After scaling and shifting we may assume that $I = (-1, +1)$. For each $k \in \mathbb{Z}$, set

$$A_k = \{x \in I; \psi(x) < k\}.$$

Note that A_k is nondecreasing, $\lim_{k\to-\infty}|A_k| = 0$ and $\lim_{k\to+\infty}|A_k| = 2$. Thus, there exists some $k \in \mathbb{Z}$ such that

$$|A_k| \leq 1 \text{ and } |A_{k+1}| > 1. \tag{1}$$

Applying Corollary 1 with $A = A_k$ and with $A = A_{k+1}$ we find (using (1))

$$|A_k| \leq |A_k|\ |{}^cA_k| \leq \left(C^* \varepsilon \int_A \int_{{}^cA_k} \frac{dxdy}{|x-y|^{2-\varepsilon}} \right)^{1/\varepsilon} \tag{2}$$

and

$$|{}^cA_{k+1}| \leq |A_{k+1}|\ |{}^cA_{k+1}| \leq \left(C^* \varepsilon \int_{A_{k+1}} \int_{{}^cA_{k+1}} \frac{dxdy}{|x-y|^{2-\varepsilon}} \right)^{1/\varepsilon}. \tag{3}$$

On the other hand

$$|\psi(x) - \psi(y)| \geq 1 \text{ for a.e. } x \in A_k, y \in {}^cA_k$$

and

$$|\psi(x) - \psi(y)| \geq 1 \text{ for a.e. } x \in A_{k+1}, y \in {}^cA_{k+1}.$$

Therefore

$$\begin{aligned} |\{x \in I; \psi(x) \neq k\}| &= |A_k| + |{}^cA_{k+1}| \\ &\leq 2\left(C^* \varepsilon \int_I \int_I \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{2-\varepsilon}} dxdy \right)^{1/\varepsilon}. \end{aligned}$$

Lemma 3. *If $\alpha > 0, a < b < x, A \subset (a,b)$ is measurable, then*

$$\int_{(a,b)\setminus A} \frac{dy}{(x-y)^\alpha} \geq \int_a^{b-|A|} \frac{dy}{(x-y)^\alpha}$$

and similarly, if $x < a < b$, then

$$\int_{(a,b)\setminus A} \frac{dy}{(y-x)^\alpha} \geq \int_{a+|A|}^b \frac{dy}{(y-x)^\alpha}.$$

The proof of Lemma 3 is elementary and left to the reader.

Proof of Theorem 2. Let $\psi_\varepsilon : \Omega = (-1,+1) \to \mathbb{R}$ be any measurable function such that $u_\varepsilon = e^{i\psi_\varepsilon}$. We have to prove that for all $\varepsilon < 1/2$,

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} \geq c\varepsilon^{-1/2} \tag{4}$$

for some absolute constant c to be determined.

We argue by contradiction and assume that for some $\varepsilon < 1/2$

$$\|\psi_\varepsilon\|_{H^{(1-\varepsilon)/2}(\Omega)} < \eta\varepsilon^{-1/2}. \tag{5}$$

We will reach a contradiction if η is less than some absolute constant. Set

$$\psi = \frac{1}{2\pi}(\psi_\varepsilon - \varphi_\varepsilon)$$

so that $\psi : \Omega \to \mathbb{Z}$; recall that $u_\varepsilon = e^{i\varphi_\varepsilon}$ and the function φ_ε is defined by

$$\varphi_\varepsilon(x) = \begin{cases} 0 & \text{for } -1 < x < 0, \\ 2\pi x/\delta & \text{for } 0 < x < \delta, \\ 2\pi & \text{for } \delta < x < 1, \end{cases}$$

where $\delta = e^{-1/\varepsilon}$.

A straightforward computation (using the fact that ψ takes its values into $\mathbb{Z}$) shows that

$$|\psi(x) - \psi(y)| \le |\psi_\varepsilon(x) - \psi_\varepsilon(y)| \text{ for a.e. } x, y \in \left(-1, \frac{2\delta}{3}\right) \tag{7}$$

and

$$|\psi(x) - \psi(y)| \le |\psi_\varepsilon(x) - \psi_\varepsilon(y)| \text{ for a.e. } x, y \in \left(\frac{\delta}{3}, 1\right). \tag{8}$$

Applying Lemma 2 with $I = (-1, \frac{2\delta}{3})$ and $I = (\frac{\delta}{3}, 1)$, together with (5), (7) and (8) yields the existence of $\ell, m \in \mathbb{Z}$ such that

$$\left|\left\{x \in \left(-1, \frac{2\delta}{3}\right); \psi(x) \neq \ell\right\}\right| \le 2(C^*\eta^2)^{1/\varepsilon}$$

and

$$\left|\left\{x \in \left(x \in \frac{\delta}{3}, 1\right); \psi(x) \neq m\right\}\right| \le 2(C^*\eta^2)^{1/\varepsilon}.$$

We choose η in such a way that

$$4(C^*\eta^2)^{1/\varepsilon} < \delta/3, \text{ for } \varepsilon < 1/2,$$

for example

$$\eta^2 < 1/4eC^*. \tag{9}$$

It follows that $\ell = m$. Without loss of generality (after adding a constant to ψ_ε) we may assume that

$$\ell = m = 0. \tag{10}$$

Therefore

$$\psi_\varepsilon(x) = \varphi_\varepsilon(x) \text{ for } x \in [(-1, 0)\backslash A] \cup [(\delta, 1)\backslash B] \tag{11}$$

where

$$A = \{x \in (-1, 0); \psi(x) \neq 0\}$$

and

$$B = \{x \in (\delta, 1); \psi(x) \neq 0\}$$

with

$$|A| < \delta/6, |B| < \delta/6. \tag{12}$$

¿From (11) and the definition of φ_ε we have

$$\begin{aligned}
\varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dxdy &\geq \varepsilon \int_{-1}^0 dx \int_0^1 \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dy \\
&\geq \varepsilon \int_{(-1,0)\setminus A} dx \int_{(\delta,1)\setminus B} \frac{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dy \\
&\geq \varepsilon \int_{(-1,0)\setminus A} dx \int_{(\delta,1)\setminus B} \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}}.
\end{aligned}$$

Applying Lemma 3 and (5) we find

$$\begin{aligned}
\eta^2 > \varepsilon \int_\Omega \int_\Omega \frac{|\psi_\varepsilon(x) - \psi_\varepsilon(y)|^2}{|x-y|^{2-\varepsilon}} dxdy &\geq \varepsilon \int_{-1}^{-|A|} dx \int_{\delta+|B|}^1 \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}} \\
&\geq \varepsilon \int_{-1}^{-\delta/6} dx \int_{\delta+\delta/6}^1 \frac{4\pi^2 dy}{|x-y|^{2-\varepsilon}} = 4\pi^2(1-e^{-1}) + o(1)
\end{aligned}$$

as $\varepsilon \to 0$. We obtain a contradiction for an appropriate choice of η.

APPENDIX. Proof of square function inequality

Let $\{\mathcal{F}_n\}_{n=0,1,2,\ldots}$ be refining finite partitions such that

$$\begin{aligned}
&\#\mathcal{F}_n = K^n \\
&|Q| = K^{-n} \quad \text{if } Q \text{ is an } \mathcal{F}_n\text{-atom}
\end{aligned}$$

(If $\Omega = [0,1]^d, K = 2^d$).

Denote $\mathbb{E}_n$ the $\mathcal{F}_n$-expectation

$$\Delta_n f = \mathbb{E}_n f - \mathbb{E}_{n-1} f \qquad \text{(we used the notation } \widetilde{\Delta}_n f \text{ in Section 4)}$$

$$Sf = \left(\sum |\Delta_n f|^2 \right)^{1/2} \qquad \text{(the square function)}$$

$$|f| \underset{a.e.}{\leq} f^* = \sup |\mathbb{E}_n f| \qquad \text{(the maximal function)}$$

Proposition 1.

$$\text{mes}\,[|f| > \lambda \|Sf\|_\infty] < e^{-c\lambda^2} \qquad (\lambda \geq 1) \tag{1}$$

where $c = c(K) > 0$ is a constant.

Proposition 2. *(good-λ inequality)*

$$\text{mes}\,[f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{n-1}[|\Delta_n f|] < \varepsilon\lambda] < e^{-\frac{c}{\varepsilon^2}}\,\text{mes}\,[f^* > \lambda] \qquad (0 < \varepsilon < 1) \tag{2}$$

Proposition 3.

$$\|f^*\|_q \leq C\sqrt{q}\|Sf\|_q \quad \textit{for } q \geq 2 \tag{3}$$

We follow essentially [4].

Proof of Proposition 1.

One verifies that there is a constant $A = A(K)$ such that if φ is $\mathcal{F}_n$-measurable and $\mathbb{E}_{n-1}\varphi = 0$, then

$$\mathbb{E}_{n-1}[e^{\varphi - A\varphi^2}] \leq 1. \tag{4}$$

Hence

$$\mathbb{E}_{n-1}[e^{\Delta_n f - A(\Delta_n f)^2}] \leq 1 \tag{5}$$

and, denoting $S_n f = \left(\sum_{m\leq n} |\Delta_m f|^2\right)^{1/2}$,

$$\begin{aligned}\int e^{\mathbb{E}_n f - A(S_n f)^2} &= \int e^{\mathbb{E}_{n-1} f - A(S_{n-1} f)^2} \mathbb{E}_{n-1}[e^{\Delta_n f - A(\Delta_n f)^2}] \\ &\leq \int e^{\mathbb{E}_{n-1}} f - A(S_{n-1} f)^2 \qquad \text{(by (5))} \\ &\leq 1.\end{aligned}$$

Thus

$$\int e^{f - A(Sf)^2} \leq 1. \tag{6}$$

Assume $\|Sf\|_\infty \leq 1$. Applying (6) to tf ($t > 0$ a parameter), we get

$$\int e^{tf} \leq e^{At^2}$$

$$\text{mes}\ [f > \lambda] \leq e^{At^2 - t\lambda}$$

and for appropriate choice of t

$$\text{mes}\ [f > \lambda] < e^{-\frac{\lambda^2}{4A}}.$$

This proves (1).

Proof of Proposition 2.

This is a standard stopping time argument.

Consider a collection of maximal atoms $\{Q_\alpha\} \subset \bigcup \mathcal{F}_n$ s.t. if Q_α is an $\mathcal{F}_n$-atom, then $|\mathbb{E}_n f| > \lambda$ on Q_α. Thus $Q_\alpha \cap Q_\beta = \phi$ for $\alpha \neq \beta$. Fix α. From the maximality

$$|\mathbb{E}_{n-1} f| \leq \lambda \text{ on } Q_\alpha. \tag{7}$$

Therefore

$$\begin{aligned}&[f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda] \cap Q_\alpha \subset \\ &[(f - \mathbb{E}_n f)^* > (1-\varepsilon)\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda] \cap Q_\alpha = (8)\end{aligned}$$

For $m > n$, denote χ_m the indicator function of the set

$$Q_\alpha \cap \Bigg[\Bigg(\sum_{\ell=n+1}^{m-1} |\Delta_\ell f|^2\Bigg)^{1/2} < \varepsilon\lambda\Bigg] \cap \Bigg[\mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda\Bigg] \cap \bigcap_{n \leq \ell < m} [|\mathbb{E}_\ell f - \mathbb{E}_n f| \leq (1-\varepsilon)\lambda] = (9).$$

Thus

$$\chi_m = \mathbb{E}_{m-1}\ \chi_m$$

and

$$g = \sum_{m>n} \chi_m \Delta_m f$$

is an $\{\mathcal{F}_m | m \geq n\}$-martingale on Q_α.

¿From the definition of χ_m, we have clearly

$$S(g) = \Bigg(\sum_{m>n} \chi_m |\Delta_m f|^2\Bigg)^{1/2} < \varepsilon\lambda + \varepsilon\lambda \lesssim \varepsilon\lambda \tag{10}$$

and

$$|g| > (1-\varepsilon)\lambda \text{ on the set (8).}$$

¿From Proposition 1 and (10)

$$\text{mes } [x\ \varepsilon Q_\alpha \big|\ |g| > (1-\varepsilon)\lambda] < e^{-\frac{c}{\varepsilon^2}} |Q_\alpha| \tag{11}$$

hence

$$\text{mes } (8) \lesssim e^{-\frac{c}{\varepsilon^2}} |Q_\alpha|. \tag{12}$$

Summing (12) over α implies

$$\text{mes } [f^* > 2\lambda, Sf < \varepsilon\lambda, \sup \mathbb{E}_{m-1}[|\Delta_m f|] < \frac{1}{K}\varepsilon\lambda] < e^{-\frac{c}{\varepsilon^2}} \sum |Q_\alpha| \leq e^{-\frac{c}{\varepsilon^2}} \text{mes } [f^* > \lambda]$$

which is (2).

Proof of Proposition 3.

$$
\begin{aligned}
\|f^*\|_q^q &= q\int \lambda^{q-1}\text{mes }\ [f^* > \lambda]d\lambda \\
&= 2^q q\int \lambda^{q-1}\text{mes }\ [f^* > 2\lambda]d\lambda \\
&\le 2^q q\int \lambda^{q-1}\{\text{mes }\ [Sf \ge \varepsilon\lambda] + \text{mes }\ [\sup \mathbb{E}_{n-1}[|\Delta_n f|] \ge \frac{\varepsilon}{K}\lambda] + e^{-\frac{c}{\varepsilon^2}}\text{mes }\ [f^* > \lambda]\} \\
&< \left(\frac{2}{\varepsilon}\right)^q (\|Sf\|_q^q + K^q\|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q) + 2^q e^{-\frac{c}{\varepsilon^2}}\|f^*\|_q^q \qquad (13)
\end{aligned}
$$

Take $\frac{1}{\varepsilon} \sim \sqrt{q}$ so that the last term in (13) is at most $\frac{1}{2}\|f^*\|_q^q$. Thus

$$
\|f^*\|_q < C\sqrt{q}(\|Sf\|_q + \|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q). \qquad (14)
$$

Also

$$
\begin{aligned}
\|\sup \mathbb{E}_{n-1}[|\Delta_n f|]\|_q &\le \left(\sum_n \|\mathbb{E}_{n-1}[|\Delta_n f|]\|_q^q\right)^{1/q} \\
&\le \left(\sum_n \|\Delta_n f\|_q^q\right)^{1/q} \\
&\le \|Sf\|_q. \qquad (15)
\end{aligned}
$$

and (3) follows from (14), (15).

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