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Notation.

1_E the function defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$;

$\overline{E}$ the closure of the subset E of the topological space X;

$C(E, F)$ the space of continuous functions from the topological space E to the topological space F;

$C_{\mathrm{b}}(E, F)$ the Banach space of continuous, bounded functions from the topological space E to the Banach space F, equipped with the topology of uniform convergence;

$C_{\mathrm{c}}(E, F)$ the space of continuous functions $E \to F$ compactly supported in E.

$\mathcal{L}(E, F)$ the Banach space of linear, continuous operators from the Banach space E to the Banach space F, equipped with the norm topology;

$\mathcal{L}(E)$ the space $\mathcal{L}(E, E)$;

$X^\star$ the topological dual of the space X;

$X \hookrightarrow Y$ if $X \subset Y$ with continuous injection;

Ω an open subset of $\mathbb{R}^N$;

$\overline{\Omega}$ the closure of Ω in $\mathbb{R}^N$;

$\partial\Omega$ the boundary of Ω, that is $\partial\Omega = \overline{\Omega} \setminus \Omega$;

$\omega \subset\subset \Omega$ if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact;

$\partial_t u = u_t = \dfrac{\partial u}{\partial t} = \dfrac{du}{dt}$;

$\partial_i u = u_{x_i} = \dfrac{\partial u}{\partial x_i}$;

$\partial_r u = u_r = \dfrac{\partial u}{\partial r} = \dfrac{1}{r} x \cdot \nabla u$, where $r = |x|$;

$D^\alpha = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$;

$\nabla u = (\partial_1 u, \cdots, \partial_N u)$;

$\triangle = \displaystyle\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$;

$\mathcal{F}$ the Fourier transform in $\mathbb{R}^N$, defined by* $\mathcal{F}u(\xi) = \displaystyle\int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} u(x)\, dx$;

$\overline{\mathcal{F}} = \mathcal{F}^{-1}$ given by $\overline{\mathcal{F}}v(x) = \displaystyle\int_{\mathbb{R}^N} e^{2\pi i \xi \cdot x} v(\xi)\, d\xi$;

$\widehat{u} = \mathcal{F}u$;

$C_{\mathrm{c}}(\Omega)$ $= C_{\mathrm{c}}(\Omega, \mathbb{R})$ (or $C_{\mathrm{c}}(\Omega, \mathbb{C})$);

$C_{\mathrm{b}}(\Omega)$ $= C_{\mathrm{b}}(\Omega, \mathbb{R})$ (or $C_{\mathrm{b}}(\Omega, \mathbb{C})$);

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* with this definition of the Fourier transform, $\|\mathcal{F}\|_{\mathcal{L}(L^2)} = 1$, $\mathcal{F}(u \star v) = \mathcal{F}u\mathcal{F}v$ and $\mathcal{F}(D^\alpha u) = (2\pi i)^{|\alpha|} \prod_{j=1}^N x_j^{\alpha_j} \mathcal{F}u$.

$C^m_{\rm b}(\Omega) = \{u \in C_{\rm b}(\Omega);\, D^\alpha u \in C_{\rm b}(\Omega)$ for all $\alpha \in \mathbb{N}^N$ such that $|\alpha| \le m\}$, equipped with the norm $\|u\|_{C^m_{\rm b}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^\infty}$;

$C(\overline{\Omega})$ the space of continuous functions $\overline{\Omega} \to \mathbb{R}$ (or $\overline{\Omega} \to \mathbb{C}$). When Ω is bounded, $C(\overline{\Omega})$ is a Banach space when equipped with the L^∞ norm;

$C_{\rm b,u}(\overline{\Omega})$ the Banach space of uniformly continuous and bounded functions $\overline{\Omega} \to \mathbb{R}$ (or $\overline{\Omega} \to \mathbb{C}$) equipped with the topology of uniform convergence;

$C^m_{\rm b,u}(\overline{\Omega})$ the Banach space of functions $u \in C_{\rm b,u}(\overline{\Omega})$ such that $D^\alpha u \in C_{\rm b,u}(\overline{\Omega})$, for every multi-index α such that $|\alpha| \le m$. $C^m_{\rm b,u}(\overline{\Omega})$ is equipped with the norm of $W^{m,\infty}(\Omega)$;

$C_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $L^\infty(\Omega)$;

$C^{m,\alpha}(\overline{\Omega})$ for $0 \le \alpha \le 1$, the Banach space of functions $u \in C^m_{\rm b,u}(\overline{\Omega})$ such that

$$\|u\|_{C^{m,\alpha}} = \|u\|_{W^{m,\infty}} + \sup\left\{ \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha};\, x, y \in \Omega,\, |\beta| = m \right\} < \infty.$$

$\mathcal{D}(\Omega) = C^\infty_{\rm c}(\Omega)$ the Fréchet space of C^∞ functions $\Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) compactly supported in Ω, equipped with the topology of uniform convergence of all derivatives on compact subsets of Ω;

$\mathcal{D}'(\Omega)$ the space of distributions on Ω, that is the topological dual of $\mathcal{D}(\Omega)$;

$\mathcal{S}(\mathbb{R}^N)$ the Schwartz space, that is the space of $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$ (or $C^\infty(\mathbb{R}^N, \mathbb{C})$) such that for every nonnegative integer m and every multi-index α,

$$p_{m,\alpha}(u) = \sup_{x\in\mathbb{R}^N} (1 + |x|^2)^{m/2} |D^\alpha u(x)| < \infty.$$

$\mathcal{S}(\mathbb{R}^N)$ is a Fréchet space when equipped with the seminorms $p_{m,\alpha}$;

$\mathcal{S}'(\mathbb{R}^N)$ the space of tempered distributions on $\mathbb{R}^N$, that is the topological dual of $\mathcal{S}(\mathbb{R}^N)$. $\mathcal{S}'(\mathbb{R}^N)$ is a subspace of $\mathcal{D}'(\mathbb{R}^N)$;

p' the conjugate of p given by $\frac{1}{p} + \frac{1}{p'} = 1$;

$L^p(\Omega)$ the Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that $\int_\Omega |u(x)|^p\, dx < \infty$ if $1 \le p < \infty$, or $\operatorname{ess\,sup}_\Omega |u| < \infty$ if $p = \infty$. $L^p(\Omega)$ is equipped with the norm

$$\|u\|_{L^p} = \begin{cases} \left(\int_\Omega |u(x)|^p\, dx \right)^{1/p}, & \text{if } p < \infty; \\ \operatorname{ess\,sup}_\Omega |u|, & \text{if } p = \infty. \end{cases}$$

$W^{m,p}(\Omega)$ the Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index α with $|\alpha| \le m$. $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^p}.$$

$W^{m,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$;

$W^{-m,p'}(\Omega)$ the dual of $W_0^{m,p}(\Omega)$;

$H^m(\Omega) = W^{m,2}(\Omega)$ $H^m(\Omega)$ is equipped with the equivalent norm

$$\|u\|_{H^m} = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{1/2}.$$

$H^m(\Omega)$ is a Hilbert space for the scalar product

$$(u, v)_{H^m} = \int_\Omega \mathrm{Re}(u(x)\overline{v(x)}) \, dx.$$

$H_0^m(\Omega) = W_0^{m,2}(\Omega)$;

$H^{-m}(\Omega) = W^{-m,2}(\Omega)$;

$\mathcal{D}(I, X) = C_c^\infty(I, X)$ the Fréchet space of C^∞ functions $I \to X$ compactly supported in I, equipped with the topology of uniform convergence of all derivatives on compact subintervals of I;

$L^p(I, X)$ the Banach space of (classes of) measurable functions $u : I \to X$ such that $\displaystyle\int_I \|u(t)\|_X^p \, dt < \infty$ if $1 \le p < \infty$, or $\operatorname{ess\,sup}_I \|u(t)\|_X < \infty$ if $p = \infty$. $L^p(I, X)$ is equipped with the norm

$$\|u\|_{L^p} = \begin{cases} \left(\displaystyle\int_I \|u(t)\|_X^p \, dt \right)^{1/p}, & \text{if } p < \infty; \\ \operatorname{ess\,sup}_I \|u(t)\|_X, & \text{if } p = \infty. \end{cases}$$

$W^{m,p}(I, X)$ the Banach space of (classes of) measurable functions $u : I \to X$ such that $\dfrac{d^j u}{dt^j} \in L^p(I, X)$ for every $0 \le j \le m$. $W^{m,p}(I, X)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{j=1}^m \|\frac{d^j u}{dt^j}\|_{L^p}.$$

$C_{\mathrm{b,u}}(\overline{I}, X)$ the Banach space of uniformly continuous and bounded functions $\overline{I} \to X$, equipped with the topology of uniform convergence;

$C^m_{\mathrm{b,u}}(\overline{I}, X)$ the Banach space of functions $u \in C_{\mathrm{b,u}}(\overline{I}, X)$ such that $\dfrac{d^j u}{dt^j} \in C_{\mathrm{b,u}}(\overline{I}, X)$, for every $0 \le j \le m$. $C^m_{\mathrm{b,u}}(\overline{I}, X)$ is equipped with the norm of $W^{m,\infty}(I, X)$;

$C^{m,\alpha}(\overline{I}, X)$ for $0 \le \alpha \le 1$, the Banach space of functions $u \in C^m_{\mathrm{b,u}}(\overline{I}, X)$ such that

$$\|u\|_{C^{m,\alpha}} = \|u\|_{W^{m,\infty}} + \sup \left\{ \frac{|\frac{d^m u(t)}{dt^m} - \frac{d^m u(s)}{dt^m}|}{|t - s|^\alpha}; \, s, t \in I \right\} < \infty.$$

$C(\overline{I}, X)$ the space of continuous functions $\overline{I} \to X$. When I is bounded, $C(\overline{I}, X)$ is a Banach space when equipped with the norm of $L^\infty(I, X)$.

$D(A)$ the domain of the operator A.

$R(A)$ the range of the operator A.

$J_\lambda(A)$ $= (I + \lambda A)^{-1}$, when A is an m-accretive operator.

A_λ $= A(I + \lambda A)^{-1}$, the Yosida approximation of the m-accretive operator A.

Chapter 1. Linear semigroups of contractions; the Hille-Yosida theory and some applications.

1.1. m-accretive operators. Throughout this section, X is a Banach space, endowed with the norm $\| \, \|$.

1.1.1. Unbounded operators in Banach spaces.

Definition 1.1.1. *A linear unbounded operator in X is a pair (D, A), where D is a linear subspace of X and A is a linear mapping $D \to X$. If $\sup\{\|Ax\|;\ x \in D, \|x\| \le 1\} < \infty$, A is bounded. If $\sup\{\|Ax\|;\ x \in D, \|x\| \le 1\} = \infty$, A is not bounded.*

Remark 1.1.2. It is clear that A is bounded if, and only if there exists a closed linear subspace Y of X such that $D \subset Y$ and an operator $\overline{A} \in \mathcal{L}(Y, X)$ such that $Ax = \overline{A}x$, for all $x \in D$.

Definition 1.1.3. *Let (D, A) be a linear unbounded operator in X. The domain $D(A)$ of A is the set*

$$D(A) = D,$$

the range $R(A)$ of A is the set

$$R(A) = A(D),$$

the nullspace $N(A)$ of A is the set

$$N(A) = \{u \in D(A);\ Au = 0\},$$

and the graph $G(A)$ of A is the set

$$G(A) = \{(x, f) \in X \times X;\ x \in D \text{ and } f = Ax\}.$$

$D(A)$, $R(A)$ and $N(A)$ are linear subspaces of X, and $G(A)$ is a linear subspace of $X \times X$. If $G(A)$ is closed in $X \times X$, we say that A is closed.

Remark 1.1.4. The pair (D, A) is often called "the operator A with domain $D(A) = D$" or just "the operator A". However, one must be aware that an operator is not only defined by the values Ax, but also with its domain. In other words, when one defines an operator, it is **absolutely necessary** to define its domain. In particular, the same "formula" can define several operators, depending on what the domain is. For example, let $X = L^2(\mathbb{R}^N)$. Let A_1 be defined by $D(A_1) = X$ and $A_1 u = u$, for all $u \in X$ (A_1 is the identity on X) and let A_2 be defined by $D(A_2) = \{u \in H^1(\mathbb{R}^N);\ u(x) = 0 \text{ for almost all } |x| \ge 1\}$ and $A_2 u = u$, for all $u \in D(A_2)$. Both A_1 and A_2 are defined by the same formula, but A_1 and A_2 have different properties. For example, the domain of A_1 is dense in X, while the domain of A_2 is not. The graph of A_1 is closed in $X \times X$, while the graph of A_2 is not. It is a good exercise for the reader to determine $\overline{D(A)}$ and $\overline{G(A)}$. A_1 is m-accretive, while A_2 is not (see below).That example is rather trivial, but we will see some nontrivial examples in Section 1.2.

Remark 1.1.5. When there is no risk of confusion, a *linear unbounded operator* in X is just called a *linear operator* in X or even an *operator* in X.

1.1.2. m-accretive operators in Banach spaces.

Definition 1.1.6. *An operator A in X with domain $D(A)$ is accretive*[1] *if*

$$\|x + \lambda Ax\| \geq \|x\|,$$

for all $x \in D(A)$ and all $\lambda > 0$.

Definition 1.1.7. *An operator A in X is m-accretive*[1] *if the following holds:*

(i) *A is accretive,*

(ii) *for all $\lambda > 0$ and all $f \in X$, there exists $x \in D(A)$ such that $x + \lambda Ax = f$.*

Lemma 1.1.8. If A is an m-accretive operator in X, then for every $\lambda > 0$ and every $f \in X$, there exists a unique solution $x \in D(A)$ of equation

$$x + \lambda Ax = f.$$

In addition $\|x\| \leq \|f\|$. In particular, given $\lambda > 0$, the mapping $f \mapsto x$ is a contraction $X \to X$, and is one to one $X \to D(A)$.

Proof. The result follows immediately from Definitions 1.1.6 and 1.1.7. □

Definition 1.1.9. *Let A be an m-accretive operator in X. Given $\lambda > 0$, the mapping $f \mapsto x$ defined in Lemma 1.1.8 is denoted by $J_\lambda(A)$ (or J_λ when there is no risk of confusion), or $(I + \lambda A)^{-1}$. We have $J_\lambda \in \mathcal{L}(X)$, $\|J_\lambda\|_{\mathcal{L}(X)} \leq 1$, and $R(J_\lambda) = D(A)$. J_λ is called the resolvent of A*

Proposition 1.1.10. *If A is an m-accretive operator in X, then the graph $G(A)$ of A is closed in $X \times X$.*

Proof. Since the operator J_1 is continuous, its graph is closed, and since $R(J_1) = D(A)$, this means that the set $\{(x, f) \in X \times X;\, x \in D(A) \text{ and } f = x + Ax\}$ is closed in $X \times X$. Therefore, the set $\{(x, f) \in X \times X;\, x \in D(A) \text{ and } f = Ax\}$ is closed in $X \times X$. This proves the result. □

Remark 1.1.11. Let A be an m-accretive operator in X (or, more generally, a closed operator), and suppose X is reflexive. Consider a family $(x_\varepsilon)_{\varepsilon > 0} \subset D(A)$. If $x_\varepsilon \rightharpoonup x$ in X as $\varepsilon \downarrow 0$, and if Ax_ε is bounded in X, then $x \in D(A)$ and $Ax_\varepsilon \rightharpoonup Ax$ in X as $\varepsilon \downarrow 0$. Indeed, there exists a sequence $\varepsilon_n \downarrow 0$ and $y \in X$ such that $Ax_{\varepsilon_n} \rightharpoonup y$ in X, as $n \to \infty$. In particular, $(x_{\varepsilon_n}, Ax_{\varepsilon_n}) \rightharpoonup (x, y)$ in $X \times X$, as $n \to \infty$. On the other hand, since $G(A)$ is closed in $X \times X$, it is also closed for the weak topology of $X \times X$; and so, $x \in D(A)$ and $y = Ax$. Finally, one shows easily with the same argument that the whole family $(Ax_\varepsilon)_{\varepsilon > 0}$ converges to Ax.

[1] *Some authors say that A is dissipative (respectively, m-dissipative), if and only if $-A$ is accretive (respectively, m-accretive).*

Corollary 1.1.12. *Let A be an m-accretive operator in X. For every $x \in D(A)$, let $\|x\|_{D(A)} = \|x\| + \|Ax\|$ and $|||x|||_{D(A)} = \|x + Ax\|$. It follows that*

(i) *$\| \|_{D(A)}$ is a norm on $D(A)$, and $(D(A), \| \|_{D(A)})$ is a Banach space. $\| \cdot \|_{D(A)}$ is called the graph norm;*

(ii) *$D(A) \hookrightarrow X$ (with the graph norm);*

(iii) *the restriction of A to $D(A)$ is continuous $D(A) \to X$, (with the graph norm) and $\|A\|_{\mathcal{L}(D(A),X)} \leq 1$.*

(iv) *$||| \, |||_{D(A)}$ is an equivalent norm on $D(A)$;*

(v) *J_1 is an isomorphism from X to $D(A)$ (with the graph norm).*

Proof. It is clear that $\| \|_{D(A)}$ is a norm on $D(A)$. Furthermore, the mapping

$$D(A) \to X \times X$$
$$g : x \mapsto (x, Ax)$$

satisfies $\|g(x)\|_{X \times X} = \|x\|_{D(A)}$. Since $g(D(A)) = G(A)$, which is closed by Proposition 1.1.10, it follows that $(D(A), \| \|_{D(A)})$ is a Banach space. This proves (i). (ii) follows from inequality $\|x\| \leq \|x\|_{D(A)}$, while (iii) follows from inequality $\|Ax\| \leq \|x\|_{D(A)}$. It is clear that $|||x|||_{D(A)} \leq \|x\|_{D(A)}$, and also

$$\|x\|_{D(A)} \leq 2\|x\| + |||x|||_{D(A)} \leq 3\|x\|_{D(A)},$$

since A is accretive. This proves (iv). Finally, since $R(J_1) = D(A)$ by Lemma 1.1.8, it is immediate that $|||J_1 x|||_{D(A)} = \|x\|$, for all $x \in X$. Thus, J_1 is an isometry from X onto $D(A)$ equipped with the equivalent norm $||| \, |||_{D(A)}$. This completes the proof. □

Remark 1.1.13. Throughout the rest of this book, we will always consider $D(A)$ as the Banach space $(D(A), \| \|_{D(A)})$.

Corollary 1.1.14. *If A is an m-accretive operator in X, then*

(i) *$\|J_1 x\|_{D(A)}$ defines a norm on X, which is equivalent to the original norm $\| \|$;*

(ii) *$J_\lambda \in \mathcal{L}(X, D(A))$, for every $\lambda > 0$.*

Proof. It follows from Corollary 1.1.12 (iv) that $|||J_1 x|||_{D(A)} = \|x\|$. Hence (i). Given $\lambda > 0$ and $x \in X$, we have $\lambda A J_\lambda x = x - J_\lambda x$; and so,

$$\|J_\lambda x\|_{D(A)} = \|J_\lambda x\| + \frac{1}{\lambda}\|x - J_\lambda x\| \leq \left(1 + \frac{2}{\lambda}\right)\|x\|.$$

Hence (ii). □

Definition 1.1.15. *Let A be an m-accretive operator in X, and let J_λ be as in Definition 1.1.9. For every $x \in X$ and $\lambda > 0$, one defines $A_\lambda x \in X$ by $A_\lambda x = A J_\lambda x$. A_λ is called the Yosida approximation of A.*

Lemma 1.1.16. *Let A be an m-accretive operator in X, and let A_λ be as above. The following properties hold:*

(i) $A_\lambda x = \dfrac{x - J_\lambda x}{\lambda}$, *for every* $x \in X$;

(ii) $A_\lambda \in \mathcal{L}(X)$ *and* $\|A_\lambda\|_{\mathcal{L}(X)} \le \dfrac{2}{\lambda}$, *for all* $\lambda > 0$;

(iii) $A_\lambda x = J_\lambda A x$, *for every* $x \in D(A)$;

(iv) $(J_\lambda)_{|D(A)} \in \mathcal{L}(D(A))$ *and* $\|(J_\lambda)_{|D(A)}\|_{\mathcal{L}(D(A))} \le 1$, *for every* $\lambda > 0$;

(v) A_λ *is* m*-accretive.*

Proof. Let $x \in X$ and $z = J_\lambda x$. Since $z + \lambda A z = x$, we have $\lambda A_\lambda x = \lambda A z = x - z$. This proves (i), and (ii) follows immediately. Finally, if $x \in D(A)$ and $z = J_\lambda x$, then

$$z + \lambda A z = x.$$

Since both x and z belong to $D(A)$, it follows that $Az \in D(A)$ and that

$$Az + \lambda A(Az) = Ax.$$

Now let $w = J_\lambda A x$. Since

$$w + \lambda A w = Ax,$$

we have

$$(w - Az) + \lambda(w - Az) = 0.$$

Since A is accretive, it follows that $w = Az$, which proves (iii). Next,

$$\|J_\lambda x\|_{D(A)} = \|J_\lambda x\| + \|A_\lambda x\| = \|J_\lambda x\| + \|A J_\lambda x\| \le \|x\| + \|Ax\| = \|x\|_{D(A)},$$

from which (iv) follows. Consider now $\mu > 0$. Given $x \in X$, it follows from (i) that

$$x + \mu A_\lambda x = \Big(1 + \frac{\mu}{\lambda}\Big) x - \frac{\mu}{\lambda} J_\lambda x;$$

and so,

$$\|x + \mu A_\lambda x\| \ge \Big(1 + \frac{\mu}{\lambda}\Big) \|x\| - \frac{\mu}{\lambda}\|J_\lambda x\| \ge \Big(1 + \frac{\mu}{\lambda}\Big) \|x\| - \frac{\mu}{\lambda}\|x\| = \|x\|.$$

Therefore, A_λ is accretive. Since $A_\lambda \in \mathcal{L}(X)$, it follows that A_λ is m-accretive (see Remark 1.1.22). □

Remark 1.1.17. If A is an m-accretive operator in X, and if X is reflexive, then one can show that $D(A)$ is dense in X. See Corollary 1.1.37 for the case of Hilbert spaces and Exercise 1.8.2 for the general case.

Remark 1.1.18. If X is a Hilbert space, then one can improve the estimate in (ii) above. In this case, $\|A_\lambda\|_{\mathcal{L}(X)} \le 1/\lambda$. Indeed, given $x \in X$, let $f = J_\lambda x$, so that $f + \lambda A f = x$. Taking the scalar product with Af, we obtain $\lambda \|Af\|^2 + (x, Af) \le \|x\|\,\|Af\|$, and the result follows from Lemma 1.1.36 below. However, in the general case, one can have $\|A_\lambda\|_{\mathcal{L}(X)} = 2/\lambda$ for all $\lambda > 0$ (see Exercise 1.8.1).

The purpose of the following proposition is to show that J_λ is a good approximation of the identity, and that the (bounded) operator A_λ is a good approximation of the (unbounded) operator A, as $\lambda \downarrow 0$.

Proposition 1.1.19. *Let A be an m-accretive operator in X. If $D(A)$ is dense in X, then*

(i) $\|J_\lambda x - x\| \le \lambda \|Ax\|$, *for all $\lambda > 0$ and all $x \in D(A)$;*

(ii) $\|J_\lambda x - x\| \underset{\lambda\downarrow 0}{\longrightarrow} 0$, *for all $x \in X$;*

(iii) $\|A_\lambda x - Ax\| \underset{\lambda\downarrow 0}{\longrightarrow} 0$, *for all $x \in D(A)$;*

(iv) $\|J_\lambda x - x\|_{D(A)} \underset{\lambda\downarrow 0}{\longrightarrow} 0$, *for all $x \in D(A)$.*

Proof. Let $x \in D(A)$. Since $J_\lambda x - x = -\lambda A_\lambda x$, (i) follows from Lemma 1.1.16 (iii). Since $\|J_\lambda - I\|_{\mathcal{L}(X)} \le 2$ and $D(A)$ is dense in X, (ii) follows from Proposition A.1.4. Given $x \in D(A)$, it follows from (ii) that $J_\lambda Ax - Ax \underset{\lambda\downarrow 0}{\longrightarrow} 0$ in X. (iii) follows, since $J_\lambda Ax = A_\lambda x$, by Lemma 1.1.16. Finally, (iv) follows from (ii) and (iii). □

Remark 1.1.20. Property (i) holds as well if $D(A)$ is not dense. Therefore, if A is an m-accretive operator, then $J_\lambda x \to x$ as $\lambda \downarrow 0$, for every $x \in D(A)$, hence for every $x \in \overline{D(A)}$ (see Proposition A.1.4).

Finally, the following proposition gives a quite useful characterization of m-accretive operators.

Proposition 1.1.21. *If A is an accretive operator in X, then the following properties are equivalent:*

(i) *A is m-accretive,*

(ii) *there exists $\lambda_0 > 0$ such that for all $f \in X$, there exists a solution $x \in D(A)$ of equation $x + \lambda_0 Ax = f$.*

Proof. It is clear that (i)⇒(ii). Let us show that (ii)⇒(i). Since A is accretive, it follows from property (ii) that given $f \in X$, there exists a unique $x \in D(A)$ such that $x + \lambda_0 Ax = f$. In addition, $\|x\| \le \|f\|$. Therefore, the mapping $f \mapsto$ is continuous $X \to X$, and its norm is ≤ 1. Let us denote this operator by J. Let now $\lambda > 0$ and $f \in X$. Note that the equation

$$x + \lambda Ax = f,$$

is equivalent to

$$x + \lambda_0 Ax = \frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) x.$$

This last equation is equivalent to

$$x = F(x),$$

where

$$F(x) = J\left(\frac{\lambda_0}{\lambda} f + \left(1 - \frac{\lambda_0}{\lambda}\right) x\right).$$

Note that F is Lipschitz continuous $X \to X$ with a Lipschitz constant $L \le \left|1 - \frac{\lambda_0}{\lambda}\right|$. Therefore, if $\lambda > \lambda_0/2$, then $L < 1$; and so, it follows from Theorem A.1.1 that there exists $x \in X$ such that $x = F(x)$. Therefore, for every $\lambda > \lambda_0/2$ and every $f \in X$, there exists $x \in D(A)$ such that $x + \lambda Ax = f$. Iterating n times this argument, it follows that for every $\lambda > \lambda_0/2^n$ and every $f \in X$, there exists $x \in D(A)$ such that $x + \lambda Ax = f$. Since n is arbitrary, the result follows. □

Remark 1.1.22. Let A be an accretive operator in X. In order to check that A is m-accretive, we have in principle to solve equation $x + \lambda Ax = f$ for all $f \in X$ and all $\lambda > 0$. Proposition 1.1.15 means that in fact, we only have to solve the equation for all $f \in X$ and **some** $\lambda > 0$. It follows in particular that if $A \in \mathcal{L}(X)$ is accretive, then A is m-accretive. Indeed, if $\lambda \|A\|_{\mathcal{L}(X)} < 1$, then $R(I + \lambda A) = X$.

Corollary 1.1.23. *Let A and B be two operators in X. If $R(I + A) = X$, if B is accretive and if $G(A) \subset G(B)$, then $A = B$ and A is m-accretive.*

Proof. Let $(x, f) \in G(B)$, and let $g = f + x$. In particular, $x \in D(B)$ and $x + Bx = g$. Since $R(I + A) = X$, there exists $y \in D(A)$ such that $y + Ay = g$. Since $G(A) \subset G(B)$, it follows that $y \in D(B)$ such that $y + By = g$. In particular, $(x - y) + B(x - y) = 0$. Therefore, $y = x$, since B is accretive. It follows that $(x, f) \in G(A)$. Therefore, $A = B$. Finally, A is accretive and $R(I + A) = X$; and so, A is m-accretive by Proposition 1.1.21. □

Corollary 1.1.24. *Let A and B be two m-accretive operators in X. If $G(A) \subset G(B)$, then $A = B$.*

1.1.3. Accretive operators and duality maps; sums of accretive operators. We recall the definition of the duality map F. For every $x \in X$, we define the duality set $F(x) \subset X^\star$ by

$$F(x) = \{\xi \in X^\star;\ \|\xi\|_{X^\star} = \|x\| \text{ and } \langle \xi, x\rangle_{X^\star,X} = \|x\|^2\}.$$

It follows from the Hahn-Banach theorem that $F(x) \neq \emptyset$.

Lemma 1.1.25. *Let A be a linear operator in X. The following properties are equivalent:*

(i) *A is accretive;*

(ii) *for all $x \in D(A)$ there exists $\xi \in F(x)$ such that $\langle \xi, Ax\rangle_{X^\star,X} \geq 0$.*

Proof. Assume A is accretive. Let $x \in D(A)$ and set $y = Ax$. We have $\|x + \lambda y\| \geq \|x\|$ for all $\lambda > 0$. Given $\lambda > 0$, let $\xi_\lambda \in F(x + \lambda y)$ and set $f_\lambda = \xi_\lambda / \|\xi_\lambda\|$. We have

$$\|x\| \leq \|x + \lambda y\| = \langle f_\lambda, x + \lambda y\rangle_{X^\star,X} = \langle f_\lambda, x\rangle_{X^\star,X} + \lambda \langle f_\lambda, y\rangle_{X^\star,X} \leq \|x\| + \lambda \langle f_\lambda, y\rangle_{X^\star,X}.$$

In particular, $\langle f_\lambda, y\rangle_{X^\star,X} \geq 0$ and $\liminf\limits_{\lambda \downarrow 0} \langle f_\lambda, x\rangle_{X^\star,X} \geq \|x\|$. On the other hand, since $\|f_\lambda\| \leq 1$, there exist a sequence $\lambda_n \downarrow 0$ and $f \in X^\star$ such that $\|f\| \leq 1$, $\lim\limits_{n\to\infty} \langle f_{\lambda_n}, x\rangle_{X^\star,X} = \langle f, x\rangle_{X^\star,X}$ and $\lim\limits_{n\to\infty} \langle f_{\lambda_n}, y\rangle_{X^\star,X} = \langle f, y\rangle_{X^\star,X}$. It follows that $\langle f, y\rangle_{X^\star,X} \geq 0$ and that $\langle f, x\rangle_{X^\star,X} \geq \|x\|$. Since $\langle f, x\rangle_{X^\star,X} \leq \|x\|$, we obtain $\langle f, x\rangle_{X^\star,X} = \|x\|$. Setting $\xi = \|x\| f$, we deduce $\xi \in F(x)$ and $\langle \xi y\rangle_{X^\star,X} \geq 0$, hence (ii). Conversely, assume (ii) holds. Let $x \in D(A)$ and let $\xi \in F(x)$ be such that $\langle \xi, Ax\rangle_{X^\star,X} \geq 0$. Set $f = x + \lambda Ax$. It follows that

$$\langle \xi, f\rangle_{X^\star,X} = \langle \xi, x\rangle_{X^\star,X} + \lambda \langle \xi, Ax\rangle_{X^\star,X} \geq \langle \xi, x\rangle_{X^\star,X}.$$

Therefore,

$$\|x\|^2 = \langle \xi, x\rangle_{X^\star,X} \leq \langle \xi, f\rangle_{X^\star,X} \leq \|\xi\|\,\|f\| = \|x\|\,\|f\|;$$

and so, A is accretive, which completes the proof. □

Lemma 1.1.26. *Let A be an m-accretive operator in X. Then $\langle \xi, Ax\rangle_{X^\star,X} \geq 0$, for every $x \in D(A)$ and every $\xi \in F(x)$.*

Proof. Let $x \in D(A)$ and $\xi \in F(x)$. For every $\lambda > 0$, we have

$$\langle \xi, (I-\lambda A)^{-1}x\rangle_{X^\star,X} \leq \|x\|\,\|(I-\lambda A)^{-1}x\| \leq \|x\|^2 = \langle \xi, x\rangle_{X^\star,X};$$

and so,

$$\langle \xi, x-(I-\lambda A)^{-1}x\rangle_{X^\star,X} \geq 0.$$

Dividing the above inequality by λ and letting $\lambda \downarrow 0$, it follows from Remark 1.1.27 that $\langle \xi, Ax\rangle_{X^\star,X} \geq 0$. This completes the proof. □

Corollary 1.1.27. *Let A and B be linear operators in X. Define the operator $A+B$ by $D(A+B) = D(A)\cap D(B)$ and $(A+B)x = Ax+Bx$. If A is m-accretive and if B is accetive, then $A+B$ is accretive.*

Proof. The result follows immediately from Lemmas 1.1.25 and 1.1.26 above. □

1.1.4. Restriction and extrapolation. In this section we show that, given an m-accretive operator with a dense domain, one can restrict it to a smaller space, or extend it to a larger space in such a way that the restricted or extended operator is m-accretive. These considerations will be useful in the next sections for characterizing the "weak solutions".

Theorem 1.1.28. *Let A be an m-accretive operator in X with dense domain and let X_1 be the Banach space $(D(A), \|\,\|_{D(A)})$. The operator $A_{(1)}$ in X_1 defined by*

$$\begin{cases} D(A_{(1)}) = \{x \in X_1;\, Ax \in X_1\}, \\ A_{(1)}x = Ax, \text{ for all } x \in D(A_{(1)}); \end{cases}$$

is m-accretive in X_1, and $D(A_{(1)})$ is dense in X_1.

Proof. Consider $x \in D(A_{(1)})$, $f \in X_1$ and $\lambda > 0$ such that

$$x + \lambda A_{(1)}x = f.$$

In particular

$$x + \lambda Ax = f. \tag{1.1.1}$$

It follows that $Ax \in D(A)$ and that

$$Ax + \lambda A(Ax) = Af. \tag{1.1.2}$$

Since A is accretive, it follows from (1.1.1) and (1.1.2) that $\|x\| \leq \|f\|$ and that $\|Ax\| \leq \|Af\|$. Therefore, $\|x\|_{X_1} \leq \|f\|_{X_1}$, and $A_{(1)}$ is accretive.

Let now $\lambda > 0$ and $f \in X_1$, and let $x = J_\lambda f$. It follows that

$$x + \lambda Ax = f.$$

In particular, $Ax \in D(A)$ (i.e. $x \in D(A_{(1)})$) and

$$x + \lambda A_{(1)} x = f;$$

and so, $A_{(1)}$ is m-accretive.

Finally, let $x \in X_1$ and let $x_\lambda = J_\lambda x$. One verifies as above that $x_\lambda \in D(A_{(1)})$. Furthermore, it follows from Proposition 1.1.19 (iv) that

$$x_\lambda \underset{\lambda \downarrow 0}{\longrightarrow} x, \text{ in } X_1.$$

Therefore, $D(A_{(1)})$ is dense in X_1. This completes the proof. □

Remark 1.1.29. Here are some observations concerning Theorem 1.1.28.

(i) One can iterate Theorem 1.1.28 and construct a family $(X_n)_{n\in\mathbb{N}}$ of Banach spaces such that

$$\cdots \hookrightarrow X_{n+1} \hookrightarrow X_n \hookrightarrow \cdots \hookrightarrow X_0 = X,$$

all embeddings being dense, and a family $(A_{(n)})_{n\in\mathbb{N}}$ of operators such that $A_{(n)}$ is m-accretive in X_n with domain X_{n+1} and $A_{(n)}x = Ax$ for all $x \in X_{n+1}$. Note that if A is bounded, then $X_n = X$ for all $n \in \mathbb{N}$, while if A is not bounded the family $(X_n)_{n\in\mathbb{N}}$ is strictly decreasing.

(ii) It follows from Corollary 1.1.14 that $X_1 = J_1(X)$ and that $\|J_1 x\|_{X_1} \approx \|x\|$. One verifies easily by iteration that $X_n = J_1^n(X)$ for every nonnegative integer n and that $\|J_1^n x\|_{X_n} \approx \|x\|$.

Remark 1.1.30. Given an operator A on X, one can define powers of A as follows. One define A^2 by

$$\begin{cases} D(A^2) = \{x \in D(A);\, Ax \in D(A)\}, \\ A^2 x = A(Ax), \text{ for } x \in D(A^2). \end{cases}$$

More generally, one defines by induction the operator A^n, for $n \geq 2$ by

$$\begin{cases} D(A^n) = \{x \in D(A^{n-1});\, A^{n-1}x \in D(A)\}, \\ A^n x = A(A^{n-1}x), \text{ for } x \in D(A^n). \end{cases}$$

One verifies quite easily that the spaces X_n defined in Remark 1.1.29 coincide with $D(A^n)$, with equivalent norms if $D(A^n)$ is equipped with the norm $\|x\|_{D(A^n)} = \sum_{j=0}^{n} \|A^j x\|$. It follows in praticular from Remark 1.1.29 (i) that if A is an m-accretive operator with dense domain, then $D(A^m)$ is dense in X for every nonnegative integer m.

Theorem 1.1.31. *If A is an m-accretive operator in X with dense domain, then there exist a Banach space X_{-1} and an operator $A_{(-1)}$ in X_{-1} such that*

(i) *$X \hookrightarrow X_{-1}$, with dense embedding;*

(ii) *for all* $x \in X$, *the norm of* x *in* X_{-1} *is equal to* $\|J_1 x\|$;

(iii) $A_{(-1)}$ *is* m*-accretive in* X_{-1};

(iv) $D(A_{(-1)}) = X$, *with equivalent norms;*

(v) *for all* $x \in D(A)$, $A_{(-1)}x = Ax$.

In addition, X_{-1} *and* $A_{(-1)}$ *verifying (i) to (v) are unique.*

Proof. Let $|||x||| = \|J_1 x\|$, for all $x \in X$. It is clear that $|||\ |||$ is a norm on X and that $|||x||| \le \|x\|$. Let X_{-1} be the completion of X for the norm $|||\ |||$. Note that X_{-1} is unique, and that (i) and (ii) hold. Furthermore, note that

$$AJ_1 x = x - J_1 x, \text{ for all } x \in X.$$

It follows that (see Lemma 1.1.16)

$$J_1 Ax = x - J_1 x, \text{ for all } x \in D(A);$$

and so,

$$|||Ax||| \le |||x||| + \|x\| \le 2\|x\|, \text{ for all } x \in D(A).$$

Therefore, one can construct by continuity a unique operator $\overline{A} \in \mathcal{L}(X, X_{-1})$ such that $\overline{A}x = Ax$ for all $x \in D(A)$ and

$$|||\overline{A}x||| \le 2\|x\|, \text{ for all } x \in X. \tag{1.1.3}$$

Define the operator $A_{(-1)}$ in X_{-1} by

$$\begin{cases} D(A_{(-1)}) = X, \\ A_{(-1)}x = \overline{A}x, \text{ for all } x \in X. \end{cases}$$

It is clear that (v) holds and that we have the algebraic identity $D(A_{(-1)}) = X$. The equivalence of the norms follows easily from (1.1.3), which proves (iv). Consider next $\lambda > 0$. Let $x \in D(A)$ and $v = J_1 x$. We have

$$v + \lambda Av = J_1(x + \lambda Ax).$$

Since A is accretive, it follows that

$$|||x + \lambda Ax||| = \|v + \lambda Av\| \ge \|v\| = |||x|||.$$

By density of $D(A)$ in X and continuity of $\overline{A}$, we obtain

$$|||x + \lambda \overline{A}x||| \ge |||x|||, \text{ for all } x \in X;$$

and so, $A_{(-1)}$ is accretive. Consider now $f \in X_{-1}$. Let $f_n \in X$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in X_{-1}, and let $x_n = J_1 f_n$. In particular, x_n is a Cauchy sequence in X. Let x be its limit. Since

$$f_n = x_n + Ax_n = x_n + \overline{A}x_n,$$

it follows that

$$f = x + \overline{A}x.$$

Applying Proposition 1.1.21, it follows that $A_{(-1)}$ is m-accretive in X_{-1}. Finally, uniqueness of $A_{(-1)}$ follows from uniqueness of $\overline{A}$. This completes the proof. □

Remark 1.1.32. One can iterate Theorem 1.1.31 and construct a family $(X_{-n})_{n\in\mathbb{N}}$ of Banach spaces such that

$$X_0 = X \hookrightarrow \cdots \hookrightarrow X_{-n+1} \hookrightarrow X_{-n} \hookrightarrow \cdots,$$

all embeddings being dense, and a family $(A_{(-n)})_{n\in\mathbb{N}}$ of operators such that $A_{(-n)}$ is m-accretive in X_{-n} with domain X_{-n+1} and $A_{(-n)}x = Ax$ for all $x \in D(A)$. Note that if A is bounded, then $X_{-n} = X$ for all $n \in \mathbb{N}$, while if A is not bounded the family $(X_{-n})_{n\in\mathbb{N}}$ is strictly increasing. Applying now Remark 1.1.29, we obtain the family

$$\cdots \hookrightarrow X_{n+1} \hookrightarrow X_n \hookrightarrow \cdots \hookrightarrow X_0 = X \hookrightarrow \cdots \hookrightarrow X_{-n+1} \hookrightarrow X_{-n} \hookrightarrow \cdots,$$

all embeddings being dense, and the family $(A_{(n)})_{n\in\mathbb{Z}}$ of operators such that $A_{(n)}$ is m-accretive in X_n with domain X_{n+1} and $A_{(n)}x = A_{(j)}x$ for all $x \in X_n \cap X_j$.

Remark 1.1.33. Here are a few simple observations about Theorem 1.1.31 and Remark 1.1.32.

(i) Note that the restrictions and extrapolations commute. In particular, $(X_1)_{-1} = (X_{-1})_1 = X$ and $(A_{(1)})_{(-1)} = (A_{(-1)})_{(1)} = A$. This follows immediately from Corollary 1.1.14.

(ii) Note also that X_{-n} is the completion of X for the norm $\|J_\lambda^n x\|$. In particular, J_λ^n can be extended by continuity, to an isomorphism $X_{-n} \to X$. One verifies easily that for every $x \in D(A_{(-n)}) = X_{-n+1}$, $A_{(-n)}x$ is the limit in X_{-n} of $A(J_\lambda^n x)$. Note that $J_\lambda^n x \in D(A)$.

Corollary 1.1.34. *With the notation of Theorem 1.1.31, if $x \in X$ is such that $A_{(-1)}x \in X$, then $x \in D(A)$ and $Ax = A_{(-1)}x$.*

Proof. Set $f = x + A_{(-1)}x \in X$. Since A is m-accretive, there exists $y \in D(A)$ such that $y + Ay = f$; and so, $y + A_{(-1)}y = f$. Since $A_{(-1)}$ is accretive, it follows that $x = y \in D(A)$. Hence the result. □

Corollary 1.1.35. *If A is an m-accretive operator in X with dense domain, then*

(i) $\|J_\lambda x - x\|_{X_{-1}} \le 2\lambda\|x\|$, *for all $x \in X$;*

(ii) *if $(x_\lambda)_{\lambda>0}$ is a bounded family in X and if X is reflexive, then $J_\lambda x_\lambda - x_\lambda \rightharpoonup 0$ in X, as $\lambda \downarrow 0$.*

Proof. (i) follows from Proposition 1.1.19 (i) applied to $A_{(-1)}$ and from (1.1.3). (ii) follows from (i) and Lemma A.1.9. □

1.1.5. The case of Hilbert spaces. Self-adjoint and skew-adjoint operators.

Throughout this section, we assume that X is a Hibert space, and we denote by $(\ ,\ )$ its scalar product. We have the following characterization of accretive operators.

Lemma 1.1.36. *If A is a linear operator in X, then the following properties are equivalent:*

(i) *A is accretive;*

(ii) *$(Ax, x) \geq 0$, for all $x \in D(A)$.*

Proof. Assume that A is accretive and let $x \in D(A)$. For all $\lambda > 0$, we have

$$(Ax, x) + \frac{\lambda}{2}\|Ax\|^2 = \frac{\|x + \lambda Ax\|^2 - \|x\|^2}{2\lambda} \geq 0.$$

(ii) follows by letting $\lambda \downarrow 0$. Conversely, assume that (ii) holds, and let $\lambda > 0$ and $x \in D(A)$. We have

$$\|x + \lambda Ax\|^2 = \|x\|^2 + 2\lambda(Ax, x) + \lambda^2\|Ax\|^2 \geq \|x\|^2;$$

and so, A is accretive. □

Corollary 1.1.37. *If A is an m-accretive operator in X, then $D(A)$ is dense in X.*

Proof. Let z be in the orthogonal of $D(A)$ in X, and let $x = J_1 z \in D(A)$. We have

$$0 = (z, x) = (x + Ax, x) = (Ax, x) + \|x\|^2 \geq \|x\|^2;$$

and so, $x = 0$, thus $z = 0$. Therefore, $D(A)$ is dense. □

Remark 1.1.38. One verifies easily that the spaces X_n defined in Remark 1.1.32 are all Hilbert spaces. In particular, the scalar product in X_1 is defined by $(x, y)_{X_1} = (x, y) + (Ax, Ay)$, for all $x, y \in X_1$, and the scalar product in X_{-1} is defined by $(x, y)_{X_{-1}} = (J_1 x, J_1 y)$, for all $x, y \in X$.

Given an operator A in X with **dense** domain, we recall that its adjoint A^* is defined as follows. We set

$$D(A^*) = \{x \in X;\ \exists C < \infty, \forall y \in D(A), |(Ay, x)| \leq C\|y\|\}.$$

Given $x \in D(A)$, the linear mapping

$$\begin{cases} D(A) \to \mathbb{R} \\ y \mapsto (Ay, x) \end{cases}$$

can be extended by continuity to a linear, continuous mapping $X \to \mathbb{R}$. This defines an element of $X^\star = X$, which we denote by $A^* x$. It is well known that if $B \in \mathcal{L}(X)$, then $(A + B)^* = A^* + B^*$. In particular, $(I + A)^* = I + A^*$. Finally, we recall (see Brezis [17], Corollary II.17) that $(R(A))^\perp = N(A^*) = \{x \in D(A^*);\ A^* x = 0\}$.

Remark 1.1.39. Note that if A is m-accretive in X, it follows from Corollary 1.1.37 that $D(A)$ is dense in X; and so, A^* is well defined.

Lemma 1.1.40. *If A is an operator in X with dense domain and if A^* is its adjoint, then*

(i) *$G(A^*) = \{(x, f) \in X \times X;\ (f, y) = (x, g), \forall (y, g) \in G(A)\}$, i.e. $(x, f) \in G(A^*)$ if and only if $(-f, x) \in G(A)^\perp$;*

(ii) *$G(A^*)$ is closed in $X \times X$.*

Proof. Let

$$Z = \{(x, f) \in X \times X;\ (f, y) = (x, g), \forall (y, g) \in G(A)\}.$$

Let $(x, f) \in Z$. Since

$$(x, Ay) = (f, y),$$

for all $y \in D(A)$, we deduce

$$|(x, Ay)| \leq \|f\|\, \|y\|.$$

It follows that $x \in D(A^*)$ and that $f = A^* x$. Therefore, $Z \subset G(A^*)$. Consider now $x \in D(A^*)$, and let $f = A^* x$. We have

$$(f, y) = (x, Ay),$$

for all $x \in D(A)$. This means that

$$(f, y) = (x, g),$$

for all $(y, g) \in G(A)$; and so, $G(A^*) \subset Z$. This proves (i), and (ii) follows immediately. □

Propositon 1.1.41. *If A is an m-accretive operator in X, then*

(i) *A^* is m-accretive in X;*

(ii) *$(I + \lambda A^*)^{-1} = \big((I + \lambda A)^{-1}\big)^*$, for all $\lambda > 0$;*

(iii) *$(A^*)_\lambda = (A_\lambda)^*$, for all $\lambda > 0$;*

(iv) *$e^{-t(A^*)_\lambda} = (e^{-tA_\lambda})^*$, for all $\lambda > 0$ and $t \in \mathbb{R}$.*

Proof. Let us first show that A^* is accretive. Let $x \in D(A^*)$ and $\lambda > 0$. Applying Lemma 1.1.16, we obtain

$$(A^* x, J_\lambda x) = (x, A J_\lambda x) = (x, A_\lambda x) = \frac{1}{\lambda}(\|x\|^2 - (x, J_\lambda x)) \geq 0.$$

Letting $\lambda \downarrow 0$, it follows from Lemma 1.1.36 that A^* is accretive. Consider now $\lambda > 0$ and let $L_\lambda = \big((I + \lambda A)^{-1}\big)^* \in \mathcal{L}(X)$. Let $z \in X$ and $x = L_\lambda z$. For every $(y, g) \in G(A)$, we have

$$\begin{aligned}
(x, g) &= \frac{1}{\lambda}((x, y + \lambda g) - (x, y)) = \frac{1}{\lambda}((L_\lambda z, (I + \lambda A) y) - (x, y)) \\
&= \frac{1}{\lambda}((z, (I + \lambda A)^{-1}(I + \lambda A) y) - (x, y)) = \frac{1}{\lambda}(z - x, y).
\end{aligned}$$

It follows that $\left(x, \dfrac{z - x}{\lambda}\right) \in G(A^*)$. Therefore, $x \in D(A^*)$ and $x + \lambda A^* x = z$. This proves (i) and (ii). Applying Lemma 1.1.16, we obtain

$$(A^*)_\lambda = \frac{I - (I + \lambda A^*)^{-1}}{\lambda} = \frac{I - \big((I + \lambda A)^{-1}\big)^*}{\lambda} = \frac{\big(I - \big(I + \lambda A\big)^{-1}\big)^*}{\lambda} = (A_\lambda)^*.$$

Hence (iii). (iv) follows easily. □

Proposition 1.1.42. *Let A be a closed, accretive operator in X with dense domain. If $N(I + A^*) = \{0\}$, then A is m-accretive. In particular, if A^* is accretive, then A is m-accretive.*

Proof. We have

$$(R(I + A))^\perp = N((I + A)^*) = N(I + A^*) = \{0\};$$

and so, $R(I+A)$ is dense in X. We now show that $R(I+A)$ is closed. Let $(f_n)_{n\in\mathbb{N}} \subset R(I+A)$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in X. We have $f_n = x_n + Ax_n$ where $x_n \in D(A)$. Since A is accretive, it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Let x be its limit. Note that $(Ax_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since $G(A)$ is closed, it follows that $x \in D(A)$ and that $f = (I+A)x$; and so, $R(I+A)$ is closed. Therefore, $R(I+A) = X$, and it follows from Proposition 1.1.21 that A is m-accretive. □

Definition 1.1.43. *An operator A in X with dense domain is symmetric (respectively skew-symmetric) if $G(A) \subset G(A^*)$ (respectively $G(A) \subset G(-A^*)$). An operator A in X with dense domain is self-adjoint (respectively skew-adjoint) if $A = A^*$ (respectively $A = -A^*$).*

Remark 1.1.44. It follows from Definition 1.1.43 that A is symmetric if and only if $(Ax, y) = (x, Ay)$ for all $x, y \in D(A)$. As well, A is skew-symmetric if and only if $(Ax, y) = -(x, Ay)$ for all $x, y \in D(A)$. It is also clear that if A is self-adjoint (respectively, skew-adjoint), then A is symmetric (respectively, skew-symmetric). However, the converse is not true (see Exercises 1.8.4 and 1.8.5). Indeed, the identity $A = \pm A^*$ is an identity between operators. It means that $G(A) = \pm G(A^*)$, or as well that $D(A) = D(A^*)$ and that $Ax = \pm A^*x$, for all $x \in D(A)$.

Corollary 1.1.45. *If A is a densely defined operator in X, then the following properties hold:*

(i) *if A is skew-adjoint, then A and $-A$ are m-accretive and $(Ax, x) = 0$ for all $x \in D(A)$;*

(ii) *if A is self-adjoint and accretive, then A is m-accretive.*

Proof. (i) If $x \in D(A)$, then $(Ax, x) = (x, A^*x) = -(x, Ax)$; and so, $(Ax, x) = 0$. In particular, it follows from Lemma 1.1.36 that A and $-A$ are accretive. Since $G(A) = -G(A^*)$ is closed by Lemma 1.1.40, property (i) follows from Proposition 1.1.42

(ii) $A^* = A$ is accretive. Furthermore, $G(A) = G(A^*)$ is closed by Lemma 1.1.40. Applying Proposition 1.1.42, it follows that A is m-accretive. Hence (ii). □

Corollary 1.1.46. *If A is an m-accretive operator in X, then the following properties are equivalent:*

(i) *A is self-adjoint;*

(ii) *$(Ax, y) = (x, Ay)$, for all $x \in D(A)$.*

Proof. Assume that A is self-adjoint. Then, $G(A) = G(A^*)$, from which (ii) follows. Conversely, assume that A verifies (ii). This means that $G(A) \subset G(A^*)$. Let $(x, f) \in G(A^*)$, and let $g = x + A^*x = x + f$. Since A is m-accretive, there exists $y \in D(A)$ such that $g = y + Ay$. Since $G(A) \subset G(A^*)$, it follows that

$y \in D(A^*)$ and that $g = y + A^*y$; and so, $x = y$, since A^* is accretive by Proposition 1.1.41. Therefore, $G(A^*) \subset G(A)$; and so, $A = A^*$. □

Corollary 1.1.47. *If A is an m-accretive operator in X, then the following properties are equivalent:*

(i) *A is skew-adjoint;*

(ii) *$(Ax, x) = 0$, for all $x \in D(A)$;*

(iii) *$-A$ is m-accretive.*

Proof. It follows from Corollary 1.1.45 that (i)⇒(ii) and that (i)⇒(iii). It remains to show that (iii)⇒(ii) and (ii)⇒(i). Assume that A and $-A$ are m-accretive. Applying Lemma 1.1.36 to both A and $-A$, it follows that $(Ax, x) = 0$, for all $x \in D(A)$. Hence (ii). Finally, assume that (ii) holds, and let $x, y \in D(A)$. We have

$$(Ax, y) + (x, Ay) = (A(x + y), x + y) - (Ax, x) - (Ay, y) = 0.$$

This means that $G(A) \subset G(-A^*)$. Next, consider $(x, f) \in G(-A^*)$ and let $g = x - A^*x = x + f$. Since A is m-accretive, there exists $y \in D(A)$ such that $g = y + Ay$. Since $G(A) \subset G(-A^*)$, it follows that $y \in D(A^*)$ and that $g = y - A^*y$; and so, $x = y$, since $-A^*$ is accretive by (ii) and Proposition 1.1.41. Thus, $x \in D(A)$ and $Ax = -A^*x$. Therefore, $G(-A^*) \subset G(A)$; and so, $G(-A^*) = G(A)$. Therefore (i) holds, which completes the proof. □

Corollary 1.1.48. *Let A be an m-accretive operator, and let $A_{(n)}$ be the operators defined in Remark 1.1.32, for $n \in \mathbb{Z}$. If A is self-adjoint (respectively, skew-adjoint), then $A_{(n)}$ is self-adjoint (respectively, skew-adjoint).*

Proof. Assume that A is self-adjoint, the proof being similar if A is skew-adjoint. Arguing by induction, we only need to show that $A_{(1)}$ and $A_{(-1)}$ are self-adjoint. Given $x, y \in D(A_{(1)})$, it follows from Remark 1.1.38 that

$$(A_{(1)}x, y)_{X_1} = (Ax, y) + (A(Ax), Ay).$$

Therefore, $(A_{(1)}x, y)_{X_1} = (A_{(1)}y, x)_{X_1}$. Since $A_{(1)}$ is m-accretive, it follows from Corollary 1.1.46 that $A_{(1)}$ is self-adjoint. One shows as well that $A_{(-1)}$ is self-adjoint, by applying Remarks 1.1.33 and 1.1.38. □

Proposition 1.1.49. *If A is a densely defined operator, then the following properties hold.*

(i) *If A is self-adjoint and if ℓ is a nonnegative integer, then $A^{2\ell}$ is self-adjoint and accretive (hence m-accretive).*

(ii) *If A is self-adjoint and accretive and if ℓ is a positive integer, then $A^{2\ell+1}$ is self-adjoint and accretive (hence m-accretive).*

(iii) *If A is skew-adjoint and if ℓ is a positive integer, then $(-1)^\ell A^{2\ell}$ is self-adjoint and accretive (hence m-accretive).*

(iv) *If A is skew-adjoint and if ℓ is a positive integer, then $A^{2\ell+1}$ is skew-adjoint (hence m-accretive).*

Proof. Let A be as in (i), (ii), (iii) or (iv), so that in particular $A^* = \varepsilon A$ with $\varepsilon = \pm 1$. We proceed in five steps.

Step 1. If m is a nonnegative integer, then

$$\|A^j x\| \le \|A^m x\|^{\frac{j}{m}} \|x\|^{\frac{m-j}{m}}, \tag{1.1.4}$$

for all $0 \le j \le m$ and for all $x \in D(A^m)$.

We argue by induction on m. (1.1.4) is clearly true for $m = 1$. Assume (1.1.4) for some $m_0 \ge 1$, and let $x \in D(A^{m_0+1})$. We have

$$\|A^{m_0} x\|^2 = (A^{m_0} x, A^{m_0} x) = \varepsilon (A^{m_0-1} x, A^{m_0+1} x) \le \|A^{m_0-1} x\| \, \|A^{m_0+1} x\|.$$

Since by (1.1.4), $\|A^{m_0-1} x\| \le \|A^{m_0} x\|^{\frac{m_0-1}{m_0}} \|x\|^{\frac{1}{m_0}}$, we deduce that

$$\|A^{m_0} x\| \le \|A^{m_0+1} x\|^{\frac{m_0}{m_0+1}} \|x\|^{\frac{1}{m_0+1}}. \tag{1.1.5}$$

Let now $0 \le j \le m_0$. Since by (1.1.4), $\|A^j x\| \le \|A^{m_0} x\|^{\frac{j}{m_0}} \|x\|^{\frac{m_0-j}{m_0}}$, we obtain by applying (1.1.5) $\|A^j x\| \le \|A^{m_0+1} x\|^{\frac{j}{m_0+1}} \|x\|^{\frac{m_0+1-j}{m_0+1}}$, which is (1.1.4) for $m = m_0 + 1$ and $0 \le j \le m_0$. The case $j = m_0 + 1$ being trivial, this shows (1.1.4) for $m = m_0 + 1$.

Step 2. If m is a nonnegative integer, then A^m is closed. Indeed, suppose $(x_n)_{n \ge 0} \in D(A^m)$ satisfies $x_n \underset{n\to\infty}{\longrightarrow} x$ and $A^m x_n \underset{n\to\infty}{\longrightarrow} y$ in X. By Step 1, $\|A^j x_n\|$ is bounded for all $j \le m$. In particular, $\|A x_n\|$ is bounded. Since A is closed, it follows from Remark 1.1.11 that $x \in D(A)$ and $A x_n \rightharpoonup A x$. Since $\|A(A x_n)\|$ is bounded, we may apply again Remark 1.1.11, and it follows that $x \in D(A^2)$ and $A^2 x_n \rightharpoonup A^2 x$. By induction, we deduce easily that $x \in D(A^m)$ and $A^m x_n \rightharpoonup A^m x$. Thus $y = A^m x$ and A^m is closed.

Step 3. Let m be a nonnegative integer and let $\sigma = \varepsilon^{\frac{m}{2}}$ if m is even, $\sigma = 1$ if m is odd and $\varepsilon = 1$, $\sigma = \pm 1$ if m is odd and $\varepsilon = -1$. It follows that σA^m is accretive. Consider $x, y \in D(A^m)$. If $m = 2\ell$, we write

$$(\sigma A^m x, y) = \sigma \varepsilon^\ell (A^\ell x, A^\ell y) = (A^\ell x, A^\ell y). \tag{1.1.6}$$

If $m = 2\ell + 1$, we write

$$(\sigma A^m x, y) = \sigma \varepsilon^\ell (A(A^\ell x), A^\ell y) = \varepsilon^\ell (A(A^\ell x), A^\ell y). \tag{1.1.7}$$

If m is even, we deduce from (1.1.6) that $(\sigma A^m x, x) = \|A^\ell x\|^2 \ge 0$. If m is odd and $\varepsilon = 1$, then $A \ge 0$ and we deduce from (1.1.7) that $(\sigma A^m x, x) = (A(A^\ell x), A^\ell x) \ge 0$. If m is odd and $\varepsilon = -1$, then A is skew-adjoint and we deduce from (1.1.7) that $(\sigma A^m x, x) = 0$.

Step 4. Let m be a nonnegative integer and let σ be as in Step 3. It follows that $N(I + \sigma (A^m)^*) = \{0\}$. Indeed, consider $y \in N(I + \sigma (A^m)^*)$. (Note that A^m is densely defined by Remark 1.1.30, so that $I + \sigma (A^m)^*$ is well-defined.) We have

$$(y, x + \sigma A^m x) = 0, \tag{1.1.8}$$

for all $x \in D(A^m)$. Fix $z \in X$ and, given $\lambda > 0$, let $x_\lambda = (I + \varepsilon \lambda A)^{-m} z \in D(A^m)$. (Note that if $\varepsilon = -1$, then $-A$ is also m-accretive). We now let $x = x_\lambda$ in (1.1.8). Since $A^* = \varepsilon A$ and $((I + \varepsilon \lambda A)^{-1})^* = (I + \lambda A)^{-1}$, we find

$$\sigma \varepsilon^m (A^m y_\lambda, z) = -(y, x_\lambda), \tag{1.1.9}$$

where $y_\lambda = (I + \lambda A)^{-m} y$. Since $\|x_\lambda\| \le \|z\|$. Since z is arbitrary, we deduce in particular that

$$\|A^m y_\lambda\| \le \|y\|.$$

Note also that $y_\lambda \underset{\lambda \downarrow 0}{\longrightarrow} y$. Since A^m is closed by Step 2, we deduce from Remark 1.1.11 that $y \in D(A^m)$ and that $A^m y_\lambda \rightharpoonup A^m y$. Letting $\lambda \downarrow 0$ in (1.1.9), we deduce

$$(y + \sigma \varepsilon^m A^m y, z) = 0.$$

Since $z \in X$ is arbitrary, we have then $y + \sigma \varepsilon^m A^m y = 0$. Setting $\widetilde{\sigma} = \sigma \varepsilon^m$, we see that $\widetilde{\sigma}$ is as in Step 3, and we conclude that $y = 0$.

Step 5. Conclusion. Let m be a nonnegative integer and let σ be as in Step 3. It follows from Steps 2, 3 and 4 that σA^m is densely defined, closed, accretive, and that $N(I + \sigma (A^m)^*) = \{0\}$. It follows from Proposition 1.1.42 that σA^m is m-accretive. If $\varepsilon = 1$, then it follows from (1.1.6) and (1.1.7) that σA^m is symmetric, hence self-adjoint by Corollary 1.1.46. If $\varepsilon = -1$ and m is even, then it follows from (1.1.6) that σA^m is symmetric, hence self-adjoint by Corollary 1.1.46. Finally, if $\varepsilon = -1$ and m is odd, then it follows from (1.1.7) that $(\sigma A^m x, x) = 0$ for all $x \in D(A^m)$. Therefore, σA^m is skew-adjoint by Corollary 1.1.47. This completes the proof. □

Let A be an m-accretive operator in X and let A^* be its adjoint. It follows from Proposition 1.1.41 that A^* is also m-accretive. In particular, $D((A^*)^n)$ is dense in X, for every nonnegative integer n. Therefore, if $D((A^*)^n)$ is equipped with the norm $\|x\|_{D((A^*)^n)} = \sum_{j=1}^{n} \|(A^*)^j x\|$, then $D((A^*)^n) \hookrightarrow X \hookrightarrow D((A^*)^n)^\star$ with dense embeddings. We have the following results.

Proposition 1.1.51. *If A is as above and $(X_{-n})_{n \ge 0}$ are the spaces defined in Remark 1.1.32, then $X_{-n} = D((A^*)^n)^\star$ with equivalent norms.*

Proof. It suffices to show that $\|x\|_{X_{-n}} \approx \|x\|_{D((A^*)^n)^\star}$. By density, we may assume that $x \in X$. It follows from Remark 1.1.33 (ii), Proposition 1.1.42, Remark 1.1.30 and Remark 1.1.29 (ii) that

$$\begin{aligned}
\|x\|_{X_{-n}} = \|J_1(A)^n x\| &= \sup_{\|y\|=1} (J_1(A)^n x, y)_{X,X} \\
&= \sup_{\|y\|=1} (x, J_1(A^*)^n y)_{X,X} = \sup_{\|y\|=1} (x, J_1(A^*)^n y)_{D((A^*)^n)^\star, D((A^*)^n)} \\
&= \sup_{\|z\|_{D((A^*)^n)}=1} (x, z)_{D((A^*)^n)^\star, D((A^*)^n)} = \|x\|_{D((A^*)^n)^\star}.
\end{aligned}$$

Hence the result. □

Corollary 1.1.52. *Let A be a self-adjoint accretive or a skew-adjoint operator in X and let $(X_n)_{n \in \mathbb{Z}}$ be the spaces introduced in Remark 1.1.32. Then, $X_{-n} = X_n^\star$ with equivalent norms, for every $n \in \mathbb{Z}$.*

Proof. Consider $n \ge 0$. It follows from Remark 1.1.30 that $X_n = D(A^n) = D((A^*)^n)$; and so, $X_{-n} = X_n^\star$ by Proposition 1.1.51. Since the speces X_n are Hilbert spaces (see Remark 1.1.38), they are reflexive. Therefore, $X_{-n}^\star = X_n^{\star\star} = X_n$. Hence the result. □

Finally, we establish a useful property of self-adjoint operators in complex Hilbert spaces. Let X is a $\mathbb{C}$-linear vector space, endowed with a norm $\| \|$ that makes it a *real* Banach space. We recall that X is a complex Hilbert space if there exists a mapping $b : X \times X \to \mathbb{C}$ with the following properties:

$$\begin{cases} b(\lambda x + \mu y, z) = \lambda b(x, z) + \mu b(y, z), \text{ for all } x, y, z \in X \text{ and all } \lambda, \mu \in \mathbb{R}; \\ b(y, x) = \overline{b(x, y)}, \text{ and all } x, y \in X; \\ b(ix, y) = ib(x, y), \text{ and all } x, y \in X; \\ b(x, x) = \|x\|^2, \text{ and all } x \in X. \end{cases}$$

It follows easily that X equipped with the scalar product

$$(x, y) = \mathrm{Re}(b(x, y)),$$

is a *real* Hilbert space.

Lemma 1.1.53. *Let X be a complex Hilbert space, and let A be an operator in X. Assume that A is $\mathbb{C}$-linear, and let iA be defined by*

$$\begin{cases} D(iA) = D(A), \\ (iA)x = iAx, \text{ for all } x \in D(A). \end{cases}$$

If $D(A)$ is dense in X, then A^* is $\mathbb{C}$-linear and $(iA)^* = -iA^*$.

Proof. Let $(x, f) \in G(A^*)$. For all $(y, g) \in G(A)$ and all $\lambda \in \mathbb{C}$, we have

$$(\lambda f, y) = (f, \overline{\lambda} y) = (x, A(\overline{\lambda} y)) = (x, \overline{\lambda} Ay) = (\lambda x, Ay).$$

It follows that $(\lambda x, \lambda f) \in G(A^*)$; and so, A^* is $\mathbb{C}$-linear. Furthermore, Given $(x, f) \in G(A^*)$ and $(y, g) \in G(A)$, we have

$$(-if, y) = (f, iy) = (x, A(iy)) = (x, ig).$$

It follows that $(x, if) \in G((iA)^*)$; and so, $G(-iA^*) \subset G((iA)^*)$. Applying that result to iA, we find $G(-i(iA)^*) \subset G(-A^*)$. By $\mathbb{C}$-linearity, it follows that $G((iA)^*) \subset G(-iA^*)$; and so, $G((iA)^*) = G(-iA^*)$. Hence the result. □

Corollary 1.1.54. *Let X be a complex Hilbert space, and let A be an operator in X. If A is $\mathbb{C}$-linear, then the following properties are equivalent:*

(i) *A is self-adjoint;*

(ii) *iA is skew-adjoint.*

Proof. Assume that A is self-adjoint. It follows from Lemma 1.1.53 that

$$(iA)^* = -iA^* = -iA;$$

and so, iA is skew-adjoint. Conversely, if iA is skew-adjoint, then

$$A^* = (-i(iA))^* = i(iA)^* = -i(iA) = A;$$

and so, A is self-adjoint. □

1.2. Examples of m-accretive partial differential operators. In this section, we describe some examples of partial differential operators that are related to classical evolution equations.

1.2.1. First order operators. Here are a few examples related to transport equations.

Example 1. A first order operator in $\mathbb{R}$. Let $X = C_{\mathrm{b}}(\mathbb{R})$, and define the operator A in X by

$$\begin{cases} D(A) = \{u \in C^1(\mathbb{R}) \cap X;\, u' = \dfrac{du}{dx} \in X\}, \\ Au = u', \text{ for } u \in D(A). \end{cases} \tag{1.2.1}$$

We have the following result.

Proposition 1.2.1. *If A is defined by (1.2.1), then both A and $-A$ are m-accretive.*

Proof. Let us first show that A is accretive. Let $\lambda > 0$ and let $(u, f) \in D(A) \times X$ verify $u + \lambda Au = f$. It follows that

$$u + \lambda u' = f, \text{ for all } x \in \mathbb{R}. \tag{1.2.2}$$

Let

$$Lf(x) = \frac{1}{\lambda} \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}} f(s)\, ds. \tag{1.2.3}$$

We have

$$|Lf(x)| \le \frac{1}{\lambda} \|f\|_{L^\infty} \int_{-\infty}^{x} e^{\frac{s-x}{\lambda}}\, ds = \|f\|_{L^\infty}.$$

Therefore,

$$\|Lf\|_{L^\infty} \le \|f\|_{L^\infty}. \tag{1.2.4}$$

Note that the general solution of (1.2.2) is given by

$$u(x) = Lf(x) + ae^{\frac{x}{\lambda}}.$$

Since both u and Lf are bounded, it follows that $a = 0$. Therefore, $u = Lf$, and it follows from (1.2.4) that A is accretive.

Consider now $\lambda > 0$ and $f \in X$. It follows from (1.2.4) that $Lf \in X$. Furthermore, $Lf \in C^1(\mathbb{R})$, and Lf verifies equation (1.2.2). Therefore, $Lf \in D(A)$ and $Lf + \lambda(Lf)' = f$. Therefore, A is m-accretive. One shows as well that $-A$ is m-accretive. □

Remark 1.2.2. Note that in the above example, $D(A)$ is **not** dense in X. For example, $u(x) = \sin(x^2)$ belongs to X. However, one checks easily that if $z \in C^1(\mathbb{R})$ verifies $\|z - u\|_{L^\infty} \le 1/4$, then $\sup_{x\in\mathbb{R}} |z'(x)| = \infty$, and so $z \notin D(A)$. Therefore, u cannot be approximated by elements of $D(A)$.

Remark 1.2.3. One can modify the above example as follows.

(i) Let $X = L^\infty(\mathbb{R})$, and let A be defined by

$$\begin{cases} D(A) = W^{1,\infty}(\mathbb{R}), \\ Au = u', \text{ for } u \in D(A). \end{cases} \tag{1.2.5}$$

Then, both A and $-A$ are m-accretive. The proof is essentially the same as the proof of Proposition 1.2.1. Note that in the above example also, one can show easily that $D(A)$ is **not** dense in X.

(ii) Let now $X = C_0(\mathbb{R})$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1(\mathbb{R}) \cap X;\ u' \in X\}, \\ Au = u', \text{ for } u \in D(A). \end{cases} \tag{1.2.6}$$

Then, both A and $-A$ are m-accretive, with dense domain. Since $\mathcal{D}(\mathbb{R}) \subset D(A)$, it follows that $D(A)$ is dense in X. The rest of the proof follows that of Proposition 1.2.1.

(iii) Consider now $1 \le p < \infty$, let $X = L^p(\mathbb{R})$, and let A be defined by

$$\begin{cases} D(A) = W^{1,p}(\mathbb{R}), \\ Au = u', \text{ for } u \in D(A). \end{cases} \tag{1.2.7}$$

Then, both A and $-A$ are m-accretive, with dense domain. If $p = 2$, then A is skew-adjoint. Since $\mathcal{D}(\mathbb{R}) \subset D(A)$, it follows that $D(A)$ is dense in X. In order to show that A is m-accretive, and following the proof of Proposition 1.2.1, we only have to show that $L \in \mathcal{L}(L^p)$, and that $\|L\|_{\mathcal{L}(L^p)} \le 1$. Let p' be the conjugate of p. It follows from Hölder's inequality that

$$\begin{aligned} |Lf(x)| &= \left| \frac{1}{\lambda} \int_{-\infty}^0 e^{\frac{s}{\lambda}} f(x+s)\, ds \right| \le \frac{1}{\lambda} \int_{-\infty}^0 e^{\frac{s}{\lambda p'}} \left(e^{\frac{s}{\lambda}} |f(x+s)|^p \right)^{1/p} ds \\ &\le \lambda^{-\frac{1}{p}} \int_{-\infty}^0 e^{\frac{s}{\lambda}} |f(x+s)|^p\, ds; \end{aligned}$$

and so,

$$\int_{\mathbb{R}} |Lf(x)|^p\, dx \le \frac{1}{\lambda} \left(\int_{-\infty}^0 e^{\frac{s}{\lambda}}\, ds \right) \left(\int_{\mathbb{R}} |f(s)|^p\, ds \right) = \|f\|_{L^p}^p,$$

which is the desired estimate. One shows by the same method that $-A$ is m-accretive. Finally, when $p = 2$, it follows from Corollary 1.1.47 that A is skew-adjoint.

Example 2. A first order operator in a bounded interval. Consider $X = \{u \in C([0,1]);\ u(0) = 0\}$, equipped with the sup norm. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in C^1([0,1]);\ u(0) = u'(0) = 0\}, \\ Au = u', \text{ for } u \in D(A). \end{cases} \tag{1.2.8}$$

We have the following result.

Proposition 1.2.4. *The operator A defined by (1.2.8) is m-accretive with dense domain.*

Proof. Following the proof of Proposition 1.2.1, one shows easily that, given $f \in X$ and $\lambda > 0$, the unique solution $u \in D(A)$ of equation

$$u + \lambda u' = f,$$

is given by

$$u(x)=\frac{1}{\lambda}\int_0^x e^{\frac{s-x}{\lambda}}f(s)\,ds,$$

from which it follows that A is m-accretive. It remains to show that $D(A)$ is dense in X. Consider $u\in X$ and $\delta>0$, and let $u_\delta\in X$ be defined by $u_\delta(x)=0$, on $[0,\delta]$ and $u_\delta(x)=u(x-\delta)$ for $x\ge\delta$. We have $\|u_\delta-u\|\to 0$ in X, as $\delta\downarrow 0$. Given $\varepsilon>0$, let δ be small enough so that $\|u_\delta-u\|\le\varepsilon/2$. Let $v_\delta\in C_{\rm c}(\mathbb{R})$ be defined by $v_\delta(x)=0$, for $x\le 0$, $v_\delta(x)=u_\delta(x)$, for $0\le x\le 1$, $v_\delta(x)=(2-x)u_\delta(1)$, for $1\le x\le 2$, and $v_\delta(x)=0$, for $x\ge 2$. Given a sequence ρ_n of mollifiers, we have (see Brezis [17], Proposition IV.21, p.70) $\rho_n*v_\delta\to v_\delta=u_\delta$, uniformly on $[0,1]$. Therefore, for n large enough, we have $\|u-(\rho_n*v_\delta)_{|[0,1]}\|\le\varepsilon$. On the other hand, it is clear that $(\rho_n*v_\delta)_{|[0,1]}\in D(A)$ for n large enough. This completes the proof. □

Remark 1.2.5. One can modify the above example as follows.

(i) Let $X=L^\infty(0,1)$, and let A be defined by

$$\begin{cases} D(A)=\{u\in W^{1,\infty}(0,1);\ u(0)=0\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}$$

Then, A is m-accretive. The proof is an adaptation of the proof of Proposition 1.2.4. Note that $D(A)$ is not dense in X.

(ii) Let $1\le p<\infty$, let $X=L^p(0,1)$, and let A be defined by

$$\begin{cases} D(A)=\{u\in W^{1,p}(0,1);\ u(0)=0\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}\tag{1.2.9}$$

Then, A is m-accretive with dense domain. Since $\mathcal{D}(0,1)\subset D(A)$, it follows that $D(A)$ is dense in X. The rest of the proof is an adaptation of the proof of Proposition 1.2.4 (see also Remark 1.2.3 (iii)).

(iii) Let $X=\{u\in C([0,1]);\ u(0)=u(1)\}$, and let A be defined by

$$\begin{cases} D(A)=\{u\in C^1([0,1]);\ u(0)=u(1) \text{ and } u'(0)=u'(1)\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}$$

Then both A and $-A$ are m-accretive with dense domain.

(iv) Let $1\le p<\infty$, let $X=L^p(0,1)$, and let A be defined by

$$\begin{cases} D(A)=\{u\in W^{1,p};\ u(0)=u(1)\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}$$

Then, both A and $-A$ are m-accretive with dense domain. If $p=2$, then A is skew-adjoint.

Example 3. First order operators on $\mathbb{R}_+$. One can modify the above examples by considering operators on the half line. The proofs of the corresponding results are almost the same as in the case of the whole line or a bounded interval. For example, let $X=C_0(\mathbb{R}_+)=\{u\in C([0,\infty));\ u(0)=0 \text{ and } \lim\limits_{x\to\infty}u(x)=0\}$, and let A be defined by

$$\begin{cases} D(A)=\{u\in C^1([0,\infty))\cap X;\ u'\in X\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}$$

We have the following result.

Proposition 1.2.6. *If A is as above, then A is m-accretive with dense domain.*

Remark 1.2.7. One can modify the above example as follows.

(i) Let $p = \infty$, let $X = L^\infty(\mathbb{R}_+)$, and let A be defined by

$$\begin{cases} D(A) = \{u \in W^{1,\infty}(\mathbb{R}_+);\ u(0) = 0\}, \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

Then, A is m-accretive, and $D(A)$ is not dense in X.

(ii) Let $1 \le p < \infty$, let $X = L^p(\mathbb{R}_+)$, and let A be defined by

$$\begin{cases} D(A) = \{u \in W^{1,p}(\mathbb{R}_+);\ u(0) = 0\}, \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

Then, A is m-accretive with dense domain.

One can modify the above examples by considering the operator $-u'$ instead of u'. For example, let $X = \{u \in C([0,\infty));\ \lim\limits_{x\to\infty} u(x) = 0\}$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty));\ \lim\limits_{x\to\infty} u(x) = \lim\limits_{x\to\infty} u'(x) = 0\}, \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

We have the following result.

Proposition 1.2.8. *If A is as above, then A is m-accretive with dense domain.*

Remark 1.2.9. One can modify the above example as follows.

(i) Let $X = C_{\mathrm{b}}([0,\infty))$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty)) \cap X;\ u' \in X\}, \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

Then, A is m-accretive, and $D(A)$ is **not** dense in X.

(ii) Let $X = L^\infty(0,\infty)$, and let A be defined by

$$\begin{cases} D(A) = W^{1,\infty}(0,\infty), \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

Then, A is m-accretive, and $D(A)$ is **not** dense in X.

(iii) Let $1 \le p < \infty$, $X = L^p(0,\infty)$, and let A be defined by

$$\begin{cases} D(A) = W^{1,p}(0,\infty), \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

Then, A is m-accretive with dense domain.

Remark 1.2.10. Note that when A is as in Proposition 1.2.4, Remarks 1.2.9, 1.2.5 and 1.2.7, $-A$ is **not** m-accretive.

Example 4. A first order operator in $\mathbb{R}^N$. Let $X = C_{\mathrm{b}}(\mathbb{R}^N)$, and let $a \in \mathbb{R}^N$. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in X;\, a \cdot \nabla u \in X\}, \\ Au = a \cdot \nabla u = \displaystyle\sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j}, \text{ for } u \in D(A). \end{cases} \tag{1.2.10}$$

The condition $a \cdot \nabla u \in X$ is understood in the sense of distributions. We have the following result.

Proposition 1.2.11. *If A is defined by (1.2.10), then both A and $-A$ are m-accretive.*

The proof relies on the following two lemmas.

Lemma 1.2.12. *Let $\lambda > 0$ and $1 \le p \le \infty$. If $u \in L^p(\mathbb{R}^N)$ verifies*

$$u + \lambda a \cdot \nabla u = 0, \text{ in } \mathcal{D}'(\mathbb{R}^N),$$

then $u = 0$ almost everywhere.

Proof. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers (see Brezis [17], p.70), and let $u_n = \rho_n * u$. We have $u_n \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and

$$u_n + \lambda a \cdot \nabla u_n = 0, \text{ in } \mathbb{R}^N.$$

Given $x \in \mathbb{R}^N$, let

$$h(t) = e^t u_n(x + \lambda a t), \text{ for } t \in \mathbb{R}.$$

It follows that

$$h'(t) = e^t (u_n(x + \lambda a t) + \lambda a \cdot \nabla u_n(x + \lambda a t)) = 0;$$

and so, h is constant. Letting $t \to -\infty$, and since u_n is bounded, it follows that $h \equiv 0$. In particular, $u_n(x) = 0$. Since x is arbitrary, we have $u_n \equiv 0$. Hence the result, since $u_n \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, as $n \to \infty$. □

Lemma 1.2.13. *Given $\lambda > 0$ and $f \in C_{\mathrm{b}}(\mathbb{R}^N)$, let*

$$Lf(x) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{s}{\lambda}} f(x - as)\, ds.$$

Then,

$$Lf + \lambda a \cdot \nabla(Lf) = f, \tag{1.2.11}$$

in $\mathcal{D}'(\mathbb{R}^N)$. In addition,

$$\|Lf\|_{L^p} \le \|f\|_{L^p}, \tag{1.2.12}$$

for all $1 \le p \le \infty$ such that $f \in L^p(\mathbb{R}^N)$.

Proof. Define

$$Mf(x) = \frac{1}{\lambda} \int_0^\infty e^{-\frac{s}{\lambda}} f(x + as)\, ds,$$

for $f \in C_{\rm b}(\mathbb{R}^N)$. It follows easily from Fubini's theorem that for every $f \in C_{\rm b}(\mathbb{R}^N)$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\langle Lf, \varphi \rangle = \int_{\mathbb{R}^N} f M\varphi \, dx.$$

Furthermore,

$$\begin{aligned} M(\lambda a \cdot \nabla \varphi)(x) &= \int_0^\infty e^{-\frac{s}{\lambda}} a \cdot \nabla \varphi(x + as) \, ds = \int_0^\infty e^{-\frac{s}{\lambda}} \frac{d}{ds} (\varphi(x + as)) \, ds \\ &= -\varphi(x) + M\varphi(x); \end{aligned}$$

and so,

$$\begin{aligned} \langle Lf, \varphi \rangle &= \int_{\mathbb{R}^N} f M\varphi \, dx = \int_{\mathbb{R}^N} f M(\lambda a \cdot \nabla \varphi) \, dx + \langle f, \varphi \rangle \\ &= \langle Lf, \lambda a \cdot \nabla \varphi \rangle + \langle f, \varphi \rangle = \langle -\lambda a \cdot \nabla (Lf) + f, \varphi \rangle. \end{aligned}$$

Hence (1.2.11). Finally,

$$|Lf(x)| \le \frac{1}{\lambda} \|f\|_{L^\infty} \int_0^\infty e^{-\frac{s}{\lambda}} \, ds = \|f\|_{L^\infty}.$$

(1.2.12) follows for $p = \infty$. For $p < \infty$, we have

$$|Lf(x)| \le \frac{1}{\lambda} \int_0^\infty e^{-\frac{s}{\lambda p'}} \left(e^{-\frac{s}{\lambda}} |f(x - as)|^p \right)^{1/p} \, ds \le \lambda^{-\frac{1}{p}} \left(\int_0^\infty e^{-\frac{s}{\lambda}} |f(x - as)|^p \, ds \right)^{1/p}.$$

Therefore,

$$\int_{\mathbb{R}^N} |Lf|^p \le \frac{1}{\lambda} \|f\|_{L^p}^p \int_0^\infty e^{-\frac{s}{\lambda}} \, ds = \|f\|_{L^p}^p.$$

This completes the proof. □

Proof of Proposition 1.2.11. Let us first show that A is accretive. Let $\lambda > 0$, $f \in X$ and $u \in D(A)$ verify $u + \lambda Au = f$. Let $w = Lf$, where L is defined in Lemma 1.2.13. It follows that

$$(u - w) + a \cdot \nabla (u - w) = 0, \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Applying Lemma 1.2.12, we find $u = w$, and accretivity follows from (1.2.12). Finally, given $\lambda > 0$ and $f \in X$, it is clear that $u = Lf$ belongs to $D(A)$ and it follows from Lemma 1.2.13 that $u + \lambda Au = f$. Therefore, A is m-accretive. One shows as well that $-A$ is m-accretive. □

Remark 1.2.14. Note that in Proposition 1.2.11, $D(A)$ is not dense in X.

Remark 1.2.15. One can modify slightly the above example as follows.

(i) Let $X = C_0(\mathbb{R}^N)$, and let $a \in \mathbb{R}^N$. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in X; \, a \cdot \nabla u \in X\}, \\ Au = a \cdot \nabla u = \displaystyle\sum_{j=1}^N a_j \frac{\partial u}{\partial x_j}, \text{ for } u \in D(A). \end{cases} \tag{1.2.13}$$

Then, both A and $-A$ are m-accretive with dense domain. The proof is easily adapted from the proof of Proposition 1.2.11.

(ii) Let $X = L^\infty(\mathbb{R}^N)$, and let $a \in \mathbb{R}^N$. Define the operator A in X by (1.2.13). Then, both A and $-A$ are m-accretive, and $D(A)$ is not dense in X. The proof is easily adapted from the proof of Proposition 1.2.11.

(iii) Let $X = L^p(\mathbb{R}^N)$, $1 \le p < \infty$, and let $a \in \mathbb{R}^N$. Define the operator A in X by (1.2.13). Then, both A and $-A$ are m-accretive, with dense domain. If $X = L^2(\mathbb{R}^N)$, then A is skew-adjoint. The proof is easily adapted from the proof of Proposition 1.2.11. Skew-adjointness of A when $p = 2$ follows from Corollary 1.1.47.

Remark 1.2.16. Note that for all the examples of Section 1.2.1, one can work either in the spaces of real-valued functions, or in the spaces of complex-valued functions.

1.2.2. The Laplacian with Dirichlet boundary condition. The following examples are important in the study of the heat equation.

Example 1. **H^{-1} theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set $X = H^{-1}(\Omega)$, and define the operator A on X by

$$\begin{cases} D(A) = H_0^1(\Omega), \\ Au = -\triangle u, \text{ for all } u \in D(A). \end{cases} \tag{1.2.14}$$

We equip $H_0^1(\Omega)$ with the usual norm $(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}$. We have the following result.

Proposition 1.2.17. *The operator A defined by (1.2.14) is self-adjoint, accretive, and $\|\cdot\|_{D(A)}$ is equivalent to $\|\cdot\|_{H^1}$. In particular, A is m-accretive with dense domain.*

Proof. It follows from Lemma A.4.3 that for every $f \in X$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$-\triangle u + u = f, \text{ in } X.$$

Let us denote by J the operator $f \mapsto u$. It follows from Remark A.4.4 (i) that J is an isometry from X onto $H_0^1(\Omega)$. In particular,

$$(u, v)_{H^{-1}} = (Ju, Jv)_{H_0^1}. \tag{1.2.15}$$

Let $u, v \in H_0^1(\Omega)$. It follows from (A.3.17) and (A.3.14) that

$$\begin{aligned} (u, Jv)_{H_0^1} &= \int_\Omega \nabla u \cdot \nabla (Jv)\, dx + (u, Jv)_{L^2} \\ &= \langle u, -\triangle (Jv)\rangle_{H_0^1, H^{-1}} + \langle u, Jv\rangle_{H_0^1, H^{-1}} = \langle u, v\rangle_{H_0^1, H^{-1}} = (u, v)_{L^2}. \end{aligned} \tag{1.2.16}$$

Furthermore, it follows from (1.2.15) that

$$\begin{aligned} (-\triangle u, v)_{H^{-1}} &= (-\triangle u + u, v)_{H^{-1}} - (u, v)_{H^{-1}} = (J(-\triangle u + u), Jv)_{H_0^1} - (u, v)_{H^{-1}} \\ &= (-\triangle (Ju) + Ju, Jv)_{H_0^1} - (u, v)_{H^{-1}} = (u, Jv)_{H_0^1} - (u, v)_{H^{-1}}. \end{aligned}$$

Applying (1.2.16), it follows that

$$(-\triangle u, v)_{H^{-1}} = (u, v)_{L^2} - (u, v)_{H^{-1}}. \tag{1.2.17}$$

In particular, for every $u \in H_0^1(\Omega)$, we have

$$(Au, u)_{H^{-1}} = \|u\|_{L^2}^2 - \|u\|_{H^{-1}}^2 \ge 0,$$

by (A.3.16); and so, A is accretive, by Lemma 1.1.36. Given $f \in X$, it follows from the preceding observations that $u = Jf \in D(A)$ and that $u + Au = f$. Therefore, A is m-accretive (Proposition 1.1.21). Finally, it follows from (1.2.17) that

$$(Au, v)_{H^{-1}} = (u, Av)_{H^{-1}},$$

for all $u, v \in D(A)$. Applying Corollary 1.1.46, it follows that A is self-adjoint. Finally, it follows from Corollary 1.1.12 that $\|u\|_{D(A)} \approx \|u - \triangle u\|_{H^{-1}}$, on $D(A)$. By Remark A.4.4 (i), we obtain $\|u\|_{D(A)} \approx \|u\|_{H_0^1}$, on $D(A)$. This completes the proof. □

We now describe some useful properties of $(I + \lambda A)^{-1}$.

Proposition 1.2.18. *Let A be defined by (1.2.14) and let $J_\lambda = (I + \lambda A)^{-1}$ for $\lambda > 0$. The following properties hold:*

(i) *$J_\lambda \in \mathcal{L}(H^{-1}(\Omega))$ and $\|J_\lambda\|_{\mathcal{L}(H^{-1})} \le 1$, for every $\lambda > 0$;*

(ii) *$J_\lambda \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$, for every $\lambda > 0$;*

(iii) *$(J_\lambda)_{|H_0^1(\Omega)} \in \mathcal{L}(H_0^1(\Omega))$ and $\|(J_\lambda)_{|H_0^1(\Omega)}\|_{\mathcal{L}(H_0^1)} \le 1$, for every $\lambda > 0$;*

(iv) *$J_\lambda u \underset{\lambda \downarrow 0}{\longrightarrow} u$ in $H^{-1}(\Omega)$, for every $u \in H^{-1}(\Omega)$;*

(v) *$J_\lambda u \underset{\lambda \downarrow 0}{\longrightarrow} u$ in $H_0^1(\Omega)$, for every $u \in H_0^1(\Omega)$.*

Proof. (i) follows from Definition 1.1.9, (ii) follows from Corollary 1.1.14, (iii) follows from Lemma 1.1.16, (iv) and (v) follow from Proposition 1.1.19. □

Proposition 1.2.19. *Let A be defined by (1.2.14), let $J_\lambda = (I + \lambda A)^{-1}$ for $\lambda > 0$ and let $1 \le p < \infty$. For every $u \in H^{-1}(\Omega) \cap L^p(\Omega)$, the following properties hold:*

(i) *$J_\lambda u \in L^p(\Omega)$ and $\|J_\lambda u\|_{L^p} \le \|u\|_{L^p}$, for every $\lambda > 0$;*

(ii) *$J_\lambda u \underset{\lambda \downarrow 0}{\longrightarrow} u$ in $L^p(\Omega)$.*

Proof. (i) follows from Theorem A.4.11 and definition of J_λ. The proof of (ii) is more delicate. Note that, in view of (i) and Proposition A.1.4, we only have to establish the result for $u \in \mathcal{D}(\Omega)$. Therefore, consider $u \in \mathcal{D}(\Omega)$, and assume that u is supported in $\Omega_R = \{x \in \Omega; |x| < R\}$. Set $u_\lambda = J_\lambda u$. It follows that $u_\lambda \in H_0^1(\Omega)$ and

$$-\lambda \triangle u_\lambda + u_\lambda = u.$$

Define $v(x) = 2\|u\|_{L^\infty} e^{\sqrt{1+R^2}} e^{-\sqrt{1+|x|^2}}$. One has $v \ge 2|u|$, and one verifies easily that $v \ge \triangle v$. It follows that v is a supersolution of the above equation, and that $-v$ is a subsolution, for $0 < \lambda \le 1/2$. Applying Corollary A.4.27 and Proposition A.3.34, we obtain

$$|u_\lambda| \le v \in L^p(\Omega), \text{ almost everywhere in } \Omega. \tag{1.2.18}$$

We now argue by contradiction, and we assume that there exists a sequence $\lambda_n \downarrow 0$ and $\varepsilon > 0$ such that $\|J_{\lambda_n} u - u\|_{L^p} \ge \varepsilon$. It follows from Proposition 1.2.18 that $J_{\lambda_n} u \underset{n \to \infty}{\longrightarrow} u$ in $H_0^1(\Omega)$. In particular, it follows from Corollary A.3.10 that there exists a subsequence, which we still denote by λ_n, such that $J_{\lambda_n} u \underset{n\to\infty}{\longrightarrow} u$ almost everywhere in Ω. Applying (1.2.18) and the dominated convergence theorem, we find $J_{\lambda_n} u \underset{n\to\infty}{\longrightarrow} u$ in $L^p(\Omega)$, which is a contradiction. This completes the proof. □

Remark 1.2.20. Under the assumptions of Proposition 1.2.19, assume that $u \in L^\infty(\Omega)$. Then, it follows from Theorem A.4.11 that $J_\lambda u \in L^\infty(\Omega)$, for every $\lambda > 0$. However, note that in general $J_\lambda u \not\to u$ in $L^\infty(\Omega)$, as $\lambda \downarrow 0$. Indeed, assuming that Ω is bounded, it follows from Corollary A.4.17 that every limit point in $L^\infty(\Omega)$ of the family $(J_\lambda u)_{\lambda>0}$ belongs to $C_{\rm b}(\Omega)$; and so, $\liminf\limits_{\lambda\downarrow 0} \|J_\lambda u - u\|_{L^\infty} > 0$ if $u \notin C_{\rm b}(\Omega)$. On the other hand, if Ω verifies the assumptions of Theorem A.4.28 and if $u \in C_0(\Omega)$, then it follows from Theorem A.4.28 that $J_\lambda u \in C_0(\Omega)$, and an obvious adaptation of the proof of Proposition 1.2.19 shows that $J_\lambda u \underset{\lambda\downarrow 0}{\longrightarrow} u$ in $C_0(\Omega)$.

Example 2. **L^2 theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set $X = L^2(\Omega)$, and define the operator A on X by

$$\begin{cases} D(A) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}, \\ Au = -\triangle u, \ \text{for all } u \in D(A). \end{cases} \tag{1.2.19}$$

We have the following result.

Proposition 1.2.21. *The operator A defined by (1.2.19) is self-adjoint and accretive. In particular, A is m-accretive with dense domain. In addition, $D(A) \hookrightarrow H^1_0(\Omega)$; and in particular, if Ω is bounded, then $D(A) \hookrightarrow L^2(\Omega)$ with compact injection.*

Proof. Let $u, v \in D(A)$. It follows from formulas (A.3.14) and (A.3.17) that

$$(Au, v)_{L^2} = -(\triangle u, v)_{L^2} = -\langle v, \triangle u\rangle_{H^1_0, H^{-1}} = \int_\Omega \nabla u \cdot \nabla v. \tag{1.2.20}$$

In particular, $(Au, u)_{L^2} \ge 0$, for all $u \in D(A)$; and so, A is accretive, by Lemma 1.1.36. Given $f \in L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, it follows from Proposition 1.2.17 that there exists $u \in H^1_0(\Omega)$ such that

$$u - \triangle u = f, \ \text{in } H^{-1}(\Omega).$$

In particular, we have $\triangle u = u - f \in L^2(\Omega)$; and so, $u \in D(A)$ and $u + Au = f$. Therefore, A is m-accretive (Proposition 1.1.21). Furthermore, it follows from (1.2.20) that

$$(Au, v)_{L^2} = (u, Av)_{L^2},$$

for all $u, v \in D(A)$. Applying Corollary 1.1.46, it follows that A is self-adjoint. Finally, given $u \in D(A)$, it follows from (1.2.20) that

$$\|u\|^2_{H^1} = \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2} = (Au, u)_{L^2} + \|u\|^2_{L^2} \le \|u\|_{D(A)} \|u\|_{L^2}.$$

This completes the proof. □

Proposition 1.2.22. *Let A be as in Proposition 1.2.18. If Ω has a bounded boundary of class C^2, then $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, with equivalent norms.*

Proof. This follows from Theorem A.4.8. □

Remark 1.2.23. Here are some simple regularity properties of the domain of A.

(i) It follows from Proposition A.4.10 that $D(A) \subset H^2_{\mathrm{loc}}(\Omega)$, without any restriction on Ω.

(ii) An iterative application of Proposition A.4.10 shows that $D(A^n) \subset H^{2n}_{\mathrm{loc}}(\Omega)$ and that $D(A^n) \hookrightarrow H^{2n}(\Omega')$ for every $\Omega' \subset\subset \Omega$ (cf. Remark 1.1.30). Furthermore, $D(A^n) = \{u \in H^{2n}_{\mathrm{loc}}(\Omega);\ \triangle^j u \in H^1_0(\Omega),$ for $0 \le j \le n-1$ and $\triangle^n u \in L^2(\Omega)\}$. In particular, $\bigcap_{n\ge 1} D(A^n) \subset C^\infty(\Omega)$ (cf. Theorem A.3.40).

(iii) Applying Theorem A.4.8 one obtains as well that, if Ω has a bounded boundary of class C^{2n}, then $D(A^n) \hookrightarrow H^{2n}(\Omega)$, and $D(A^n) = \{u \in H^{2n}(\Omega);\ \triangle^j u \in H^1_0(\Omega),$ for $0 \le j \le n-1\}$. In particular, if Ω has a bounded boundary of class C^∞, then $\bigcap_{n\ge 1} D(A^n) \subset C^\infty(\overline{\Omega})$ (cf. Theorem A.3.40). Therefore, if we assume further that Ω is bounded, then $\bigcap_{n\ge 1} D(A^n) = \{u \in C^\infty(\overline{\Omega});\ u = \triangle u = \triangle^2 u = \cdots = 0 \text{ on } \partial\Omega\}$ (see Proposition A.3.23).

Remark 1.2.24. If A is defined by (1.2.19), then it follows easily from Theorem A.4.8 (uniqueness) that $(I + \lambda A)^{-1}$ coincides with the restriction to $L^2(\Omega)$ of the operator J_λ defined in Proposition 1.2.19.

Corollary 1.2.25. *Let A be defined by (1.2.19), let I be an interval of $\mathbb{R}$ and let $1 < p < \infty$. Then, the following properties hold:*

(i) $L^p(I, D(A)) \cap W^{1,p'}(I, L^2(\Omega)) \hookrightarrow C_{\mathrm{b}}(\overline{I}, H^1_0(\Omega))$;

(ii) *for every $u \in L^p(I, D(A)) \cap W^{1,p'}(I, L^2(\Omega))$, the function $t \mapsto \|\nabla u(t)\|^2_{L^2}$ belongs to $W^{1,1}(I)$, and*

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} = -2(\triangle u(t), u_t(t))_{L^2},$$

for almost all $t \in I$.

Proof. Consider $u \in C^1_c(\overline{I}, D(A))$. It follows from (1.2.20) that

$$\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} = 2(\nabla u(t), \nabla u_t(t))_{L^2} = -2(\triangle u(t), u_t(t))_{L^2},$$

for all $t \in I$. One concludes as in Corollary A.3.64. □

Remark 1.2.26. If A is defined by (1.2.19), then it follows from Proposition 1.2.21 and Proposition 1.1.49 that A^n is self-adjoint and accretive, for every positive integer n.

Example 3. **L^p theory.** Let Ω be any open subset of $\mathbb{R}^N$. We will apply Proposition 1.2.19 in order to construct a realization of the Laplacian in $L^p(\Omega)$. We begin with the following observation.

Lemma 1.2.27. *Let $1 \le p < \infty$, and for $\lambda > 0$ let J_λ be defined in Proposition 1.2.19. There exists a unique operator $I_\lambda \in \mathcal{L}(L^p(\Omega))$ such that $I_\lambda f = J_\lambda f$, for every $f \in H^{-1}(\Omega) \cap L^p(\Omega)$. In addition, the following properties hold:*

(i) $\|I_\lambda\|_{\mathcal{L}(L^p)} \le 1$, *for all $\lambda > 0$;*

(ii) *for all* $f \in L^p(\Omega)$ *and all* $\lambda > 0$, *we have* $\triangle I_\lambda f \in L^p(\Omega)$ *and* $-\lambda \triangle I_\lambda f + I_\lambda f = f$;

(iii) $R(I_\lambda) = R(I_\mu)$, *for all* $\lambda, \mu > 0$.

Proof. Since $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, it follows from Proposition 1.2.19 that J_λ has a unique extension $I_\lambda \in L^p(\Omega)$, which verifies (i). Consider now $f \in L^p(\Omega)$, and let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^p(\Omega)$. It follows that $I_\lambda f_n \underset{n\to\infty}{\longrightarrow} I_\lambda f$ in $L^p(\Omega)$, and since $-\lambda \triangle I_\lambda f_n + I_\lambda f_n = f_n$ in $H^{-1}(\Omega)$, we have $-\lambda \triangle I_\lambda f + I_\lambda f = f$ in $\mathcal{D}'(\Omega)$. Hence (ii). Finally, let $f \in H^{-1}(\Omega) \cap L^p(\Omega)$, and let $u = I_\lambda f \in H^{-1}(\Omega) \cap L^p(\Omega)$. Given $\mu > 0$, we have

$$-\mu \triangle u + u = \frac{\lambda - \mu}{\lambda} u + \frac{\mu}{\lambda} f.$$

Set $g = \dfrac{\lambda - \mu}{\lambda} u + \dfrac{\mu}{\lambda} f$ and let $v = I_\lambda g \in H^{-1}(\Omega) \cap L^p(\Omega)$. It follows that

$$-\mu \triangle (v - u) + v - u = 0, \text{ in } H^{-1}(\Omega);$$

and so, $u = v$. Therefore, $I_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) \subset I_\mu(H^{-1}(\Omega) \cap L^p(\Omega))$. Exchanging the roles of μ and λ, we find $I_\lambda(H^{-1}(\Omega) \cap L^p(\Omega)) = I_\mu(H^{-1}(\Omega) \cap L^p(\Omega))$. Since both I_λ and I_μ are continuous on $L^p(\Omega)$ and $H^{-1}(\Omega) \cap L^p(\Omega)$ is dense in $L^p(\Omega)$, (iii) follows. □

Proposition 1.2.28. *Let* $1 \le p < \infty$, *and for* $\lambda > 0$ *let* I_λ *be defined by Lemma 1.2.27. The operator* A *in* $L^p(\Omega)$ *defined by*

$$\begin{cases} D(A) = R(I_1); \\ Au = -\triangle u, \text{ for } u \in D(A); \end{cases} \tag{1.2.21}$$

is m-accretive with dense domain.

Remark. Note that for $u \in D(A)$, we have $\triangle u \in L^p(\Omega)$ by Lemma 1.2.27; and so, definition (1.2.21) makes sense.

Proof. Let $u \in D(A)$ and $\lambda > 0$, and let $f = \lambda A u + u = -\lambda \triangle u + u$. It follows from Lemma 1.2.27 (iii) that there exists $g \in L^p(\Omega)$ such that $u = I_\lambda g$. In particular, $g = -\lambda \triangle u + u$; and so, $f = g$. Applying again Lemma 1.2.27, we find $\|u\|_{L^p} \le \|f\|_{L^p}$. Therefore, A is accretive. Let now $f \in L^p(\Omega)$, and let $u = I_1 f$. It follows that $u \in D(A)$ and that $Au + u = f$; and so, A is m-accretive. Finally, let $u \in \mathcal{D}(\Omega)$, and let $f = -\triangle u + u \in \mathcal{D}(\Omega)$. It follows that $u = I_1 f$. Therefore, $\mathcal{D}(\Omega) \subset D(A)$, and it follows that $D(A)$ is dense in $L^p(\Omega)$. □

Proposition 1.2.29. *Let* $1 < p < \infty$ *and let* A *be defined by (1.2.21). If* Ω *has a bounded boundary of class* C^2, *then* $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *with equivalent norms.*

Proof. Define the operator B in $L^p(\Omega)$ by

$$\begin{cases} D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega); \\ Bu = -\triangle u, \text{ for } u \in D(B). \end{cases}$$

It follows easily from Remark A.4.18 (i) and (iii) that B is m-accretive. Consider now $u \in D(B)$ and let $f = -\triangle u + u$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^p(\Omega)$, and let $u_n = I_1 f_n$, where I_1 is defined in Lemma 1.2.27. It follows from Remark A.4.18 (i) and (ii) that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{2,p}(\Omega)$, then from Lemma 1.2.27 that $u = I_1 f$. Therefore, $u \in D(A)$, and it follows that $G(B) \subset G(A)$. Applying Corollary 1.1.23, we obtain that $A = B$. Equivalence of the norms follows from Remark A.4.18 (i). □

Remark 1.2.30. Let $p = 1$ and let A be defined by (1.2.21). If Ω has a bounded boundary of class C^2, then $D(A) = \{u \in W_0^{1,1}(\Omega);\ \triangle u \in L^1(\Omega)\}$. This follows from Remark A.4.18 (iv) (cf. the proof of Proposition 1.2.29). Note that in general $D(A)$ is not contained in $W^{2,1}(\Omega)$ (see Remark A.4.18 (ii)).

Remark 1.2.31. One can consider the case $p = \infty$. Let Ω be a **bounded** open subset of $\mathbb{R}^N$, and define the operator A on $L^\infty(\Omega)$ by

$$\begin{cases} D(A) = \{u \in L^\infty(\Omega) \cap H_0^1(\Omega);\ \triangle u \in L^\infty(\Omega)\}, \\ Au = -\triangle u, \text{ for all } u \in D(A). \end{cases}$$

It follows easily from Remark A.4.18 (iv) that A is m-accretive. Note that $D(A) \subset C(\Omega)$, as follows easily from Corollary A.4.17. Note also that if Ω satisfies the assumptions of Theorem A.4.28, we have $D(A) \subset C_0(\Omega)$. In particular $D(A)$ is **not** dense in $L^\infty(\Omega)$, and this justifies Example 4 below.

Example 4. C_0 **theory.** Let Ω be an open subset of $\mathbb{R}^N$, and define the operator A in $C_0(\Omega)$ by

$$\begin{cases} D(A) = \{u \in C_0(\Omega);\ \triangle u \in C_0(\Omega)\}; \\ Au = -\triangle u, \text{ for } u \in D(A). \end{cases} \tag{1.2.22}$$

We have the following result.

Proposition 1.2.32. *If $N \geq 2$, assume that every $x \in \partial\Omega$ has the exterior cone property. Then, the operator A defined by (1.2.22) is m-accretive with dense domain.*

Proof. m-accretiveness follows from Corollary A.4.33. Since $\mathcal{D}(\Omega) \subset D(A)$, it follows that $D(A)$ is dense. This completes the proof. □

Remark 1.2.33. Note that all the results of Section 1.2.2 hold true as well in the corresponding spaces of complex-valued functions. The proofs are the same (cf. Section A.4.6).

1.2.3. The Schrödinger operator. The following examples are related to Schrödinger's equation.

Example 1. H^{-1} **theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set $X = H^{-1}(\Omega)$, and define the operator A on X by

$$\begin{cases} D(A) = H_0^1(\Omega), \\ Au = -i\triangle u, \text{ for all } u \in D(A). \end{cases} \tag{1.2.23}$$

We have the following result.

Proposition 1.2.34. *The operator A defined by (1.2.23) is skew-adjoint, and $D(A) = H^1_0(\Omega)$ with equivalent norms. In particular, A and $-A$ are m-accretive with dense domain.*

Proof. The result follows from Proposition 1.2.17, Corollary 1.1.54 and Remark 1.2.33. □

Example 2. **L^2 theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set $X = L^2(\Omega)$, and define the operator A on X by

$$\begin{cases} D(A) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}, \\ Au = -i\triangle u, \text{ for all } u \in D(A). \end{cases} \tag{1.2.24}$$

We have the following result.

Proposition 1.2.35. *The operator A defined by (1.2.24) is skew-adjoint. In particular, A and $-A$ are m-accretive with dense domain. In addition, $D(A) \hookrightarrow H^1_0(\Omega)$.*

Proof. The result follows from Proposition 1.2.21, Corollary 1.1.54 and Remark 1.2.33. □

Remark 1.2.36. Note that if Ω has a bounded boundary of class C^2, then, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, with equivalent norms. This follows from Proposition 1.2.22.

1.2.4. The wave operator. The following examples are related to the wave equation and to Klein-Gordon equation.

Example 1. **$L^2 \times H^{-1}$ theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set

$$\mathcal{Y} = L^2(\Omega) \times H^{-1}(\Omega), \tag{1.2.25}$$

with its natural scalar product, and define the operator $\mathcal{B}$ on $\mathcal{Y}$ by

$$\begin{cases} D(\mathcal{B}) = H^1_0(\Omega) \times L^2(\Omega), \\ \mathcal{B}(u, v) = (-v, -\triangle u + u), \text{ for all } (u, v) \in D(\mathcal{B}). \end{cases} \tag{1.2.26}$$

We have the following result.

Proposition 1.2.37. *The operator $\mathcal{B}$ defined by (1.2.26) is skew-adjoint, and $\|\cdot\|_{D(B)}$ is equivalent to $\|\cdot\|_{H^1_0 \times L^2}$. In particular, $\mathcal{B}$ and $-\mathcal{B}$ are m-accretive with dense domain.*

Proof. Let $U \in D(\mathcal{B})$, and write $U = (u, v)$. Let $w \in H^1_0(\Omega)$ be the solution of $-\triangle w + w = v$ (cf. Lemma A.4.2). It follows from Remark A.4.3 (iii) and (A.3.17) that

$$\begin{aligned} (\mathcal{B}U, U)_{\mathcal{Y}} &= (-v, u)_{L^2} + (-\triangle u + u, v)_{H^{-1}} = (-v, u)_{L^2} + (u, w)_{H^1_0} \\ &= (-v, u)_{L^2} + \int_\Omega \{\nabla u \cdot \nabla w + uw\}\, dx \\ &= (-v, u)_{L^2} + \langle u, -\triangle w + w\rangle_{H^1_0, H^{-1}} = (-v, u)_{L^2} + \langle u, v\rangle_{H^1_0, H^{-1}}; \end{aligned}$$

and so, by (A.3.14),

$$(\mathcal{B}U, U)_{\mathcal{Y}} = 0. \tag{1.2.27}$$

In particular, $\mathcal{B}$ is accretive (cf. Lemma 1.1.36). Finally, given $F = (f, g) \in \mathcal{Y}$, equation $U + \mathcal{B}U = F$ is equivalent to the system

$$\begin{cases} u - v = f; \\ -\triangle u + u + v = g; \end{cases}$$

or equivalently

$$\begin{cases} -\triangle u + 2u = f + g; \\ v = u - f. \end{cases}$$

It follows from Lemma A.4.3 that there exists $u \in H_0^1(\Omega)$ solving the first equation. Then, v given by the second equation belongs to $L^2(\Omega)$. It follows that $\mathcal{B}$ is m-accretive. The result now follows from (1.2.27) and Corollary 1.1.47. □

One can extend Proposition 1.2.37 as follows. Let λ_1 be defined by

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^2, u \in H_0^1(\Omega), \int_\Omega |u|^2 = 1 \right\}. \tag{1.2.28}$$

(note that λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$ if Ω is bounded) and consider $\lambda > -\lambda_1$. Consider on $H_0^1(\Omega)$ the norm $|||\cdot|||$ defined by (A.4.2), that is

$$|||u||| = \left(\int_\Omega \{ |\nabla u|^2 + \lambda |u|^2 \} \, dx \right)^{1/2},$$

and consider on $H^{-1}(\Omega)$ the corresponding dual norm. Define the operator $\mathcal{B}$ on $\mathcal{Y}$ by

$$\begin{cases} D(\mathcal{B}) = H_0^1(\Omega) \times L^2(\Omega), \\ \mathcal{B}(u, v) = (-v, -\triangle u + \lambda u), \text{ for all } (u, v) \in D(\mathcal{B}). \end{cases} \tag{1.2.29}$$

We have the following result.

Proposition 1.2.38. *Let λ_1 be defined by (1.2.28) and let $\lambda > -\lambda_1$. The operator $\mathcal{B}$ defined by (1.2.29) is skew-adjoint, and $\| \cdot \|_{D(B)}$ is equivalent to $\| \cdot \|_{H_0^1 \times L^2}$. In particular, $\mathcal{B}$ and $-\mathcal{B}$ are m-accretive with dense domain.*

Proof. The proof is easily adapted from that of Proposition 1.2.36, by using in particular Lemma A.4.2 and Theorem A.4.5. □

Remark 1.2.39. If Ω is bounded, then it follows from Poincaré's inequality (A.3.4) that $\lambda_1 > 0$. In particular, one can take $\lambda = 0$ in Proposition 1.2.38.

Example 2. $H_0^1 \times L^2$ **theory.** Let Ω be any open subset of $\mathbb{R}^N$. Set

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega), \tag{1.2.30}$$

and define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\begin{cases} D(\mathcal{A}) = \{ (u, v) \in \mathcal{H}, \triangle u \in L^2(\Omega) \text{ and } v \in H_0^1(\Omega) \}, \\ \mathcal{A}(u, v) = (-v, -\triangle u + u), \text{ for all } (u, v) \in D(\mathcal{A}). \end{cases} \tag{1.2.31}$$

We have the following result.

Proposition 1.2.40. *The operator $\mathcal{A}$ defined by (1.2.31) is skew-adjoint. In particular, $\mathcal{A}$ and $-\mathcal{A}$ are m-accretive with dense domain. In addition, $D(\mathcal{A}) \hookrightarrow H_0^1(\Omega) \times H_0^1(\Omega)$.*

Proof. Let $U \in D(\mathcal{A})$, and write $U = (u, v)$. It follows from formulas (A.3.14) and (A.3.17) that

$$(\mathcal{A}U, U)_{\mathcal{H}} = -\int_{\Omega} \{\nabla u \cdot \nabla w + v \triangle u\} \, dx = 0. \tag{1.2.32}$$

In particular, $\mathcal{A}$ is accretive (cf. Lemma 1.1.36). Finally, given $F = (f, g) \in \mathcal{H}$, equation $U + \mathcal{A}U = F$ is equivalent to the system

$$\begin{cases} u - v = f; \\ -\triangle u + u + v = g; \end{cases}$$

or equivalently

$$\begin{cases} -\triangle u + 2u = f + g; \\ v = u - f. \end{cases}$$

It follows from Lemma A.4.3 that there exists $u \in H_0^1(\Omega)$ with $\triangle u \in L^2(\Omega)$ solving the first equation. Then, v given by the second equation belongs to $H_0^1(\Omega)$. It follows that $\mathcal{A}$ is m-accretive. The result now follows from (1.2.32) and Corollary 1.1.47. Property $D(\mathcal{A}) \hookrightarrow H_0^1(\Omega) \times H_0^1(\Omega)$ follows from Proposition 1.2.21. □

Remark 1.2.41. It follows from Theorem A.4.8 that if Ω has a bounded boundary of class C^2, then $D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ with equivalent norms.

Corollary 1.2.42. *Let $\mathcal{A}$ be defined by (1.2.31) and let $\mathcal{H}_{-1}$ and $\mathcal{A}_{(-1)}$ be defined by Theorem 1.1.31. If $\mathcal{Y}$ and $\mathcal{B}$ are defined by (1.2.25) and (1.2.26), then $\mathcal{H}_{-1} = \mathcal{Y}$ with equivalent norms, and $\mathcal{A}_{(-1)} = \mathcal{B}$.*

Proof. Let $V = (u, v) \in D(\mathcal{A}) \subset \mathcal{X} \subset \mathcal{Y}$. We have

$$\|\mathcal{A}V\|_{\mathcal{Y}}^2 = \|v\|_{L^2}^2 + \| - \triangle u + u\|_{H^{-1}}^2 = \|v\|_{L^2}^2 + \|u\|_{H_0^1}^2 = \|V\|_{\mathcal{H}}^2;$$

and so, since $\mathcal{B}$ is skew-adjoint,

$$\|(I - \mathcal{A})V\|_{\mathcal{Y}}^2 = \|(I - \mathcal{B})V\|_{\mathcal{Y}}^2 = \|\mathcal{B}V\|_{\mathcal{Y}}^2 + \|V\|_{\mathcal{Y}}^2 = \|V\|_{\mathcal{H}}^2 + \|V\|_{\mathcal{Y}}^2.$$

Given $U \in \mathcal{H}$, and applying the above inequality to $V = J_1(\mathcal{A})$, we obtain

$$\|U\|_{\mathcal{Y}}^2 = \|J_1(\mathcal{A})U\|_{\mathcal{H}}^2 + \|J_1(\mathcal{A})U\|_{\mathcal{Y}}^2.$$

Since $\mathcal{H} \hookrightarrow \mathcal{Y}$, it follows that

$$\|U\|_{\mathcal{Y}}^2 \approx \|J_1(\mathcal{A})U\|_{\mathcal{H}}^2.$$

On the other hand, note that $\mathcal{H}$ is dense in $\mathcal{Y}$, since $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \subset \mathcal{H}$; and so, $\mathcal{Y}$ is the completion of $\mathcal{H}$ for the norm $\|J_1(\mathcal{A})U\|_{\mathcal{H}}$. It follows that $\mathcal{Y}$ verifies properties (i) and (ii) of Theorem 1.1.31. Furthermore, it follows from Proposition 1.2.37 that $\mathcal{B}$ verifies properties (iii) and (iv). Finally, property (v) follows by definition. Hence the result, by Theorem 1.1.31. □

One can extend Proposition 1.2.40 and Corollary 1.2.42 as follows. Let λ_1 be defined by (1.2.28), and consider $\lambda > -\lambda_1$. Consider on $H^1_0(\Omega)$ the norm $|||\cdot|||$ defined by (A.4.2), that is

$$|||u||| = \Bigl(\int_\Omega \{|\nabla u|^2 + \lambda |u|^2\}\, dx \Bigr)^{1/2}.$$

Define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\begin{cases} D(\mathcal{A}) = \{(u,v) \in \mathcal{H}, \triangle u \in L^2(\Omega) \text{ and } v \in H^1_0(\Omega)\}, \\ \mathcal{A}(u,v) = (-v, -\triangle u + \lambda u), \text{ for all } (u,v) \in D(\mathcal{A}). \end{cases} \tag{1.2.33}$$

We have the following result.

Proposition 1.2.43. *Let λ_1 be defined by (1.2.28). If $\lambda > -\lambda_1$, then the operator $\mathcal{A}$ defined by (1.2.33) is skew-adjoint. In particular, $\mathcal{A}$ and $-\mathcal{A}$ are m-accretive with dense domain. Furthermore, $\mathcal{H}_{-1} = \mathcal{Y}$ with equivalent norms and $\mathcal{A}_{(-1)} = \mathcal{B}$, where $\mathcal{H}_{-1}$ and $\mathcal{A}_{(-1)}$ are defined by Theorem 1.1.31, and $\mathcal{Y}$ and $\mathcal{B}$ are defined by (1.2.25) and (1.2.29).*

Proof. The proof is easily adapted from the proofs of Proposition 1.2.40 and Corollary 1.2.42, by making use in particular of Lemma A.4.2 and Theorem A.4.5. □

Remark 1.2.44. If Ω is bounded, then it follows from Poincaré's inequality (A.3.7) that $\lambda_1 > 0$. In particular, one can take $\lambda = 0$ in Proposition 1.2.43.

Remark 1.2.45. Here are some simple regularity properties of the domain of $\mathcal{A}$.

(i) It follows from Proposition A.4.10 that $D(\mathcal{A}) \subset H^2_{\mathrm{loc}}(\Omega) \times H^1_0(\Omega)$, without any restriction on Ω.

(ii) One verifies easily that if $n \ge 1$, then $D(\mathcal{A}^{2n}) = \{(u,v) \in H^1_0(\Omega) \times L^2(\Omega);\ \triangle^j u, \triangle^{j-1} v \in H^1_0(\Omega)$ for $1 \le j \le n$ and $\triangle^n v \in L^2(\Omega)\}$, and that $D(\mathcal{A}^{2n+1}) = \{(u,v) \in H^1_0(\Omega) \times L^2(\Omega);\ \triangle^j u, \triangle^j v \in H^1_0(\Omega)$ for $0 \le j \le n$ and $\triangle^{n+1} u \in L^2(\Omega)\}$. An iterative application of Proposition A.4.10 shows that $D(\mathcal{A}^n) \subset H^{n+1}_{\mathrm{loc}}(\Omega) \times H^n_{\mathrm{loc}}(\Omega)$ and that $D(\mathcal{A}^n) \hookrightarrow H^{n+1}(\Omega') \times H^n(\Omega')$ for every $\Omega' \subset\subset \Omega$. In particular, $\bigcap_{n\ge1} D(\mathcal{A}^n) \subset C^\infty(\Omega) \times C^\infty(\Omega)$ (cf. Theorem A.3.40).

(iii) Applying Theorem A.4.8 one obtains as well that, if Ω has a bounded boundary of class C^{n+1}, then $D(\mathcal{A}^n) \subset H^{n+1}(\Omega) \times H^n(\Omega)$. In particular, if Ω has a bounded boundary of class C^∞, then $\bigcap_{n\ge1} D(\mathcal{A}^n) \subset C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ (cf. Theorem A.3.40). Therefore, if we assume further that Ω is bounded, then $\bigcap_{n\ge1} D(\mathcal{A}^n) = \{u \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega});\ u = \triangle u = \triangle^2 u = \cdots = 0 \text{ and } v = \triangle v = \triangle^2 v = \cdots = 0 \text{ on } \partial\Omega\}$ (see Proposition A.3.23).

Remark 1.2.46. Note that all the results of Section 1.2.4 hold true as well in the corresponding spaces of complex-valued functions. The proofs are the same (cf. Section A.4.6).

1.2.5. The Stokes operator. We introduce here the Stokes operator, which is essential in the study of the Navier-Stokes equation. For simplicity, we begin with the Stokes operator in $\mathbb{R}^N$. Let $N \ge 2$, and consider the Hilbert space

$$E = \bigl(L^2(\mathbb{R}^N)\bigr)^N.$$

A vector of E has the form $\boldsymbol{u} = (u_1, \dots, u_N)$. We write

$$\nabla \cdot \boldsymbol{u} = \mathrm{div}\boldsymbol{u} = \sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i},$$

and

$$\triangle \boldsymbol{u} = (\triangle u_1, \dots, \triangle u_N).$$

Let

$$X = \{\boldsymbol{u} \in E;\ \nabla \cdot \boldsymbol{u} = 0\}.$$

Here, the condition $\nabla \cdot \boldsymbol{u} = 0$ is understood in the sense of distributions. It is clear that X is a closed subspace of E. Therefore, X is also a Hilbert space with the scalar product of E, that is

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} u_i v_i \, dx.$$

We define the Stokes operator A by

$$\begin{cases} D(A) = \{\boldsymbol{u} \in \big(H^2(\mathbb{R}^N)\big)^N \cap X;\ \triangle \boldsymbol{u} \in X\}; \\ A\boldsymbol{u} = -\triangle \boldsymbol{u}, \ \text{for } \boldsymbol{u} \in D(A). \end{cases}$$

We have the following result.

Theorem 1.2.47. *The operator A defined above is self-adjoint and accretive, hence m-accretive with dense domain.*

Proof. We first show that $R(I + A) = X$. Let $\boldsymbol{f} \in X$. In particular, $f_i \in L^2(\mathbb{R}^N)$ for every $i \in \{1, \dots, N\}$. Therefore, it follows from Propositions 1.2.21 and 1.2.22 that there exists $u_i \in H^2(\mathbb{R}^N)$ such that $-\triangle u_i + u_i = f_i$. Setting $\boldsymbol{u} = (u_1, \dots, u_n)$, we have $\boldsymbol{u} \in \big(H^2(\mathbb{R}^N)\big)^N$ and $-\triangle \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f}$. On the other hand, setting $v_i = \dfrac{\partial u_i}{\partial x_i} \in H^1(\mathbb{R}^N)$ and $g_i = \dfrac{\partial f_i}{\partial x_i} \in H^{-1}(\mathbb{R}^N)$, it follows from equation $-\triangle u_i + u_i = f_i$ that $-\triangle v_i + v_i = g_i$. Note that $\nabla \cdot \boldsymbol{u} = \sum_{j=1}^{N} v_i$ and $\sum_{j=1}^{N} g_i = \nabla \cdot \boldsymbol{f} = 0$. Setting $w = \nabla \cdot \boldsymbol{u} \in H^{-1}(\mathbb{R}^N)$, it follows that $-\triangle w + w = 0$. This implies that $w = 0$ (see Proposition 1.2.17); and so, $\boldsymbol{u} \in X$. Furthermore, $\triangle \boldsymbol{u} = \boldsymbol{u} - \boldsymbol{f} \in X$; and so, $\boldsymbol{u} \in D(A)$ and $\boldsymbol{u} + A\boldsymbol{u} = \boldsymbol{f}$, which implies that $R(I + A) = X$. Finally, it follows from Proposition 1.2.21 that $(A\boldsymbol{u}, \boldsymbol{u}) \geq 0$ and $(A\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, A\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in D(A)$, which completes the proof. □

Remark 1.2.48. Let $1 < p < \infty$. One can consider the Stokes operator in L^p. More precisely, let $E = \big(L^p(\mathbb{R}^N)\big)^N$, and set $X = \{\boldsymbol{u} \in E;\ \nabla \cdot \boldsymbol{u} = 0\}$. X is a closed subspace of E, therefore, X is also a Banach space. We define the Stokes operator A (in L^p) by

$$\begin{cases} D(A) = \{\boldsymbol{u} \in \big(W^{2,p}(\mathbb{R}^N)\big)^N \cap X;\ \triangle \boldsymbol{u} \in X\}; \\ A\boldsymbol{u} = -\triangle \boldsymbol{u}, \ \text{for } \boldsymbol{u} \in D(A). \end{cases}$$

Arguing as above, one shows that A is m-accretive with dense domain.

The definition of the Stokes operator in a domain Ω is more technical. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. Let $E = \big(L^2(\Omega)\big)^N$, and let

$$F = \{\boldsymbol{u} \in (\mathcal{D}(\Omega))^N;\ \nabla \cdot \boldsymbol{u} = 0\}.$$

Let X be the closure of F in E. It is clear that X is a closed subspace of E. Therefore, X is also a Hilbert space with the scalar product of E, that is

$$(\boldsymbol{u}, \mathbf{v}) = \sum_{i=1}^{N} \int_\Omega u_i v_i \, dx.$$

One can show that the trace $\boldsymbol{u} \cdot \nu$ makes sense for every $\boldsymbol{u} \in E$ such that $\nabla \cdot \boldsymbol{u} \in L^2(\Omega)$ (where $\nu(x) \in \mathbb{R}^N$ is the outward unit normal vector at the point $x \in \partial\Omega$), and that

$$X = \{\boldsymbol{u} \in E;\ \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u} \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

Furthermore, the orthogonal $X^\perp$ of X in E is $X^\perp = \{\boldsymbol{u} \in E;\ \exists p \in H^1(\Omega), \boldsymbol{u} = \nabla p\}$ (see Temam [94], Theorems 1.4 and 1.5, pp.15—16). Let $P : E \to X$ be the orthogonal projection on X. We define the Stokes operator A by

$$\begin{cases} D(A) = \big(H^2(\Omega) \cap H^1_0(\Omega)\big)^N \cap X; \\ A\boldsymbol{u} = -P(\triangle \boldsymbol{u}), \text{ for } \boldsymbol{u} \in D(A). \end{cases}$$

We have the following result (see Fujita and Kato [46] for a proof).

Theorem 1.2.49. *The operator A defined above is self-adjoint and accretive, hence m-accretive with dense domain.*

Remark 1.2.50. It is clear from what precedes that $(\boldsymbol{u}, \boldsymbol{f}) \in D(A) \times X$ verify $A\boldsymbol{u} = \boldsymbol{f}$ if and only if there exists $p \in H^1(\Omega)$ such that $-\triangle \boldsymbol{u} + \nabla p = \boldsymbol{f}$. It is clear that p is determined up to a constant; and so, if we define

$$Y = \{p \in H^1(\Omega);\ \int_\Omega p = 0\},$$

then given $(\boldsymbol{u}, \boldsymbol{f}) \in D(A) \times X$ such that $A\boldsymbol{u} = \boldsymbol{f}$, there exists a unique $p \in H^1(\Omega)$ such that $-\triangle \boldsymbol{u} = \boldsymbol{f} + \nabla p$. Moreover, the mapping $\boldsymbol{f} \mapsto (\boldsymbol{u}, p)$ is continous $X \to H^2(\Omega) \times H^1(\Omega)$, as follows from Temam [94, Proposition I.2.2].

Remark 1.2.51. Let $1 < p < \infty$. As above, one can consider the Stokes operator in L^p. See McCracken [80] Fujiwara and Morimoto [47] and Giga [51].

1.2.6. The Airy operator. We introduce here the Airy operator, which is essential in the study of the Korteweg-De Vries equation. Let $X = L^2(\mathbb{R})$, and define the operator A on X by

$$\begin{cases} D(A) = H^3(\mathbb{R}); \\ Au = u_{xxx} = \dfrac{d^3 u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

We have the following result.

Theorem 1.2.52. *The operator A defined above is skew-adjoint. In particular, both A and $-A$ are m-accretive with dense domain.*

Proof. Consider the operator B on $L^2(\mathbb{R})$ defined by

$$\begin{cases} D(B) = H^1(\mathbb{R}), \\ Bu = u', \text{ for } u \in D(B). \end{cases}$$

It follows that $A = B^3$. Since B is skew-adjoint (see Remark 1.2.3 (iii)), it follows from Proposition 1.1.49 (iv) that A is skew-adjoint. □

Remark 1.2.53. It is not difficult to show that the space X_{-1} and the operator $A_{(-1)}$ introduced in Theorem 1.1.31 are given by $X_{-1} = H^{-3}(\mathbb{R})$, and

$$\begin{cases} D(A_{(-1)}) = L^2(\mathbb{R}); \\ A_{(-1)}u = u_{xxx} = \dfrac{d^3u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

In particular, the operator $A_{(-1)}$ defined above is skew-adjoint.

Remark 1.2.54. One can modify the above example as follows. Let $m \in \mathbb{Z}$ be an integer, and let $X = H^m(\mathbb{R})$ (as a matter of fact, m could be any real number). Define the operator A on X by

$$\begin{cases} D(A) = H^{m+3}(\mathbb{R}); \\ Au = u_{xxx} = \dfrac{d^3u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

As above, one shows that the operator A is skew-adjoint. In particular, both A and $-A$ are m-accretive with dense domain. Note that the space X_{-1} and the operator $A_{(-1)}$ introduced in Theorem 1.1.31 are given by $X_{-1} = H^{m-3}(\mathbb{R})$, and

$$\begin{cases} D(A_{(-1)}) = H^m(\mathbb{R}); \\ A_{(-1)}u = u_{xxx} = \dfrac{d^3u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

The above definitions make sense since if $u \in H^m(\mathbb{R})$ for some $m \in \mathbb{Z}$, then $u_{xxx} \in H^{m-3}(\mathbb{R})$ (here u_{xxx} is defined in the sense of distributions).

We next consider the Airy operator with periodic boundary conditions. Let ℓ be a positive real number, and set $X = L^2(0, \ell)$. Define the operator A on X by

$$\begin{cases} D(A) = \{u \in H^3(0, \ell);\ u(0) = u(\ell), u'(0) = u'(\ell), u''(0) = u''(\ell)\} \\ Au = u_{xxx} = \dfrac{d^3u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

We observe that the definition of $D(A)$ makes sense, since $H^3(0, \ell) \hookrightarrow C^2([0, \ell])$. We have the following result.

Theorem 1.2.55. *The operator A defined above is skew-adjoint. In particular, both A and $-A$ are m-accretive with dense domain.*

Proof. Consider the operator B on $L^2(0,\ell)$ defined by

$$\begin{cases} D(B) = \{u \in H^1(0,\ell);\, u(0) = u(\ell)\}, \\ Bu = u', \text{ for } u \in D(B). \end{cases}$$

It follows that $A = B^3$. Since B is skew-adjoint (see Remark 1.2.5 (iv)), it follows from Proposition 1.1.49 (iv) that A is skew-adjoint. □

1.3. The Hille-Yosida-Phillips theorem. This section is devoted to the study of the linear evolution equation $\dfrac{du}{dt} + Au = 0$, where A is an m-accretive operator with dense domain.

1.3.1. The semigroup generated by $-A$, where A is an m-accretive operator. Throughout this section, X is a Banach space, endowed with the norm $\|\cdot\|$. We begin with the following lemma.

Lemma 1.3.1. *If A is an m-accretive operator in X with dense domain, then for every $\lambda > 0$, the operator $A_\lambda \in \mathcal{L}(X)$ introduced in Definition 1.1.15 enjoys the following properties:*

(i) $\|e^{-tA_\lambda}\|_{\mathcal{L}(X)} \le 1$, *for all $t \ge 0$ and all $\lambda > 0$;*

(ii) $\|e^{-tA_\lambda}x - e^{-tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\|$, *for all $x \in X$, all $t \ge 0$ and all $\lambda, \mu > 0$.*

Proof. Consider the operator J_λ introduced in Definition 1.1.9, and let $x \in X$. By Lemma 1.1.16, we have

$$e^{-tA_\lambda}x = e^{-\frac{tI}{\lambda} + \frac{tJ_\lambda}{\lambda}}x = e^{-\frac{t}{\lambda}}e^{\frac{tJ_\lambda}{\lambda}}x;$$

and so,

$$\|e^{-tA_\lambda}x\| \le e^{-\frac{t}{\lambda}}\|e^{\frac{tJ_\lambda}{\lambda}}x\| \le e^{-\frac{t}{\lambda}}e^{\frac{t\|J_\lambda\|_{\mathcal{L}(X)}}{\lambda}}\|x\| \le e^{-\frac{t}{\lambda}}e^{\frac{t}{\lambda}}\|x\| \le \|x\|.$$

Hence property (i). Consider now $\lambda, \mu > 0$. It follows easily from Definition 1.1.15 and Lemma 1.1.16 that A_λ and A_μ commute; and so,

$$e^{-stA_\lambda}e^{-(1-s)tA_\mu}x = e^{-tA_\mu}e^{-st(A_\lambda - A_\mu)}x,$$

for all $x \in X$, $t \ge 0$ and $s \in [0,1]$. Applying Proposition A.1.19, we obtain

$$\begin{aligned} \frac{d}{ds}\left\{e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\right\} &= -te^{-tA_\lambda}e^{-st(A_\lambda - A_\mu)}(A_\lambda x - A_\mu x) \\ &= -te^{-stA_\lambda}e^{-(1-s)tA_\mu}(A_\lambda x - A_\mu x). \end{aligned}$$

In particular, it follows from property (i) that

$$\left\|\frac{d}{ds}\left\{e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\right\}\right\| \le t\|A_\lambda x - A_\mu x\|.$$

Therefore,

$$\|e^{-tA_\lambda}x - e^{-tA_\mu}x\| = \left\|\int_0^1 \frac{d}{ds}\left\{e^{-stA_\lambda}e^{-(1-s)tA_\mu}x\right\}ds\right\| \le t\|A_\lambda x - A_\mu x\|.$$

Hence (ii). This completes the proof. □

Corollary 1.3.2. *Let A be an m-accretive operator in X, with dense domain. Given $\lambda > 0$, consider the operator $A_\lambda \in \mathcal{L}(X)$ introduced in Definition 1.1.15. There exists a family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ such that*

(i) $\|T(t)\|_{\mathcal{L}(X)} \leq 1$, *for all* $t \geq 0$*;*

(ii) $e^{-tA_\lambda}x \underset{\lambda\downarrow 0}{\longrightarrow} T(t)x$ *for all* $x \in X$*, uniformly on bounded subsets of* $[0,\infty)$*.*

Proof. Let $T_\lambda(t) = e^{-tA_\lambda}$. It follows from Lemma 1.3.1 (i) that

$$\|T_\lambda(t)\|_{\mathcal{L}(X)} \leq 1, \tag{1.3.1}$$

for all $t \geq 0$. Consider now $x \in D(A)$. It follows from Lemma 1.3.1 (ii) and Proposition 1.1.19 (iii) that, given $T > 0$, the function $T_\lambda(t)x$ is a Cauchy sequence in $C([0,T],X)$. Let $T(t)x = \lim_{\lambda\downarrow 0} T_\lambda(t)x$. It is clear that $T(t)$ is a linear mapping $D(A) \to X$. Furthermore, it follows from (1.3.1) that $\|T(t)x\| \leq \|x\|$, for all $x \in D(A)$. Since $D(A)$ is dense in X, it follows that $T(t)$ can be extended to an operator of $\mathcal{L}(X)$, which we still denote by $T(t)$. Property (ii) now follows from Proposition A.1.4, and property (i) follows from (1.3.1). This completes the proof. □

Remark 1.3.3. The family $(T(t))_{t\geq 0}$ constructed in Corollary 1.3.2 is sometimes denoted by e^{-tA}. Note that if A is bounded, this is consistent with the usual definition of the exponential, as follows immediately from Proposition A.1.19 and Proposition 1.3.4 below.

Proposition 1.3.4. *Let A be an m-accretive operator in X, with dense domain, and consider the family $(T(t))_{t\geq 0}$ constructed in Corollary 1.3.2. For every $x \in D(A)$ and every $t \geq 0$, the following properties hold:*

(i) $\left\|\dfrac{T(t)x - x}{t}\right\| \leq \|Ax\|$, *for all* $t > 0$*;*

(ii) *the mapping* $t \mapsto T(t)x$ *belongs to* $C([0,\infty), D(A)) \cap C^1([0,\infty), X)$*;*

(iii) $AT(t)x = T(t)Ax$, *for all* $t \geq 0$*.*

In addition, the function $u(t) = T(t)x$ is the unique solution of the problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \text{ for all } t \geq 0; \\ u(0) = x; \end{cases} \tag{1.3.2}$$

in the space $C([0,\infty), D(A)) \cap C^1([0,\infty), X)$.

Proof. Consider $x \in D(A)$. With the notation introduced in the proof of Corollary 1.3.2, let $u(t) = T(t)x$, $u_\lambda(t) = T_\lambda(t)x$ and $v_\lambda(t) = -u'_\lambda(t) = A_\lambda u_\lambda(t) = T_\lambda(t)A_\lambda x$. We have

$$v_\lambda(t) - T(t)Ax = T_\lambda(t)(A_\lambda x - Ax) + (T_\lambda(t)Ax - T(t)Ax);$$

and so, by Corollary 1.3.2 and Proposition 1.1.19 (iii),

$$\|v_\lambda(t) - T(t)Ax\| \leq \|A_\lambda x - Ax\| + \|T(t)Ax - T_\lambda(t)Ax\| \underset{\lambda\downarrow 0}{\longrightarrow} 0,$$

uniformly on bounded intervals. Passing to the limit, as $\lambda \downarrow 0$ in identity

$$u_\lambda(t) = x - \int_0^t v_\lambda(s)\, ds,$$

we obtain

$$u(t) = x - \int_0^t T(s)Ax\, ds.$$

Hence (i). It follows also that $u \in C^1([0,\infty), X)$ and that

$$\frac{du}{dt} = -T(t)Ax. \tag{1.3.3}$$

Let now $w_\lambda(t) = J_\lambda u_\lambda(t)$, where J_λ is introduced Definition 1.1.9. It follows from Corollary 1.1.14 (ii) and Proposition 1.1.19 (ii) that $w_\lambda(t) \in D(A)$ and $w_\lambda(t) \underset{\lambda\downarrow 0}{\longrightarrow} u(t)$ in X, for every $t \geq 0$. Note also that $Aw_\lambda(t) = v_\lambda(t)$; and so, $(w_\lambda, Aw_\lambda) \underset{\lambda\downarrow 0}{\longrightarrow} (u(t), T(t)Ax)$ in $X \times X$. Since $G(A)$ is closed (cf. Proposition 1.1.10), it follows that $u(t) \in D(A)$, and that

$$Au(t) = T(t)Ax. \tag{1.3.4}$$

(1.3.3) and (1.3.4) yield property (iii). In addition, it follows from (1.3.4) that $Au \in C([0,\infty), X)$; and so, $u \in C([0,\infty), D(A))$. Hence property (ii). Furthermore, it follows from (1.3.3) and (1.3.4) that u solves problem (1.3.2). It remains to establish uniqueness. Consider a solution $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X)$ of (1.3.2). Given $t > 0$, let $z(s) = T(t-s)u(s)$ for $s \in [0,t]$. It follows that $z \in C([0,t], D(A)) \cap C^1([0,t], X)$, and that

$$\frac{dz}{ds} = T(t-s)\left(\frac{du}{ds} + Au\right) = 0;$$

and so, $z(t) = z(0)$. This means that $u(t) = T(t)x$. Since $t > 0$ is arbitrary, the result follows. □

1.3.2. Semigroups and their generators. We begin by introducing semigroups of contractions, and their generators.

Definition 1.3.5. A family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called a *semigroup of contractions* if it satisfies the following properties:

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$, for all $s, t \geq 0$;

(iii) the mapping $t \mapsto T(t)x$ is continuous $[0,\infty) \to X$, for all $x \in X$;

(iv) $\|T(t)\|_{\mathcal{L}(X)} \leq 1$, for all $t \geq 0$.

Remark. Note that in our definition, we include the continuity of the mapping $t \mapsto T(t)x$. A number of authors do not inclufde this in their definition ant then they use the terminology "C_0 semigroups of contractions".

Definition 1.3.6. Let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup of contractions. The *generator* L of $(T(t))_{t\geq 0}$ is the linear operator in X defined by

(i) $D(L) = \left\{ x \in X;\ \frac{T(t)x - x}{t} \text{ has a limit in } X \text{ as } t \downarrow 0 \right\}$;

(ii) $Lx = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$, for all $x \in D(L)$.

Remark 1.3.7. Note that if $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is a semigroup of contractions, then for every $x \in X$, the function $t \mapsto \|T(t)x\|$ is nonincreasing on $[0, \infty)$. Indeed, $\|T(t+s)x\| = \|T(s)T(t)x\| \leq \|T(t)x\|$.

The introduction of m-accretive operators is justified by the following result.

Proposition 1.3.8. *If $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is a semigroup of contractions in X and if L is its generator, then $-L$ is m-accretive with dense domain.*

The proof of Proposition 1.3.8 relies on the following lemma.

Lemma 1.3.9. *If $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ is a semigroup of contractions in X and if L is its generator, then the following properties hold:*

(i) *given $x \in X$ and $t > 0$, set $I(t, x) = \int_0^t T(s)x\, ds$. Then, $I(t, x) \in D(L)$ and $LI(t, x) = T(t)x - x$;*

(ii) *given $x \in X$, set $Jx = \int_0^\infty e^{-t} T(t)x\, dt$. Then, $Jx \in D(L)$ and $Jx - LJx = x$.*

Proof. Given $h > 0$, we have

$$\frac{T(h) - I}{h} I(t, x) = \frac{1}{h} \int_h^{t+h} T(t)x\, dt - \frac{1}{h} \int_0^t T(t)x\, dt = \frac{1}{h} \int_t^{t+h} T(t)x\, dt - \frac{1}{h} \int_0^h T(t)x\, dt.$$

Letting $h \downarrow 0$, it follows that $I(t, x) \in D(L)$ and that $LI(t, x) = T(t)x - x$. Hence (i). On the other hand, we have

$$\begin{aligned}
\frac{T(h) - I}{h} Jx &= \frac{1}{h} \int_0^\infty e^{-t} (T(t+h)x - T(t)x)\, dt \\
&= \frac{1}{h} \int_h^\infty e^{-(t-h)} T(t)x\, dt - \frac{1}{h} \int_0^\infty e^{-t} T(t)x\, dt \\
&= \frac{e^h - 1}{h} \int_0^\infty e^{-t} T(t)x\, dt - e^{-h} \frac{1}{h} \int_0^h e^{-t} T(t)x\, dt.
\end{aligned}$$

Letting $h \downarrow 0$, we obtain

$$\lim_{h \downarrow 0} \frac{T(h) - I}{h} Jx = Jx - x,$$

in X. It follows that $Jx \in D(L)$ and that $Jx - LJx = x$. Hence (ii). □

Proof of Proposition 1.3.8 Let $x \in D(L)$ and $\lambda, h > 0$. We have

$$x - \lambda \frac{T(h)x - x}{h} = \left(1 + \frac{\lambda}{h}\right) x - \frac{\lambda}{h} T(h)x;$$

and so,

$$\|x - \lambda \frac{T(h)x - x}{h}\| \geq \left(1 + \frac{\lambda}{h}\right) \|x\| - \frac{\lambda}{h} \|x\| = \|x\|.$$

Letting $h \downarrow 0$ in the above inequality, it follows that $-L$ is accretive. Furthermore, given $f \in X$, let $x = Jf$ where J is defined in Lemma 1.3.9. It follows that $x \in D(L)$ and that $x - Lx = f$. Therefore, $-L$ is

m-accretive (see Proposition 1.1.21). Finally, given $x \in X$ and $\varepsilon > 0$, consider $x_\varepsilon = \dfrac{1}{\varepsilon} I(\varepsilon, x)$, where $I(\varepsilon, x)$ is defined in Lemma 1.3.9. It is clear that $x_\varepsilon \underset{\varepsilon \downarrow 0}{\longrightarrow} x$ in X. Since $x_\varepsilon \in D(L)$, it follows that $D(L)$ is dense in X. This completes the proof. □

Conversely, the introduction of semigroups of contractions is justified by the following result.

Proposition 1.3.10. *Let A be an m-accretive operator in X with dense domain. The family $(T(t))_{t\ge 0} \subset \mathcal{L}(X)$ introduced in Corollary 1.3.2 enjoys the following properties:*

(i) *$(T(t))_{t\ge 0}$ is a semigroup of contractions in X;*

(ii) *the generator of $(T(t))_{t\ge 0}$ is $-A$;*

(iii) *if a semigroup of contractions $(S(t))_{t\ge 0}$ admits $-A$ as its generator, then $S(t) = T(t)$ for all $t \ge 0$.*

Proof. It follows from Corollary 1.3.2 that $\|T(t)\|_{\mathcal{L}(X)} \le 1$. Furthermore, it follows from Proposition 1.3.4 that $T(t+s)x = T(t)T(s)x$, for all $s, t \ge 0$ and all $x \in D(A)$. Since $D(A)$ is dense, we find $T(t+s) = T(t)T(s)$. Furthermore, it follows from Corollary 1.3.2 that the function $T(t)x$ is continuous $[0,\infty) \to X$, for all $x \in X$. Hence (i). Let L be the generator of $(T(t))_{t\ge 0}$, and consider $x \in D(A)$. Applying Proposition 1.3.4, we obtain

$$T(t)x = x - \int_0^t T(s)Ax.$$

It follows that $x \in D(L)$ and that $Lx = -Ax$. In other words, $G(A) \subset G(-L)$. Since both $-L$ and A are m-accretive, it follows from Corollary 1.1.24 that $A = -L$. Hence (ii). Finally, assume that another semigroup of contractions $(S(t))_{t\ge 0}$ admits $-A$ as its generator. Consider $x \in D(A)$, and let $u(t) = S(t)x$. Given $t \ge 0$ and $h > 0$, we have

$$\frac{u(t+h) - u(t)}{h} = \frac{S(h) - I}{h} u(t) = S(t)\frac{S(h)x - x}{h} \underset{h\downarrow 0}{\longrightarrow} -S(t)Ax.$$

It follows that $u(t) \in D(A)$ and that $\dfrac{d^+u}{dt}$ exists, for all $t \ge 0$, and that

$$Au(t) = S(t)Ax = \frac{d^+u}{dt}.$$

Therefore, $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X)$ (see Theorem A.1.16), and u solves equation (1.3.2). It follows from Proposition 1.3.4 that $S(t)x = T(t)x$, for all $t \ge 0$. By density, we find $T(t) = S(t)$. This completes the proof. □

Remark 1.3.11. Property (iii) of Proposition 1.3.10 means that if A is an m-accretive operator, then the semigroup of contractions generated by $-A$ is unique. In particular, there is a one-to-one and onto correspondance between semigroups of contractions and m-accretive operators with dense domain.

By applying Propositions 1.3.8 and 1.3.10, we obtain the following result (the Hille-Yosida-Phillips theorem).

Theorem 1.3.12. *A linear operator A in X is the generator of a semigroup of contractions in X if and only if $-A$ is m-accretive with dense domain.*

We now establish an invariance result. That result will be helpful for showing that, when the operator and the initial data have some symmetry properties, then the solutions of (1.3.2) have the same properties.

Proposition 1.3.13. *Let A be an m-accretive operator in X with dense domain, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A$. Let $L \in \mathcal{L}(X)$ be such that $L_{|D(A)} \in \mathcal{L}(D(A))$. If $ALx = LAx$ for all $x \in D(A)$, then $T(t)L = LT(t)$ for all $t \geq 0$. In particular, if $Lx = 0$, then $LT(t)x = 0$ for all $t \geq 0$.*

Proof. Let $x \in D(A)$, and let $u(t) = T(t)x$. Then, u solves problem (1.3.2). If we set $v(t) = Lu(t)$, we have $v \in C([0, \infty), D(A)) \cap C^1([0, \infty), X)$, $\dfrac{dv}{dt} + Av = 0$, and $v(0) = Lx$. Therefore, $v(t) = T(t)Lx$; and so, $T(t)Lx = LT(t)x$, for all $x \in D(A)$. The result now follows by density. □

Corollary 1.3.14. *Let A be an m-accretive operator in X with dense domain, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A$. If J_λ is the operator introduced in Definition 1.1.9, then $T(t)J_\lambda = J_\lambda T(t)$ for all $\lambda > 0$ and all $t \geq 0$.*

Proof. It follows from Lemma 1.1.16 that one can apply Proposition 1.3.13 with $L = J_\lambda$. Hence the result. □

We conclude this section with a characterization of the domain of m-accretive operators in reflexive Banach spaces.

Proposition 1.3.15. *Let A be an m-accretive operator in X and let $(T(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A$. If X is reflexive, then every $x \in X$ such that*

$$\sup_{h>0} \frac{1}{h} \|T(h)x - x\| < \infty$$

belongs to $D(A)$. In particular, $D(A) = \{x \in X;\, \exists C, \|T(h)x - x\| \leq Ch, \text{ for all } h > 0\}$.

Proof. Let x be as in the statement, and let $u(t) = T(t)x$. Given $0 \leq s < t$, we have

$$\|u(t) - u(s)\| = \|T(s)(T(t-s)x - x)\| \leq \|T(t-s)x - x\| \leq C(t-s).$$

It follows that u is Lipschitz continuous $[0, \infty) \to X$. Since u is also bounded, we have $u \in W^{1,\infty}((0, \infty), X)$ (cf. Corollary A.2.38). In particular, there exists $t_n \downarrow 0$ such that u is differentiable at every t_n and $\|u'(t_n)\| \leq C$ (Theorem A.2.30 and Corollary A.2.23). In particular,

$$\frac{u(t_n + h) - u(t_n)}{h} = \frac{T(h) - I}{h} T(t_n)x$$

has a limit as $h \downarrow 0$, for every $n \in \mathbb{N}$. This implies that $T(t_n)x \in D(A)$, and $\|AT(t_n)x\| \leq C$, for all $n \in \mathbb{N}$. In particular, there exists a subsequence, which we still denote by $(t_n)_{n\in\mathbb{N}}$ and $y \in X$ such that $AT(t_n)x \rightharpoonup y$, as $n \to \infty$. Since $T(t_n)x \to x$, as $n \to \infty$, it follows that $(T(t_n)x, AT(t_n)x) \rightharpoonup (x, y)$ in $X \times X$. Since the graph of A is closed, it is also closed for the weak topology; and so, $x \in D(A)$. Hence the result. □

Remark. When X is not reflexive, the conclusion of the above proposition fails. For example, let $X = L^1(\mathbb{R})$, let A be the operator defined in Remark 1.4.2 (i) below and let $(T(t))_{t\in\mathbb{R}}$ be the group of isometries generated by $-A$. Let $x = 1_{(0,1)}$. It follows from Remark 1.4.2 (i) below that $T(t)x = 1_{(t,t+1)}$. In particular, $\|T(h)x - x\| \le 2h$, for all $h > 0$. On the other hand, note that $D(A) = W^{1,1}(\mathbb{R}) \subset C(\mathbb{R})$; and so, $x \notin D(A)$.

1.3.3. Regularity properties. In this section, we show that certain subspaces of X are invariant under the action of semigroups of contractions. We begin with a simple result.

Proposition 1.3.16. *Let A be an m-accretive operator in X, and let $(T(t))_{t\ge 0}$ be the semigroup of contractions in X generated by $-A$. If $T_{(1)}(t) = T(t)_{|D(A)}$ and if $A_{(1)}$ is the operator defined by Theorem 1.1.28, then $(T_{(1)}(t))_{t\ge 0}$ is a semigroup of contractions in $D(A)$ and its generator is $-A_{(1)}$.*

Proof. It follows from Proposition 1.3.4 that $T(t)$ maps $D(A)$ into itself. In addition, for every $t \ge 0$ and $x \in D(A)$, we have

$$\|T(t)x\|_{D(A)} = \|T(t)x\| + \|AT(t)x\| = \|T(t)x\| + \|T(t)Ax\| \le \|x\| + \|Ax\| = \|x\|_{D(A)}.$$

Therefore, $T(t)_{|D(A)} \in \mathcal{L}(D(A))$, and $\|T(t)_{|D(A)}\|_{\mathcal{L}(D(A))} \le 1$. In particular, the definition of $(T_{(1)}(t))_{t\ge 0}$ makes sense. Furthermore, it follows from Proposition 1.3.4 that $(T_{(1)}(t))_{t\ge 0}$ is a semigroup of contractions in $D(A)$. Let L be its generator, and consider $x \in D(A_{(1)}) = D(A^2)$. We have

$$\frac{T_{(1)}(h)x - x}{h} = \frac{T(h)x - x}{h} \underset{h\downarrow 0}{\longrightarrow} -Ax,$$

in X. Furthermore, $Ax \in D(A)$; and so, by Proposition 1.3.4

$$A\frac{T_{(1)}(h)x - x}{h} = \frac{T_{(1)}(h)Ax - Ax}{h} \underset{h\downarrow 0}{\longrightarrow} -A(Ax),$$

in X. Therefore,

$$\frac{T_{(1)}(h)x - x}{h} = \frac{T(h)x - x}{h} \underset{h\downarrow 0}{\longrightarrow} -Ax,$$

in $D(A)$; and so, $x \in D(L)$ and $Lx = -Ax$. It follows that $G(A_{(1)}) \subset G(-L)$. Since both $-L$ and $A_{(1)}$ are m-accretive in $D(A)$, it follows from Corollary 1.1.24 that $A_{(1)} = -L$. Hence the result. □

Corollary 1.3.17. *Let A be an m-accretive operator in X, and let $(T(t))_{t\ge 0}$ be the semigroup of contractions in X generated by $-A$. Given a positive integer n, consider the space X_n and the operator $A_{(n)}$ defined by Remark 1.1.29. If $T_{(n)}(t) = T(t)_{|X_n}$ for $t \ge 0$, then $(T_{(n)}(t))_{t\ge 0}$ is a semigroup of contractions in X_n and its generator is $-A_{(n)}$.*

Proof. This follows from applying iteratively Proposition 1.3.16 and Remark 1.1.29. □

Corollary 1.3.18. *Let A be an m-accretive operator in X, and consider the spaces $(X_n)_{n\geq 0}$ defined by Remark 1.1.29. Let $x \in D(A)$, and let $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), X)$ be the solution of problem (1.3.2). If $x \in X_n$ for some $n \geq 1$, then*

$$u \in \bigcap_{j=0}^{n} C_b^j([0,\infty), X_{n-j}). \tag{1.3.5}$$

Furthermore,

$$\frac{d^j u}{dt^j} = (-1)^j T(t) A^j x = (-1)^j A^j u(t), \tag{1.3.6}$$

for all $t \geq 0$ and all $0 \leq j \leq n$, and

$$\frac{d}{dt}\left(\frac{d^j u}{dt^j}\right) + A\left(\frac{d^j u}{dt^j}\right) = 0, \tag{1.3.7}$$

for all $t \geq 0$ and all $0 \leq j \leq n-1$. In particular, if $x \in \bigcap_{n\geq 0} D(A^n)$, we have $u \in C^\infty([0,\infty), X_n)$, for all $n \geq 0$.

Proof. Let us first establish (1.3.5) and (1.3.6). We argue by induction. The case $n = 1$ follows from Proposition 1.3.4. Assume now that the result holds up to some $n \geq 1$. Let $x \in X_{n+1}$. In particular, we have $A^j x \in X_{n-j+1}$, for every $0 \leq j \leq n+1$; and so, $u \in C_{\mathrm{b}}([0,\infty), X_{n+1})$, by Corollary 1.3.17. Furthermore, it follows from (1.3.6) that $\frac{d^j u}{dt^j} = (-1)^j T(t) A^j x$, for every $0 \leq j \leq n$. Applying Corollary 1.3.17 and Proposition 1.3.4, it follows that $\frac{d^j u}{dt^j} \in C([0,\infty), X_{n-j+1}) \cap C^1([0,\infty), X_{n-j})$, for every $0 \leq j \leq n$. Therefore, (1.3.5) holds at order $n+1$. It follows easily that (1.3.6) also holds at order $n+1$, by applying Proposition 1.3.4 (iii). Finally, (1.3.7) is a consequence of (1.3.5) and (1.3.6), by by applying Proposition 1.3.4 (iii). This completes the proof. □

1.3.4. Weak solutions and extrapolation. If $x \in D(A)$, then $u(t) = T(t)x$ is the solution of problem (1.3.2) (cf. Proposition 1.3.4). On the other hand, if $x \in X \setminus D(A)$, then $u \notin C([0,\infty), D(A))$, and in particular, u cannot solve (1.3.2) on $[0,\infty)$. The object of this section is to show that u solves a weak form of problem (1.3.2).

Lemma 1.3.19. *Let A be an m-accretive operator in X, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions in X generated by $-A$. Consider the space X_{-1} and the operator $A_{(-1)}$ defined by Theorem 1.1.31. If $(T_{(-1)}(t))_{t\geq 0}$ is the semigroup of contractions in X_{-1} generated by $A_{(-1)}$, then $T_{(-1)}(t)_{|X} = T(t)$ for all $t \geq 0$.*

Proof. This follows from Remark 1.1.33, Proposition 1.3.16 and Proposition 1.3.10 (iii). □

Corollary 1.3.20. *Let A be an m-accretive operator in X, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions in X generated by $-A$. Consider the space X_{-1} and the operator $A_{(-1)}$ defined by Theorem 1.1.31. Let $x \in X$, and set $u(t) = T(t)x$, for all $t \geq 0$. Then, u is the unique solution of problem*

$$\begin{cases} \dfrac{du}{dt} + A_{(-1)} u = 0; \\ u(0) = x; \end{cases}$$

in the space $C([0,\infty), X) \cap C^1([0,\infty), X_{-1})$.

Proof. This follows from Proposition 1.3.4, applied to the operator $A_{(-1)}$, and from Lemma 1.3.19. □

Corollary 1.3.21. *Let A be an m-accretive operator in X, and let $(T(t))_{t\ge 0}$ be the semigroup of contractions in X generated by $-A$. Given $n \ge 0$, consider the space X_{-n} and the operator $A_{(-n)}$ defined by Remark 1.1.32. If $(T_{(-n)})_{t\ge 0}$ is the semigroup of contractions in X_{-n} generated by $A_{(-n)}$, then $T_{(-n)}(t)_{|X_{-j}} = T_{(-j)}(t)$ for all $0 \le j \le n$ and all $t \ge 0$.*

Proof. The result follows by applying iteratively Lemma 1.3.19 and Remark 1.1.33. □

Corollary 1.3.22. *Let A be an m-accretive operator in X, and let $(T(t))_{t\ge 0}$ be the semigroup of contractions in X generated by $-A$. Given $n \ge 0$, consider the space X_{-n} and the operator $A_{(-n)}$ defined by Remark 1.1.32, and let $(T_{(-n)})_{t\ge 0}$ be the semigroup of contractions in X_{-n} generated by $A_{(-n)}$. Let $x \in X$, and set $u(t) = T(t)x$, for $t \ge 0$. Then, $u \in C^n_b([0,\infty), X_{-n})$ for all $n \ge 0$. In addition,*

$$\frac{d^n u}{dt^n} = (-1)^n T_{(-n)}(t) A^n_{(-n)} x = (-1)^n A^n_{(-n)} u(t),$$

and

$$\frac{d}{dt}\left(\frac{d^{n-1}u}{dt^{n-1}}\right) + (-1)^{n+1} A_{(-n)} \left(\frac{d^{n-1}u}{dt^{n-1}}\right) = 0,$$

for all $t \ge 0$ and all $n \ge 1$.

Proof. The result follows by applying Corollary 1.3.18 to the operator $A_{(-n)}$, for every $n \ge 0$. □

1.3.5. Groups of isometries. We will show that, under some appropriate assumptions, some semigroups of contractions can be embedded in larger families of operators. We begin with the following definition.

Definition 1.3.23. A family $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is called a *group of isometries* if it satisfies the following properties:

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$, for all $s, t \in \mathbb{R}$;

(iii) the mapping $t \mapsto T(t)x$ is continuous $\mathbb{R} \to X$, for all $x \in X$;

(iv) $\|T(t)x\| = \|x\|$, for all $t \in \mathbb{R}$ and all $x \in X$.

Remark 1.3.24. Here are some immediate consequences of Definition 1.3.23.

(i) If $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a group of isometries, then $(T(t))_{t\ge 0}$ is a semigroup of contractions. In addition, if one sets $S(t) = T(-t)$, for all $t \in \mathbb{R}$, then $(S(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is also a group of isometries; and so, $(S(t))_{t\ge 0}$ is a semigroup of contractions.

(ii) Recall that in a Banach space an isometry, i.e. a linear map $T : X \to X$ such that $\|Tx\| = \|x\|$ for all $x \in X$ need not be surjective. For example, $T\varphi(t) = \varphi(t+h)$ on $X = L^p(0,\infty)$ with $h > 0$. Note also that if $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a group of isometries, then $T(t)X = X$, for all $t \in \mathbb{R}$. Indeed, we have $T(t)X \subset X$. On the other hand, given $t \in \mathbb{R}$ and $x \in X$, we have $x = T(t)y$ with $y = T(-t)x$; and so, $X \subset T(t)X$. Conversely, if $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ is a semigroup of contractions such that $T(t)$ is a surjective isometry for all $t \geq 0$, then $(T(t))_{t\in\mathbb{R}}$ cen be embedded in a group of isometries $(S(t))_{t\in\mathbb{R}}$. Indeed, set $S(t) = T(t)$ for $t \geq 0$ and $S(t) = (T(-t))^{-1}$ for $t < 0$.

Lemma 1.3.25. *Let $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ be a group of isometries. If L is the generator of the semigroup of contractions $(T(t))_{t\geq 0}$, and if $\widetilde{L}$ is the generator of the semigroup of contractions $(S(t))_{t\geq 0}$, where $S(t) = T(-t)$, then $L = -\widetilde{L}$. In particular, both L and $-L$ are m-accretive with dense domain.*

Proof. Let $x \in D(L)$. Given $h > 0$, we have

$$\frac{S(h)x - x}{h} = \frac{T(-h)x - x}{h} = -T(-h)\frac{T(h)x - x}{h} \underset{h\downarrow 0}{\longrightarrow} -Lx.$$

It follows that $x \in D(\widetilde{L})$ and that $\widetilde{L}x = -Lx$; and so, $G(L) \subset G(-\widetilde{L})$. As well, given $x \in D(\widetilde{L})$ and $h > 0$, we have

$$\frac{T(h)x - x}{h} = -T(h)\frac{T(-h)x - x}{h} = -T(h)\frac{S(h)x - x}{h} \underset{h\downarrow 0}{\longrightarrow} -\widetilde{L}x.$$

It follows that $x \in D(L)$ and that $Lx = -\widetilde{L}x$; and so, $G(\widetilde{L}) \subset G(-L)$. Therefore, $\widetilde{L} = -L$. Hence the result, by Proposition 1.3.8. □

Lemma 1.3.26. *Let A be an m-accretive with dense domain, such that $-A$ is m-accretive. Let $(T(t))_{t\geq 0}$ be the the semigroup of contractions in X generated by $-A$, and let $(S(t))_{t\geq 0}$ be the the semigroup of contractions in X generated by A. Define $(U(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ by*

$$U(t) = \begin{cases} T(t), & \text{if } t \geq 0; \\ S(-t), & \text{if } t \geq 0. \end{cases}$$

Then, $(U(t))_{t\in\mathbb{R}}$ is a group of isometries.

Proof. Given $x \in D(A)$, let $u(t) = U(t)x$ for $t \in \mathbb{R}$. Applying Proposition 1.3.4 to both A and $-A$, we see that u is the unique solution in $C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R} \setminus \{0\}, X)$ of the equation $u' + Au = 0$ for all $t \neq 0$ with the initial condition $u(0) = x$. Since

$$\frac{d^+u}{dt}(0) = \frac{d^-u}{dt}(0) = Ax,$$

we see that in fact $u \in C^1(\mathbb{R}, X)$. It follows that $U(t+s)x = U(t)U(s)x$ for all $t, s \in \mathbb{R}$ and all $x \in D(A)$; hence for all $x \in X$ by density. Next, since by construction $U(t)$ is a contraction for all $t \in \mathbb{R}$, we have

$$\|U(t)x\| \leq \|x\| = \|U(-t)U(t)x\| \leq \|U(t)x\|,$$

for all $t \in \mathbb{R}$ and all $x \in X$. Therefore, $U(t)$ is an isometry. The other properties are immediate. □

Applying Lemmas 1.3.25 and 1.3.26, one obtains the following result.

Proposition 1.3.27. *If $(T(t))_{t\geq 0}$ is a semigroup of contractions in X with the generator $-A$, then the following properties are equivalent:*

(i) *$-A$ is m-accretive;*

(ii) *there exists a group of isometries $(U(t))_{t\in\mathbb{R}}$ such that $T(t) = U(t)$, for all $t \geq 0$.*

Corollary 1.3.28. *Let $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ be a group of isometries, and let $-A$ be the generator of the semigroup of contractions $(T(t))_{t\geq 0}$. Then, for every $x \in D(A)$, the function $u(t) = T(t)x$, $t \in \mathbb{R}$ is the unique solution of problem*

$$\begin{cases} \dfrac{du}{dt} + Au = 0; \\ u(0) = x; \end{cases}$$

in the space $C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)$.

Proof. It follows from Proposition 1.3.27, Lemma 1.3.26 and Proposition 1.3.4 that $u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R} \setminus \{0\}, X)$, that $\dfrac{du}{dt} = -Au$ for all $t \neq 0$, and that

$$\frac{d^+u}{dt}(0) = \frac{d^-u}{dt}(0) = Ax.$$

The result follows easily. □

Remark 1.3.29. Consider a group of isometries $(T(t))_{t\in\mathbb{R}} \subset \mathcal{L}(X)$, and let $x \in X$. It follows immediately from the group property and Corollary 1.3.28 that if $T(t_0)x \in D(A)$ for some $t_0 \in \mathbb{R}$, then $T(t)x \in D(A)$ for all $t \in \mathbb{R}$. Therefore, if $x \notin D(A)$, then $T(t)x \notin D(A)$ for all $t \in \mathbb{R}$.

1.3.6. The case of Hilbert spaces. Throughout this section, we assume that X is a Hilbert space, endowed with the scalar product $(\cdot,\cdot)$. We will apply the results of Section 1.1.5 to obtain further properties.

Lemma 1.3.30. *If $(T(t))_{t\geq 0}$ is a semigroup of contractions with the generator $-A$, then*

(i) *$(T(t)^*)_{t\geq 0}$ is a semigroup of contractions;*

(ii) *the generator of $(T(t)^*)_{t\geq 0}$ is $-A^*$.*

Proof. It follows from Proposition 1.1.41 and Corollary 1.1.37 that A^* is m-accretive with dense domain. Let $(S(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A^*$. Applying Corollary 1.3.2 and Proposition 1.1.41, we obtain easily

$$S(t)x = \lim_{\lambda\downarrow 0} e^{-t(A^*)_\lambda}x = \lim_{\lambda\downarrow 0}(e^{-tA_\lambda})^*x = (T(t))^*x,$$

for all $t \geq 0$ and all $x \in X$. Hence the result. □

Remark 1.3.31. Here are some comments on Lemma 1.3.30. Let $(T(t))_{t\geq 0}$ be a semigroup of contractions in a general Banach space X, and let A be its generator.

(i) One may always consider $T(t)^*$. The family $(T(t)^*)_{t\geq 0}$ satisfies properties (i), (ii) and (iv) of Definition 1.3.5. This is easily verified. However, property (iii) (continuity) may fail. For example, let $X = L^1(\mathbb{R})$ and let $(T(t)^*)_{t\geq 0}$ be defined by $T(t)\varphi(x) = \varphi(x-t)$ (see Remark 1.4.2 (i)). $(T(t)^*)_{t\geq 0}$ is defined on $X^\star = L^\infty(\mathbb{R})$ by $T(t)^*\varphi(x) = \varphi(x+t)$, and one verifies easily that $(T(t)^*)_{t\geq 0}$ is not continuous on $X^\star$.

(ii) Since $D(A)$ is dense in X, one may consider the operator A^* on $X^\star$, and A^* is m-accretive (see Exercise 1.8.3). If $D(A^*)$ is dense in $X^\star$ then the proof of Lemma 1.3.30 shows that $(T(t)^*)_{t\geq 0}$ is indeed a semigroup of contractions in $X^\star$ and that its generator is $-A^*$.

(iii) In particular, if X is reflexive, then A^* is m-accretive with dense domain (see Exercise 1.8.2); and so, $(T(t)^*)_{t\geq 0}$ is a semigroup of contraction, and its generator is $-A^*$.

Corollary 1.3.32. *If A be a self-adjoint, accretive operator in X and if $(T(t))_{t\geq 0}$ is the semigroup of contractions generated by $-A$, then $T(t) = (T(t))^*$ for all $t \geq 0$.*

Proof. It follows from Corollary 1.1.45 that A is m-accretive with dense domain. The result now follows from Lemma 1.3.30. □

Corollary 1.3.33. *If A is a skew-adjoint operator in X, then there exists a group of isometries $(T(t))_{t\in\mathbb{R}}$ such that $-A$ is the generator of the semigroup of contractions $(T(t))_{t\geq 0}$. In addition, $(T(t))^* = T(-t)$, for all $t \in \mathbb{R}$.*

Proof. It follows from Corollary 1.1.47 that A and $-A$ are m-accretive with dense domain. The result now follows easily from Proposition 1.3.27 and Lemmas 1.3.26 and 1.3.30. □

Finally, we describe below a fundamental property of self-adjoint operators.

Theorem 1.3.34. *Let A be a self-adjoint, accretive operator in X, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A$. For every $x \in X$, the function $u(t) = T(t)x$ for $t \geq 0$ has the following properties:*

(i) *$u \in C([0,\infty), X) \cap C((0,\infty), D(A)) \cap C^1((0,\infty), X)$ and u is the unique solution of problem*

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \text{ for all } t > 0; \\ u(0) = x; \end{cases} \tag{1.3.8}$$

in that class;

(ii) *$\|Au(t)\| \leq \dfrac{1}{t\sqrt{2}}\|x\|$, for all $t > 0$. Moreover, the function $t \mapsto \sqrt{t}\|Au(t)\|$ belongs to $L^2(0,\infty)$ and $\displaystyle\int_0^\infty s\|Au(s)\|^2\,ds \leq \frac{1}{4}\|x\|^2$;*

(iii) *$(Au(t), u(t)) \leq \dfrac{1}{2t}\|x\|^2$, for all $t > 0$. Moreover, the function $t \mapsto (Au(t), u(t))$ belongs to $L^2(0,\infty)$ and $\displaystyle\int_0^\infty (Au(t), u(t))\,ds \leq \frac{1}{2}\|x\|^2$;*

(iv) *if* $x \in D(A)$, *then also* $\|Au(t)\|^2 \le \dfrac{1}{2t}(Ax, x)$, *for all* $t > 0$. *Moreover,* $Au \in L^2((0,\infty), X)$ *and* $\|Au\|^2_{L^2((0,\infty),X)} \le \dfrac{1}{2}(Ax, x)$.

Proof. Let $x \in X$, and let $u(t) = T(t)x$. Given $\lambda > 0$, let A_λ be the operator introduced in Definition 1.1.15, and set $u_\lambda(t) = e^{-tA_\lambda}x$. It follows from Lemma 1.1.16 and Proposition 1.1.42 that A_λ is self-adjoint and acretive. Therefore, $(e^{-tA_\lambda})_{t\ge 0}$ is a semigroup of contractions. Applying Remark 1.3.7, we obtain the following property:

$$\text{The mapping } t \mapsto \|u_\lambda'(t)\| = \|e^{-tA_\lambda}A_\lambda x\| \text{ is nonincreasing.} \tag{1.3.9}$$

In addition, the following identities hold:

$$\frac{d}{dt}\|u_\lambda(t)\|^2 = -2(A_\lambda u_\lambda(t), u_\lambda(t)), \text{ for all } t \ge 0, \tag{1.3.10}$$

$$\frac{d}{dt}(A_\lambda u_\lambda(t), u_\lambda(t)) = 2(A_\lambda u_\lambda(t), u_\lambda'(t)) = -2\|u_\lambda'(t)\|^2, \text{ for all } t \ge 0. \tag{1.3.11}$$

It follows from (1.3.11) that $(A_\lambda u_\lambda(t), u_\lambda(t))$ is a nonincreasing function of t; and so, integrating (1.3.10) between 0 and $t > 0$, we obtain

$$t(A_\lambda u_\lambda(t), u_\lambda(t)) \le \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))\, ds \le \frac{1}{2}\|x\|^2. \tag{1.3.12}$$

Applying (1.3.9) and integrating (1.3.11) between 0 and $t > 0$, we obtain

$$2t\|u_\lambda'(t)\|^2 \le 2\int_0^t \|u_\lambda'(s)\|^2\, ds = (A_\lambda x, x) - (A_\lambda u_\lambda(t), u_\lambda(t)) \le (A_\lambda x, x), \tag{1.3.13}$$

where the last inequality follows from Lemma 1.1.36. As well, multiplying (1.3.11) by t and integrating, we find

$$\begin{aligned} t^2\|u_\lambda'(t)\|^2 &\le 2\int_0^t s\|u_\lambda'(s)\|^2\, ds \le -2\int_0^t s\frac{d}{ds}(A_\lambda u_\lambda(s), u_\lambda(s))\, ds \\ &\le \int_0^t (A_\lambda u_\lambda(s), u_\lambda(s))\, ds. \end{aligned}$$

Applying (1.3.12), it follows that

$$2t^2\|u_\lambda'(t)\|^2 \le \|x\|^2. \tag{1.3.14}$$

Consider now $t > 0$. It follows from Corollary 1.3.2 and Proposition 1.1.19 that

$$J_\lambda u_\lambda(t) \underset{\lambda\downarrow 0}{\longrightarrow} u(t),$$

in X. On the other hand, it follows from (1.3.14) that $A(J_\lambda u_\lambda(t)) = u_\lambda'(t)$ is bounded in X. Applying Remark 1.1.11, we find $u(t) \in D(A)$ and $A_\lambda u_\lambda(t) \rightharpoonup Au(t)$, as $\lambda \downarrow 0$. Property (i) now follows by applying Proposition 1.3.4 with the initial value $u(\varepsilon)$, for arbitrary $\varepsilon > 0$, and letting $\varepsilon \downarrow 0$. Other properties are obtained by passing to the limit in (1.3.12), (1.3.13) and (1.3.14) and using the weak lower-semicontinuity of the norm, then equation (1.3.8). □

Corollary 1.3.35. *Let* A *be a self-adjoint, accretive operator in* X, *let* $(T(t))_{t\ge 0}$ *be the semigroup of contractions generated by* $-A$, *and consider the spaces* $(X_n)_{n\ge 0}$ *defined by Remark 1.1.29. Let* $x \in X$ *and set* $u(t) = T(t)x$. *Then,* $u \in C^\infty((0,\infty), X_n)$, *for every* $n \ge 0$. *In addition,*

$$\|A^n u(t)\| \le \left(\frac{n}{\sqrt{2}}\right)^n \frac{1}{t^n}\|x\|, \tag{1.3.15}$$

for all $n \geq 1$ and all $t > 0$.

Proof. Consider the operators $A_{(n)}$ defined by Remark 1.1.29. It follows from Corollary 1.1.48 and Remark 1.1.29 that $A_{(n)}$ is a self-adjoint, accretive operator in X_n. Consider $t > 0$. It follows from Theorem 1.3.34 that $u(t/n) \in X_1$ and that

$$\|Au(t/n)\| \leq \frac{n}{t\sqrt{2}}\|x\|.$$

Applying now Theorem 1.3.34 to the operator $A_{(1)}$, one obtains as well that $u(2t/n) = T(t/n)u(t/n) \in X_2$ and that

$$\|A^2u(2t/n)\| \leq \Big(\frac{n}{t\sqrt{2}}\Big)^2\|x\|.$$

By induction, one finds $u(t) \in X_n$ and

$$\|A^n u(t)\| \leq \Big(\frac{n}{t\sqrt{2}}\Big)^n\|x\|.$$

Hence (1.3.15). Since t and n are arbitrary, the result now follows from Corollary 1.3.18, applied to $u(t+\varepsilon)$, $\varepsilon > 0$. □

Remark 1.3.36. Corollary 1.3.35 describes a *smoothing effect.* For every $x \in X$ and every $t > 0$, $T(t)x$ belongs to $\bigcap_{n\geq 0} D(A^n)$. This property displays the irreversible character of equation (1.3.2), when A is self-adjoint and accretive. More precisely, if $y \in X \setminus \bigcap_{n\geq 0} D(A^n)$, there does not exist any pair $(x,t) \in X \times (0,\infty)$ such that $y = T(t)x$. This is in great contrast with the case of skew adjoint operators, for which $T(t)X = X$.

1.3.7. Analytic semigroups. Throughout this section, we assume that X is a **complex** Banach space. We recall that every real Banach space has a canonical complexification, and that conversely, any complex Banach space has an underlying real Banach space structure. Let A be a linear unbounded operator in X (considered as a real Banach space), and assume that A is $\mathbb{C}$-linear (i.e. $\lambda x \in D(A)$ and $A(\lambda x) = \lambda Ax$ for all $\lambda \in \mathbb{C}$ and $x \in D(A)$). The numerical range $S(A)$ of A is the set

$$S(A) = \{\langle \xi, Ax\rangle_{X^\star,X};\ x \in D(A), \|x\| = 1, \xi \in F(x)\}.$$

Here, $\langle\cdot,\cdot\rangle_{X^\star,X}$ is the **complex** duality bracket between $X^\star$ and X, and F is the duality mapping. Assume that A is m-accretive with dense domain, and let $(T(t))_{t\geq 0}$ be the semigroup of contractions generated by $-A$. Since A is $\mathbb{C}$-linear, it follows easily that $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$, with X considered as a complex Banach space. Given $0 < \theta \leq \pi$, we define the sector C_θ by

$$C_\theta = \{z \in \mathbb{C} \setminus \{0\};\ -\theta < \arg z < \theta\},$$

so that $\overline{C_\theta} = \{0\} \cup \{z \in \mathbb{C} \setminus \{0\};\ -\theta \leq \arg z \leq \theta\}$.

Definition 1.3.37. *Let $(T(t))_{t\geq 0}$ be as above. We say that $(T(t))_{t\geq 0}$ is an analytic semigroup if there exists $0 < \theta \leq \pi$ and a mapping $\widetilde{T} : \overline{C_\theta} \to \mathcal{L}(X)$ with the following properties:*

(i) *$T(t) = \widetilde{T}(t)$ for all $t \geq 0$;*

(ii) $\widetilde{T}(z_1+z_2)=\widetilde{T}(z_1)\widetilde{T}(z_2)$ *for all* $z_1,z_2\in C_\theta$;

(iii) $\displaystyle\lim_{z\in\overline{C_\theta},z\to 0}\widetilde{T}(z)x=x$ *for all* $x\in X$;

(iv) *the mapping* $z\mapsto\widetilde{T}(z)$ *is holomorphic* $C_\theta\to\mathcal{L}(X)$.

We have the following characterization of analytic semigroups (see Pazy [85], Theorem 5.2, p. 61).

Theorem 1.3.38. *Let A be a $\mathbb{C}$-linear, m-accretive operator with dense domain and let $(T(t))_{t\ge 0}$ be the semigroup of contractions generated by $-A$. The following properties are equivalent:*

(i) $(T(t))_{t\ge 0}$ *is an analytic semigroup;*

(ii) *the mapping $t\mapsto T(t)$ is differentiable $(0,\infty)\to\mathcal{L}(X)$ and there exists a constant C such that $\|tT'(t)\|_{\mathcal{L}(X)}\le C$ for all $t\in(0,1]$.*

Remark 1.3.39. It follows in particular from Theorems 1.3.38 and 1.3.34 that if A is a self-adjoint, accretive operator in a complex Hilbert space X, then the semigroup of contractions generated by $-A$ is analytic. (Note that A has a canonical $\mathbb{C}$-linear, self-adjoint and accretive extension).

Finally, we have the following useful sufficient condition (see Haraux [58], Theorem 7.5, p. 116).

Theorem 1.3.40. *Let A be a $\mathbb{C}$-linear, m-accretive operator with dense domain and let $(T(t))_{t\ge 0}$ be the semigroup of contractions generated by $-A$. If the numerical range of A verifies $\overline{S(A)}\subset\overline{C_\theta}$ for some $0<\theta<\pi/2$, then $(T(t))_{t\ge 0}$ is an analytic semigroup.*

1.4. Examples of semigroups generated by partial differential operators. In this section, we apply the results of the preceding section to the examples described in Section 1.2.

1.4.1. First order equations. We consider the operators introduced in Section 1.2.1. We first study the one-dimensional case.

Let $X=C_0(\mathbb{R})$, and let A be defined by

$$\begin{cases} D(A)=\{u\in C^1(\mathbb{R})\cap X;\ u'\in X\},\\ Au=u',\ \text{for } u\in D(A).\end{cases}\tag{1.4.1}$$

It follows from Remark 1.2.3 (ii) that both A and $-A$ are m-accretive with dense domain. It follows from the results of Section 1.3.5 that $-A$ generates a group of isometries $(T(t))_{t\in\mathbb{R}}$ in X. For every $\varphi\in D(A)$, $u(t)=T(t)\varphi$ is the unique solution in $C(\mathbb{R},D(A))\cap C^1(\mathbb{R},X)$ of the problem

$$\begin{cases} u_t+u_x=0,\quad t,x\in\mathbb{R};\\ u(0,x)=\varphi(x),\quad x\in\mathbb{R}.\end{cases}$$

Furthermore, we have the following characterization of $(T(t))_{t\in\mathbb{R}}$.

Proposition 1.4.1. *If A is as above and if $(T(t))_{t\in\mathbb{R}}$ is the group of isometries generated by $-A$, then*

$$T(t)\varphi(x)=\varphi(x-t),\ \textit{for all}\ t,x\in\mathbb{R},\tag{1.4.2}$$

for every $\varphi \in X$.

Proof. Given $\varphi \in D(A)$, define $v(t)$, for $t \in \mathbb{R}$, by

$$v(t,x) = \varphi(x-t), \text{ for } x \in \mathbb{R}.$$

One verifies easily that $v \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)$, that $v(0) = \varphi$ and that

$$\frac{dv}{dt} + Av = 0,$$

for all $t \in \mathbb{R}$. Applying Corollary 1.3.28, it follows that $v(t) = T(t)\varphi$. The result now follows by density. □

Remark 1.4.2. One has similar results for the other one-dimensional examples of Section 1.2.1. In particular, one has the following results.

(i) Consider $1 \le p < \infty$, let $X = L^p(\mathbb{R})$, and let A be defined by

$$\begin{cases} D(A) = W^{1,p}(\mathbb{R}), \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Remark 1.2.3 (iii) that both A and $-A$ are m-accretive with dense domain, and it follows from the results of Section 1.3.5 that $-A$ generates a group of isometries $(T(t))_{t\in\mathbb{R}}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by formula (1.4.2).

(ii) Consider $X = \{u \in C([0,1]);\, u(0) = 0\}$, equipped with the sup norm. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in C^1([0,1]);\, u(0) = u'(0) = 0\}, \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.4 that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\ge 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by

$$T(t)\varphi(x) = \begin{cases} 0, \text{ if } x \le \min\{t,1\}; \\ \varphi(x-t), \text{ if } \min\{t,1\} \le x \le 1. \end{cases} \tag{1.4.3}$$

Note that in particular, $T(t) = 0$ for $t \ge 1$.

(iii) Consider $1 \le p < \infty$, and let $X = L^p(0,1)$. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in W^{1,p}(0,1);\, u(0) = 0\}; \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Remark 1.2.5 (ii) that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\ge 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by formula (1.4.3). Note that here also, $T(t) = 0$ for $t \ge 1$.

(iv) Let $X = \{u \in C([0,1]);\, u(0) = u(1)\}$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,1]);\, u(0) = u(1) \text{ and } u'(0) = u'(1)\}, \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Remark 1.2.5 (iii) that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\geq 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by the following formula. Define $\widetilde{\varphi} \in C(\mathbb{R})$ by $\widetilde{\varphi}(x) = \varphi(x - E(x))$, for all $x \in \mathbb{R}$. Here, $E(x)$ denotes the integer part of x, i.e. $E(x) = m$ if $m \leq x < m + 1$. Note that $\widetilde{\varphi}$ is periodic with period 1. Then, $T(t)$ is given by

$$T(t)\varphi(x) = \widetilde{\varphi}(x - t),$$

for all $t, x \in \mathbb{R}$. Note that here, $T(t + m) = T(t)$ for $t \in \mathbb{R}$ and all $m \in \mathbb{Z}$.

(v) Let $X = C_0(\mathbb{R}_+) = \{u \in C([0,\infty);\, u(0) = 0 \text{ and } \lim\limits_{x\to\infty} u(x) = 0\}$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty)) \cap X;\, u' \in X\}, \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.6 that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\geq 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by

$$T(t)\varphi(x) = \begin{cases} 0, \text{ if } x \leq t; \\ \varphi(x - t), \text{ if } x \geq t. \end{cases} \tag{1.4.4}$$

Note that $T(t)$ is an isometry in X. However, clearly $T(t)$ is not surjective, and thus $(T(t))_{t\geq 0}$ cannot be embedded in a group of isometries (see Remark 1.3.24 (ii)).

(vi) Let $X = L^p(0,\infty)$. Define the operator A in X by

$$\begin{cases} D(A) = \{u \in W^{1,p}(0,\infty);\, u(0) = 0\}; \\ Au = u', \text{ for } u \in D(A). \end{cases}$$

It follows from Remark 1.2.7 (ii) that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\geq 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by formula (1.4.4). Note that here also, $T(t)$ is an isometry in X.

(vii) Let $X = \{u \in C([0,\infty);\, \lim\limits_{x\to\infty} u(x) = 0\}$, and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty));\, \lim\limits_{x\to\infty} u(x) = \lim\limits_{x\to\infty} u'(x) = 0\}, \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.8 that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\geq 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that $T(t)$ is given by

$$T(t)\varphi(x) = \varphi(x + t), \text{ for all } x \leq 0 \text{ and all } t \geq 0, \tag{1.4.5}$$

for every $\varphi \in X$. Note that in general $\|T(t)\varphi\| < \|\varphi\|$ for all $t > 0$. The reader can easily construct a φ with this property.

(viii) Let $1 \le p < \infty$, $X = L^p(0,\infty)$, and let A be defined by

$$\begin{cases} D(A) = W^{1,p}(0,\infty), \\ Au = -u', \text{ for } u \in D(A). \end{cases}$$

It follows from Remark 1.2.9 (iii) that A is m-accretive with dense domain, and it follows from the results of Section 1.3.2 that $-A$ generates a semigroup of contractions $(T(t))_{t\ge 0}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by formula (1.4.5). Note that here also, in general $\|T(t)\varphi\| < \|\varphi\|$ for all $t > 0$.

Let now $X = C_0(\mathbb{R}^N)$. Given $a \in \mathbb{R}^N$, consider the operator A defined by

$$\begin{cases} D(A) = \{u \in X;\, a \cdot \nabla u \in X\}, \\ Au = a \cdot \nabla u, \text{ for } u \in D(A). \end{cases} \tag{1.4.6}$$

It follows from Remark 1.2.15 (i) that both A and $-A$ are m-accretive with dense domain, and it follows from the results of Section 1.3.5 that $-A$ generates a group of isometries $(T(t))_{t\in\mathbb{R}}$ in X. Then, we have the following result.

Proposition 1.4.3. *If A be as above and if $(T(t))_{t\in\mathbb{R}}$ is the group of isometries generated by $-A$, then*

$$T(t)\varphi(x) = \varphi(x - ta), \tag{1.4.7}$$

for all $\varphi \in X$ and all $t \in \mathbb{R}$.

Proof. The proof is easily adapted from that of Proposition 1.4.1. □

Remark 1.4.4. Consider $1 \le p < \infty$, and let $X = L^p(\mathbb{R}^N)$. Define the operator A in X by (1.4.6). It follows from Remark 1.2.15 (iii) that both A and $-A$ are m-accretive with dense domain, and it follows from the results of Section 1.3.5 that $-A$ generates a group of isometries $(T(t))_{t\in\mathbb{R}}$ in X. Arguing as in the proof of Proposition 1.4.1, one shows easily that, for every $\varphi \in X$, $T(t)\varphi$ is given by formula (1.4.7).

1.4.2. The heat equation. Throughout this section, Ω is an arbitrary open subset of $\mathbb{R}^N$. For some of the results, we will make further assumptions on Ω which we will specify. We will apply the results of Section 1.3 to the examples of Section 1.2.2, in order to solve the initial value problem for the heat equation $u_t = \triangle u$.

Consider the operator A defined on $H^{-1}(\Omega)$ by (1.2.14), that is

$$\begin{cases} D(A) = H_0^1(\Omega), \\ Au = -\triangle u, \text{ for all } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.17 that A is self adjoint and accretive. Therefore, $-A$ is the generator of a semigroup of contractions on $H^{-1}(\Omega)$, which we denote by $(T(t))_{t\geq 0}$. On the other hand, the operator B defined on $L^2(\Omega)$ by (1.2.19), that is

$$\begin{cases} D(B) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}, \\ Bu = -\triangle u, \text{ for all } u \in D(B), \end{cases}$$

is self adjoint and accretive by Proposition 1.2.21. Therefore, $-B$ is the generator of a semigroup of contractions on $L^2(\Omega)$, which we denote by $(S(t))_{t\geq 0}$.

Lemma 1.4.5. *With the above notation, $T(t)\varphi = S(t)\varphi$ for all $t \geq 0$ and all $\varphi \in L^2(\Omega)$.*

Proof. Since $G(B) \subset G(A)$ as subsets of $H^{-1}(\Omega) \times H^{-1}(\Omega)$, the result follows immediately from Proposition 1.3.4 when $\varphi \in D(B) \subset D(A)$. Since both $S(t)$ and $T(t)$ are continous $L^2(\Omega) \to H^{-1}(\Omega)$, the result follows, by density of $D(B)$ in $L^2(\Omega)$. □

Remark 1.4.6. Since $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ coincide where they are both defined, that is on $L^2(\Omega)$, we will by denote also by $(T(t))_{t\geq 0}$ the semigroup of contractions generated by $-B$.

Remark 1.4.7. By Corollary 1.3.32, $T(t)$ is self-adjoint in $L^2(\Omega)$ for all $t \geq 0$, i.e.

$$(T(t)\varphi, \psi)_{L^2} = (\varphi, T(t)\psi)_{L^2},$$

for all $t \geq 0$ and all $\varphi, \psi \in L^2(\Omega)$.

Theorem 1.4.8. *Let A and $(T(t))_{t\geq 0}$ be as above. Given $\varphi \in H^{-1}(\Omega)$, set $u(t) = T(t)\varphi$ for $t \geq 0$. Then, the following properties hold:*

(i) $u \in C([0,\infty), H^{-1}(\Omega)) \cap C((0,\infty), H^1_0(\Omega)) \cap C^1((0,\infty), H^{-1}(\Omega))$, *and u is the unique solution of problem*

$$\begin{cases} u_t - \triangle u = 0 \text{ for all } t > 0, \\ u(0) = \varphi; \end{cases} \tag{1.4.8}$$

in that class. Moreover, $\triangle^n u \in C^\infty((0,\infty), H^1_0(\Omega))$ for every nonnegative integer n;

(ii) $u \in C^\infty((0,\infty) \times \Omega)$*;*

(iii) *if $\varphi \in L^2(\Omega)$, then $u \in C([0,\infty), L^2(\Omega))$. If $\varphi \in H^1_0(\Omega)$, then $u \in C([0,\infty), H^1_0(\Omega)) \cap C^1([0,\infty), H^{-1}(\Omega))$. If moreover $\triangle\varphi \in L^2(\Omega)$, then $\triangle u \in C([0,\infty), L^2(\Omega))$ and $u \in C^1([0,\infty), L^2(\Omega))$.*

Proof. Since $D(A) = H^1_0(\Omega)$ with equivalent norms (Proposition 1.2.17), property (i) follows from Theorem 1.3.34 and Corollary 1.3.35. Next, it follows from property (i), Corollary 1.3.35 and Lemma 1.4.5 that $u \in C^\infty((0,\infty), D(B^n))$ for every nonnegative integer n; and so property (ii) follows from Remark 1.2.23 (ii) and Sobolev's embedding theorem. Finally, property (iii) follows from Lemma 1.4.5. □

When Ω satisfies certain regularity assumptions, the semigroup has better regularity properties. Some of these properties are described in the following proposition.

Theorem 1.4.9. *Let A and $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. Given $\varphi \in H^{-1}(\Omega)$, set $u(t) = T(t)\varphi$ for $t \geq 0$. Then the following properties hold:*

(i) *if Ω has a bounded boundary of class C^2 and if $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, then $u \in C([0,\infty), H^2(\Omega)) \cap C^1([0,\infty), L^2(\Omega))$;*

(ii) *if Ω has a bounded boundary of class C^{2m} for some positive integer m, then $u \in C^\infty((0,\infty), H^{2m}(\Omega))$;*

(iii) *if Ω has a bounded boundary of class C^∞, then $u \in C^\infty([\varepsilon,\infty) \times \overline{\Omega})$, for every $\varepsilon > 0$;*

(iv) *if Ω is bounded with boundary of class C^∞, and if $\varphi \in C^\infty(\overline{\Omega})$ satisfies the compatibility relations $u = \triangle u = \cdots \triangle^n u = \cdots = 0$ on $\partial\Omega$, then $u \in C^\infty([0,\infty) \times \overline{\Omega})$.*

Proof. If $\varphi \in D(B)$, then it follows from Lemma 1.4.5 that $u \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(\Omega))$. Since $D(B) \hookrightarrow H^2(\Omega)$ whenever Ω has a bounded boundary of class C^2 (Remark 1.2.23), property (i) follows. On the other hand, $u \in C^\infty((0,\infty), D(B^n))$ for every nonnegative integer n (see the proof of Theorem 1.4.8). Therefore, properties (ii) and (iii) follow from Remark 1.2.23 and Sobolev's embedding theorem. Finally, if $\varphi \in \bigcap_{n\geq 0} D(B^n)$, then it follows from Corollary 1.3.17 that $u \in \bigcap_{n\geq 0} C^\infty([0,\infty), D(B^n))$. Hence property (iv), by applying Remark 1.2.23 and Sobolev's embedding theorem. □

Remark 1.4.10. Note that the compatibility relations of property (iv) of Theorem 1.4.9 are necessary conditions if $u \in C^\infty([0,\infty) \times \overline{\Omega})$. Indeed, we have $u = \dfrac{du}{dt} = \cdots = \dfrac{d^n u}{dt^n} = \cdots = 0$ on $(0,\infty) \times \partial\Omega$. Since $\dfrac{d^n u}{dt^n} = \triangle^n u$, the compatibility relations follow.

We next describe some pointwise estimates that are consequences of Theorem 1.3.34.

Theorem 1.4.11. *Let A and $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. Given $\varphi \in L^2(\Omega)$, set $u(t) = T(t)\varphi$ for $t \geq 0$. Then the following properties hold:*

(i) $\|\triangle u(t)\|_{L^2} \leq \dfrac{1}{t\sqrt{2}}\|\varphi\|_{L^2}$ *for all $t > 0$ and* $\displaystyle\int_0^\infty s\|\triangle u(s)\|^2\,ds \leq \frac{1}{4}\|\varphi\|_{L^2}^2$;

(ii) $\|\nabla u(t)\|_{L^2} \leq \dfrac{1}{\sqrt{2t}}\|\varphi\|_{L^2}$ *for all $t > 0$ and* $\displaystyle\int_0^\infty \|\nabla u(s)\|^2\,ds \leq \frac{1}{2}\|\varphi\|_{L^2}^2$;

(iii) *if $\varphi \in H_0^1(\Omega)$, then* $\|\triangle u(t)\|_{L^2} \leq \dfrac{1}{\sqrt{2t}}\|\nabla\varphi\|_{L^2}$ *for all $t > 0$ and* $\displaystyle\int_0^\infty \|\triangle u(s)\|^2\,ds \leq \frac{1}{2}\|\nabla\varphi\|_{L^2}^2$.

Proof. By density, we need only consider the case $\varphi \in \mathcal{D}(\Omega)$. In this case, the resuls follow from Theorem 1.3.34, Lemma 1.4.5 and identity (A.3.17). □

The following result, which is a form of the weak maximum principle for the heat equation, is essential for the study of both the linear and the nonlinear heat equations.

Theorem 1.4.12. *Let $T > 0$, $1 < p < \infty$ and $f \in L^1_{\text{loc}}((0,T), H^{-1}(\Omega))$. Assume $u \in C([0,T], L^2(\Omega)) \cap L^p((0,T), H^1(\Omega)) \cap W^{1,p'}((0,T), H^{-1}(\Omega))$ solves equation*

$$u_t - \triangle u = f, \text{ for almost all } t \in (0,T),$$

and that

(i) *there exists* $v \in L^p((0,T), H_0^1(\Omega))$ *such that* $u(t) \le v(t)$ *almost everywhere in* Ω *for almost all* $t \in (0,T)$*;*

(ii) $f = g + h$*, with* $g \in L^1_{\text{loc}}((0,T), H^{-1}(\Omega))$*,* $g(t) \le 0$ *for almost all* $t \in (0,T)$*, and* $h \in L^1_{\text{loc}}((0,T), L^2(\Omega))$*,* $h(t) \le C|u(t)|$ *almost everywhere in* Ω *for almost all* $t \in (0,T)$ *where* C *is independent of* t*;*

(iii) $u(0) \le 0$ *almost everywhere in* Ω*.*

It follows that $u(t) \le 0$ *almost everywhere in* Ω *for all* $t \in (0,T)$*.*

Proof. Since $u^+(t) \in H_0^1(\Omega)$ for almost all $t \in (0,T)$ by Proposition A.3.34, it follows that

$$\langle u_t(t), u^+(t)\rangle_{H^{-1},H_0^1} - \langle \triangle u(t), u^+(t)\rangle_{H^{-1},H_0^1} = \langle f(t), u^+(t)\rangle_{H^{-1},H_0^1},$$

for almost all $t \in (0,T)$. It follows from Corollary A.3.15 and formula (A.3.17) that $\langle \triangle u(t), u^+(t)\rangle_{H^{-1},H_0^1} \le 0$, and it follows from assumption (ii) and formula (A.3.14) that

$$\langle f(t), u^+(t)\rangle_{H^{-1},H_0^1} \le \langle h(t), u^+(t)\rangle_{H^{-1},H_0^1} \le C\int_\Omega |u(t)|u^+(t)\,dx = C\int_\Omega u^+(t)^2\,dx.$$

Therefore, applying Corollary A.3.68, we obtain

$$\frac{d}{dt}\int_\Omega u^+(t)^2\,dx \le C\int_\Omega u^+(t)^2\,dx,$$

for almost all $t \in (0,T)$. Integrating the above inequality and using assumption (iii), we obtain

$$\int_\Omega u^+(t)^2\,dx \le C\int_0^t\int_\Omega u^+(s)^2\,dx\,ds,$$

for all $t \in (0,T)$; and so, $u^+(t) \equiv 0$ by Gronwall's lemma. Hence the result. □

Remark 1.4.13. Here are some comments concerning Theorem 1.4.12.

(i) Assumption (ii) can be slightly weakened. In fact, in the proof of Theorem 1.4.12 we only need that $\langle f(t), u^+(t)\rangle_{H^{-1},H_0^1} \le C\|u^+(t)\|_{L^2}^2$ for almost all $t \in (0,T)$, where C is independent of t.

(ii) Assumption (i) means that $u \le 0$ on $\partial\Omega$. This assumption is essential, as the following example shows. Take $\Omega = (-1,1)$ and $u(t,x) = t + \frac{x^2}{2} - \frac{1}{2}$. It is clear that u satisfies all the assumptions of Theorem 1.4.12 (with $f = 0$) except (i), but $u(t)$ takes some positive values on Ω for each $t > 0$.

(iii) One must be very careful about the regularity assumptions on u, which are essential. Consider for example the equation

$$\begin{cases} u_t - \triangle u = 0, \\ u_{|\partial\Omega} = 0, \\ u(0) = -1, \end{cases}$$

in a bounded, smooth domain $\Omega \subset \mathbb{R}^N$. Theorem 1.4.12 asserts that $u(t) \le 0$ in Ω. On the other hand, we have $u \in C^\infty([\varepsilon,\infty) \times \overline{\Omega})$ and if we set $v(t) = u_t(t)$, then

$$\begin{cases} v_t - \triangle v = 0, \\ v_{|\partial\Omega} = 0, \\ v(0) = 0. \end{cases}$$

Note that the condition $v(0) = 0$ makes sense since $v \in C([0, \infty), H^{-2}(\Omega))$ (because $v = \triangle u$ and $u \in C([0, \infty), L^2(\Omega))$. On the other hand, we have $v(t) \geq 0$ in Ω, $v(t) \not\equiv 0$ for all $t > 0$ (see Exercise 1.8.18). In particular, the conclusion of Theorem 1.4.12 does not hold, due to the lack of regularity of v at $t = 0$.

(iv) Let $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8 and let $\varphi \in \mathcal{D}(\Omega)$. If $\varphi \geq 0$ in Ω, then $T(t)\varphi \geq 0$ in Ω for all $t \geq 0$. This follows from Theorem 1.4.12 applied to $u = -T(t)\varphi$ with $f \equiv 0$ and $v = u$. By density, it follows that $T(t)\varphi \geq 0$ almost everywhere in Ω for all $t \geq 0$ and all $\varphi \in L^2(\Omega)$ such that $\varphi \geq 0$ almost everywhere in Ω.

When $\Omega = \mathbb{R}^N$, one can compute $T(t)\varphi$ in terms of a kernel, as shows the following result.

Proposition 1.4.14. *Suppose that $\Omega = \mathbb{R}^N$ and let $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. For every $t > 0$, define the function $S_t \in \mathcal{S}(\mathbb{R}^N)$ by $S_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}$ for $x \in \mathbb{R}^N$. If $\varphi \in L^2(\mathbb{R}^N)$ and if $u(t) = T(t)\varphi$ for $t \geq 0$, then the following properties hold:*

(i) *$T(t)\varphi = S_t \star \varphi$ for all $t > 0$;*

(ii) *if $\varphi \in L^p(\mathbb{R}^N)$ for some $1 \leq p \leq \infty$, then $u(t) \in L^q(\mathbb{R}^N)$ for all $p \leq q \leq \infty$, and*

$$\|u(t)\|_{L^q} \leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varphi\|_{L^p}$$

for all $t > 0$.

Proof. Let $\varphi \in L^2$ and set $u(t) = S_t \star \varphi$, for $t > 0$. We will check that $u(t) \in D(A)$ for all $t > 0$, $u \in C([0, \infty), L^2(\mathbb{R}^N)) \cap C^1((0, \infty), L^2(\mathbb{R}^N))$, $u(0) = \varphi$ and $u_t = \triangle u$ for all $t > 0$. This will show property (i) (see Theorem 1.3.34 (i)). The regularity properties are easily verified. Furthermore, a direct calculation shows that $\partial_t S_t - \triangle S_t = 0$ for all $t > 0$, which shows that u verifies the equation $u_t - \triangle u = 0$ for all $t > 0$. Therefore, it remains to show that $u(t) \to \varphi$ in $L^2(\mathbb{R}^N)$ as $t \downarrow 0$. Note that

$$\|S_t\|_{L^1} = \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} e^{-|x|^2}\, dx = \left(\frac{1}{\pi} \int_{\mathbb{R}^2} e^{-|x|^2}\, dx\right)^{N/2} = \left(2\int_0^\infty e^{-r^2} r\, dr\right)^{N/2} = 1.$$

It follows from Young's inequality that $\|S_t \star \varphi\|_{L^2} \leq \|\varphi\|_{L^2}$. Therefore, by density and Lemma A.1.4, we need only show the result for $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Suppose now that $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Note that

$$S_t \star \varphi(x) = \frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} e^{-|z|^2} \varphi(x - 2\sqrt{t}z)\, dz.$$

It follows from the above formula and the dominated convergence theorem that $S_t \star \varphi(x) \to \varphi(x)$ as $t \downarrow 0$. Let R be large enough so that $\mathrm{Supp}(\varphi) \subset \{|x| \leq R\}$. For $|x| \geq 2R$ and $|y| \leq R$, we have $|x - y| \geq \frac{1}{2}|x|$. For $|x| \geq 2R$, it follows that

$$|S_t \star \varphi(x)| = \frac{1}{(4\pi t)^{N/2}} \left| \int_{\{|x|\leq R\}} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy \right| \leq \frac{1}{(4\pi t)^{N/2}} \int_{\{|x|\leq R\}} e^{-\frac{|x|^2}{16t}} |\varphi(y)|\, dy \leq C t^{-N/2} e^{-\frac{|x|^2}{16t}}.$$

Therefore, there exists $\varepsilon > 0$ such that $|S_t \star \varphi(x)| \leq C e^{-\varepsilon|x|^2}$ for $|x| \geq 2R$. Since $\|S_t \star \varphi\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$, we have as well $|S_t \star \varphi(x)| \leq C e^{-\varepsilon|x|^2}$ for all $x \in \mathbb{R}^N$. By the dominated convergence theorem, we obtain that $S_t \star \varphi \to \varphi$ in $L^2(\mathbb{R}^N)$ as $t \downarrow 0$, which shows (i). Property (ii) follows from property (i) and Young's inequality, since an easy calculation shows that $\|S_t\|_{L^r} = r^{-\frac{N}{2r}}(4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})} \le (4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})}$. □

Theorem 1.4.15. *Let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.8 for a general domain Ω, let $\varphi \in L^2(\Omega)$ and let $u(t) = T(t)\varphi$. If $\varphi \in L^p(\Omega)$ for some $1 \le p \le \infty$, then $u(t) \in L^q(\Omega)$ for all $p \le q \le \infty$, and*

$$\|u(t)\|_{L^q} \le (4\pi t)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{L^p}$$

for all $t > 0$.

Proof. Suppose first that $p < \infty$. By density, we need only prove the result for $\varphi \in H_0^1(\Omega) \cap L^p(\Omega)$. Define $\psi \in H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ by

$$\psi(x) = \begin{cases} |\varphi(x)| \text{ if } x \in \Omega, \\ 0 \text{ if } x \notin \Omega, \end{cases}$$

and let $v(t) = S_t \star \psi$ where S_t is defined in Proposition 1.4.14. It follows from Theorem 1.4.8 that $v \in C([0,\infty), H^1(\mathbb{R}^N)) \cap C^1([0,\infty), H^{-1}(\mathbb{R}^N))$ and $v_t - \triangle v = 0$ in $H^{-1}(\mathbb{R}^N)$. In addition, $v(t) \ge 0$; and so, if we set $w(t) = v(t)_{|\Omega}$, then $w \in C([0,\infty), H^1(\Omega)) \cap C^1([0,\infty), H^{-1}(\Omega))$, $w_t - \triangle v = 0$ in $H^{-1}(\Omega)$ and $w(t) \ge 0$. It follows easily that $z_1(t) = u(t) - w(t)$ and $z_2(t) = -u(t) - w(t)$ verify the assumptions of Theorem 1.4.12 (take $v = |u|$). Therefore, $z_1(t), z_2(t) \le 0$, which implies that $|u(t)| \le w(t)$ almost everywhere on Ω for all $t \ge 0$. In particular, $\|u(t)\|_{L^q(\Omega)} \le \|w(t)\|_{L^q(\Omega)} \le \|v(t)\|_{L^q(\mathbb{R}^N)}$ and the result follows from Proposition 1.4.14. If $p = \infty$, apply the result for finite p, with $q = \infty$, then let $p \uparrow \infty$. □

Remark. The $L^p - L^q$ estimates of Theorem 1.4.15 can be also obtained by a technique of multipliers. Since that technique can be applied to certain nonlinear problems to which the comparison argument is not applicable, we describe it in Section 1.7.3 below.

Remark 1.4.16. It follows in particular from Theorem 1.4.15 that for arbitrary domains Ω, $T(t)$ is continuous $L^p(\Omega) \to L^q(\Omega)$, for every $t > 0$ and every $1 \le p \le q \le \infty$ with $p < \infty$. In particular, if Ω has finite measure, then, given $t > 0$ and $1 \le p < \infty$, $T(t)$ is continuous $L^p(\Omega) \to L^q(\Omega)$ for every $1 \le q \le \infty$. However, if for example $\Omega = \mathbb{R}^N$, then $T(t)$ does not map $L^p(\Omega)$ to $L^q(\Omega)$ if $q < p$. Indeed, let $1 < p < \infty$ and let

$$\varphi(x) = \frac{1}{(1+|x|)^{N/p}\mathrm{Log}(2+|x|)}, \text{ for } x \in \mathbb{R}^N.$$

One verifies easily that $\varphi \in L^q(\mathbb{R}^N)$ for $q \ge p$ and that $\varphi \notin L^q(\mathbb{R}^N)$ for $q < p$. On the other hand, given $t > 0$,

$$T(t)\varphi(x) = (4\pi t)^{-N/2}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy \ge (4\pi t)^{-N/2}e^{-\frac{1}{4t}}\int_{\{|x-y|\le 1\}}\varphi(y)\,dy.$$

One verifies easily that $\inf\limits_{x\in\mathbb{R}^N}\inf\limits_{|x-y|\le 1}\dfrac{\varphi(y)}{\varphi(x)} > 0$. Therefore, there exists $\varepsilon > 0$ such that $T(t)\varphi \ge \varepsilon t^{-N/2}e^{-\frac{1}{4t}}\varphi$; and so, $T(t)\varphi \notin L^q(\mathbb{R}^N)$ for $q < p$.

Corollary 1.4.17. *Let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.8, let $\varphi \in L^2(\Omega)$ and let $u(t) = T(t)\varphi$. If $\varphi \in L^p(\Omega)$ for some $1 \le p < \infty$, then $u \in C([0,\infty), L^p(\Omega))$.*

Proof. By density and Theorem 1.4.15, we need only consider the case $\varphi \in \mathcal{D}(\Omega)$. Suppose that φ is supported in the ball of $\mathbb{R}^N$ with center 0 and radius $R < \infty$, and define $w \in \mathcal{S}(\mathbb{R}^N)$ by $w(x) = \|\varphi\|_{L^\infty} e^{\sqrt{1+R^2}} e^{-\sqrt{1+|x|^2}}$. An easy calculation shows that $\triangle w \le w$; and so, if we set $z(t,x) = e^t w(x)$, then $z_t - \triangle z \ge 0$. Therefore, if we apply Theorem 1.4.12 to $u_1 = u - z_{|\Omega}$ and $u_2 = -u - z_{|\Omega}$ (take $v = |u|$), then we obtain $|u(t)| \le z(t)$. Therefore, given $T > 0$, we have $|u(t,x)| \le e^T w(x)$ almost everywhere in Ω for all $t \in [0,T]$. Continuity of $u(t)$ in $L^p(\Omega)$ now follows from continuity in $L^2(\Omega)$ and the dominated convergence theorem. □

Remark 1.4.18. It follows from Corollary 1.4.17 that for every $1 \le p < \infty$, $(T(t))_{t\ge 0}$ can be uniquely extended by continuity to a semigroup of contractions in $L^p(\Omega)$, which we still denote by $(T(t))_{t\ge 0}$. On the other hand, the operator A in $L^p(\Omega)$ defined by (1.2.21) also defines a semigroup of contractions in $L^p(\Omega)$. It turns out that the two semigroups coincide, as shows the following result.

Proposition 1.4.19. *Let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.8, let $1 \le p < \infty$ and let $(S(t))_{t\ge 0}$ be the semigroup of contractions in $L^p(\Omega)$ generated by $-A$, where A is the operator defined by (1.2.21). Then, $T(t)\varphi = S(t)\varphi$ for every $\varphi \in L^2(\Omega) \cap L^p(\Omega)$ and every $t \ge 0$.*

Proof. By density, we need only show the result for $\varphi \in \mathcal{D}(\Omega)$. Set $\psi = -\triangle\varphi + \varphi$, so that $\varphi = J_1\psi$ where J_1 is as in Lemma 1.2.27. If $u(t) = T(t)\varphi$, then it follows from Corollary 1.3.18 that $u_t(t) = T(t)\triangle\varphi$; and so, by Corollary 1.4.17, that $u \in C^1([0,\infty), L^p(\Omega))$. Since $u_t = \triangle u$, we have also $\triangle u \in C([0,\infty), L^p(\Omega))$. Furthermore, if $v(t) = T(t)\psi \in C([0,\infty), L^p(\Omega))$, then it follows from Corollary 1.3.14 that $u(t) = T(t)J_1\psi = J_1 T(t)\psi = J_1 v(t)$. Therefore, $u \in C([0,\infty), D(A)) \cap C^1([0,\infty), L^p(\Omega))$. Since $u_t = \triangle u = -Au$, it follows from Proposition 1.3.4 that $u(t) = S(t)\psi$. Hence the result. □

$(T(t))_{t\ge 0}$ is in fact an analytic semigroup in $L^p(\Omega)$ for all $1 < p < \infty$. This property is the object of the following result.

Proposition 1.4.20. *Let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.8 and let $1 < p < \infty$. Then $(T(t))_{t\ge 0}$ is an analytic semigroup in $L^p(\Omega)$.*

If Ω has a bounded boundary of class C^2, then in addition

$$\|T(t)\varphi\|_{W^{2,p}} \le C\left(1 + \frac{1}{t}\right)\|\varphi\|_{L^p}, \tag{1.4.9}$$

and

$$\|T(t)\varphi\|_{W^{1,p}} \le C\left(1 + \frac{1}{\sqrt{t}}\right)\|\varphi\|_{L^p}, \tag{1.4.10}$$

for all $t > 0$ and all $\varphi \in L^p(\Omega)$.

Proof. Note that by Theorem 1.4.11 and Remark 1.2.23, we need only prove the result for $p \ne 2$. We proceed in three steps.

Step 1. We show that $(T(t))_{t\ge 0}$ is analytic in $L^p(\Omega)$ for $p > 2$. Let A be the operator in $L^p(\Omega)$ defined by (1.2.21), so that $-A$ is the generator of $(T(t))_{t\ge 0}$ considered as a semigroup of contractions in $L^p(\Omega)$.

(See Proposition 1.4.19.) We extend A to complex valued functions by $\mathbb{C}$-linearity. It is clear that A is also m-accretive in $L^p(\Omega,\mathbb{C})$, and that the semigroup generated by $-A$ is the natural extension of $(T(t))_{t\geq 0}$ to $L^p(\Omega,\mathbb{C})$. For the rest of the proof, we denote by $L^p(\Omega)$ and $W^{1,p}(\Omega)$ the spaces of complex valued functions. Let $\langle\cdot,\cdot\rangle_{L^p,L^{p'}}$ denote the complex duality bracket between $L^p(\Omega)$ and $L^{p'}(\Omega)$. Let $f\in\mathcal{D}(\Omega)$, and set $u=I_1 f$, with the notation of Lemma 1.2.27. We have $u\in H^1_0(\Omega)\cap L^\infty(\Omega)$. Let $M=\|u\|_{L^\infty}$ and let $F:\mathbb{C}\to\mathbb{C}$ be defined by

$$F(z)=\begin{cases}|z|^{p-2}z, & \text{if } |z|\leq M,\\ M^{p-2}z, & \text{if } |z|\geq M,\end{cases}$$

so that F is Lipschitz continuous. We have in particular $F(u)=|u|^{p-2}u$, and it follows easily from Corollary A.3.29 (see also Section A.3.7) that $|u|^{p-2}u\in H^1_0(\Omega)$. Furthermore, it follows from Theorem A.3.12 and an easy calculation that

$$\nabla(|u|^{p-2}\overline{u})=\begin{cases}|u|^{p-4}\Big(|u|^2|\nabla u|^2+\dfrac{p-2}{2}(\overline{u}^2\nabla u^2+|u|^2|\nabla u|^2)\Big) \text{ if } u\neq 0,\\ 0 \text{ if } u=0,\end{cases}$$

almost everywhere. Therefore,

$$\operatorname{Re}(\nabla u\cdot\nabla(|u|^{p-2}\overline{u}))\geq |u|^{p-2}|\nabla u|^2,$$

and

$$|\operatorname{Im}(\nabla u\cdot\nabla(|u|^{p-2}\overline{u}))|\leq \frac{p-2}{2}|u|^{p-2}|\nabla u|^2,$$

almost everywhere. Setting $\mathcal{C}_\theta=\{z\in\mathbb{C};\ -\theta<\arg z<\theta\}$ with $\tan\theta=\dfrac{p-2}{2}$, we have $\nabla u\cdot\nabla(|u|^{p-2}\overline{u})\in\overline{\mathcal{C}}_\theta$ almost everywhere. Therefore,

$$\langle Au,|u|^{p-2}u\rangle_{L^p,L^{p'}}=\int_\Omega \nabla u\cdot\nabla(|u|^{p-2}\overline{u})\,dx\in\overline{\mathcal{C}}_\theta.$$

Since I_1 is an isomorphism from $L^p(\Omega)$ onto $D(A)$ (Theorem 1.1.12), and since the mapping $u\mapsto|u|^{p-2}u$ is continuous $L^p(\Omega)\to L^{p'}(\Omega)$, it now follows by density of $\mathcal{D}(\Omega)$ in $L^p(\Omega)$ that

$$\langle Au,|u|^{p-2}u\rangle_{L^p,L^{p'}}\in\overline{\mathcal{C}}_\theta,$$

for all $u\in D(A)$. The result now follows from Theorem 1.3.39, since the duality map in L^p is given by $F(u)=\|u\|_{L^p}^{2-p}|u|^{p-2}u$.

Step 2. We show that $(T(t))_{t\geq 0}$ is analytic in $L^p(\Omega)$ for $1<p<2$. Let now $1<p<2$. Given $\varphi,\psi\in\mathcal{D}(\Omega)$, we set $u(t)=T(t)\varphi$ and $v(t)=T(t)\psi$. Note that $\varphi\in D(A)$, so that $v\in C([0,\infty),D(A))\cap C^1([0,\infty),L^p(\Omega))$. We have for $t>0$

$$(u(t),\psi)_{L^p,L^{p'}}=(u(t),\psi)_{L^2,L^2}=(\varphi,v(t))_{L^2,L^2}=(\varphi,v(t))_{L^p,L^{p'}},$$

since $T(t)$ is self-adjoint in $L^2(\Omega)$ (see Corollary 1.3.32). Therefore, it follows from Step 1 and Theorem 1.3.38 that

$$|(u'(t),\psi)_{L^p,L^{p'}}|=|(\varphi,v'(t))_{L^p,L^{p'}}|\leq\frac{C}{t}\|\varphi\|_{L^p}\|\psi\|_{L^{p'}}.$$

Since $u'(t) = -Au(t)$, we obtain

$$\|Au(t)\|_{L^p} \le \frac{C}{t}\|\varphi\|_{L^p}.$$

Let now $\varphi \in L^p(\Omega)$, let $(\varphi_n)_{n\ge 1} \subset \mathcal{D}(\Omega)$ be such that $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^p(\Omega)$, and set $u(t) = T(t)\varphi$ and $u_n(t) = T(t)\varphi_n$. Fix $t > 0$. It follows from the above inequality that there exists a subsequence $(n_k)_{k\ge 1}$ and $\xi \in L^p(\Omega)$ with $\|\xi\|_{L^p} \le C/t$ such that $Au_{n_k}(t) \rightharpoonup \xi$. Since the graph of A is closed, it is also closed for the weak topology, so that $u(t) \in D(A)$ and $\|Au(t)\|_{L^p} \le \dfrac{C}{t}\|\varphi\|_{L^p}$. It follows easily that the mapping $t \mapsto T(t)$ is differentiable $(0,\infty) \to \mathcal{L}(L^p(\Omega))$ and that $\|tT'(t)\|_{\mathcal{L}(L^p(\Omega))} \le C$ for all $t > 0$; and so, $(T(t))_{t\ge 0}$ is analytic in $L^p(\Omega)$ by Theorem 1.3.38.

Step 3. We show formulas (1.4.9) and (1.4.10). Formula (1.4.9) follows from Steps 1 and 2 and Theorem 1.3.38. Formula (1.4.10) follows from (1.4.9) and the Gagliardo-Nirenberg inequality (A.3.10). This completes the proof. □

Remark 1.4.21. Here are some comments on the above results.

(i) It follows from Proposition 1.4.20 that if $\varphi \in L^p(\Omega)$ for some $1 < p < \infty$ and if $u(t) = T(t)\varphi$, then $\triangle^n u \in C^\infty((0,\infty), L^p(\Omega))$ for every integer $n \ge 0$. Moreover $\|\partial_t^m \triangle^n u(t)\| \le Ct^{-(m+n)}\|\varphi\|_{L^p}$ for all $t > 0$, where C is a constant depending on m and n. By applying Theorem 1.4.15, we obtain that if $\varphi \in L^p(\Omega)$ for some $1 \le p < \infty$, then $\triangle^n u \in C^\infty((0,\infty), L^q(\Omega))$, for all $p \le q < \infty$, $q > 1$. Moreover,

$$\|\partial_t^m \triangle^n u(t)\|_{L^q} \le Ct^{-(m+n)} t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\varphi\|_{L^p},$$

for all $t > 0$, where C is a constant depending on m and n.

(ii) The conclusion of Corollary 1.4.17 does not hold for $p = \infty$. To see this, consider $\Omega' \subset\subset \Omega$ and let $\varphi = 1_{\Omega'} \in L^2(\Omega) \cap L^\infty(\Omega)$. Given $t > 0$, it follows in particular from Theorem 1.4.8 that $u(t) \in C(\Omega)$. Therefore, $\|u(t) - \varphi\|_{L^\infty} \ge 1/2$; and so, $u \notin C([0,\infty), L^\infty(\Omega))$. However, we have the following L^∞ regularity result.

Proposition 1.4.22. *Let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.8, let $\varphi \in L^p(\Omega)$ for some $1 \le p < \infty$, and let $u(t) = T(t)\varphi$. Then $\triangle^m u \in C^\infty((0,\infty), L^\infty(\Omega))$, for every nonnegative integer m. Moreover,*

$$\|\partial_t^m \triangle^n u(t)\|_{L^\infty} \le Ct^{-(m+n)} t^{-\frac{N}{2p}}\|\varphi\|_{L^p},$$

for all $t > 0$, where C is a constant depending on m and n.

Proof. Note that we need only prove the continuity properties, since the estimates then follow from Remark 1.4.21 (i), by letting $q \uparrow \infty$. We proceed in two steps.

Step 1. $u \in C((0,\infty), L^\infty(\Omega))$. Let $\varepsilon > 0$ and let $t_0, t \ge \varepsilon$. We have $u(t)-u(t_0) = T(\varepsilon)(u(t-\varepsilon)-u(t_0-\varepsilon))$; and so, by Theorem 1.4.15,

$$\|u(t) - u(t_0)\|_{L^\infty} \le (4\pi\varepsilon)^{-N/2p}\|u(t-\varepsilon) - u(t_0-\varepsilon)\|_{L^p}.$$

Since $u \in C([0,\infty), L^p(\Omega))$, it follows that $u \in C([\varepsilon,\infty), L^\infty(\Omega))$. Hence the result, since ε is arbitrary.

Step 2. Conclusion. Given $\varepsilon > 0$, it follows from Remark 1.4.21 (i) that $\triangle^{m+n} u(\varepsilon) \in L^p(\Omega)$ for every nonnegative integers m, n. Since $\dfrac{d^n}{dt^n} \triangle^m u(t+\varepsilon) = T(t) \triangle^{m+n} u(\varepsilon)$ (see Corollary 1.3.18), it follows from Step 1 that $\triangle^m u \in C^n((\varepsilon, \infty), L^\infty(\Omega))$. Hence the result, since ε, m and n are arbitrary. □

Under the assumptions of Proposition 1.2.32, the operator A defined by (1.2.22) defines a semigroup of contractions in $C_0(\Omega)$. The following result shows that this semigroup coincides with the semigroup generated by the heat equation in $L^2(\Omega)$, on $L^2(\Omega) \cap C_0(\Omega)$

Proposition 1.4.23. *If $N \geq 2$, suppose that every $x \in \partial\Omega$ has the exterior cone property. Let $(T(t))_{t \geq 0}$ be as in Theorem 1.4.8 and let $(S(t))_{t \geq 0}$ be the semigroup of contractions in $C_0(\Omega)$ generated by $-A$, where A is the operator defined by (1.2.22). Then, $T(t)\varphi = S(t)\varphi$ for every $\varphi \in L^2(\Omega) \cap C_0(\Omega)$ and every $t \geq 0$.*

Proof. By density (see Proposition A.3.58), we need only show the result for $\varphi \in \mathcal{D}(\Omega)$. Set $u(t) = S(t)\varphi$ and let $T > 0$. Since $\mathcal{D}(\Omega) \subset \bigcap_{n \geq 0} D(A^n)$, it follows from Corollary 1.3.18 that $\triangle^m u \in C^\infty([0,\infty), C_0(\Omega))$ for every nonnegative integer m. In particular, given any $\Omega' \subset\subset \Omega$, we have $u_{|\Omega'} \in C^\infty([0,\infty), H^2(\Omega'))$ (see Proposition A.4.10). On the other hand, since $u \in C([0,\infty), C_0(\Omega))$, $\bigcup_{0 \leq t \leq T} \{u(t)\}$ is in a compact subset of $C_0(\Omega)$. It follows easily (apply Lemma A.3.48) that for every $\varepsilon > 0$, there exists a compact subset K of Ω such that $(u(t) - \varepsilon)^+$ is supported in K. Consider now $\Omega' \subset\subset \Omega$ such that $K \subset \Omega'$. It follows that $(u - \varepsilon)^+ \in C([0,T], H_0^1(\Omega'))$. Therefore, by Lemma A.3.63,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega'} [(u(t) - \varepsilon)^+]^2 &= 2 \int_{\Omega'} (u(t) - \varepsilon)^+ u_t = 2 \int_{\Omega'} (u(t) - \varepsilon)^+ \triangle u \\ &= -2 \int_{\Omega'} \nabla (u(t) - \varepsilon)^+ \cdot \nabla u = -2 \int_{\Omega'} |\nabla (u(t) - \varepsilon)^+|^2; \end{aligned}$$

and so, there exists a constant C such that

$$\|(u - \varepsilon)^+\|_{L^\infty(0,T;L^2(\Omega'))} + \|\nabla (u - \varepsilon)^+\|_{L^2(0,T;L^2(\Omega'))} \leq C \|(\varphi - \varepsilon)^+\|_{L^2(\Omega')}.$$

Since $(u - \varepsilon)^+$ is supported in Ω', this implies

$$\|(u - \varepsilon)^+\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla (u - \varepsilon)^+\|_{L^2(0,T;L^2(\Omega))} \leq C \|(\varphi - \varepsilon)^+\|_{L^2(\Omega)}. \tag{1.4.11}$$

Note that $(u - \varepsilon)^+ \uparrow u^+$ as $\varepsilon \downarrow 0$. Therefore, it follows from (1.4.11) and the monotone convergence theorem that $u^+(t) \in L^2(\Omega)$ and that $(u(t) - \varepsilon)^+ \underset{\varepsilon \downarrow 0}{\longrightarrow} u^+(t)$ in $L^2(\Omega)$ for every $t \in [0,T]$. Since $(u - \varepsilon)^+$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by (1.4.11), it follows from Theorem A.2.20, that $u^+ \in L^\infty(0,T;L^2(\Omega))$ and that

$$\int_0^t (u(t) - \varepsilon)^+ \theta(t)\, dt \rightharpoonup \int_0^t u(t)^+ \theta(t)\, dt,$$

in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ for every $\theta \in C_c((0,T))$. It follows also from (1.4.11) that $(u - \varepsilon)^+$ is bounded in $L^2(0,T;H_0^1(\Omega))$. By Remark A.2.19 (i), there exists a sequence $(\varepsilon_n)_{n \geq 0}$ and $v \in L^2(0,T;H_0^1(\Omega))$ such that $(u - \varepsilon_n)^+ \rightharpoonup v$ in $L^2(0,T;H_0^1(\Omega))$, from which it follows (see Lemma A.2.21) that $u^+ = v \in L^2(0,T;H_0^1(\Omega))$. Applying this result to $-u$, we obtain as well that $u^- \in L^2(0,T;H_0^1(\Omega))$; and so, $u = u^+ - u^- \in L^2(0,T;H_0^1(\Omega))$. Applying this result to $v(t) = \dfrac{d^n u}{dt^n} = S(t)\triangle^n \varphi$ for arbitrary integers n, it follows that $u \in W^{n,2}((0,T), H_0^1(\Omega))$ for any n. Therefore, $u \in C^\infty([0,T], H_0^1(\Omega))$. Since T is arbitrary, we obtain

$u \in C^\infty([0,\infty), H_0^1(\Omega))$, and since $u_t = \triangle u$ for all $t \geq 0$ and $u(0) = \varphi$, it follows from Theorem 1.3.34 that $u(t) = T(t)\varphi$. Hence the result. □

Corollary 1.4.24. *If $N \geq 2$, suppose that every $x \in \partial\Omega$ has the exterior cone property, and let $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. If $\varphi \in L^2(\Omega)$ and if $u(t) = T(t)\varphi$, then $u \in C^\infty((0,\infty), C_0(\Omega))$, and in particular $u \in C([\varepsilon,\infty) \times \overline{\Omega})$ for every $\varepsilon > 0$. If furthermore $\varphi \in C_0(\Omega)$, then $u \in C([0,\infty), C_0(\Omega))$, and in particular $u \in C([0,\infty) \times \overline{\Omega})$.*

Proof. We proceed in several steps, and we use the notation of Proposition 1.4.23.

Step 1. If $\varphi \in C_0(\Omega)$, then $u \in C([0,\infty), C_0(\Omega))$. This follows immediately from Proposition 1.4.23, since $S(t)\varphi \in C([0,\infty), C_0(\Omega))$ by construction.

Step 2. If $\varphi \in L^2(\Omega)$, then $u \in C((0,\infty), C_0(\Omega))$. Consider $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $L^2(\Omega)$, and set $u_n(t) = T(t)\varphi_n$. Given $\varepsilon > 0$, it follows from Proposition 1.4.14 that $\|u(\varepsilon) - u_n\varepsilon)\|_{L^\infty} \leq (4\pi\varepsilon)^{-N/4}\|\varphi - \varphi_n\|_{L^2}$. Since $u_n(\varepsilon) \in C_0(\Omega)$, it follows easily that $u(\varepsilon) \in C_0(\Omega)$ and the result follows from Step 1.

Step 3. Conclusion. By Step 1, it remains to show that, given $\varphi \in L^2(\Omega)$, we have $u \in C^\infty((0,\infty), C_0(\Omega))$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. It follows from Theorem 1.4.8 that $\triangle^n u(\varepsilon) \in L^2(\Omega)$. Therefore, by Step 2 and formula (1.3.6), $\dfrac{d^n u}{dt^n} \in C((\varepsilon,\infty), C_0(\Omega))$. This completes the proof, since n and ε are arbitrary. □

By Proposition 1.4.23, $(T(t))_{t\geq 0}$ can be extended to a semigroup of contractions in $C_0(\Omega)$ provided Ω is smooth enough. Set

$$M(\Omega) = C_0(\Omega)^*.$$

We now extend $(T(t))_{t\geq 0}$ by duality to $M(\Omega)$.

Theorem 1.4.25. *If $N \geq 2$, suppose that every $x \in \partial\Omega$ has the exterior cone property. Given any $\varphi \in M(\Omega)$, we define $S(t)\varphi \in M(\Omega)$ by*

$$\langle S(t)\varphi, \psi\rangle_{M(\Omega),C_0} = \langle \varphi, T(t)\psi\rangle_{M(\Omega),C_0}, \tag{1.4.12}$$

for all $\varphi \in C_0(\Omega)$. ($S(t)\varphi$ is well defined by Proposition 1.4.23.). The following properties hold.

(i) *$S(t)\varphi = T(t)\varphi$ for all $t \geq 0$, if $\varphi \in L^p(\Omega)$ for some $1 \leq p < \infty$, or if $\varphi \in C_0(\Omega)$.*

(ii) *If $u(t) = S(t)\varphi$ for all $t \geq 0$, then $u \in C((0,\infty)L^p(\Omega))$ for all $1 \leq p < \infty$ and $u \in C((0,\infty)C_0(\Omega))$. Moreover,*

$$\|u(t)\|_{L^p} \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})}\|\varphi\|_{M(\Omega)}, \tag{1.4.13}$$

for all $t > 0$ and all $1 \leq p \leq \infty$.

(iii) *If u is as above, then $u \in C^\infty((0,\infty) \times \Omega) \cap C^\infty((0,\infty), H_0^1(\Omega))$ and u satisfies the equation*

$$u_t - \triangle u = 0, \tag{1.4.14}$$

in $(0,\infty)\times\Omega$, *and*

$$\int_\Omega u(t)\varphi\psi \underset{t\downarrow 0}{\longrightarrow} \int_\Omega \varphi\psi, \tag{1.4.15}$$

for all $\psi\in C_0(\Omega)$.

(iv) *If* $v\in C((0,\infty),H^1_0(\Omega))\cap C((0,\infty),L^1(\Omega))$ *satisfies the equation (1.4.14) in* $H^{-1}(\Omega)$ *for all* $t>0$, *and satisfies the initial condition (1.4.15), then* $v(t)=S(t)\varphi$ *for all* $t\ge 0$.

(v) *If* Ω *is bounded and if* $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ *in* $M(\Omega)$ *weak-*$\star$, *then* $S(t)\varphi_n \underset{n\to\infty}{\longrightarrow} S(t)\varphi$ *for all* $t>0$, *in* $C_0(\Omega)$.

Remark 1.4.26. In view of property (i) of Theorem 1.4.25, we will denote by $(T(t))_{t\ge 0}$ the family $(S(t))_{t\ge 0}$.

Remark 1.4.27. Note that $(T(t))_{t\ge 0}$ is **not** a semigroup of contractions on $M(\Omega)$, because it does not satisfy the continuity property $S(t)\varphi \underset{t\downarrow 0}{\longrightarrow} \varphi$ in $M(\Omega)$ for all $\varphi\in M(\Omega)$. Indeed, let $\varphi\in M(\Omega)$ and suppose that $T(t)\varphi \underset{t\downarrow 0}{\longrightarrow} \varphi$ in $M(\Omega)$. Since $T(t)\varphi\in L^1(\Omega)$ for all $t>0$ and $L^1(\Omega)$ is a closed subspace of $M(\Omega)$, it follows that $\varphi\in L^1(\Omega)$ and $T(t)\varphi \underset{t\downarrow 0}{\longrightarrow} \varphi$ in $L^1(\Omega)$. Thus, if $\varphi\in M(\Omega)$ but $\varphi\notin L^1(\Omega)$, then $T(t)\varphi$ does not converge to φ in $M(\Omega)$.

Remark 1.4.28. Let $x_0\in\Omega$ and let δ_{x_0} be the Dirac mass at x_0. We have $\delta_{x_0}\in M(\Omega)$, so that $T(t)\delta_{x_0}$ is well defined.

Remark 1.4.29. Note that the boundedness assumption in property (v) is essential. Consider for example $\Omega=\mathbb{R}$ and let $\varphi_n = 1_{(n,n+1)}$. We have $\varphi_n \underset{n\to\infty}{\longrightarrow} 0$ in $M(\mathbb{R})$ weak-$\star$. On the other hand, $T(t)\varphi_n(x) = T(t)\varphi_0(x-n)$ does not have any strong limit in $L^p(\Omega)$ for any $p>1$.

Proof of Theorem 1.4.25. (i) Let $\varphi,\psi\in\mathcal{D}(\Omega)$. It follows from Remark 1.4.7 that

$$\langle S(t)\varphi,\psi\rangle_{M(\Omega),C_0} = \langle \varphi,T(t)\psi\rangle_{M(\Omega),C_0} = (\varphi,T(t)\psi)_{L^2} = (T(t)\varphi,\psi)_{L^2} = \langle T(t)\varphi,\psi\rangle_{M(\Omega),C_0}.$$

(i) now follows by density and the estimate of Theorem 1.4.15.

(ii) Let $\varphi\in M(\Omega)$ and $\psi\in\mathcal{D}(\Omega)$. We have by (1.4.12) and Theorem 1.4.15

$$|\langle T(t)\varphi,\psi\rangle_{M(\Omega),C_0}| \le \|\varphi\|_{M(\Omega)}\|T(t)\psi\|_{L^\infty} \le (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})}\|\varphi\|_{M(\Omega)}\|\psi\|_{L^{p'}}, \tag{1.4.16}$$

for all $t>0$ and all $1\le p\le\infty$. Let $p>11$. We deduce from (1.4.16) that $S(t)\varphi\in L^p(\Omega)$ for all $t>0$ and that estimate (1.4.13) holds for all $p>1$. Letting now $p\downarrow 1$ we deduce that $S(t)\varphi\in L^1(\Omega)$ for all $t>0$ and that (1.4.13) holds for $p=1$. The regularity properties of (ii) now follow from Corollaries 1.4.17 and 1.4.24.

(iii) The regularity of u and the equation (1.4.14) follow from (ii) and Theorem 1.4.8. Let $\psi\in C_0(\Omega)$. We have

$$\int_\Omega u(t)\psi = \langle S(t)\varphi,\psi\rangle_{M(\Omega),C_0} = \langle\varphi,T(t)\psi\rangle_{M(\Omega),C_0} \underset{t\downarrow 0}{\longrightarrow} \langle\varphi,\psi\rangle_{M(\Omega),C_0},$$

since $T(t)\psi \underset{t\downarrow 0}{\longrightarrow} \psi$ in $C_0(\Omega)$. This proves (1.4.15)

(iv) Let v be as in (iv). Given $0<\varepsilon<t<\infty$, we have by Theorem 1.4.8 $v(t)=T(t-\varepsilon)v(\varepsilon)$; and so,

$$\langle v(t),\psi\rangle_{M(\Omega),C_0} = \langle v(\varepsilon),T(t-\varepsilon)\psi\rangle_{M(\Omega),C_0},$$

for every $\psi \in C_0(\Omega)$. We let $\varepsilon \downarrow 0$, and we observe that by (1.4.15) $v(\varepsilon) \underset{\varepsilon\downarrow 0}{\longrightarrow} \varphi$ in $M(\Omega)$ weak-$\star$ and that $T(t-\varepsilon)\psi \underset{\varepsilon\downarrow 0}{\longrightarrow} T(t)\psi$ in $C_0(\Omega)$. Therefore,

$$\langle v(t), \psi\rangle_{M(\Omega),C_0} = \langle \varphi, T(t)\psi\rangle_{M(\Omega),C_0} = \langle S(t)\varphi, \psi\rangle_{M(\Omega),C_0},$$

which implies that $u(t) = S(t)\varphi$.

(v) Suppose $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ in $M(\Omega)$ weak-$\star$. It follows from (1.4.12) that $S(t)\varphi_n \underset{n\to\infty}{\longrightarrow} S(t)\varphi$ in $M(\Omega)$ weak-$\star$, and in particular in $\mathcal{D}'(\Omega)$. Note that $\sup_{n\geq 0} \|\varphi_n\|_{M(\Omega)} < \infty$. Therefore, it follows from (1.4.13) and Theorem 1.4.11 that $(S(t)\varphi_n)_{n\geq 0}$ belongs to a bounded subset of $H^1_0(\Omega)$, hence to a compact subset of $L^2(\Omega)$. In particular, $S(t)\varphi_n \underset{n\to\infty}{\longrightarrow} S(t)\varphi$ in $L^2(\Omega)$ for every $t > 0$. It follows from Theorem 1.4.15 that $S(t)\varphi_n = S(t/2)S(t/2)\varphi_n \underset{n\to\infty}{\longrightarrow} S(t/2)S(t/2)\varphi = S(t)\varphi$ in $L^\infty(\Omega)$. □

We describe below some decay properties as $t \to \infty$ of $(T(t))_{t\geq 0}$.

Theorem 1.4.30. *Assume that $|\Omega| < \infty$, let $\lambda_1 > 0$ be defined by (1.2.28), and let $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. Then, $\|T(t)\|_{\mathcal{L}(L^2)} \leq e^{-\lambda_1 t}$ for all $t \geq 0$.*

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ and set $u(t) = T(t)\varphi$. It follows from Theorem 1.4.8 and Corollary A.3.54 that

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\langle u(t), u_t(t)\rangle_{H^1_0,H^{-1}} = 2\langle u(t), \triangle u(t)\rangle_{H^1_0,H^{-1}} = -2\|\nabla u(t)\|^2_{L^2};$$

and so, by (1.2.28) and Proposition A.4.34,

$$\frac{d}{dt}\|u(t)\|^2_{L^2} \leq -2\lambda_1\|u(t)\|^2_{L^2}.$$

Integrating the above differential inequality, we obtain $\|u(t)\|^2_{L^2} \leq e^{-2\lambda_1 t}\|\varphi\|^2_{L^2}$. The result now follows by density. □

Remark 1.4.31. Instead of assuming $|\Omega| < \infty$ in Theorem 1.4.30, one may assume as well that Ω is bounded in one direction, or more generally that Ω verifies the assumptions of Remark A.3.38 (i).

Corollary 1.4.32. *Assume that $|\Omega| < \infty$, let $\lambda_1 > 0$ be defined by (1.2.28), and let $(T(t))_{t\geq 0}$ be as in Theorem 1.4.8. For every $1 \leq p \leq \infty$, there exists a constant C_p such that $\|T(t)\varphi\|_{L^p} \leq C_p e^{-\lambda_1 t}\|\varphi\|_{L^p}$, for every $t \geq 0$ and every $\varphi \in L^p(\Omega)$. In addition, one can take $C = e^{\frac{\lambda_1|\Omega|^{2/N}}{4\pi}}$.*

Proof. Suppose first that $p < \infty$. By density, we need only show the result for $\varphi \in \mathcal{D}(\Omega)$. Consider $t_0 > 0$. For $t \leq t_0$, it follows from that

$$\|T(t)\varphi\|_{L^p} \leq \|\varphi\|_{L^p} = e^{\lambda_1 t_0}e^{-\lambda_1 t_0}\|\varphi\|_{L^p} \leq e^{\lambda_1 t_0}e^{-\lambda_1 t}\|\varphi\|_{L^p}.$$

Next, if $p > 2$, it follows from Theorems 1.4.15 and 1.4.30 that

$$\begin{aligned}\|T(t)\varphi\|_{L^p} &\leq (4\pi t_0)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\|T(t-t_0)\varphi\|_{L^2} \\ &\leq (4\pi t_0)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}e^{\lambda_1 t_0}e^{-\lambda_1 t}\|\varphi\|_{L^2} \\ &\leq \Big((4\pi t_0)^{-N/2}|\Omega|\Big)^{\frac{1}{2}-\frac{1}{p}} e^{\lambda_1 t_0}e^{-\lambda_1 t}\|\varphi\|_{L^p}.\end{aligned}$$

If $p < 2$, one obtains as well

$$\begin{aligned}\|T(t)\varphi\|_{L^p} &\le |\Omega|^{-\left(\frac{1}{2}-\frac{1}{p}\right)}\|T(t)\varphi\|_{L^2}\\ &\le |\Omega|^{-\left(\frac{1}{2}-\frac{1}{p}\right)}e^{\lambda_1 t_0}e^{-\lambda_1 t}\|T(t_0)\varphi\|_{L^2}\\ &\le \Big((4\pi t_0)^{-N/2}|\Omega|\Big)^{-\left(\frac{1}{2}-\frac{1}{p}\right)} e^{\lambda_1 t_0}e^{-\lambda_1 t}\|\varphi\|_{L^p},\end{aligned}$$

and the result follows by taking $t_0 = \dfrac{\lambda_1|\Omega|^{2/N}}{4\pi}$. For $p = \infty$, apply the result for finite p then make $p \uparrow \infty$. (Note that, for example, $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, so that $T(t)\varphi$ is well defined for $\varphi \in L^\infty(\Omega)$.) □

Remark 1.4.33. Note that one can never take $C_p = 1$ if $p \neq 2$ (see Cazenave and Haraux [29], Corollaire 3.5.10 and the remark that follows).

When Ω is bounded, one can express the solution of the heat equation in terms of the decomposition of the initial value on the basis of $L^2(\Omega)$ of the eigenvectors of $-\triangle$ in $H_0^1(\Omega)$. More precisely, we have the following result.

Proposition 1.4.34. *Let $(\lambda_n)_{n\ge1}$ be the family of eigenvalues of $-\triangle$ in $H_0^1(\Omega)$, let $(\varphi_n)_{n\ge1}$ be a Hilbert basis of $L^2(\Omega)$ made of eigenvectors (see Section A.4.5), and let $(T(t))_{t\ge0}$ be as in Theorem 1.4.8. Given $\varphi \in L^2(\Omega)$, set $a_n = (\varphi, \varphi_n)_{L^2}$ for all $n \ge 1$, so that $\varphi = \sum_{n=1}^{\infty} a_n\varphi_n$, and let $u(t) = T(t)\varphi$. Then, $u(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\varphi_n$, for all $t \ge 0$.*

Proof. Consider an integer $k \ge 1$. Given a family $(a_n)_{1\le n\le k}$, let $\varphi = \sum_{n=1}^{k} a_n\varphi_n$ and set

$$u(t) = \sum_{n=1}^{k} a_n e^{-\lambda_n t}\varphi_n,$$

for $t \ge 0$. Since $(\varphi_n)_{n\ge1} \subset H_0^1(\Omega)$, it follows that $u \in C^\infty([0,\infty), H_0^1(\Omega))$ and that $u(0) = \varphi$. Furthermore,

$$\frac{du}{dt} = -\sum_{n=1}^{k} a_n e^{-\lambda_n t}\lambda_n\varphi_n = \sum_{n=1}^{k} a_n e^{-\lambda_n t}\triangle\varphi_n = \triangle u;$$

and so, $u(t) = T(t)\varphi$ by Theorem 1.4.8. The result follows easily, since the set

$$\bigcup_{k\ge1}\left\{\sum_{n=1}^{k} a_n\varphi_n;\ (a_n)_{1\le n\le k} \subset \mathbb{R}^k\right\}$$

is dense in $L^2(\Omega)$. □

Remark 1.4.35. Note that the results of this section are true as well in the corresponding spaces of complex-valued functions, as follows easily by considering $\mathrm{Re}\,u$ and $\mathrm{Im}\,u$. The only exception is Theorem 1.4.12 that does not make sense anymore. (See Section A.4.6 and Remark 1.2.33.)

Remark 1.4.36. We can apply Proposition 1.3.13 to show that if Ω and φ have some symmetry properties, then $T(t)\varphi$ has the same properties. For example, assume that $\Omega \subset \mathbb{R}^N$ is symmetric with respect to the hyperplane $\{x_N = 0\}$. Let ω be the corresponding symmetry, i.e. $\omega x = (x_1, \dots, x_{N-1}, -x_N)$. Define the operator $L \in \mathcal{L}(L^2(\Omega))$ by $Lu(x) = u(x) - u(\omega x)$. Considering the operator A on $L^2(\Omega)$ defined by $D(A) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}$ and $Au = \triangle u$ for $u \in D(A)$, straightforward calculations show that A and L fulfill the assumptions of Proposition 1.3.13. Therefore, if $\varphi(x_1, \dots, x_N) = \varphi(x_1, \dots, x_{N-1}, -x_N)$ almost everywhere, then $u(t) = T(t)\varphi$ verifies $u(t, x_1, \dots, x_N) = u(t, x_1, \dots, x_{N-1}, -x_N)$ almost everywhere for all $t \geq 0$. By considering $Lu(x) = u(x) + u(\omega x)$, one shows as well that if $\varphi(x_1, \dots, x_N) = -\varphi(x_1, \dots, x_{N-1}, -x_N)$ almost everywhere, then $u(t) = T(t)\varphi$ verifies $u(t, x_1, \dots, x_N) = -u(t, x_1, \dots, x_{N-1}, -x_N)$ almost everywhere for all $t \geq 0$. By considering the rotations, one shows that if Ω is a ball centered at the origin and if φ is spherically symmetric, then $u(t)$ is also spherically symmetric.

1.4.3. Schrödinger's equation. Throughout this section, Ω is an arbitrary open subset of $\mathbb{R}^N$. For some of the results, we will make further assumptions on Ω which we will specify. We will apply the results of Section 1.3 to the examples of Section 1.2.3, in order to solve the initial value problem for the Schrödinger equation $iu_t + \triangle u = 0$.

Consider the operator A defined on $H^{-1}(\Omega)$ by (1.2.23), that is

$$\begin{cases} D(A) = H^1_0(\Omega), \\ Au = -i\triangle u, \text{ for all } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.34 that A is skew adjoint. Therefore, $-A$ is the generator of a group of isometries on $H^{-1}(\Omega)$, which we denote by $(T(t))_{t\in\mathbb{R}}$. On the other hand, the operator B defined on $L^2(\Omega)$ by (1.2.24), that is

$$\begin{cases} D(B) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}, \\ Bu = -i\triangle u, \text{ for all } u \in D(B), \end{cases}$$

is skew adjoint by Proposition 1.2.35. Therefore,$-B$ is the generator of a group of isometries on $L^2(\Omega)$, which we denote by $(S(t))_{t\in\mathbb{R}}$.

Lemma 1.4.37. *With the above notation, $T(t)\varphi = S(t)\varphi$ for all $t \in \mathbb{R}$ and all $\varphi \in L^2(\Omega)$.*

Proof. Since $G(B) \subset G(A)$ as subsets of $H^{-1}(\Omega) \times H^{-1}(\Omega)$, the result follows from Proposition 1.3.4(see the proof of Lemma 1.4.5). □

Remark 1.4.38. Since $(T(t))_{t\in\mathbb{R}}$ and $(S(t))_{t\in\mathbb{R}}$ coincide where they are both defined, that is on $L^2(\Omega)$, we will by denote also by $(T(t))_{t\in\mathbb{R}}$ the group of isometries generated by $-B$.

Theorem 1.4.39. *Let A and $(T(t))_{t\in\mathbb{R}}$ be as above. Given $\varphi \in H^1_0(\Omega)$, set $u(t) = T(t)\varphi$ for $t \in \mathbb{R}$. Then, the following properties hold:*

(i) *$u \in C(\mathbb{R}, H^1_0(\Omega)) \cap C^1(\mathbb{R}, H^{-1}(\Omega))$ and u is the unique solution of problem*

$$\begin{cases} iu_t + \triangle u = 0 \text{ for all } t \in \mathbb{R}, \\ u(0) = \varphi; \end{cases} \tag{1.4.17}$$

in that class; moreover, $\|u(t)\|_{H^{-1}} = \|\varphi\|_{H^{-1}}$, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ and $\|\nabla u(t)\|_{L^2} = \|\nabla\varphi\|_{L^2}$ for every $t \in \mathbb{R}$;

(ii) *if furthermore* $\triangle\varphi \in L^2(\Omega)$, *then* $u \in C^1(\mathbb{R}, L^2(\Omega))$ *and* $\triangle u \in C(\mathbb{R}, L^2(\Omega))$; *moreover,* $\|\triangle u(t)\|_{L^2} = \|\triangle\varphi\|_{L^2}$ *for every* $t \in \mathbb{R}$;

(iii) *if furthermore there exists a positive integer* m *such that* $\triangle^j\varphi \in H^1_0(\Omega)$ *for all* $0 \le j \le m-1$ *and* $\triangle^m\varphi \in L^2(\Omega)$, *then* $\triangle^j u \in C^{m-j}(\mathbb{R}, L^2(\Omega))$ *for all* $0 \le j \le m$; *moreover,* $\|\triangle^j u(t)\|_{L^2} = \|\triangle^j\varphi\|_{L^2}$ *for every* $t \in \mathbb{R}$ *and every* $1 \le j \le m$.

Proof. Since $D(A) = H^1_0(\Omega)$ with equivalent norms (Proposition 1.2.17), the first part of property (i) follows from Corollary 1.3.28. Since $(T(t))_{t\in\mathbb{R}}$ is a group of isometries in both $H^{-1}(\Omega)$ and $L^2(\Omega)$, conservation of the H^{-1} and L^2 norms follow. On the other hand, it follows from Proposition 1.3.4 that $T(t)A\varphi = AT(t)\varphi$. Therefore, $T(t)\triangle\varphi = \triangle u(t)$; and so, Therefore, $T(t)(-\triangle\varphi + \varphi) = -\triangle u(t) + u(t)$. It follows that $\|-\triangle u(t) + u(t)\|_{H^{-1}} = \|-\triangle\varphi + \varphi\|_{H^{-1}}$. By property (iii) of Remark A.4.4, this implies that $\|u(t)\|_{H^1_0} = \|\varphi\|_{H^1_0}$. Since $\|u(t)\|_{L^2}$ is conserved, conservation of $\|\nabla u(t)\|_{L^2}$ follows, which completes the proof of (i). Property (ii) follows from Lemma 1.4.37 and Corollary 1.3.28 applied to the operator B. Finally, property (iii) follows from Remarks 1.1.30 and 1.2.23 (ii), and Corollary 1.3.18. □

Remark 1.4.40. One can apply Corollary 1.3.20 (or even Corollary 1.3.22) to obtain existence and uniqueness of solutions to (1.4.11) when $\varphi \in L^2(\Omega)$ or $\varphi \in H^{-1}(\Omega)$ (or even in larger spaces). However, the spaces in which lies the solution may be complicated. For example, if $\varphi \in L^2(\Omega)$, the solution is unique in the class $C(\mathbb{R}, L^2(\Omega)) \cap C^1(\mathbb{R}, Y)$ where $Y = (D(B))^\star$ in the duality $D(B) \hookrightarrow L^2(\Omega) \hookrightarrow (D(B))^\star$. Note that $\triangle$ defines a continuous mapping $L^2(\Omega) \to Y$ by $\langle \triangle u, v\rangle = \mathrm{Re}\int_\Omega u\triangle\overline{v}\,dx$, for all $u \in L^2(\Omega)$ and all $v \in D(B)$.

When Ω satisfies certain regularity assumptions, we have better regularity properties. Some of these properties are described in the following result.

Theorem 1.4.41. *Let* A *and* $(T(t))_{t\in\mathbb{R}}$ *be as in Theorem 1.4.39. Given* $\varphi \in H^{-1}(\Omega)$, *set* $u(t) = T(t)\varphi$ *for* $t \in \mathbb{R}$. *Then, the following properties hold:*

(i) *if* Ω *has a bounded boundary of class* C^2 *and if* $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$, *then* $u \in C(\mathbb{R}, H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$;

(ii) *if* Ω *is bounded with boundary of class* C^∞, *and if* $\varphi \in C^\infty(\overline{\Omega})$ *satisfies the compatibility relations* $u = \triangle u = \cdots \triangle^n u = \cdots = 0$ *on* $\partial\Omega$, *then* $u \in C^\infty(\mathbb{R} \times \overline{\Omega})$.

Proof. If $\varphi \in D(B)$, then it follows from Lemma 1.4.5 that $u \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(\Omega))$. Since $D(B) \hookrightarrow H^2(\Omega)$ whenever Ω has a bounded boundary of class C^2 (Remark 1.2.36), property (i) follows. Finally, if $\varphi \in \bigcap_{n\ge 0} D(B^n)$, then it follows from Corollary 1.3.17 that $u \in \bigcap_{n\ge 0} C^\infty([0,\infty), D(B^n))$. Hence property (ii), by applying Remark 1.2.23 and Sobolev's embedding theorem. □

Remark 1.4.42. Note that the compatibility relations of property (ii) of Theorem 1.4.41 are necessary conditions if $u \in C^\infty([0,\infty) \times \overline{\Omega})$. Indeed, we have $u = \frac{du}{dt} = \cdots = \frac{d^n u}{dt^n} = \cdots = 0$ on $(0,\infty) \times \partial\Omega$. Since $\frac{d^n u}{dt^n} = i^n\triangle^n u$, the compatibility relations follow.

Remark 1.4.43. When Ω is bounded, one can express the solution of Schrödinger's equation in terms of the decomposition of the initial value on the basis of $L^2(\Omega)$ made of the eigenvectors of $-\triangle$ in $H_0^1(\Omega)$. More precisely, let $(\lambda_n)_{n\ge 1}$ be the family of eigenvalues of $-\triangle$ in $H_0^1(\Omega)$, let $(\varphi_n)_{n\ge 1}$ be a Hilbert basis of $L^2(\Omega)$ made of eigenvectors (see Section A.4.5), and let $(T(t))_{t\ge 0}$ be as in Theorem 1.4.39. Given $\varphi \in L^2(\Omega)$, set $a_n = (\varphi, \varphi_n)_{L^2}$ for all $n \ge 1$, so that $\varphi = \sum_{n=1}^{\infty} a_n \varphi_n$, and let $u(t) = T(t)\varphi$. Then, $u(t) = \sum_{n=1}^{\infty} a_n e^{-i\lambda_n t} \varphi_n$, for all $t \ge 0$. See the proof of Proposition 1.4.34.

Remark 1.4.44. We can apply Proposition 1.3.13 to show that if Ω and φ have some symmetry properties, then $T(t)\varphi$ has the same properties (see Remark 1.4.36).

1.4.4. Schrödinger's equation in $\mathbb{R}^N$.

We devote a section to Schrödinger's equation in the special case $\Omega = \mathbb{R}^N$, since in this case many more properties of the equation are known.

Lemma 1.4.45. *Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$. If $u \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^N))$ is defined by*

$$\widehat{u(t)}(\xi) = \widehat{\varphi}(\xi) e^{-4\pi^2 it|\xi|^2}, \text{ for all } \xi \in \mathbb{R}^N,$$

then

$$u(t) = S_t \star \varphi, \text{ for all } t \neq 0,$$

with

$$S_t(x) = (4\pi it)^{-N/2} e^{\frac{i|x|^2}{4t}},$$

where we set $i^{-N/2} = e^{-\frac{iN\pi}{8}}$.

The proof relies on the following lemma.

Lemma 1.4.46. *Let $z \in \mathbb{C}$ verify $\mathrm{Re} z > 0$. If $\rho \in \mathcal{S}(\mathbb{R}^N)$ is defined by*

$$\widehat{\rho}(\xi) = e^{-4\pi^2 z|\xi|^2}, \text{ for } \xi \in \mathbb{R}^N,$$

then

$$\rho(x) = (4\pi z)^{-N/2} e^{-\frac{|x|^2}{4z}}, \text{ for } x \in \mathbb{R}^N,$$

where we set $z^{-N/2} = |z|^{-N/2} e^{-\frac{iN\theta}{2}}$, if $z = |z|e^{i\theta}$ with $-\pi/2 < \theta < \pi/2$.

Proof. Let

$$\varphi(\theta) = \int_{\mathbb{R}^N} exp(-\pi e^{i\theta}|\xi|^2)\, d\xi, \text{ for } -\pi/2 < \theta < \pi/2.$$

Integration over $\mathbb{R}^N$ of identity $\nabla \cdot \big(exp(-\pi e^{i\theta}|\xi|^2)\xi\big) = exp(-\pi e^{i\theta}|\xi|^2)[N - 2\pi e^{i\theta}|\xi|^2]$ shows that φ solves the differential equation $\varphi'(\theta) = -\dfrac{Ni}{2}\varphi(\theta)$; and so, $\varphi(\theta) = e^{-\frac{iN\theta}{2}}\varphi(0)$. Since it is well known that $\varphi(0) = 1$, we find $\varphi(\theta) = e^{-\frac{iN\theta}{2}}$. Therefore, if we set $z = |z|e^{i\theta}$ with $-\pi/2 < \theta < \pi/2$, then

$$\int_{\mathbb{R}^N} e^{-4\pi^2 z|\xi|^2}\, d\xi = (4\pi|z|)^{-N/2}\varphi(\theta) = (4\pi z)^{-N/2}. \tag{1.4.18}$$

Next, note that

$$\rho(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi} e^{-4\pi^2 z|\xi|^2}\, d\xi.$$

Integrating over $\mathbb{R}^N$ identity

$$\nabla \cdot (e^{2\pi i x\cdot\xi} e^{-4\pi^2 z|\xi|^2} x) = 4\pi i z \left(\frac{|x|^2}{2z} + 2\pi i x \cdot \xi \right) e^{2\pi i x\cdot\xi} e^{-4\pi^2 z|\xi|^2},$$

where the divergence is with respect to ξ, it follows that

$$x \cdot \nabla \rho(x) = -\frac{|x|^2}{2z} \rho(x).$$

Therefore, if we fix $x \in \mathbb{R}^N$ and if we set

$$f(s) = \rho(sx), \text{ for } s \ge 0,$$

then f solves the ordinary differential equation $f'(s) = -\frac{|x|^2}{2z} s f(s)$; and so,

$$\rho(x) = f(1) = e^{-\frac{|x|^2}{4z}} f(0) = e^{-\frac{|x|^2}{4z}} \int_{\mathbb{R}^N} e^{-4\pi^2 z|\xi|^2}\, d\xi.$$

The result now follows from (1.4.18). □

Proof of Lemma 1.4.45. It is clear that the function $t \mapsto \widehat{u(t)}$ is continuous $\mathbb{R} \to \mathcal{S}(\mathbb{R}^N)$; and so, $u \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^N))$. Consider $t \neq 0$. Given $\varepsilon > 0$, define u_ε by

$$\widehat{u_\varepsilon(t)}(\xi) = \widehat{\varphi}(\xi) e^{-4\pi^2 (i+\varepsilon) t |\xi|^2}, \text{ for all } \xi \in \mathbb{R}^N.$$

It is also clear that $u_\varepsilon \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^N))$. It follows from the dominated convergence theorem that $\widehat{u_\varepsilon(t)}$ converges to $\widehat{u(t)}$ in $L^2(\mathbb{R}^N)$, as $\varepsilon \downarrow 0$; and so, $u_\varepsilon(t)$ converges to $u(t)$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$. In particular, there exists a sequence $\varepsilon_n \downarrow 0$ such that $u_{\varepsilon_n}(t)$ converges to $u(t)$ almost everywhere in $\mathbb{R}^N$, as $n \to \infty$. Furthermore,

$$u_\varepsilon(t) = \mathcal{F}^{-1}\left(\widehat{\varphi}(\cdot) e^{-4\pi^2 (i+\varepsilon) t |\cdot|^2} \right) = \varphi \star \mathcal{F}^{-1}\left(e^{-4\pi^2 (i+\varepsilon) t |\cdot|^2} \right);$$

and so, by Lemma 1.4.46, $u_\varepsilon(t) = K_\varepsilon(t) \star \varphi$, where $K_\varepsilon(t) = (4\pi(i+\varepsilon))^{-N/2} e^{-\frac{|x|^2}{4(i+\varepsilon)}}$. Therefore, by the dominated convergence theorem, $u_\varepsilon(t)$ converges pointwise to $S_t \star \varphi$, as $\varepsilon \downarrow 0$. Hence the result. □

Corollary 1.4.47. *If $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and if $u(t) = T(t)\varphi$ for all $t \in \mathbb{R}$, then*

$$u(t) = S_t \star \varphi, \text{ for all } t \neq 0,$$

with S_t as in Lemma 1.4.45.

Proof. Set $v(t) = \widehat{u}(t)$, for all $t \in \mathbb{R}$. We have $v \in C(\mathbb{R}; L^2(\mathbb{R}^N))$. Furthermore, since $i u_t + \triangle u = 0$, it follows that

$$i v_t(t,\xi) - 4\pi^2 v(t,\xi) = 0.$$

Integration of the above differential equation in t for every $\xi \in \mathbb{R}^N$ yields

$$\widehat{u}(t,\xi) = v(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widetilde{\varphi}. \tag{1.4.19}$$

The result now follows from Lemma 1.4.45. □

Remark 1.4.48. It follows immediately from formula (1.4.19) that for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we have $T(\cdot)\varphi \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}^N))$. By duality, $T(t)$ can be extended to $\mathcal{S}'(\mathbb{R}^N)$, and $T(\cdot)\varphi \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^N))$ for every $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. Furthermore, if $\varphi \in H^s(\mathbb{R}^N)$ for some $s \in \mathbb{R}$, then $u(t) = T(t)\varphi$ verifies $u \in \bigcap\limits_{j=0}^{\infty} C^j(\mathbb{R}, H^{s-2j}(\mathbb{R}^N))$, as follows immediately from formula (1.4.19) and the definition of the Sobolev spaces $H^s(\mathbb{R}^N)$ by the Fourier transform.

Corollary 1.4.49. *For every $t \neq 0$, define the dilation operator D_t by $D_t u(x) = (4\pi t)^{-N/2} u\left(\dfrac{x}{4\pi t}\right)$ and the multiplier M_t by $M_t(x) = e^{i\frac{|x|^2}{4t}}$. Then,*

$$T(t)\varphi = i^{-N/2} M_t D_t \left(\mathcal{F}(M_t\varphi)\right),$$

for all $t \neq 0$ and all $\varphi \in L^2(\mathbb{R}^N)$.

Proof. By density, we need only establish the result for $\varphi \in \mathcal{S}(\mathbb{R}^N)$. In this case, it follow from Corollary 1.4.47 that

$$\begin{aligned} T(t)\varphi(x) &= (4\pi i t)^{-N/2} \int_{\mathbb{R}^N} e^{i\frac{|x-y|^2}{4t}} \varphi(y)\, dy \\ &= i^{-N/2}(4\pi t)^{-N/2} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^N} e^{-2\pi i\left(\frac{x}{4\pi t}\right)\cdot y} e^{i\frac{|y|^2}{4t}} \varphi(y)\, dy. \end{aligned}$$

Hence the result. □

The above formulation is the basic step for establishing the following fundamental estimate for Schrödinger's equation in $\mathbb{R}^N$:

Theorem 1.4.50. *Let $2 \le p \le \infty$. If $\varphi \in L^2(\mathbb{R}^N) \cap L^{p'}(\mathbb{R}^N)$ and $u(t) = T(t)\varphi$, then $u \in C(\mathbb{R} \setminus \{0\}, L^p(\mathbb{R}^N))$. Moreover,*

$$\|u(t)\|_{L^p} \le (4\pi|t|)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)} \|\varphi\|_{L^{p'}}, \tag{1.4.20}$$

for all $t \neq 0$.

Proof. It follows from Corollary 1.4.49 that

$$\|u(t)\|_{L^p} = \|D_t(\mathcal{F}M_t\varphi)\|_{L^p}.$$

An easy calculation shows that $\|D_t u\|_{L^p} = (4\pi t)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_{L^p}$; and so,

$$\|u(t)\|_{L^p} = (4\pi t)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)} \|\mathcal{F}M_t\varphi\|_{L^p}.$$

Therefore,

$$\|u(t)\|_{L^p} \le (4\pi t)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)} \|M_t\varphi\|_{L^{p'}} = (4\pi t)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)} \|\varphi\|_{L^{p'}},$$

which proves estimate (1.4.20). To prove continuity, consider $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{S}(\mathbb{R}^N)$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N) \cap L^{p'}(\mathbb{R}^N)$ as $n \to \infty$. We have $T(\cdot)\varphi_n \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^N))$; and so, $T(\cdot)\varphi_n \in C(\mathbb{R}; L^p(\mathbb{R}^N))$. Furthermore, it follows from (1.4.20) applied to $\varphi_n - \varphi$ that $T(\cdot)\varphi_n - u(\cdot)$ converges to 0 in $C((-\infty, -\varepsilon] \cup [\varepsilon, \infty); L^p(\mathbb{R}^N))$ for every $\varepsilon > 0$. Hence the result. □

Lemma 1.4.51. *Consider $1 \le p, q \le \infty$. If $\mathcal{F} \in \mathcal{L}(L^p(\mathbb{R}^N), L^q(\mathbb{R}^N))$, then $2 \le q \le \infty$ and $p = q'$.*

Proof. Given $z \in \mathbb{C}$ such that $\mathrm{Re} z > 0$, let $\rho_z(x) = (4\pi z)^{-N/2} e^{-\frac{|x|^2}{4z}}$. It follows from Lemma 1.4.46 that $\widehat{\rho_z}(\xi) = e^{-4\pi^2 z|\xi|^2}$. Therefore, if $z = a + ib$ with $a > 0$, then $|\rho_z(x)| = \Big(4\pi(a^2+b^2)^{1/2}\Big)^{-N/2} e^{-\frac{a|x|^2}{4(a^2+b^2)}}$ and $|\widehat{\rho_z}(\xi)| = e^{-4\pi^2 a|\xi|^2}$. An easy calculation shows that

$$\frac{\|\widehat{\rho_z}\|_{L^q}}{\|\rho_z\|_{L^p}} = (q^{-1/q} p^{1/p})^{N/2} (4\pi a)^{-\frac{N}{2}(\frac{1}{p}+\frac{1}{q}-1)} \left(\frac{a^2}{a^2+b^2}\right)^{\frac{N}{2}(\frac{1}{2}-\frac{1}{p})}. \tag{1.4.21}$$

Note that if (i) holds, then the right-hand side of (1.4.21) must remain bounded independently of $a > 0$ and $b \in \mathbb{R}$. First fix $a > 0$ and make $b \to \infty$. If $\dfrac{1}{2} - \dfrac{1}{p} > 0$, then the right-hand side of (1.4.21) goes to ∞, which is a contradiction. Therefore, $p \le 2$. Take now $b = 0$. If $\dfrac{1}{p} + \dfrac{1}{q} \ne 1$, then the right-hand side of (1.4.21) goes to ∞ as $a \uparrow \infty$ or as $a \downarrow 0$. Therefore, $p = q'$, which completes the proof. □

Remark 1.4.52. It follows from Theorem 1.4.50 that for every $t \ne 0$, one can extend by continuity $T(t)$ to an operator of $\mathcal{L}(L^{p'}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for every $2 \le p \le \infty$. It follows easily from Proposition 1.4.47 and Lemma 1.4.51 that if p, q are such that $T(t) \in \mathcal{L}(L^q(\mathbb{R}^N), L^p(\mathbb{R}^N))$ for some $t \ne 0$, then one must have $2 \le p \le \infty$ and $q = p'$. This is a major difference between Schrödinger's equation and the heat equation (c.f. Theorem 1.4.15). Furthermore, note that estimate (1.4.20) does not hold in a bounded domain (if $p > 2$). Indeed, if $T(t) \in \mathcal{L}(L^{p'}(\Omega), L^p(\Omega))$ with $|\Omega| < \infty$, then in particular $T(t) \in \mathcal{L}(L^{p'}(\Omega), L^2(\Omega))$. It follows that $I = T(-t)T(t) \in \mathcal{L}(L^{p'}(\Omega), L^2(\Omega))$, which is absurd.

The following result, which is known as Strichartz estimate (see Strichartz [93]), is a consequence of estimate (1.4.20). The proof, which makes use of estimates for the nonhomogeneous problem, is given in Section 1.6 below. (We do not give the original proof of [93], but we follow the much simpler proof of Ginibre and Velo [56].) Before stating the estimate, we make the following definition.

Definition 1.4.53. *We say that a pair (q, r) of real numbers is admissible if the following holds:*

(i) $2 \le r < \dfrac{2N}{N-2}$ *($2 \le r \le \infty$ if $N = 1$, $2 \le r \le \infty$ if $N = 2$);*

(ii) $\dfrac{2}{q} = N\left(\dfrac{1}{2} - \dfrac{1}{r}\right)$ *(and so, $2 \le q \le \infty$).*

In particular, the pair $(\infty, 2)$ is always admissible.

Theorem 1.4.54. *For every $\varphi \in L^2(\mathbb{R}^n)$ and for every admissible pair (q, r), the function $t \mapsto T(t)\varphi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^N)) \cap C(\mathbb{R}, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C, depending only on q such that*

$$\|T(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r)} \le C\|\varphi\|_{L^2},$$

for every $\varphi \in L^2(\mathbb{R}^N)$.

Remark 1.4.55. Theorem 1.4.54 describes a quite remarkable smoothing effect. Indeed, for all $t \in \mathbb{R}$, $T(t)L^2 = L^2$. In particular, given $t \neq 0$ and $p \in (2, \frac{2N}{N-2})$, there exists a dense subset E_p of L^2 such that $T(t)\varphi \notin L^p$, for every $\varphi \in E_p$. However, it follows from Theorem 1.4.54 that for every $\varphi \in L^2$, $T(t)\varphi \in L^p$, for almost all $t \in \mathbb{R}$. Note that by the preceding observation, the restriction "for almost all $t \in \mathbb{R}$" cannot be reduced to "for all $t \neq 0$" in general. Note also that by considering a sequence $(p_n)_{n\geq 0} \subset [2, \frac{2n}{n-2})$ such that $p_n \to \frac{2N}{N-2}$ as $n \to \infty$, it follows that given $\varphi \in L^2$, there exists a set $N_\varphi \subset \mathbb{R}$ of measure 0 such that for every $t \in \mathbb{R} \setminus N_\varphi$, one has $T(t)\varphi \in L^p$ for every $p \in [2, \frac{2N}{N-2})$ ($p \in [2, \infty]$, if $N = 1$).

Remark 1.4.56. We do not know if the estimate of Theorem 1.4.54 holds in the limiting case $r = \frac{2N}{N-2}$, $q = 2$. However, a similar estimate holds with the space and time integration reversed. More precisely, we have

$$\left(\int_{\mathbb{R}^N} \left(\int_{-\infty}^{+\infty} |u(t,x)|^2 \, dt \right)^{\frac{N}{N-2}} dx \right)^{\frac{N-2}{2N}} \leq C\|\varphi\|_{L^2},$$

for every $\varphi \in L^2(\mathbb{R}^N)$, that is $\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N, L^2(\mathbb{R}))} \leq C\|\varphi\|_{L^2}$ (see Ruiz and Vega [90]).

Corollary 1.4.57. *Let* $\varphi \in H^1(\mathbb{R}^N)$ *and let* $r \in (2, \frac{2N}{N-2})$ *(*$r \in (2,\infty)$*, if* $N = 2$*,* $r \in (2, \infty]$*, if* $N = 1$*). Then,* $\|T(t)\varphi\|_{L^r} \to 0$*, as* $t \to \pm\infty$*.*

Proof. Let q be such that (q, r) is an admissible pair (see Definition 1.4.5). It follows from Gagliardo-Nirenberg's inequality (see Theorem A.3.44 and Remark A.3.45) that there exists C such that for every $t, s \in \mathbb{R}$,

$$\|u(t) - u(s)\|_{L^r} \leq C\|u(t) - u(s)\|_{H^1}^{\frac{2}{q}} \|u(t) - u(s)\|_{L^2}^{\frac{q-2}{q}}.$$

Since $\varphi \in H^1(\mathbb{R}^N)$, it follows from Theorem 1.4.39 that $u(t)$ is bounded in $H^1(\mathbb{R}^N)$; and so,

$$\|u(t) - u(s)\|_{L^r} \leq C\|u(t) - u(s)\|_{L^2}^{\frac{q-2}{q}}.$$

Furthermore, $u_t = i\triangle u$ is bounded in $H^{-1}(\mathbb{R}^N)$; and so, there exists C such that (see Lemma A.3.60)

$$\|u(t) - u(s)\|_{L^2} \leq C|t - s|^{1/2}.$$

Therefore,

$$\|u(t) - u(s)\|_{L^r} \leq C|t - s|^{\frac{q-2}{2q}}.$$

In particular, $u : \mathbb{R} \to L^r(\mathbb{R}^N)$ is uniformly continuous. The result now follows from the property $u \in L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ (Theorem 1.4.54), since $q < \infty$. □

We now study a different smoothing effect. One verifies easily with the formula of Corollary 1.4.47 that for every $\varphi \in L^2(\mathbb{R}^N)$ with compact support, the function $(t,x) \mapsto T(t)\varphi(x)$ is analytic in $(0, +\infty) \times \mathbb{R}^N$. In other words, $T(t)$ being essentially the Fourier transform (see Corollary 1.4.49), maps functions having a nice decay as $|x| \to \infty$ to smooth functions. We now establish precise estimates describing this smoothing effect, which will enable us to prove similar results in the nonlinear case. Let us first introduce some notation. For $j \in \{1, \cdots, N\}$, let P_j be the partial differential operator on $\mathbb{R}^{N+1}$ defined by

$$P_j u(t,x) = (x_j + 2it\partial_j)u(t,x) = x_j u(t,x) + \frac{\partial_j u}{\partial x_j}(t,x). \tag{1.4.22}$$

For a multi-index α, we define the partial differential operator P_α on $\mathbb{R}^{N+1}$ by

$$P_\alpha = \prod_{i=1}^{N} P_i^{\alpha_i}.$$

Furthermore, for $x \in \mathbb{R}^N$, we set

$$x^\alpha = \prod_{i=1}^{N} x_i^{\alpha_i}.$$

Consider a smooth function $u : \mathbb{R}^{N+1} \to \mathbb{C}$. A direct calculation shows that

$$P_j u(t,x) = 2ite^{i\frac{|x|^2}{4t}} \frac{\partial}{\partial_{x_j}} \left(e^{-i\frac{|x|^2}{4t}} u \right),$$

from which it follows by an obvious iteration argument that

$$P_\alpha u(t,x) = (2it)^{|\alpha|} e^{i\frac{|x|^2}{4t}} D^\alpha \left(e^{-i\frac{|x|^2}{4t}} u \right). \tag{1.4.23}$$

We have the following result.

Theorem 1.4.58. *Let α be a multi-index. Let $\varphi \in L^2(\mathbb{R}^N)$ be such that $x^\alpha \varphi \in L^2(\mathbb{R}^N)$, and let $u(t) = T(t)\varphi \in C(\mathbb{R}, L^2(\mathbb{R}^N))$. The following properties hold:*

(i) *$T(t)x^\alpha \varphi = P_\alpha u(t)$, and in particular $P_\alpha u \in C(\mathbb{R}, L^2(\mathbb{R}^N))$ and $\|P_\alpha u(t)\|_{L^2} = \|x^\alpha \varphi\|_{L^2}$ for all $t \in \mathbb{R}$.*

(ii) *$D^\alpha \left(e^{-i\frac{|x|^2}{4t}} u(t) \right) \in C(\mathbb{R} \setminus \{0\}, L^2(\mathbb{R}^N))$ and*

$$(2|t|)^{|\alpha|} \|D^\alpha \left(e^{-i\frac{|x|^2}{4t}} u(t) \right) \|_{L^2} = \|x^\alpha \varphi\|_{L^2}.$$

for every $t \neq 0$.

Proof. By density, we need only establish the result for $\varphi \in \mathcal{S}(\mathbb{R}^N)$, in which case both u and $P_\alpha u$ belong to $C(\mathbb{R}, \mathcal{S}(\mathbb{R}^N))$ (see Remark 1.4.48). A direct calcularion shows that $[P_\alpha, i\partial_t + \triangle] = 0$, where $[\cdot, \cdot]$ is the commutator bracket. Therefore, $P_\alpha u$ is also a solution of Schrödinger's equation; and so, $P_\alpha u(t) = T(t)P_\alpha u(0)$ for all $t \in \mathbb{R}$. (i) follows, since $P_\alpha u(0) = x^\alpha \varphi$. Property (ii) follows from property (i) and identity (1.4.23). □

Remark 1.4.59. Property (i) of Theorem 1.4.58 means that $T(t)x^\alpha = P_\alpha T(t)$.

Corollary 1.4.60. *Let $\varphi \in L^2(\mathbb{R}^N)$, and assume that for some nonnegative integer m, we have $(1+|x|^m)\varphi \in L^2(\mathbb{R}^N)$. Then, $e^{-i\frac{|x|^2}{4t}} u(t) \in C(\mathbb{R} \setminus \{0\}, H^m(\mathbb{R}^N))$, and if k is the integer part of $m/2$, we have*

$$u \in \bigcap_{0 \le j \le k} C^j(\mathbb{R} \setminus \{0\}, H^{m-2j}_{\text{loc}}(\mathbb{R}^N)).$$

In particular, if $(1+|x|^m)\varphi \in L^2(\mathbb{R}^N)$ *for every nonnegative integer* m*, then* $u \in C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^N)$.

Proof. The H^m-regularity of $e^{-i\frac{|x|^2}{4t}}u(t)$ follows from Theorem 1.4.58. Since $e^{i\frac{|x|^2}{4t}} \in C^\infty(\mathbf{R}\setminus\{0\}\times\mathbf{R}^n)$, it follows that $u \in C(\mathbf{R}\setminus\{0\}, H^m_{loc}(\mathbf{R}^n))$. The regularity of the time derivatives follows from the equation.□

Corollary 1.4.61. *Let* $\varphi \in L^2(\mathbb{R}^N)$ *be such that* $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)$*, and let* $u(t) = T(t)\varphi$*. The following properties hold:*

(i) *The function* $t \mapsto (x+2it\nabla)u(t,x)$ *belong to* $L^q(\mathbb{R}, L^r(\mathbb{R}^N))$ *for every admissible pair* (q,r)*;*

(ii) *for every* $r \in [2, \dfrac{2N}{N-2})$ $(r \in [2,\infty)$*, if* $N = 2$*,* $r \in [2,\infty]$*, if* $N = 1)$*, we have* $u \in C(\mathbb{R}/\{0\}, L^r(\mathbb{R}^N))$ *and there exists* C*, depending only on* r *and* N *such that*

$$\|u(t)\|_{L^r} \le C|t|^{-N\left(\frac{1}{2}-\frac{1}{r}\right)}(\|\varphi\|_{L^2} + \|x\varphi\|_{L^2}),$$

for every $t \neq 0$.

Proof. It follows from identity (1.4.22) and Remark 1.4.59 that

$$(x+2it\nabla)u(t,x) = T(t)\psi,$$

where $\psi(x) = x\varphi(x)$; and so, property (i) follows from Theorem 1.4.54. Consider now the function $v(t,x) = e^{-i\frac{|x|^2}{4t}}u(t,x)$. It follows from Theorem 1.4.58 that $\nabla v \in C(\mathbb{R}/\{0\}, L^2(\mathbb{R}^N))$ and that

$$\|\nabla v(t)\|_{L^2} \le C|t|^{-1}\|x\varphi\|_{L^2}.$$

The result now follows from Gagliardo-Nirenberg's inequality, since $|u(t,x)| \equiv |v(t,x)|$. □

Finally, we describe a third kind of smoothing effect, of Sobolev type. It says that for every $\varphi \in L^2(\mathbb{R}^N)$, $T(t)\varphi$ belongs to $H^{1/2}_{\mathrm{loc}}(\mathbb{R}^N)$ for almost all $t \in \mathbb{R}$. It was discovered simultaneously by Constantin and Saut [30], Sjölin [92] and Vega [95]. See also Ben Artzi and Devinatz [11], Ben Artzi and Klainerman [12], Kato and Yajima [67] for further developments, as well as Kenig, Ponce and Vega [70] for a related smoothing effect. A typical result in this direction is the following (see Ben Artzi and Klainerman [12] for a rather simple proof).

Theorem 1.4.62. *There exists a constant* C *such that for every* $\varphi \in L^2(\mathbb{R}^N)$*,* $u(t) = T(t)\varphi$ *verifies*

$$\int_{-\infty}^{+\infty}\int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)}|Pu(t,x)|^2\,dx\,dt \le C\|\varphi\|^2_{L^2},$$

where $P = (I-\triangle)^{1/4}$ *is the pseudo-differential operator defined by* $\widehat{Pu}(\xi) = (1+4\pi^2|\xi|^2)^{1/4}\widehat{u}(\xi)$.

1.4.5. The wave equation. Throughout this section, Ω is an arbitrary open subset of $\mathbb{R}^N$. For some of the results, we will make further assumptions on Ω which we will specify. We will apply the results of Section 1.3 to the examples of Section 1.2.4, in order to solve the initial value problem for the wave equation

$u_{tt} - \triangle u = 0$ or the Klein-Gordon equation $u_{tt} - \triangle u + m^2 u = 0$. Consider $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ with its natural scalar product, and define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\begin{cases} D(\mathcal{A}) = \{(u,v) \in \mathcal{H}, \triangle u \in L^2(\Omega) \text{ and } v \in H_0^1(\Omega)\}, \\ \mathcal{A}(u,v) = (-v, -\triangle u + u), \text{ for all } (u,v) \in D(\mathcal{A}). \end{cases}$$

It follows from Proposition 1.2.40 that $\mathcal{A}$ is skew-adjoint. Therefore, $-\mathcal{A}$ is the generator of a group of isometries $(T(t))_{t\in\mathbb{R}}$. Furthermore, it follows from Corollary 1.2.42 that, with the notation of Theorem 1.1.31, $\mathcal{H}_{-1} = L^2(\Omega) \times H^{-1}(\Omega)$ with equivalent norms, and $\mathcal{A}_{(-1)}$ is the operator $\mathcal{B}$ defined by

$$\begin{cases} D(\mathcal{B}) = H_0^1(\Omega) \times L^2(\Omega), \\ \mathcal{B}(u,v) = (-v, -\triangle u + u), \text{ for all } (u,v) \in D(\mathcal{B}). \end{cases}$$

Therefore, $(T(t))_{t\in\mathbb{R}}$ can be extended to a group of isometries on $L^2(\Omega) \times H^{-1}(\Omega)$, which we still denote by $(T(t))_{t\in\mathbb{R}}$. We have the following result.

Theorem 1.4.63. *Let $(T(t))_{t\in\mathbb{R}}$ be as above. Given $(\varphi,\psi) \in H_0^1(\Omega) \times L^2(\Omega)$, set $T(t)(\varphi,\psi) = (u(t), v(t))$. The following properties hold:*

(i) *$u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \cap C^2(\mathbb{R}, H^{-1}(\Omega))$ and u is the unique solution of the problem*

$$\begin{cases} u_{tt} - \triangle u + u = 0 \text{ for all } t \in \mathbb{R}, \\ u(0) = \varphi, \quad u_t(0) = \psi, \end{cases} \tag{1.4.24}$$

in that class. Furthermore, $v = u_t$ and

$$\int_\Omega \{u_t(t,x)^2 + |\nabla u(t,x)|^2 + u(t,x)^2\}\, dx = \int_\Omega \{\psi(x)^2 + |\nabla \varphi(x)|^2 + \varphi(x)^2\}\, dx, \tag{1.4.25}$$

for all $t \in \mathbb{R}$;

(ii) *if furthermore $\triangle\varphi \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)$, then in addition $\triangle u \in C(\mathbb{R}, L^2(\Omega))$ and $u \in C^1(\mathbb{R}, H_0^1(\Omega)) \cap C^2(\mathbb{R}, L^2(\Omega))$ and*

$$\int_\Omega \{|\nabla u_t(t,x)|^2 + \triangle u(t,x)^2 + |\nabla u(t,x)|^2\}\, dx = \int_\Omega \{|\nabla \psi(x)|^2 + \triangle \varphi(x)^2 + |\nabla \varphi(x)|^2\}\, dx, \tag{1.4.26}$$

for all $t \in \mathbb{R}$.

Proof. Since $(\varphi,\psi) \in D(\mathcal{B})$, it follows that $(u,v) \in C(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega) \times H^{-1}(\Omega))$ and that (u,v) is the unique solution in that class of the equation $(u_t, v_t) + \mathcal{B}(u,v) = 0$ with the initial condition $(u,v)(0) = (\varphi,\psi)$. Therefore, (u,v) is the unique solution of the system

$$\begin{cases} u_t = v, \\ v_t - \triangle u + u = 0, \\ u(0) = \varphi, \quad v(0) = \psi. \end{cases}$$

The first part of property (i) follows. Since $v = u_t$, the conservation law (1.4.25) follows from the property $\|(u,v)(t)\|_{H_0^1 \times L^2} = \|(\varphi,\psi)\|_{H_0^1 \times L^2}$ ($(T(t))_{t\in\mathbb{R}}$ is a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$). Under the assumptions of (ii), we have $(\varphi,\psi) \in D(\mathcal{A})$; and so, $(u,v) \in C(\mathbb{R}, D(\mathcal{A})) \cap C^1(\mathbb{R}, \mathcal{H})$, from which the first part of property (ii) follows. It remains to establish the conservation law (1.4.26). Note that $T(t)\mathcal{A} = \mathcal{A}T(t)$ (see Proposition 1.3.4). Therefore, $\|\mathcal{A}(u,v)(t)\|_{H^1_0\times L^2} = \|\mathcal{A}(\varphi,\psi)\|_{H^1_0\times L^2}$; and so,

$$\int_\Omega \{|\nabla u_t|^2 + u_t^2 + |u - \triangle u|^2\}\,dx = \int_\Omega \{|\nabla \psi|^2 + \psi^2 + |\varphi - \triangle \varphi|^2\}\,dx.$$

It follows that

$$\int_\Omega \{|\nabla u_t|^2 + u_t^2 + u^2 - 2u\triangle u + \triangle u^2\}\,dx = \int_\Omega \{|\nabla \psi|^2 + \psi^2 + \varphi^2 - 2\varphi\triangle\varphi + \triangle \varphi^2\}\,dx.$$

The result now follows from the above identity, after integrating by part the terms $-2u\triangle u$ and $-2\varphi\triangle\varphi$ (see identity (A.3.17)) and substracting identity (1.4.25). □

Remark 1.4.64. One can obtain higher order regularity and higher order conservation laws by applying Remark 1.2.45 and 1.1.30, and Corollary 1.3.18.

When Ω satisfies certain regularity assumptions, we have better regularity properties. Some of these properties are described in the following result.

Theorem 1.4.65. *Let $(T(t))_{t\in\mathbb{R}}$ be as above. Given $(\varphi,\psi) \in H^1_0(\Omega)\times L^2(\Omega)$, set $(u(t),v(t)) = T(t)(\varphi,\psi)$ for $t\in\mathbb{R}$. The following properties hold:*

(i) *If Ω has a bounded boundary of class C^2 and if $\varphi \in H^2(\Omega)$ and $\psi \in H^1_0(\Omega)$, then $u \in C(\mathbb{R}, H^2(\Omega)\cap H^1_0(\Omega))\cap C^1(\mathbb{R}, H^1_0(\Omega))\cap C^2(\mathbb{R}, L^2(\Omega))$;*

(ii) *if Ω is bounded with boundary of class C^∞, and if $\varphi,\psi \in C^\infty(\overline{\Omega})$ satisfy the compatibility relations $u = \triangle u = \cdots \triangle^n u = \cdots = 0$ and $v = \triangle v = \cdots \triangle^n v = \cdots = 0$ on $\partial\Omega$, then $u \in C^\infty(\mathbb{R}\times\overline{\Omega})$.*

Proof. Property (i) follows from Theorem 1.4.63 (ii) and Remark 1.2.41. If $(\varphi,\psi) \in \bigcap_{n\ge 0} D(\mathcal{A}^n)$, then it follows from Corollary 1.3.17 that $(u,v) \in \bigcap_{n\ge 0} C^\infty([0,\infty), D(\mathcal{A}^n))$. Hence property (ii), by applying Remark 1.2.45. □

Remark 1.4.66. Note that the compatibility relations of property (ii) of Theorem 1.4.65 are necessary conditions if $u \in C^\infty([0,\infty)\times\overline{\Omega})$. Indeed, we have $u = \dfrac{du}{dt} = \cdots = \dfrac{d^n u}{dt^n} = \cdots = 0$ on $(0,\infty)\times\partial\Omega$. Since $\dfrac{d^{2n}u}{dt^{2n}} = (\triangle - I)^n u$ and $\dfrac{d^{2n+1}u}{dt^{2n+1}} = \dfrac{d^{2n}v}{dt^{2n}} = (\triangle - I)^n v$, the compatibility relations follow.

We now extend the previous results. Let $\lambda_1 \ge 0$ be defined by (1.2.28), and let $\lambda > -\lambda_1$. (Note that if Ω is bounded (or bounded in one direction), then it follows from Poincaré's inequality that $\lambda_1 > 0$, so that we can chose $\lambda = 0$.) Consider on $H^1_0(\Omega)$ the equivalent norm

$$|||u||| = \left(\int_\Omega \{|\nabla u|^2 + \lambda u^2\}\,dx\right)^{1/2}.$$

We still consider $\mathcal{H} = H^1_0(\Omega)\times L^2(\Omega)$, but with the scalar product associated with the above norm on $H^1_0(\Omega)$, and we define the operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\begin{cases} D(\mathcal{A}) = \{(u,v)\in\mathcal{H}, \triangle u \in L^2(\Omega) \text{ and } v \in H^1_0(\Omega)\},\\ \mathcal{A}(u,v) = (-v, -\triangle u + \lambda u), \text{ for all } (u,v)\in D(\mathcal{A}). \end{cases}$$

It follows from Proposition 1.2.43 that $\mathcal{A}$ is skew-adjoint. Therefore, $-\mathcal{A}$ is the generator of a group of isometries $(T(t))_{t\in\mathbb{R}}$. Furthermore, it follows from Proposition 1.2.43 that, with the notation of Theorem 1.1.31, $\mathcal{H}_{-1} = L^2(\Omega) \times H^{-1}(\Omega)$ with equivalent norms, and $\mathcal{A}_{(-1)}$ is the operator $\mathcal{B}$ defined by

$$\begin{cases} D(\mathcal{B}) = H_0^1(\Omega) \times L^2(\Omega), \\ \mathcal{B}(u,v) = (-v, -\triangle u + \lambda u), \text{ for all } (u,v) \in D(\mathcal{B}). \end{cases}$$

Therefore, $(T(t))_{t\in\mathbb{R}}$ can be extended to a group of isometries on $L^2(\Omega) \times H^{-1}(\Omega)$, which we still denote by $(T(t))_{t\in\mathbb{R}}$. We have the following result.

Theorem 1.4.67. *Let $(T(t))_{t\in\mathbb{R}}$ be as above. Given $(\varphi,\psi) \in H_0^1(\Omega) \times L^2(\Omega)$, set $T(t)(\varphi,\psi) = (u(t), v(t))$. The following properties hold:*

(i) *$u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \cap C^2(\mathbb{R}, H^{-1}(\Omega))$ and u is the unique solution of the problem*

$$\begin{cases} u_{tt} - \triangle u + \lambda u = 0 \text{ for all } t \in \mathbb{R}, \\ u(0) = \varphi, \quad u_t(0) = \psi, \end{cases} \tag{1.4.27}$$

in that class. Furthermore, $v = u_t$ and

$$\int_\Omega \{u_t(t,x)^2 + |\nabla u(t,x)|^2 + \lambda u(t,x)^2\}\, dx = \int_\Omega \{\psi(x)^2 + |\nabla\varphi(x)|^2 + \lambda\varphi(x)^2\}\, dx,$$

for all $t \in \mathbb{R}$;

(ii) *if furthermore $\triangle\varphi \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)$, then in addition $\triangle u \in C(\mathbb{R}, L^2(\Omega))$ and $u \in C^1(\mathbb{R}, H_0^1(\Omega)) \cap C^2(\mathbb{R}, L^2(\Omega))$ and*

$$\int_\Omega \{|\nabla u_t(t,x)|^2 + \triangle u(t,x)^2 + \lambda|\nabla u(t,x)|^2\}\, dx = \int_\Omega \{|\nabla\psi(x)|^2 + \triangle\varphi(x)^2 + \lambda|\nabla\varphi(x)|^2\}\, dx,$$

for all $t \in \mathbb{R}$.

(iii) *If Ω has a bounded boundary of class C^2 and if $\varphi \in H^2(\Omega)$ and $\psi \in H_0^1(\Omega)$, then $u \in C(\mathbb{R}, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}, H_0^1(\Omega)) \cap C^2(\mathbb{R}, L^2(\Omega))$;*

(iv) *if Ω is bounded with boundary of class C^∞, and if $\varphi, \psi \in C^\infty(\overline{\Omega})$ satisfy the compatibility relations $u = \triangle u = \cdots \triangle^n u = \cdots = 0$ and $v = \triangle v = \cdots \triangle^n v = \cdots = 0$ on $\partial\Omega$, then $u \in C^\infty(\mathbb{R} \times \overline{\Omega})$.*

Proof. The proof is easily adaptated from the proofs of Theorems 1.4.63 and 1.4.65. □

Remark 1.4.68. One can solve problem (1.4.27) for any value of the parameter λ, even for $\lambda \le -\lambda_1$. Indeed, (1.4.27) is equivalent to

$$\begin{cases} u_{tt} - \triangle u + u = (1-\lambda)u, \\ u(0) = \varphi, \quad u_t(0) = \psi. \end{cases}$$

This can be written in the form

$$U_t + \mathcal{A}U = F(U),$$

with $U=(u,v)$ and $\mathcal{A}$ the operator associated with $\lambda=1$. Here $F(u,v)=(0,(1-\lambda)u)$. Since F is linear, it is in particular globally Lipschitz, and it follows from the results of Chapter 2 that this problem can be solved. The mapping $(\varphi,\psi)\mapsto(u(t),u_t(t))$ defines a continuous group $(T(t))_{t\in\mathbb{R}}$, which is not of isometries in general (see Section 1.7.1). With this notation, all the conclusions of Theorem 1.4.67 hold for any value of λ. The conclusions of Theorem 1.4.70 and Corollary 1.4.72 below also hold for any value of λ (the proof is the same).

Remark 1.4.69. One can obtain higher order regularity and higher order conservation laws by applying Remark 1.2.45 and 1.1.30, and Corollary 1.3.18. Note also that the compatibility relations of property (ii) of Theorem 1.4.65 are necessary conditions if $u\in C^\infty([0,\infty)\times\overline{\Omega})$. Indeed, we have $u=\dfrac{du}{dt}=\cdots=\dfrac{d^nu}{dt^n}=\cdots=0$ on $(0,\infty)\times\partial\Omega$. Since $\dfrac{d^{2n}u}{dt^{2n}}=(\triangle-\lambda I)^nu$ and $\dfrac{d^{2n+1}u}{dt^{2n+1}}=\dfrac{d^{2n}v}{dt^{2n}}=(\triangle-\lambda I)^nv$, the compatibility relations follow.

One of the most important features of the wave equation is the finite speed propagation phenomenon. It says that if Ω contains the ball $B(x_0,T_0)$, then the values of the solution in the cone $\{(t,x)\in[0,T_0]\times\Omega;\ |x-x_0|+t\le T_0\}$ are determined only by the initial values in the ball $B(x_0,T_0)$. This phenomenon is described in the following theorem.

Theorem 1.4.70. *Let $(T(t))_{t\in\mathbb{R}}$ be as in Theorem 1.4.67. Let $T_0>0$ and $x_0\in\Omega$, and assume that $B(x_0,T_0):=\{|x-x_0|<T_0\}\subset\Omega$. Let $(\varphi,\psi)\in H^1_0(\Omega)\times L^2(\Omega)$ and let u be the corresponding solution of (1.4.27). If φ and ψ vanish almost everywhere on $B(x_0,T_0)$, then u vanishes almost everywhere on the cone $\bigcup_{-T_0<t<T_0}B(x_0,T_0-|t|)=\{(t,x)\in(-T_0,T_0)\times\Omega;\ |x-x_0|<T_0-|t|\}$.*

Proof. Without loss of generality, we may assume that $x_0=0$. Assume first that $u\in C^2([0,T_0]\times\Omega)$, so that the equation $u_{tt}-\triangle u+\lambda u=0$ holds everywhere in $[0,T_0]\times\Omega$. Multiplying the equation by u_t, we obtain

$$\frac{\partial}{\partial t}\Big(\frac12 u_t^2\Big)-\nabla\cdot(u_t\nabla u)+\frac{\partial}{\partial t}\Big(\frac12|\nabla u|^2\Big)+\lambda uu_t=0,$$

which we rewrite in the form

$$\frac12\frac{\partial}{\partial t}(u_t^2+|\nabla u|^2+u^2)=\nabla\cdot(u_t\nabla u)+(1-\lambda)uu_t.$$

Given $0<t<T_0$, we integrate the above identity on $B(0,T_0-t)$. It follows that

$$\frac12\int_{B(0,T_0-t)}\frac{\partial}{\partial t}(u_t^2+|\nabla u|^2+u^2)=(T_0-t)^{N-1}\int_{S^{N-1}}\nu\cdot[u_t\nabla u)((T_0-t)\xi)]\,d\xi+\int_{B(0,T_0-t)}(1-\lambda)uu_t,$$

where S^{N-1} is the unit sphere of $\mathbb{R}^N$ and ν is the outward unit vector at S^{N-1}. Given a smooth function $\phi(t,x)$, note that

$$\begin{aligned}\frac{d}{dt}\int_{B(0,T_0-t)}\phi(t,x)\,dx&=\frac{d}{dt}\int_0^{T_0-t}r^{N-1}dr\int_{S^{N-1}}\phi(t,r\xi)\,d\xi\\&=\int_{B(0,T_0-t)}\frac{\partial\phi}{\partial t}(t,x)\,dx-(T_0-t)^{N-1}\int_{S^{N-1}}\phi(t,(T_0-t)\xi)\,d\xi.\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_{B(0,T_0-t)}(u_t^2+|\nabla u|^2+u^2)=$$
$$-\frac{1}{2}(T_0-t)^{N-1}\int_{S^{N-1}}(u_t^2+|\nabla u|^2+u^2-2u_t\nabla u\cdot\nu)(t,(T_0-t)\xi)\,d\xi+(1-\lambda)\int_{B(0,T_0-t)}uu_t.$$

Since $|2u_t\nabla u\cdot\nu|\le 2|u_t\nabla u|\le u_t^2+|\nabla u|^2$, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{B(0,T_0-t)}(u_t^2+|\nabla u|^2+u^2)\le(1-\lambda)\int_{B(0,T_0-t)}uu_t\le\frac{|1-\lambda|}{2}\int_{B(0,T_0-t)}(u_t^2+u^2).$$

Integrating the above differential inequality, we obtain

$$\begin{aligned}\int_{B(0,T_0-t)}(u_t^2+|\nabla u|^2+u^2)(t,x)\,dx&\le e^{|1-\lambda|t}\int_{B(0,T_0)}(u_t^2+|\nabla u|^2+u^2)(0,x)\,dx\\&=e^{|1-\lambda|t}\int_{B(0,T_0)}(\psi^2+|\nabla\varphi|^2+\varphi^2)(x)\,dx,\end{aligned}\tag{1.4.28}$$

for all $t\in[0,T_0]$. Let now $m\ge 2+\frac{N}{2}$, so that $H^m_{\mathrm{loc}}(\Omega)\subset C^2(\Omega)$. It follows from Remark 1.2.45 that $D(\mathcal{A}^m)\subset C^2(\Omega)^2$. Given $(\varphi,\psi)\in H^1_0(\Omega)\times L^2(\Omega)$, it follows from Remarks 1.1.29 and 1.1.30 that there exists a sequence $(\varphi_n,\psi_n)\subset D(\mathcal{A}^{m+2})$ such that $(\varphi_n,\psi_n)\to(\varphi,\psi)$ in $H^1_0(\Omega)\times L^2(\Omega)$ as $n\to\infty$. On the other hand, if we set $(u_n,v_n)=T(t)(\varphi_n,\psi_n)$, it follows from Corollary 1.3.18 that $(u_n,v_n)\in C^2([0,T_0],D(\mathcal{A}^m))$, so that $u_n\in C^2([0,T_0]\times\Omega)$. Applying (1.4.28) to u_n and letting $n\to\infty$, it follows that inequality (1.4.28) holds for every $(\varphi,\psi)\in H^1_0(\Omega)\times L^2(\Omega)$ (note that $u_n\to u$ in $C([0,T_0],H^1_0(\Omega))\cap C^1([0,T_0],L^2(\Omega))$). The result for $t\ge 0$ follows, since the right-hand side of (1.4.28) vanishes when φ and ψ are as in the statement of the theorem (with $x_0=0$). Since the equation is time reversible (i.e. it is invariant under the change of variable $t\mapsto -t$), it follows that $u(-t)$ is the solution of the problem (1.4.27) corresponding to the initial values $(\varphi,-\psi)$, and this proves the result for $t\le 0$. □

Remark 1.4.71. Theorem 1.4.70 says that if (φ_1,ψ_1) and (φ_2,ψ_2) coincide almost everywhere in $B(x_0,T_0)$, then the corresponding solutions u_1,u_2 of problem (1.4.27) coincide almost everywhere in the cone $\{(t,x)\in(0,T_0)\times\Omega;\ |x-x_0|+|t|<T_0\}$. To see this, take $(\varphi,\psi)=(\varphi_2-\varphi_1,\psi_2-\psi_1)$. In other words, it means that $u(t,x)$ depends only on the values of φ and ψ in the ball $B(x,|t|)$, as long as $B(x,|t|)\subset\Omega$. When $N\ge 3$ is odd, there is even a stronger property in the case $\lambda=0$ (see Remark 1.4.68); namely, $u(t,x)$ depends only on the values of φ and ψ in the sphere $S(x,|t|)=\{(t,x);\ |x|=|t|\}$, as long as $S(x,|t|)\subset\Omega$. This property is called Huygens' principle (see Courant and Hilbert [33]).

Corollary 1.4.72. *Let $x_0\in\Omega$, let $R>0$ be such that $B(x_0,R)\subset\Omega$ and set $T=\sup\{r>0;\ B(x_0,R+r)\subset\Omega\}$. Let $(\varphi,\psi)\in H^1_0(\Omega)\times L^2(\Omega)$ and let u be the corresponding solution of (1.4.27). If φ and ψ are supported in the ball $B(x_0,R)$, then $u(t)$ is supported in the ball $B(x_0,R+|t|)$ for all $t\in(-T,T)$.*

Proof. We proceed in two steps. We first establish the result when $\Omega=\mathbb{R}^N$, then in the general case.

Step 1. The case $\Omega=\mathbb{R}^N$. We need to show that $u(t)$ vanishes almost everywhere on the ball $B(y,\rho)$ for every $y\in\mathbb{R}^N$ and $\rho>0$ such that $|x_0-y|\ge R+|t|+\rho$. By assumption, we have $B(y,\rho+|t|)\cap B(x_0,R)=\emptyset$. It now follows from Theorem 1.4.70 that $u(t)$ vanishes almost everywhere on the ball $B(y, \rho+|t|-|t|) = B(y, \rho)$. Hence the result.

Step 2. The general case. Let φ, ψ and u be as in the statement. Define $\widetilde{\varphi}, \widetilde{\psi}$ by

$$\widetilde{\varphi} = \begin{cases} \varphi \text{ on } \Omega, \\ 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad \text{and} \quad \widetilde{\psi} = \begin{cases} \psi \text{ on } \Omega, \\ 0 \text{ on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

It is clear that $\widetilde{\varphi}, \widetilde{\psi} \in L^2(\mathbb{R}^N)$, and it follows from Proposition A.3.25 that $\widetilde{\varphi} \in H^1(\mathbb{R}^N)$. Let $\widetilde{u}(t)$ be the solution of (1.4.27) with $\Omega = \mathbb{R}^N$ with the initial values $(\widetilde{\varphi}, \widetilde{\psi})$. It follows in particular that $\widetilde{u} \in C(\mathbb{R}, H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)) \cap C^2(\mathbb{R}, H^{-1}(\mathbb{R}^N))$. Finally, let $w(t) = \widetilde{u}(t)_{|\Omega}$. It follows that $w \in C(\mathbb{R}, H^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \cap C^2(\mathbb{R}, H^{-1}(\Omega))$. Furthermore, it follows from Step 1 that $w(t)$ is supported in the ball $B(x_0, R+|t|)$ for $|t| < T$. Therefore, by Proposition A.3.28, $w \in C(\mathbb{R}, H^1_0(\Omega))$. Finally, it is clear that w solves the following problem.

$$\begin{cases} w_{tt} - \triangle w + \lambda w = 0 \\ w(0) = \varphi, \quad w_t(0) = \psi. \end{cases}$$

Therefore, $w(t) = u(t)$ for $|t| < T$, by uniqueness. This completes the proof. □

Remark 1.4.73. When $\Omega = \mathbb{R}^N$, one can establish $L^p - L^q$ estimates for the solutions of problem (1.4.27). The proofs are based on sharp Fourier analysis, and are much more difficult than the proof of Theorem 1.4.50.

(i) A typical estimate is the following. Let $\varphi = 0$, $\psi \in L^2(\mathbb{R}^N)$ and let u be the corresponding solution of (1.4.27) with $\lambda > 0$. There exists a constant C independant of φ such that

$$\|u(t)\|_{L^{\frac{2(N+1)}{N-1}}} \le C|t|^{-\frac{N-1}{N+1}} \|\psi\|_{L^{\frac{2(N+1)}{N+3}}},$$

for all $t \in \mathbb{R} \setminus \{0\}$. See Marshall, Strauss and Wainger [78] for a complete study of these properties.

(ii) One can obtain estimates of the solution of the wave equation (i.e. when $\lambda = 0$, see Remark 1.4.68) in homogeneous Besov spaces. For example,

$$\|u(t)\|_{\dot{B}^{-\beta}_{r,2}} \le C|t|^{-(N-1)(\frac{1}{2}-\frac{1}{r})} \|\varphi\|_{\dot{B}^{\beta}_{r',2}},$$

for all $t \neq 0$. Here, u is the solution of (1.4.27) with $\lambda = 0$ and $\psi = 0$, $2 \le r \le \infty$ and $\beta = \dfrac{N+1}{2}\left(\dfrac{1}{2} - \dfrac{1}{r}\right)$. See Brenner [15] and Pecher [86] for these estimates. For the Klein-Gordon equation (i.e. when $\lambda > 0$), similar estimates hold with the homoheneous Besov spaces $\dot{B}^{\beta}_{r,2}$ replaced by the Besov spaces $B^{\beta}_{r,2}$ (see Brenner [16]).

Remark 1.4.74. One can obtain for the wave equation in $\mathbb{R}^N$ (i.e. (1.4.27) with $\lambda = 0$, see Remark 1.4.68) estimates of the type described in Theorem 1.4.54. These estimates were discovered by Strichartz [93], and are called Strichartz estimates. The simplest proof makes use of the estimates in the Besov spaces, as described in Remark 1.4.73 (ii). A typical estimate is the following. Let $2 \le r \le \infty$, $2 < q \le \infty$ and $\rho \in \mathbb{R}$ satisfy

$$\frac{1}{q} = \rho - 1 + N\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{and} \quad \rho + \frac{N+1}{2}\left(\frac{1}{2} - \frac{1}{r}\right) \le 1.$$

For every $(\varphi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, the solution u of (1.4.27) with $\lambda = 0$ belongs to $L^q(\mathbb{R}, \dot{B}^\rho_{r,2}(\mathbb{R}^N))$, and there exists a constant C independant of (φ, ψ) such that

$$\|u\|_{L^q(\mathbb{R}, \dot{B}^\rho_{r,2})} \le C(\|\nabla\varphi\|_{L^2} + \|\psi\|_{L^2}).$$

(Note that for the Klein-Gordon equation, i.e. when $\lambda > 0$, similar estimates hold with the homoheneous Besov spaces $\dot{B}^\rho_{r,2}$ replaced by the Besov spaces $B^\rho_{r,2}$, see Brenner [16].) By applying Sobolev's inequalities for the homogeneous Besov spaces, one obtains estimates for space-time integrals of u. For example, if $N \ge 3$, let $\dfrac{2N}{N-2} \le r < \dfrac{2N}{N-3}$ and let q be defined by $\dfrac{1}{q} = N\left(\dfrac{1}{2} - \dfrac{1}{r}\right) - 1$. By applying the above estimate with $\rho = 0$ and since $\dot{B}^0_{r,2}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ (see Bergh and Löfström [13] or the appendix of Ginibre and Velo [56]), it follows that

$$\|u\|_{L^q(\mathbb{R}, L^r)} \le C(\|\nabla\varphi\|_{L^2} + \|\psi\|_{L^2}).$$

See Ginibre and Velo [57, Lemma 2.2] for these questions.

Remark 1.4.75. When Ω is bounded, one can express the solution of problem (1.4.27) in terms of the decomposition of the initial values on the basis of $L^2(\Omega)$ made of the eigenvectors of $-\triangle$ in $H^1_0(\Omega)$. More precisely, let $(\lambda_n)_{n\ge1}$ be the family of eigenvalues of $-\triangle$ in $H^1_0(\Omega)$, let $(\varphi_n)_{n\ge1}$ be a Hilbert basis of $L^2(\Omega)$ made of eigenvectors (see Section A.4.5). Given $(\varphi, \psi) \in H^1_0(\Omega) \times L^2(\Omega)$, set $a_n = (\varphi, \varphi_n)_{L^2}$ and $b_n = (\psi, \varphi_n)_{L^2}$ for all $n \ge 1$, so that $\varphi = \sum\limits_{n=1}^{\infty} a_n\varphi_n$ and $\psi = \sum\limits_{n=1}^{\infty} b_n\varphi_n$, and let u be the corresponding solution of (1.4.27). Then,

$$u(t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(t\sqrt{\lambda_n + \lambda}\right) + \frac{b_n}{\sqrt{\lambda_n + \lambda}} \sin\left(t\sqrt{\lambda_n + \lambda}\right)\right) \varphi_n,$$

for all $t \ge 0$. See the proof of Proposition 1.4.34.

Remark 1.4.76. We can apply Proposition 1.3.13 to show that if Ω and φ, ψ have some symmetry properties, then $T(t)(\varphi, \psi)$ has the same properties (see Remark 1.4.36).

Remark 1.4.77. Note that the results of this section (and of the following section) hold true as well in the corresponding spaces of complex-valued functions, as follows easily by considering $\mathrm{Re}u$ and $\mathrm{Im}u$ (see Section A.4.6 and Remark 1.2.46).

1.4.6. Stokes' equation. In this section, we will apply the results of Section 1.3 to the examples of Section 1.2.5, in order to solve the initial value problem for the Stokes equation. We begin with the stokes equation in $\mathbb{R}^N$. Let $N \ge 2$, and consider the Hilbert space $E = \left(L^2(\mathbb{R}^N)\right)^N$. Let

$$X = \{\boldsymbol{u} \in E;\ \nabla \cdot \boldsymbol{u} = 0\}.$$

Here, the condition $\nabla \cdot \boldsymbol{u} = 0$ is understood in the sense of distributions. X is a Hilbert space when endowed with the scalar product of E (see Section 1.2.5). We consider the Stokes operator A defined by

$$\begin{cases} D(A) = \{\boldsymbol{u} \in \left(H^2(\mathbb{R}^N)\right)^N \cap X;\ \triangle\boldsymbol{u} \in X\}; \\ A\boldsymbol{u} = -\triangle\boldsymbol{u},\ \text{for } \boldsymbol{u} \in D(A). \end{cases}$$

It follows from Theorem 1.2.47 that A is self-adjoint. Therefore, $-A$ is the generator of a semigroup of contractions in X which we denote $(T(t))_{t\geq 0}$. We have the following result.

Theorem 1.4.78. *Let $\varPhi \in X$ and set $\mathbf{u}(t) = T(t)\varPhi$. Then $\mathbf{u} \in C([0,\infty), X) \cap C^1((0,\infty), X)$, $\triangle \mathbf{u} \in C((0,\infty), X)$, and $\mathbf{u}$ is the unique solution of problem*

$$\begin{cases} \mathbf{u}_t - \triangle \mathbf{u} = 0, \text{ for all } t \geq 0, \\ \mathbf{u}(0) = \varPhi, \end{cases}$$

in that class. Moreover, $\triangle^m \mathbf{u} \in C^\infty((0,\infty), X)$ for every nonnegative integer m, and in particular $\mathbf{u} \in C^\infty((0,\infty) \times \mathbb{R}^N)^N$.

Proof. The result follows from Corollary 1.3.35, except for the last property $\mathbf{u} \in C^\infty((0,\infty) \times \mathbb{R}^N)^N$, which follows easily from Sobolev's embedding theorem. □

Remark 1.4.79. Note that all the components u_i of $\mathbf{u}$ solve the heat equation in $\mathbb{R}^N$; and so, we may apply all the results of Section 1.4.2. In particular, $(T(t))_{t\geq 0}$ verifies the conclusions of Proposition 1.4.14 with the spaces $L^p(\mathbb{R}^N)$ replaced by $(L^2(\mathbb{R}^N))^N$.

We now study the Stokes equation in a domain. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. Let $E = \big(L^2(\Omega)\big)^N$, and let $F = \{\mathbf{u} \in (\mathcal{D}(\Omega))^N;\ \nabla \cdot \mathbf{u} = 0\}$. Let X be the closure of F in E. X is also a Hilbert space with the scalar product of E. Let $P : E \to X$ be the orthogonal projection on X. We consider the Stokes operator A defined by

$$\begin{cases} D(A) = \big(H^2(\Omega) \cap H_0^1(\Omega)\big)^N \cap X; \\ A\mathbf{u} = -P(\triangle \mathbf{u}), \text{ for } \mathbf{u} \in D(A). \end{cases}$$

It follows from Theorem 1.2.49 that A is self-adjoint. Therefore, $-A$ is the generator of a semigroup of contractions in X which we denote $(T(t))_{t\geq 0}$. We have the following result.

Theorem 1.4.80. *Let $\varPhi \in X$ and set $\mathbf{u}(t) = T(t)\varPhi$. Then $\mathbf{u} \in C([0,\infty), X) \cap C^1((0,\infty), X)$, $\triangle \mathbf{u} \in C((0,\infty), X)$, and $\mathbf{u}$ is the unique solution of problem*

$$\begin{cases} \mathbf{u}_t - P(\triangle \mathbf{u}) = 0, \text{ for all } t \geq 0, \\ \mathbf{u}(0) = \varPhi, \end{cases} \tag{1.4.29}$$

in that class.

Proof. The result follows from Corollary 1.3.35. □

Remark 1.4.81. Note that $\mathbf{u} \in C([0,\infty), X) \cap C^1((0,\infty), X) \cap C((0,\infty), D(A))$ solves problem (1.4.29) if and only if there exists $p \in C((0,\infty), H^1(\Omega))$ such that

$$\begin{cases} \mathbf{u}_t - \triangle \mathbf{u} + \nabla p = 0, \text{ for all } t \geq 0, \\ \mathbf{u}(0) = \varPhi, \end{cases} \tag{1.4.30}$$

Indeed, if $\boldsymbol{u}$ solves (1.4.29), given $t > 0$ let $p \in Y \hookrightarrow H^1(\Omega)$ (see Remark 1.2.50) be such that $-\triangle \boldsymbol{u} + \nabla p = -\boldsymbol{u}_t \in X$. It follows from Remark 1.2.50 that the mapping $\boldsymbol{u}_t \mapsto p$ is continuous $X \to H^1(\Omega)$, so that $p \in C((0,\infty), H^1(\Omega))$ and $\boldsymbol{u}$ solves (1.4.30). The converse statement follows by applying the projection P to (1.4.30), since $\nabla p \in X^\perp$ (see Section 1.2.5).

Remark 1.4.82. The semigroup $(T(t))_{t\ge 0}$ verifies the same $L^p - L^q$ estimate as the heat semigroup. Namely,

$$\|T(t)\varphi\|_{L^q} \le C t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varphi\|_{L^p},$$

for all $t > 0$ and for all $1 \le p \le q \le \infty$ (see Coulhon and Lamberton [32]). The Stokes operator in L^p (see Remark 1.2.51) generates a bounded analytic semigroup (see Giga [51]). It seems that one does not know whether or not this semigroup is of contractions.

1.4.7. Airy's equation. In this section, we will apply the results of Section 1.3 to the examples of Section 1.2.6, in order to solve the initial value problem for the Airy equation $u_t + u_{xxx} = 0$. Let $X = L^2(\mathbb{R})$, and define the operator A on X by

$$\begin{cases} D(A) = H^3(\mathbb{R}); \\ Au = -u_{xxx} = -\dfrac{d^3u}{dx^3}, \text{ for } u \in D(A). \end{cases}$$

It follows from Theorem 1.2.52 that A is skew-adjoint; and so $-A$ is the generator of a group of isometries $(T(t))_{t\in\mathbb{R}}$ on X. We have the following result.

Theorem 1.4.83. *Let $(T(t))_{t\in\mathbb{R}}$ be as above. Given $\varphi \in L^2(\mathbb{R})$, set $u(t) = T(t)\varphi$ for all $t \in \mathbb{R}$. The following properties hold:*

(i) $u \in C(\mathbb{R}, L^2(\mathbb{R})) \cap u \in C^1(\mathbb{R}, H^{-3}(\mathbb{R}))$, *and u is the unique solution of the problem*

$$\begin{cases} u_t + u_{xxx} = 0, \text{ for all } t \in \mathbb{R}, \\ u(0) = \varphi, \end{cases} \tag{1.4.31}$$

in that class. Moreover, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for all $t \in \mathbb{R}$;

(ii) *If $\varphi \in H^m(\mathbb{R})$ for some integer $m \ge 0$ and if p is the integer part of $m/3$, then*

$$u \in \bigcap_{0\le j\le p} C^j(\mathbb{R}, H^{m-3p}(\mathbb{R})),$$

and equation (1.4.31) holds in $H^{m-3}(\mathbb{R})$. Moreover, $\left\|\dfrac{\partial^j}{\partial x^j} u(t)\right\|_{L^2} = \left\|\dfrac{d^j\varphi}{dx^j}\right\|_{L^2}$ for all $t \in \mathbb{R}$ and all $0 \le j \le p$.

Proof. The result follows from Theorem 1.2.52 and Remarks 1.2.53 and 1.2.54. □

Remark 1.4.84. One can show that $T(t)\varphi = S_t \star \varphi$ for all $t \ne 0$, where the kernel S_t is given by

$$S_t(x) = \frac{1}{(3t)^{1/3}} Ai\left(\frac{x}{(3t)^{1/3}}\right).$$

Here, Ai is the classical Airy function. It follows in particular from the above formula that $\|T(t)\varphi\|_{L^r} \le C|t|^{-\frac{1}{3}(1-\frac{2}{r})}\|\varphi\|_{L^{r'}}$, for all $2 \le r \le \infty$. (See Kato [63], Ginibre and Tsutsumi [55].) By applying this inequality, one obtains easily Strichatrz type estimates (the proof is an adaptation of the proof of Theorem 1.4.54). Namely,

$$\|T(\cdot)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}))} \le C\|\varphi\|_{L^2},$$

for all $2 \le r \le \infty$ and all $6 \le q \le \infty$ such that $\displaystyle\frac{2}{q} = \frac{1}{3}\Big(1 - \frac{2}{r}\Big)$.

Remark 1.4.85. Airy's equation has another remarkable smooting effect. For every $\varphi \in L^2(\mathbb{R})$, $T(t)\varphi$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R})$ for almost all $t \in \mathbb{R}$. More precisely, setting $u(t) = T(t)\varphi$, we have the estimate

$$\sup_{x\in\mathbb{R}} \Big(\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}(t,x)^2 dt \Big)^{1/2} \le C\|\varphi\|_{L^2}.$$

It follows in particular that $\|u\|_{L^2(\mathbb{R},H^1(-R,R))} \le C\|\varphi\|_{L^2}$ for all $0 < R < \infty$. (See Kato [63], Kenig, Ponce and Vega [70].)

Finally, we describe a third smoothing effect for the Airy equation.

Theorem 1.4.86. *Let $(T(t))_{t\in\mathbb{R}}$ be as above. Given $\varphi \in L^2(\mathbb{R})$, set $u(t) = T(t)\varphi$ for all $t \in \mathbb{R}$. If there exists $b > 0$ such that $e^{bx}\varphi(x) \in L^2(\mathbb{R})$, then $e^{bx}u(t) \in C^\infty((0,\infty), H^m(\mathbb{R}))$ for all nonnegative integers m. In particular, $u \in C^\infty((0,\infty) \times \mathbb{R})$.*

Proof. Let $(S(t))_{t\ge0}$ be the semigroup of the heat equation in $\Omega = \mathbb{R}$, let $(R(t))_{t\in\mathbb{R}}$ be the group of translations defined by $R(t)w(x) = w(x-t)$, and set $v(t) = e^{tb^3}S(bt)R(b^2t)T(t)\psi$ with $\psi(x) = e^{bx}\varphi(x) \in L^2(\mathbb{R})$. Since both $(R(t))_{t\ge0}$ and $(T(t))_{t\ge0}$ are semigroup of contractions in $L^2(\mathbb{R})$, it follows easily from the smoothing effect of the heat equation that $v(t) \in C^\infty((0,\infty), H^m(\mathbb{R}))$ for all nonnegative integers m. Moreover, we have

$$\|v(t)\|_{H^m} \le Ce^{tb^3}(b|t|)^{-\frac{m}{2}}\|\psi\|_{L^2}. \tag{1.4.32}$$

On the other hand, a direct calculation shows that $v_t + v_{xxx} - bv_{xx} + b^2v_x - b^3v = 0$. Therefore, if we set $z(t,x) = e^{-bx}v(t,x)$, then $z_t + z_{xxx} = 0$. In addition, $z(0) = \varphi$ in $\mathcal{D}'(\mathbb{R})$. Assume that $\varphi \in \mathcal{S}(\mathbb{R})$. One verifies (using the Fourier transform) that $T(\cdot)\psi \in C(\mathbb{R},\mathcal{S}(\mathbb{R}))$, which implies easily that $v \in C(\mathbb{R},\mathcal{S}(\mathbb{R}))$. We deduce in particular that $z \in C(\mathbb{R}, L^2(\mathbb{R}))$; and so, $z(t) \equiv u(t)$. It follows that $e^{bx}u(t) \equiv v(t)$, and we deduce from (1.4.32) that

$$\|e^{bx}u(t)\|_{H^m} \le Ce^{tb^3}(b|t|)^{-\frac{m}{2}}\|e^{bx}\varphi\|_{L^2}.$$

The general case $(1 + e^{bx})\varphi \in L^2(\mathbb{R})$ follows from the above inequality and an obvious density argument. □

1.5. Nonhomogeneous equations. Throughout this section, X is a Banach space, endowed with the norm $\|\cdot\|$, and A is an m-accretive operator in X, with dense domain. We denote by $(T(t))_{t\ge0}$ the semigroup of contractions generated by $-A$ (cf. Section 1.3).

Given $T > 0$, $x \in X$ and $f \in C([0,T], X)$, we want to solve the following problem:

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X); \\ \dfrac{du}{dt} + Au = f, \text{ for all } t \in [0,T]; \\ u(0) = x. \end{cases} \tag{1.5.1}$$

We begin with the following result (the variation of the parameter formula), which is fundamental.

Theorem 1.5.1. *(Duhamel's principle) Let $T > 0$, $x \in D(A)$ and $f \in C([0,T], X)$. If u is a solution of problem (1.5.1), then*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\, ds, \tag{1.5.2}$$

for every $t \in [0,T]$.

Proof. Note first that the mapping $s \mapsto T(t-s)f(s)$ is continuous $[0,t] \to X$; and so, formula (1.5.2) makes sense. Consider now $t \in (0,T]$, and set $w(s) = T(t-s)u(s)$ for $0 \le s \le t$. Given $s \le s+h \le t$, we have

$$\frac{w(s+h) - w(s)}{h} = T(t-s-h)\left\{ \frac{u(s+h) - u(s)}{h} - \frac{T(h) - I}{h} u(s) \right\}.$$

It follows from Definition 1.3.6 that

$$\frac{w(s+h) - w(s)}{h} \underset{h\downarrow 0}{\longrightarrow} T(t-s)(u'(s) + Au(s)) = T(t-s)f(s).$$

Since $T(t-\cdot)f(\cdot) \in C([0,t], X)$, it follows that $w \in C^1([0,t], X)$ (see Theorem A.1.16) and that $w'(s) = T(t-s)f(s)$. Integration over $[0,t]$ yields (1.5.2) for $t \in (0,T]$. On the other hand, it is clear that (1.5.2) is verified for $t = 0$. Hence the result. □

Corollary 1.5.2. *Let $T > 0$, $x \in D(A)$ and $f \in C([0,T], X)$. Then, problem (1.5.1) has at most one solution, given by formula (1.5.2).*

Given $T > 0$, $x \in X$ and $f \in C([0,T], X)$, it is clear that formula (1.5.2) defines a function $u \in C([0,T], X)$. We will now establish sufficient conditions on x and f so that u is the solution of (1.5.1). Note that it is necessary that $x \in D(A)$. However, this is not sufficient. Indeed, if $(T(t))_{t\ge 0}$ is the restriction of a group of isometries $(U(t))_{t\in\mathbb{R}}$, take $x = 0 \in D(A)$, $y \in X$, and set $f(t) = T(t)y$. It follows that the solution of (1.5.2) is $u(t) = tT(t)y$. Therefore, if $y \notin D(A)$, it follows from Remark 1.3.29 that $u \notin C([0,T], D(A))$. In particular, u does not solve (1.5.1).

Lemma 1.5.3. *Let $T > 0$, $x \in X$ and $f \in L^1((0,T), X)$. Then, formula (1.5.2) defines a function $u \in C([0,T], X)$. In addition,*

$$\|u\|_{C([0,T],X)} \le \|x\| + \|f\|_{L^1((0,T),X)},$$

for all $x \in X$ and $f \in L^1((0,T), X)$.

Proof. The result is immediate if $f \in C([0,T], X)$. The general case follows from an obvious density argument (cf. Remark A.2.18 (i)). □

Proposition 1.5.4. *Let $T > 0$, $f \in C^1([0,T];X)$ and set*

$$v(t) = \int_0^t T(t-s)f(s)\,ds, \text{ for } 0 \le t \le T. \tag{1.5.3}$$

Then, the following properties hold:

(i) $v \in C^1([0,T],X)$;

(ii) $v'(t) = f(0) + \int_0^t T(t-s)f'(s)\,ds$, *for all* $t \in [0,T]$;

(iii) $v \in C([0,T],D(A))$;

(iv) $v'(t) + Av(t) = f(t)$, *for all* $t \in [0,T]$.

Proof. Note that

$$v(t) = \int_0^t T(s)f(t-s)\,ds, \text{ for } 0 \le t \le T.$$

Properties (i) and (ii) follow easily. Let now $0 \le t < T$, and $0 < h \le T - t$. Applying (1.5.3), we obtain

$$\frac{v(t+h)-v(t)}{h} = \frac{T(h)-I}{h}v(t) + \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\,ds. \tag{1.5.4}$$

Letting $h \downarrow 0$ and using the fact that $v \in C^1([0,T],X)$, it follows that $v(t) \in D(A)$ and that $v'(t) = -Av(t) + f(t)$. Since $f - v' \in C([0,T],X)$ and $G(A)$ is closed, it follows that $v(T) \in D(A)$ and that $Av(T) = f(T) - v'(T)$. Hence (iii) and (iv). □

Proposition 1.5.5. *Let $T > 0$, $f \in C([0,T];D(A))$ and let v be defined by (1.5.3). Then, the following properties hold:*

(i) $v \in C([0,T],D(A))$;

(ii) $Av(t)(t) = \int_0^t T(t-s)Af(s)\,ds$, *for all* $t \in [0,T]$;

(iii) $v \in C^1([0,T],X)$;

(iv) $v'(t) + Av(t) = f(t)$, *for all* $t \in [0,T]$.

Proof. (i) and (ii) follow from Corollary 1.1.12 (iii), Remarks A.2.18 (vii) and A.2.15 (iii), and Proposition 1.3.4 (iii). Letting $h \downarrow 0$ in formula (1.5.4) and using the fact that $v \in C([0,T],D(A))$, (iii) and (iv) follow easily (cf. the proof of Proposition 1.5.4. □

Corollary 1.5.6. *Let $T > 0$, $x \in D(A)$, $f \in C([0,T];X)$ and let u be defined by formula (1.5.2). If either $f \in C^1([0,T],X)$ or $f \in C([0,T];D(A))$, then, the following properties hold:*

(i) *u solves problem (1.5.1);*

(ii) *if $f \in C^1([0,T],X)$, then* $u'(t) = -T(t)Ax + f(0) + \int_0^t T(t-s)f'(s)\,ds$, *for all* $t \in [0,T]$;

(iii) *if $f \in C([0,T],D(A))$, then* $Au(t) = T(t)Ax + \int_0^t T(t-s)Af(s)\,ds$, *for all* $t \in [0,T]$.

Proof. Note that $u(t) = T(t)x + v(t)$, where $v(t)$ is defined by (1.5.3). Therefore, the result follows from Propositions 1.3.4, 1.5.4 and 1.5.5. $\square$

Theorem 1.5.7. *Let $T > 0$, $x \in D(A)$, $f \in C([0,T], X)$ and let u be defined by formula (1.5.2). If one of the following properties hold:*

(i) *$f \in L^1((0,T), D(A))$;*

(ii) *$f \in W^{1,1}((0,T), X)$;*

then u is the solution of problem (1.5.1).

Proof. Assume first that (i) holds. Let $(f_n)_{n\geq 0} \subset C^1([0,T], D(A))$ converge to f in $L^1((0,T), D(A))$, and let $(u_n)_{n\geq 0}$ be the corresponding solutions of (1.5.2). It follows from Lemma 1.5.3 that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([0,T], X)$. Furthermore, it follows from Lemma 1.5.3 and Corollary 1.5.6 (iii) that $(u_n)_{n\geq 0}$ is a Cauchy sequence in $C([0,T], D(A))$. Therefore, $u \in C([0,T], D(A))$, and $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([0,T], D(A))$. On the other hand, since u_n solves (1.5.1), u_n' is a Cauchy sequence in $C([0,T], X)$; and so, $u_n' \underset{n\to\infty}{\longrightarrow} u'$ in $C([0,T], X)$ (cf. Corollary A.2.39). It follows that u solves problem (1.5.1).

If (ii) holds, let $(f_n)_{n\geq 0} \subset C^1([0,T], X)$ converge to f in $W^{1,1}((0,T), X)$, and let $(u_n)_{n\geq 0}$ be the corresponding solutions of (1.5.2). Applying Lemma 1.5.3, we find $u_n \underset{n\to\infty}{\longrightarrow} u$ in $C([0,T], X)$. On the other hand, it follows from Lemma 1.5.3 and Corollary 1.5.6 (ii) that $(u_n)_{n\geq 0}$ is a Cauchy sequence in $C^1([0,T], X)$. Therefore, $u \in C^1([0,T], X)$, and $u_n \underset{n\to\infty}{\longrightarrow}$ in $C^1([0,T], X)$. On the other hand, since u_n solves (1.5.1), Au_n is a Cauchy sequence in $C([0,T], X)$. By closedness of $G(A)$, it follows easily that $u \in C([0,T], D(A))$, and that $u_n \underset{n\to\infty}{\longrightarrow}$ in $C([0,T], D(A))$. It follows that u solves problem (1.5.1). This completes the proof. $\square$

Corollary 1.5.8. *Let $T > 0$ and $x \in D(A)$. Consider a Lipschitz continuous function $f : [0,T] \to X$, and let u be defined by formula (1.5.2). If X is reflexive, then u is the solution of problem (1.5.1).*

Proof. The result follows from Theorem 1.5.7 and Corollary A.2.38. $\square$

Remark 1.5.9. Note that it is essential that X be reflexive in Corollary 1.5.8, as shows the following example. Let $X = L^1(\mathbb{R})$, let A be the operator defined in Remark 1.4.2 (i) and let $(T(t))_{t\in\mathbb{R}}$ be the group if isometries generated by $-A$. Let $f(t) = T(t)1_{(0,1)}$, for all $t \in \mathbb{R}$. It follows from Remark 1.4.2 (i) that $f(t) = 1_{(t,t+1)}$. In particular, $\|f(t) - f(s)\| \leq 2|t-s|$, for all t, s; and so, f is Lipschitz continuous. The function u defined by (1.5.2) with $x = 0$ is

$$u(t) = \int_0^t T(t)1_{(0,1)}\, ds = tT(t)1_{(0,1)} = t1_{(t,t+1)}.$$

Note that $D(A) = W^{1,1}(\mathbb{R}) \subset C(\mathbb{R})$. Therefore, $u(t) \notin D(A)$, if $t \neq 0$. In particular, u does not solve (1.5.1).

Corollary 1.5.10. *Let $T > 0$, $x \in X$, $f \in C([0,T], X)$ and let $u \in C([0,T], X)$ be defined by (1.5.2). Consider the space X_{-1} and the operator $A_{(-1)}$ defined by Theorem 1.1.31. Then, u is the unique solution* *of the following problem:*

$$\begin{cases} u \in C([0,T],X) \cap C^1([0,T],X_{-1}); \\ \dfrac{du}{dt} + A_{(-1)}u = f, \ \textit{for all } t \in [0,T]; \\ u(0) = x. \end{cases} \tag{1.5.5}$$

Proof. Apply Theorem 1.5.7 to the operator $A_{(-1)}$. Uniqueness follows from Corollary 1.5.2, applied to the operator $A_{(-1)}$. □

Remark 1.5.11. Theorem 1.5.7 means that, given $x \in D(A)$ and $f \in C([0,T],X)$, problems (1.5.1) and (1.5.2) are equivalent under the extra assumption $f \in W^{1,1}((0,T),X) + L^1((0,T),D(A))$. It can also be useful to have equivalence of problems (1.5.1) and (1.5.2) under extra assumptions on u instead of f. This is the object of the following result.

Proposition 1.5.12. *Let $T > 0$, $x \in X$, $f \in C([0,T],X)$ and let $u \in C([0,T],X)$ be defined by (1.5.2). If one of the following assumptions holds:*

(i) $u \in C([0,T],D(A))$;

(ii) $u \in C^1([0,T],X)$;

then u solves problem (1.5.1).

Proof. Let $v(t)$ be given by (1.53). Since $u(t) = v(t) + T(t)x$, we deduce from (1.54) that

$$\frac{u(t+h) - u(t)}{h} = \frac{T(h) - I}{h} u(t) + \frac{1}{h} \int_t^{t+h} T(t+h-s) f(s)\, ds.$$

Then we argue as in the proofs of Propositions 1.54 and 1.55. □

Until now we have always assumed that f is continuous. However, note that formula (1.5.2) makes sense if we assume only $f \in L^1((0,T),X)$ (cf. Lemma 1.5.3). We have similar results in this case, which we describe below.

Lemma 1.5.13. *Let $T > 0$, $x \in X$ and $f \in L^1((0,T),X)$. If u solves the following problem:*

$$\begin{cases} u \in L^1((0,T),D(A)) \cap W^{1,1}((0,T),X); \\ \dfrac{du}{dt} + Au = f, \ \textit{for almost all } t \in (0,T); \\ u(0) = x; \end{cases} \tag{1.5.6}$$

then u is given by (1.5.2) for all $t \in [0,T]$.

Proof. Note first that $u \in W^{1,1}((0,T),X) \hookrightarrow C([0,T],X)$; and so, condition $u(0) = x$ makes sense. Also, it follows from Lemma 1.5.3 that (1.5.2) makes sense. Given $t \in (0,T]$, set $w(s) = T(t-s)u(s)$ for $0 \le s \le t$. Given $0 \le s \le t - h$, we have

$$\frac{w(s+h) - w(s)}{h} = T(t-s-h)\Big\{\frac{u(s+h) - u(s)}{h} - \frac{T(h) - I}{h} u(s)\Big\}. \tag{1.5.7}$$

Since

$$T(t-s-h)\frac{T(h)-I}{h}u(s)\underset{h\to 0}{\longrightarrow}-T(t-s)Au(s)\in L^1((0,t),X),$$

and

$$\|T(t-s-h)\frac{T(h)-I}{h}u(s)\|\le\|Au(s)\|\in L^1(0,T),$$

for almost all s (see Proposition 1.3.4), it follows from the dominated convergence theorem that

$$T(t-\cdot-h)\frac{T(h)-I}{h}u\underset{h\to 0}{\longrightarrow}-T(t-\cdot)Au,$$

in $L^1((\varepsilon,t-\varepsilon),X)$ for every $\varepsilon>0$. On the other hand, it follows easily from Corollary A.2.36 that also

$$T(t-\cdot-h)\frac{u(\cdot+h)-u(\cdot)}{h}\underset{h\to 0}{\longrightarrow}T(t-\cdot)u'(\cdot),$$

in $L^1((\varepsilon,t-\varepsilon),X)$. Therefore,

$$\frac{w(\cdot+h)-w(\cdot)}{h}\underset{h\to 0}{\longrightarrow}T(t-\cdot)(u'+Au)=T(t-\cdot)f,$$

in $L^1((\varepsilon,t-\varepsilon),X)$ for every $\varepsilon>0$. It follows easily that $w\in W^{1,1}((\varepsilon,t-\varepsilon),X)$ and that $w'=T(t-\cdot)f$. Since $T(t-\cdot)f\in L^1((0,t),X)$, it follows that $w\in W^{1,1}((0,t),X)$ and that $w'=T(t-\cdot)f$. Integrating the last identity between 0 and t, we obtain (1.5.2). □

Corollary 1.5.14. *Let $T>0$, $x\in X$ and $f\in L^1((0,T),X)$. Then, problem (1.5.6) has at most one solution, given by formula (1.5.2).*

Lemma 1.5.15. *Let $T>0$, $x\in D(A)$, $f\in L^1((0,T),D(A))$, and let u be defined by (1.5.2). Then, $u\in C([0,T],D(A))$ and u solves (1.5.6).*

Proof. It follows from Lemma 1.5.3, applied in the space $D(A)$, that $u\in C([0,T],D(A))$. Consider $(f_n)_{n\in\mathbb{N}}\subset C([0,T],D(A))$ such that $f_n\underset{n\to\infty}{\longrightarrow}f$, in $L^1((0,T),D(A))$, and let u_n be given by (1.5.2) relative to f_n. It follows from Lemma 1.5.3 (applied in the space $D(A)$) that $u_n\underset{n\to\infty}{\longrightarrow}u$ in $C([0,T],D(A))$. On the other hand, it follows from Theorem 1.5.7 that u_n solves (1.5.1). Therefore,

$$u_n'=-Au_n+f_n\underset{n\to\infty}{\longrightarrow}-Au+f,$$

in $L^1((0,T),X)$. It follows from Remark A.2.29 that $u\in W^{1,1}((0,T),X)$ and that u solves (1.5.6). □

Corollary 1.5.16. *Let $T>0$, $x\in X$, $f\in L^1((0,T),X)$, $u\in C([0,T],X)$, and consider the space X_{-1} and the operator $A_{(-1)}$ defined by Theorem 1.1.31. Then, u solves the following problem:*

$$\begin{cases} u\in C([0,T],X)\cap W^{1,1}((0,T),X_{-1}); \\ \dfrac{du}{dt}+A_{(-1)}u=f,\ \text{for almost all } t\in(0,T); \\ u(0)=x; \end{cases}\tag{1.5.8}$$

if, and only if u is given by (1.5.2).

Proof. The result follows from Lemmas 1.5.13 and 1.5.15 applied in the space X_{-1}. □

Corollary 1.5.17. *Let $T > 0$, $x \in X$, $f \in L^1((0,T),X)$ and let $u \in C([0,T],X)$ be defined by (1.5.2). If one of the following assumptions holds:*

(i) $u \in L^1((0,T),D(A))$;

(ii) $u \in W^{1,1}((0,T),X)$;

then u solves problem (1.5.6).

Proof. Assume first that (i) holds. Then, it follows from Corollary 1.5.16 that

$$\frac{du}{dt} = -A_{(-1)}u + f = -Au + f \in L^1((0,T),X);$$

and so, $u \in W^{1,1}((0,T),X)$ and u solves (1.5.1). Assume now that (ii) holds. Then, it follows from Corollary 1.5.16 that $A_{(-1)}u \in L^1((0,T),X)$. Applying Corollary 1.1.34, we find $u \in L^1((0,T),D(A))$. This completes the proof. □

Corollary 1.5.18. *Let $T > 0$, $x \in X$, $f \in L^1((0,T),X)$ and let $u \in C([0,T],X)$ be defined by (1.5.2). For every $0 \le s < T$, we have*

$$u(t+s) = T(t)u(s) + \int_0^t T(t-\sigma)f(s+\sigma)\,d\sigma, \tag{1.5.9}$$

for all $t \in [0,T-s]$. Equivalently,

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)f(\sigma)\,d\sigma, \tag{1.5.10}$$

for all $t \in [s,T]$.

Proof. It is clear that (1.5.9) and (1.5.10) are equivalent. Let $v(t)$ be equal to the right-hand side of (1.5.9). It follows from Corollary 1.5.16 that v is the unique solution of

$$\begin{cases} v \in C([0,T-s],X) \cap W^{1,1}((0,T-s),X_{-1}); \\ \dfrac{dv}{dt} + A_{(-1)}v = f, \text{ for almost all } t \in (0,T-s); \\ v(0) = u(s). \end{cases}$$

On the other hand, it it follows from Corollary 1.5.16 that $u(s+\cdot)$ also solves the above problem. Hence the result. □

Remark. Evidently, one could prove easily formula (1.5.9) by using only (1.5.2). However, our proof of Corollary 1.5.16 explains quite clearly why (1.5.9) holds.

Remark 1.5.19. Assume that the operator $-A$ is the generator of a group of isometries $(T(t))_{t\in\mathbb{R}}$. Then all the results of this section hold if one replaces the interval $[0,T]$ by the interval $[-S,T]$ with $S \ge 0$. In particular, formulas (1.5.2) and (1.5.6) hold for $-S \le t \le T$.

1.6. Specific properties of various nonhomogeneous partial differential equations. We know that the nonhomogeneous problem $u_t + Au = f$ can be solved, under appropriate assumptions on A and f (see Section 1.5). We now investigate specific estimates of u, u_t and Au in various examples.

1.6.1. The heat equation. Throughout this section, Ω is an arbitrary open subset of $\mathbb{R}^N$. Consider the operator A defined on $L^2(\Omega)$ by

$$\begin{cases} D(A) = \{u \in H^1_0(\Omega);\ \triangle u \in L^2(\Omega)\}, \\ Au = -\triangle u, \text{ for all } u \in D(A). \end{cases}$$

It follows from the results of Section 1.4.2 that $-A$ is the generator of a semigroup of contractions on $L^2(\Omega)$, which we denote by $(T(t))_{t\geq 0}$. Moreover, $(T(t))_{t\geq 0}$ is analytic and verifies the estimate $\|T(t)\varphi\|_{L^p} \leq \|\varphi\|_{L^p}$ for all $1 \leq p \leq \infty$ (in fact, $(T(t))_{t\geq 0}$ is a semigroup of contractions in $L^p(\Omega)$ for $1 \leq p < \infty$). Therefore, for every $T < 0$, $1 \leq p < \infty$ and for every $f \in L^1((0,T), L^p(\Omega))$ the function

$$u(t) = \int_0^t T(t-s)f(s)\,ds, \tag{1.6.1}$$

belongs to $C([0,T], L^p(\Omega))$ and is the weak solution of the problem

$$\begin{cases} u_t - \triangle u = f, \\ u_{|\partial\Omega} = 0, \\ u(0) = 0. \end{cases}$$

We have the following result.

Theorem 1.6.1. *Let $T > 0$, $1 < p, q < \infty$ and $f \in L^q((0,T), L^p(\Omega))$. If u is defined by (1.6.1), then $u \in W^{1,q}((0,T), L^p(\Omega))$ and $\triangle u \in L^q((0,T), L^p(\Omega))$. Moreover, there exists a constant C such that*

$$\|u_t\|_{L^q((0,T),L^p)} + \|\triangle u\|_{L^q((0,T),L^p)} \leq C\|f\|_{L^q((0,T),L^p)},$$

for all $f \in L^q((0,T), L^p(\Omega))$.

Proof. The result is an immediate consequence of the following abstract theorem. □

Theorem 1.6.2. *Let Ω be an open subset of $\mathbb{R}^N$ and let $(T(t))_{t\geq 0}$ be a semigroup of contractions on $L^2(\Omega)$ and let $-A$ be its generator. Assume further that*

(i) *$(T(t))_{t\geq 0}$ is an analytic semigroup;*

(ii) *for all $1 \leq p \leq \infty$, $\|T(t)\varphi\|_{L^p} \leq \|\varphi\|_{L^p}$ for all $t \geq 0$ and all $\varphi \in L^2(\Omega) \cap L^p(\Omega)$.*

Let $1 < p, q < \infty$ and $T > 0$. For every $f \in L^q((0,T), L^p(\Omega))$, the function

$$u(t) = \int_0^t T(t-s)f(s)\,ds$$

is well defined, $u \in C([0,T], L^p(\Omega))$. Moreover $u \in W^{1,q}((0,T), L^p(\Omega)$, $Au \in L^q((0,T), L^p(\Omega)$ and there exists a constant C such that

$$\|u_t\|_{L^q((0,T),L^p)} + \|Au\|_{L^q((0,T),L^p)} \leq C\|f\|_{L^q((0,T),L^p)},$$

for every $f \in L^q((0,T), L^p(\Omega))$.

Proof. See Coulhon and Lamberton [31] and Lamberton [74]. □

Remark 1.6.3. The type of regularity property described in the above theorem is called "maximal regularity", since it implies that all the members of the equation $u_t + Au = f$ have the same regularity.

1.6.2. The heat equation with a potential. Here we consider the equation

$$\begin{cases} u_t - \triangle u - a(t,x)u = f(t,x), \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{1.6.2}$$

under various assumptions on the potential a. We write equation (1.6.2) in the form

$$u(t) = T(t)u_0 + \int_0^t T(t-s)(a(s)u(s) + f(s))\, ds, \tag{1.6.3}$$

For simplicity, we assume throughout this section that $|\Omega| < \infty$.

If $a(t,x) \in L^\infty((0,T), L^\infty(\Omega))$ many results concerning the heat equation extend to this case. In particular, the smoothing effect $L^p(\Omega) - L^q(\Omega)$ (Theorem 1.4.15) can be obtained by a very simple comparison argument. Indeed, the solution u of (1.6.3) with $f = 0$ satisfies $|u| \le e^{Kt}v$, where $v(t) = T(t)|u_0|$ and $K = \|a\|_{L^\infty((0,T)\times\Omega)}$.

However, one can do better. A natural assumption is that $a \in L^\infty((0,T), L^\sigma(\Omega))$ with $\sigma \ge 1$, $\sigma > \dfrac{N}{2}$. Indeed, there are two reasons why this assumption is natural:

First reason. Assume for simplicity $N \ge 3$. If one multiplies the equation (1.6.2) by u, one is lead to estimate

$$\int_0^T \int_\Omega au^2.$$

This can be done using the following device: for all $\varepsilon > 0$, there exists $C(\varepsilon)$ such that

$$\int_0^T \int_\Omega au^2 \le \varepsilon \int_0^T \int_\Omega |\nabla u|^2 + C(\varepsilon) \int_0^T \int_\Omega u^2.$$

Indeed, by Hölder, we have

$$\int_\Omega au^2 \le \|a\|_{L^\sigma} \|u\|^2_{L^{\frac{2}{\sigma'}}} \le \|a\|_{L^\sigma} \|u\|^\alpha_{L^{\frac{2N}{N-2}}} \|u\|^{1-\alpha}_{L^2},$$

with $0 < \alpha < 1$, since $2 < 2\sigma' < \dfrac{2N}{N-2}$. The desired conclusion follows from Young's and Sobolev's inequalities. Finally, we choose $\varepsilon = 1/2$ and we obtain the estimates

$$\|u(t)\|_{L^2} \le e^{Ct} \|u_0\|_{L^2},$$
$$\int_0^T \int_\Omega |\nabla u|^2 \le e^{Ct} \|u_0\|^2_{L^2}.$$

Similarly, if one multiplies by powers of u one can establish $L^p - L^q$ estimates as in the proof of Proposition 1.7.3.

Second reason. The assumption that $a \in L^\infty((0,T), L^\sigma(\Omega))$ with $\sigma \ge 1$, $\sigma > \dfrac{N}{2}$ is also natural from the point of view of the integral equation (1.6.3). Indeed, note that the operator

$$u \mapsto \Psi(u)(t) = \int_0^t T(t-s)a(s)u(s)\,ds$$

is a bounded linear operator $C([0,T], L^q(\Omega)) \to C([0,T], L^q(\Omega))$, for every $q \ge \sigma'$. This follows from the fact that

$$\|T(t-s)a(s)u(s)\|_{L^q} \le (t-s)^{-\frac{N}{2\sigma}} \|a(s)\|_{L^\sigma} \|u(s)\|_{L^q},$$

and $\sigma > N/2$.

The same reasonning applies if instead $a \in L^1((0,T), L^\infty(\Omega))$, or more generally, $a \in L^1((0,T), L^\infty(\Omega)) + L^\infty((0,T), L^\sigma(\Omega))$. This space contains all the spaces of the form $L^\beta((0,T), L^\gamma(\Omega))$ with $\gamma > N/2$, $\gamma \ge 1$ and $\dfrac{1}{\beta} + \dfrac{N}{2\gamma} < 1$ (since σ can be arbitrarily chosen, $\sigma > N/2$).

Note that the above argument breaks down when σ takes the critical value $N/2$. However, we will be able to conclude in some "critical cases" when $a \in L^\beta((0,T), L^\gamma(\Omega))$ with $\dfrac{1}{\beta} + \dfrac{N}{2\gamma} = 1$ and $\gamma > N/2$, $\gamma \ge 1$. Here, the key ingredient is to use the singular convolution estimates of Marcinkiewicz. Such an a belongs to $L^1((0,T), L^\infty(\Omega)) + L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$ but we cannot replace $\dfrac{N}{2}$ by some $\sigma > \dfrac{N}{2}$.

We first give a result of the form $\|u(t)\|_{L^\infty} \le A\|u_0\|_{L^\infty} + B\|f\|_{L^\infty((0,T),L^\sigma)}$. It will be useful for later purpose to have an explicit dependence for A and B in terms of a and t.

Theorem 1.6.4. *Let $0 < T < \infty$, let $\sigma > \dfrac{N}{2}$, $\sigma \ge 1$, and let $a, f \in L^\infty((0,T), L^\sigma(\Omega))$. Given $u_0 \in L^\infty(\Omega)$, there exists a unique solution $u \in L^\infty((0,T), L^\infty(\Omega))$ of (1.6.3) on $(0,T)$, and it satisfies*

$$\|u(t)\|_{L^\infty} \le 2e^{Ct\|a\|^\alpha_{L^\infty((0,t),L^\sigma)}} \left(\|u_0\|_{L^\infty} + t^{1-\frac{N}{2\sigma}} \|f\|_{L^\infty((0,t),L^\sigma)} \right), \tag{1.6.4}$$

for all $t \in (0,T)$, with $\alpha = \dfrac{2\sigma}{2\sigma - N}$.

Moreover, uniqueness holds in the class $L^\infty((0,T), L^{\sigma'}(\Omega))$.

Proof. We first show that (1.6.3) has at most one solution in $L^\infty((0,T), L^{\sigma'}(\Omega))$. (Note that $au \in L^\infty((0,T), L^1(\Omega))$, so that the equation (1.6.3) makes sense in $L^1(\Omega)$.) Indeed, if u and v are two solutions, we have

$$\begin{aligned}
\|u(t) - v(t)\|_{L^{\sigma'}} &\le C \int_0^t (t-s)^{-\frac{N}{2}(1-\frac{1}{\sigma'})} \|a(u-v)\|_{L^1}\,ds \\
&\le C \int_0^t (t-s)^{-\frac{N}{2\sigma}} \|a\|_{L^\sigma} \|u-v\|_{L^{\sigma'}}\,ds \\
&\le C\|a\|_{L^\infty((0,T),L^\sigma)} \int_0^t (t-s)^{-\frac{N}{2\sigma}} \|u-v\|_{L^{\sigma'}}\,ds,
\end{aligned}$$

and it follows from Proposition A.5.7 that $u = v$.

We now prove that the equation (1.6.3) has a solution in $L^\infty((0,T), L^\infty(\Omega))$. We apply the contraction mapping principle to the map $\Phi : L^\infty((0,T), L^\infty(\Omega)) \to L^\infty((0,T), L^\infty(\Omega))$ defined by

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)(a(s)u(s) + f(s))\,ds, \tag{1.6.5}$$

for $t \in (0,T)$. Note that

$$\|\Phi(u)(t) - \Phi(v)(t)\|_{L^\infty} \le C\int_0^t (t-s)^{-\frac{N}{2\sigma}} \|a(s)\|_{L^\sigma} \|u(s)-v(s)\|_{L^\infty}\, ds$$
$$\le CT^{1-\frac{N}{2\sigma}} \|a\|_{L^\infty((0,T),L^\sigma)} \|u-v\|_{L^\infty((0,T),L^\infty)}.$$

Hence, Φ is a strict contraction for example if

$$C\|a\|_{L^\infty((0,T),L^\sigma)} T^{1-\frac{N}{2\sigma}} \le \frac{1}{2}.$$

Therefore, Φ has a fixed point, which is a solution of (1.6.3). In this case, the conclusion of the theorem follows with $\|u(t)\|_{L^\infty} \le 2(\|u_0\|_{L^\infty} + CT^{1-\frac{N}{2\sigma}}\|f\|_{L^\infty((0,T),L^\sigma)})$. The general case follows by a standard iteration argument. □

Remark 1.6.5. Let $0 < T < \infty$, let $\sigma > \dfrac{N}{2}$ and let $\beta \ge 1$ be such that $\dfrac{1}{\beta} < 1 - \dfrac{N}{2\sigma}$. Let $a, f \in L^\beta((0,T), L^\sigma(\Omega))$ and let $u_0 \in L^\infty(\Omega)$. Then, there exists a unique solution $u \in L^\infty((0,T), L^\infty(\Omega))$ of (1.6.3) on $(0,T)$, and it satisfies

$$\|u(t)\|_{L^\infty} \le 2e^{Ct\|a\|^\mu_{L^\beta((0,t),L^\sigma)}} \Big(\|u_0\|_{L^\infty} + Ct^{1-\frac{N}{2\sigma}-\frac{1}{\beta}} \|f\|_{L^\beta((0,t),L^\sigma)} \Big),$$

for all $t \in (0,T)$, with $\dfrac{1}{\mu} = 1 - \dfrac{N}{2\sigma} - \dfrac{1}{\beta}$. Moreover, uniqueness holds in the class $L^\infty((0,T), L^{\sigma'}(\Omega))$. This is proved by the same argument as in Theorem 1.6.4.

The next result concerns the same equation (1.6.3) except that we now consider the case $T = \infty$, and we are interested in obtaining a bound for $\|u\|_{L^\infty((0,\infty),L^\infty)}$. Note that (1.6.4) does not give any uniform bound as $t \to \infty$.

Theorem 1.6.6. *Let $\sigma > \dfrac{N}{2}$, $\sigma \ge 1$, and let $a, f \in L^\infty((0,\infty), L^\sigma(\Omega))$. Let $u_0 \in L^\infty(\Omega)$ and $u \in L^\infty_{\mathrm{loc}}([0,\infty), L^{\sigma'}(\Omega))$ verify (1.6.3) for all $t > 0$. If $u \in L^\infty((0,\infty), L^1(\Omega))$, then $u \in L^\infty((0,\infty), L^\infty(\Omega))$ and*

$$\|u\|_{L^\infty((0,\infty),L^\infty)} \le 4\|u_0\|_{L^\infty} + C\left(\|u\|_{L^\infty((0,\infty),L^1)} + \|f\|_{L^\infty((0,\infty),L^\sigma)}\right),$$

where C is independent of u.

Proof. It follows from (1.6.4) that

$$\|u\|_{L^\infty((0,T_1),L^\infty)} \le 4\left(\|u_0\|_{L^\infty} + \|f\|_{L^\infty((0,\infty),L^\sigma)}\right),$$

for $T_1 > 0$ small enough. Next, we have

$$u(t+s) = T(t)u(s) + \int_0^t T(t-\tau)(a(s+\tau)u(s+\tau) + f(s+\tau))\, d\tau.$$

Let $1 \le q \le r \le \infty$ be such that $\dfrac{1}{q} = \dfrac{1}{r} + \dfrac{1}{\sigma}$. It follows from Theorem 1.4.15 that

$$\|u(t+s)\|_{L^r} \le Ct^{-\frac{N}{2\sigma}} \|u(s)\|_{L^q} + C\int_0^t (t-\tau)^{-\frac{N}{2\sigma}} \|a(s+\tau)\|_{L^\sigma} \|u(s+\tau)\|_{L^r}\, d\tau$$
$$+ C\int_0^t (t-\tau)^{-\frac{N}{2\sigma}} \|f(s+\tau)\|_{L^q}\, d\tau.$$

Note that $q \le \sigma$, so that

$$\int_0^t (t-\tau)^{-\frac{N}{2\sigma}} \|f(s+\tau)\|_{L^q}\, d\tau \le Ct^{1-\frac{N}{2\sigma}} \|f\|_{L^\infty((0,\infty),L^\sigma)} \le Ct^{-\frac{N}{2\sigma}} \|f\|_{L^\infty((0,\infty),L^\sigma)},$$

for $t \le 1$. From the generalized Gronwall inequality (Proposition A.5.7), we deduce that there exists C such that

$$\|u(t+s)\|_{L^r} \le Ct^{-\frac{N}{2\sigma}} (\|u(s)\|_{L^q} + \|f\|_{L^\infty((0,\infty),L^\sigma)}),$$

for all $s \ge 0$ and $t \in (0,1]$.

In particular, for any $\varepsilon > 0$, there exists C such that for every $\tau \ge 0$,

$$\|u\|_{L^\infty(\tau+\varepsilon,\infty),L^r)} \le C \left(\|u\|_{L^\infty((\tau,\infty),L^q)} + \|f\|_{L^\infty((0,\infty),L^\sigma)} \right). \tag{1.6.6}$$

Finally, let $m = [\sigma]$. Starting from $\|u\|_{L^\infty((0,\infty),L^1)}$ and iterating m times the estimate (1.6.6), we find

$$\|u\|_{L^\infty((\frac{mT_1}{m+1},\infty),L^\gamma)} \le C \left(\|u\|_{L^\infty((0,\infty),L^1)} + \|f\|_{L^\infty((0,\infty),L^\sigma)} \right),$$

with $1 - \dfrac{1}{\gamma} = \dfrac{m}{\sigma}$. In particular, $\gamma \ge \sigma$. Applying finally (1.6.6) with $q = \sigma$ and $r = \infty$, we deduce

$$\|u\|_{L^\infty(T_1,\infty),L^\infty)} \le C \left(\|u\|_{L^\infty((0,\infty),L^1)} + \|f\|_{L^\infty((0,\infty),L^\sigma)} \right).$$

This completes the proof. □

We now return to the same equation (1.6.3) but on a finite interval $[0,T]$, and we assume that $u_0 \in L^q(\Omega)$, for some $q < \infty$. We study the smoothing effect.

Theorem 1.6.7. *Let $0 < T < \infty$, let $\sigma > \dfrac{N}{2}$, $\sigma \ge 1$, and let $a, f \in L^\infty((0,T), L^\sigma(\Omega))$. Given $u_0 \in L^q(\Omega)$, $1 \le q < \infty$, there exists a unique solution $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ of equation (1.6.2). Moreover, there is a constant C depending only on $N, \sigma, q, |\Omega|$ such that u satisfies*

$$\|u(t)\|_{L^\infty} \le Ce^{Ct\|a\|^\alpha_{L^\infty((0,t),L^\sigma)}} \left((t^{-\frac{N}{2q}} + 1)\|u_0\|_{L^q} + t^{1-\frac{N}{2\sigma}} \|f\|_{L^\infty((0,t),L^\sigma)} \right), \tag{1.6.7}$$

for all $t \in (0,T]$, with $\alpha = \left(1 - \dfrac{N}{2\sigma}\right)^{-1}$.

Uniqueness also holds in the class $L^\infty((0,T), L^q(\Omega))$ provided $q \ge \sigma'$ (without having to assume $u \in L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$).

Proof. By a solution $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ of equation (1.6.2), we mean that

$$\begin{cases} u(t) = T(t-\varepsilon)u(\varepsilon) + \displaystyle\int_\varepsilon^t T(t-s)(a(s)u(s) + f(s))\, ds & \text{for } 0 < \varepsilon \le t \le T, \\ u(t) \underset{t\downarrow 0}{\longrightarrow} u_0 \quad \text{in } L^q(\Omega). \end{cases} \tag{1.6.8}$$

Note that (1.6.8) makes sense for $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$. Furthermore, if $q \ge \sigma'$ and $u \in L^\infty((0,T), L^q(\Omega))$, then $au \in L^\infty((0,T), L^1(\Omega))$, so that the equation (1.6.3) makes sense in $L^1(\Omega)$ and is equivalent to (1.6.8). We now proceed in six steps.

Step 1. Uniqueness in the class $L^\infty((0,T),L^{\sigma'}(\Omega))$. The argument is the same as in the proof of Theorem 1.6.4.

Step 2. We show that for every $q \geq 1$, there exists a constant C depending only on $N,q,\sigma,|\Omega|$ such that if $u_0 \in L^\infty(\Omega)$ and if

$$CT_1^{1-\frac{N}{2\sigma}}\|a\|_{L^\infty((0,T_1),L^\sigma)} \leq 1,$$

then the solution $u \in L^\infty((0,T),L^\infty(\Omega))$ of (1.6.3) verifies

$$t^{\frac{N}{2\sigma}}\|u(t)\|_{L^\infty} \leq C(\|u_0\|_{L^q} + t\|f\|_{L^\infty((0,t),L^\sigma)}), \tag{1.6.9}$$

for all $t \in (0,T_1)$.

Indeed, consider $1 \leq \theta \leq \sigma$, and let $\rho \in [\theta,\infty]$ be such that $\dfrac{1}{\theta} = \dfrac{1}{\rho} + \dfrac{1}{\sigma}$. We have

$$\|u(t)\|_{L^\rho} \leq t^{-\frac{N}{2\sigma}}\|u_0\|_{L^\theta} + \int_0^t (t-s)^{-\frac{N}{2\sigma}}\|a(s)\|_{L^\sigma}\|u(s)\|_{L^\rho}\,ds + \int_0^t (t-s)^{-\frac{N}{2\sigma}}\|f(s)\|_{L^\theta}\,ds;$$

and so,

$$\begin{aligned}\|u(t)\|_{L^\rho} &\leq t^{-\frac{N}{2\sigma}}\|u_0\|_{L^\theta}\\ &+t^{1-\frac{N}{\sigma}}\left(\int_0^1 (1-\tau)^{-\frac{N}{2\sigma}}\tau^{-\frac{N}{2\sigma}}\,d\tau\right)\|a\|_{L^\infty((0,t)L^\sigma)}\operatorname*{ess\,sup}_{0<s<t} t^{\frac{N}{2\sigma}}\|u(s)\|_{L^\rho} + t^{1-\frac{N}{2\sigma}}|\Omega|^{\frac{1}{\rho}}\|f\|_{L^\infty((0,t)L^\sigma)}.\end{aligned}$$

Therefore, if

$$T_1^{1-\frac{N}{2\sigma}}\|a\|_{L^\infty((0,T_1),L^\sigma)}\left(\int_0^1 (1-\tau)^{-\frac{N}{2\sigma}}\tau^{-\frac{N}{2\sigma}}\,d\tau\right) \leq \frac{1}{2},$$

then

$$t^{\frac{N}{2\sigma}}\|u(t)\|_{L^\rho} \leq 2\|u_0\|_{L^\theta} + 2t|\Omega|^{\frac{1}{\rho}}\|f\|_{L^\infty((0,t),L^\sigma)}, \tag{1.6.10}$$

for all $t \in (0,T_1)$. A similar argument in the case $\theta \geq \sigma$, $\rho = \infty$ shows that if

$$T_1^{1-\frac{N}{2\sigma}}\|a\|_{L^\infty((0,T_1),L^\sigma)}\left(\int_0^1 (1-\tau)^{-\frac{N}{2\sigma}}\tau^{-\frac{N}{2\theta}}\,d\tau\right) \leq \frac{1}{2},$$

then

$$t^{\frac{N}{2\theta}}\|u(t)\|_{L^\infty} \leq 2\|u_0\|_{L^\theta} + 2t\|f\|_{L^\infty((0,t),L^\sigma)}, \tag{1.6.11}$$

for all $t \in (0,T_1)$. If $q \geq \sigma$, then (1.6.9) follows from (1.6.11). If $q < \sigma$, then let $m = [\sigma/q]$. Applying m times the estimate (1.6.10) to u_0, $u(t/m)$,..., respectively, we find

$$t^{\frac{Nm}{2\sigma}}\|u(t)\|_{L^\gamma} \leq C(\|u_0\|_{L^q} + t\|f\|_{L^\infty((0,t),L^\sigma)}),$$

with $\dfrac{1}{q} - \dfrac{1}{\gamma} = \dfrac{m}{\sigma}$. Now $\gamma \geq \sigma$, and we conclude by applying (1.6.11).

Step 3. If $u_0 \in L^\infty(\Omega)$, then for every $1 \leq q \leq \infty$ the estimate (1.6.7) holds. This is obtained by combining Step 2 and (1.6.4).

Step 4. There exists C such that for every $q \in [1,\infty]$ and every $u_0, v_0 \in L^\infty(\Omega)$, the corresponding solutions u and v of (1.6.3) verify

$$\|u(t)-v(t)\|_{L^q} \leq 2e^{Ct\|a\|^\alpha_{L^\infty((0,t),L^\sigma)}}\|u_0 - v_0\|_{L^q},$$

for all $t \in (0,T)$, with $\alpha = \dfrac{2\sigma}{2\sigma - N}$.

This is equivalent to the estimate

$$\|u(t)\|_{L^q} \le 2e^{Ct\|a\|^{\alpha}_{L^\infty((0,t),L^\sigma)}} \|u_0\|_{L^q}, \tag{1.6.12}$$

for the solution u of (1.6.3) with $f = 0$. We recall that, by Theorem 1.6.4, we have

$$\|u(t)\|_{L^\infty} \le 2e^{Ct\|a\|^{\alpha}_{L^\infty((0,t),L^\sigma)}} \|u_0\|_{L^\infty}. \tag{1.6.13}$$

We now prove, by a duality argument, that

$$\|u(t)\|_{L^1} \le 2e^{Ct\|a\|^{\alpha}_{L^\infty((0,t),L^\sigma)}} \|u_0\|_{L^1}. \tag{1.6.14}$$

Indeed, fix $t_0 \in (0,T]$, let $\psi \in \mathcal{D}(\Omega)$, and let w be the solution of

$$w(t) = T(t)\psi + \int_0^t T(t-s)(b(s)w(s))\,ds,$$

for $t \in (0,t_0)$, where $b(t) = a(t_0 - t)$. Setting $v(t) = w(t_0 - t)$ for $0 \le t \le t_0$, it follows that v solves the equation

$$\begin{cases} -v_t - \triangle v = av, \\ v_{|\partial\Omega} = 0, \\ v(t_0) = \psi. \end{cases}$$

Now if u is the solution of (1.6.3) with $f \equiv 0$ and the initial value u_0, we have

$$\begin{aligned}\Big[\int_\Omega uv\Big]_0^{t_0} &= \int_0^{t_0}\int_\Omega (uv_t + u_t v) = \int_0^{t_0}\int_\Omega (u(-\triangle v - av) + v(\triangle u + au)) \\ &= \int_0^{t_0}\int_\Omega (-u\triangle v + v\triangle u) = \int_0^{t_0}\int_\Omega (\nabla u\cdot\nabla v - \nabla v\cdot\nabla u) = 0.\end{aligned}$$

(These calculations are valid provided u_0 and a are sufficiently smooth, and then the result is obtained for general $u_0 \in L^\infty(\Omega)$ and $a \in L^\infty((0,T), L^\sigma(\Omega))$ by a density argument.) Therefore,

$$\int_\Omega u(t_0)\psi = \int_\Omega u_0 w(t_0).$$

It follows that

$$\begin{aligned}\|u(t_0)\|_{L^1} &= \sup\Big\{\int_\Omega u(t_0)\psi;\ \psi \in \mathcal{D}(\Omega) \text{ and } \|\psi\|_{L^\infty} \le 1\Big\} \\ &\le \|u_0\|_{L^1}\|w(t_0)\|_{L^\infty} \\ &\le 2e^{Cs\|a\|^{\alpha}_{L^\infty((0,t_0),L^\sigma)}}\|u_0\|_{L^1},\end{aligned}$$

where the last inequality follows from (1.6.13). Since $t_0 \in (0,T)$ is arbitrary, this proves (1.6.14). The general case $1 < q < \infty$ now follows from (1.6.13), (1.6.14) and Riesz-Thorin's interpolation theorem (Theorem A.5.11).

Step 5. Existence in the class $C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$. Let $u_0 \in L^q(\Omega)$, and let $(u_0^n)_{n\ge 0} \subset L^\infty(\Omega)$ be such that $u_0^n \underset{n\to\infty}{\longrightarrow} u_0$ in $L^q(\Omega)$. Let u^n be the corresponding solutions of (1.6.3). It follows from Steps 3 and 4 that u^n converges to a limit u in $C([0,T], L^q(\Omega))$ and in $C([\varepsilon,T], L^\infty(\Omega))$ for every $0 < \varepsilon < T$. Therefore, u solves the equation (1.6.8) and satisfies the estimate (1.6.7).

Step 6. Uniqueness in the class $C([0,T],L^q(\Omega))\cap L^\infty_{\mathrm{loc}}((0,T),L^\infty(\Omega))$. Let u and v be two solutions. Given $0<\varepsilon<T$, $u(\cdot+\varepsilon)-v(\cdot+\varepsilon)$ is the solution of the equation (1.6.3) with $f=0$ and the initial value $u(\varepsilon)-v(\varepsilon)$, respectively. It follows from Step 3 that

$$\|u(t+\varepsilon)-v(t+\varepsilon)\|_{L^\infty}\le Ce^{Ct\|a\|^\alpha_{L^\infty((0,T),L^\sigma)}}(t^{-\frac{N}{2q}}+1)\|u(\varepsilon)-v(\varepsilon)\|_{L^q},$$

for all $t\in(0,T-\varepsilon)$. Letting $\varepsilon\downarrow 0$, we obtain $u(t)=v(t)$ for all $t\in(0,T)$. This completes the proof. □

The next results concern the "critical case" $a\in L^\beta((0,T),L^\gamma(\Omega))$ with $\dfrac{1}{\beta}+\dfrac{N}{2\gamma}=1$ and $\gamma>N/2$. We begin by showing that "weak solutions" of $u_t-\triangle u=au+f$ are "almost" in L^∞.

Theorem 1.6.8. *Let $0<T<\infty$, let $\gamma>\dfrac{N}{2}$, $\gamma\ge 1$, and let $a,f\in L^\beta_{\mathrm{loc}}((0,T),L^\gamma(\Omega))$ with $\dfrac{1}{\beta}+\dfrac{N}{2\gamma}=1$. If $u\in L^\mu_{\mathrm{loc}}((0,T),L^{\gamma'}(\Omega))$, with $\mu>\beta'$, satisfies the equation*

$$\begin{cases} u_t-\triangle u=au+f,\\ u_{|\partial\Omega}=0,\end{cases}\tag{1.6.15}$$

on $(0,T)$, then $u\in L^p_{\mathrm{loc}}((0,T),L^\infty(\Omega))\cap L^\infty_{\mathrm{loc}}((0,T),L^p(\Omega))$, for every $p<\infty$.

Remark 1.6.9. Note that in general $u\notin L^\infty_{\mathrm{loc}}((0,T),L^\infty(\Omega))$ under the assumptions of Theorem 1.6.8 (see Exercise 1.8.16).

Remark 1.6.10. One limiting case (which is not allowed, see Exercise 1.8.17) in Theorem 1.6.8 is $\beta=\infty$ and $\gamma=\dfrac{N}{2}$. Assume for simplicity $f\equiv 0$, and let $a\in L^\infty((0,T),L^{\frac{N}{2}}(\Omega))$. Let $u\in L^\infty((0,T),L^q(\Omega))$ for some $q<\infty$ but possibly very large. We don't know whether u belongs to $L^p_{\mathrm{loc}}((0,T),L^p(\Omega))$ for every finite p. We believe that the answer is negative.

Proof of Theorem 1.6.8. By a solution $u\in L^\mu_{\mathrm{loc}}((0,T),L^{\gamma'}(\Omega))$ of equation (1.6.15), we mean that

$$u(t)=T(t-\varepsilon)u(\varepsilon)+\int_\varepsilon^t T(t-s)(a(s)u(s)+f(s))\,ds,\tag{1.6.16}$$

for $0<\varepsilon\le t<T$. Note that $au+f\in L^1_{\mathrm{loc}}((0,T),L^1(\Omega))$, so that (1.6.16) makes sense in $L^1(\Omega)$. We proceed in four steps.

Step 1. For every $\tau>0$, $r\in[\gamma',\infty]$ and $p\in(\beta',\infty)$, the operator Ψ defined by

$$\Psi(u)(t)=\int_0^t T(t-s)(a(s)u(s)+f(s))\,ds,$$

is bounded $L^p((0,\tau),L^r(\Omega))\to L^p((0,\tau),L^r(\Omega))$. More precisely, there exists a constant $C(p,\gamma)$ such that

$$\|\Psi(u)\|_{L^p((0,\tau),L^r)}\le C(p,\gamma)\Big(\|a\|_{L^\beta((0,\tau),L^\gamma)}\|u\|_{L^p((0,\tau),L^r)}+\tau^{\frac{1}{p}}|\Omega|^{\frac{1}{r}}\|f\|_{L^\beta((0,\tau),L^\gamma)}.\Big)\tag{1.6.17}$$

Indeed, we have by Theorem 1.4.15

$$\|\Psi(u)(t)\|_{L^r}\le\int_0^t(t-s)^{-\frac{N}{2\gamma}}\Big(\|a(s)\|_{L^\gamma}\|u(s)\|_{L^r}+|\Omega|^{\frac{1}{r}}\|f\|_{L^\gamma}\Big);$$

and so, (1.6.17) follows from the Marcinkiewicz convolution theorem (see Theorem A.5.16). Here, we have used the assumption that $p > \beta'$ for, if $p = \beta'$ then $\|a(s)\|_{L^\gamma}\|u(s)\|_{L^r}$ only belongs to $L^1(0,T)$, and the convolution of $|t|^{-\frac{1}{p}}$ with L^1 does not belong to L^p.

Step 2. If $u(t_0) \in L^\rho(\Omega)$ for some $t_0 \in (0,T)$ and some $\rho \in [\gamma', \infty]$, then $u \in L^p((t_0,t_1), L^r(\Omega))$, provided $p \in (\mu, \infty)$, $r \in [\rho, \infty]$ and $t_1 \in (t_0, T)$ are such that

$$\max\{C(p,\gamma), C(\mu,\gamma)\}\|a\|_{L^\beta((t_0,t_1)L^\gamma} < 1, \tag{1.6.18}$$

and

$$\frac{N}{2}\left(\frac{1}{\rho} - \frac{1}{r}\right) < \frac{1}{p}. \tag{1.6.19}$$

Indeed, note that $v(t) = u(t_0 + t)$ verifies

$$v(t) = T(t)u(t_0) + \int_0^t T(t-s)(a(t_0+s)v(s) + f(t_0+s))\,ds. \tag{1.6.20}$$

By (1.6.19) and Theorem 1.4.15, we have $T(\cdot)u(t_0) \in L^p((0,T-t_0), L^r(\Omega))$. Therefore, by applying (1.6.18), (1.6.17) and a fixed point argument (see the proof of Theorem 1.6.4), we deduce that the equation (1.6.20) has a solution $w \in L^p((t_0,t_1), L^r(\Omega))$. The same estimate (1.6.17) applied with $p = \mu$ and $r = \gamma'$, alongwith the assumption (1.6.18), shows uniqueness in the class $L^\mu((t_0,t_1), L^{\gamma'}(\Omega))$. Since $L^p((t_0,t_1), L^r(\Omega)) \hookrightarrow L^\mu((t_0,t_1), L^{\gamma'}(\Omega))$, we deduce that $v \equiv w$.

Step 3. $u \in L^p_{\text{loc}}((0,T), L^\infty(\Omega))$ for every $p < \infty$. Fix $\varepsilon > 0$, $\varepsilon < T$. We may always assume that $a \in L^\beta((0,T), L^\gamma(\Omega))$. there exists $0 < \tau < T$ such that

$$C(\mu,\gamma) \sup_{0 \le s \le T-\tau} \|a\|_{L^\beta((s,s+\tau),L^\gamma)} < 1. \tag{1.6.21}$$

Since $u \in L^\mu_{\text{loc}}((0,T), L^{\gamma'}(\Omega))$, we have $u(t) \in L^{\gamma'}(\Omega)$ for almost all $t \in (0,T)$. Therefore, given any $t \in (\varepsilon, T)$, there exists $t - \tau < t' \le t$ such that $u(t') \in L^{\gamma'}(\Omega)$. By (1.6.21) and Step 2, we deduce that $u \in L^\mu((t', t'+\tau), L^r(\Omega))$, for any $r \in [\gamma', \infty]$ such that

$$\frac{N}{2}\left(\frac{1}{\gamma'} - \frac{1}{r}\right) < \frac{1}{\mu}.$$

Therefore, since $\varepsilon > 0$ is arbitrary, we have $u \in L^\mu_{\text{loc}}((0,T), L^r(\Omega))$. An obvious iteration of this argument shows that $u \in L^\mu_{\text{loc}}((0,T), L^\infty(\Omega))$. Then, applying once more Step 2, we obtain $u \in L^p_{\text{loc}}((0,T), L^\infty(\Omega))$, for every $p < \infty$.

Step 4. $u \in L^\infty_{\text{loc}}((0,T), L^p(\Omega))$ for all $p < \infty$. By Step 3, we may assume that $au + f \in L^\rho((0,T), L^\gamma(\Omega))$ for every $\rho < \beta$ and $u_0 \in L^\infty(\Omega)$. We have to estimate

$$\|u(t)\|_{L^q} \le C + \int_0^t (t-s)^{-\frac{N}{2}\left(\frac{1}{\gamma} - \frac{1}{q}\right)} \|au + f\|_{L^\gamma}\,ds.$$

By Hölder, the right-hand side is bounded as soon as we can find $\rho \in [1, \beta)$ such that

$$\frac{N}{2}\left(\frac{1}{\gamma} - \frac{1}{q}\right)\rho' < 1. \tag{1.6.25}$$

Note that $\frac{N}{2\gamma}\beta' = 1$, so that $\frac{N}{2}\left(\frac{1}{\gamma} - \frac{1}{q}\right)\beta' < 1$. Hence, we can always find some $r' > \beta'$, i.e. $\rho > \beta$ satisfying (1.6.25). □

Next, we consider a situation which is still "critical". The main point is to establish an estimate for u near $t = T$, without assuming any a priori behavior of u near T.

Theorem 1.6.11. *Let $0 < T < \infty$, let $\gamma > \frac{N}{2}$, $\gamma \geq 1$, let $a, f \in L^\beta((0,T), L^\gamma(\Omega))$ with $\frac{1}{\beta} + \frac{N}{2\gamma} = 1$ and let $u_0 \in L^\infty(\Omega)$. If $u \in L^\mu_{\text{loc}}([0,T), L^\infty(\Omega))$, with $\mu > \gamma'$ is solution of the equation (1.6.3) on $(0,T)$, then $u \in L^p((0,T), L^\infty(\Omega)) \cap L^\infty((0,T), L^p(\Omega))$, for every $p < \infty$.*

Proof. Fix $p \in [\gamma', \infty)$. We consider the operator defined by (1.6.5). By (1.6.17), we deduce

$$\|\Phi(u)\|_{L^p((0,T),L^\infty)} \leq C\|u_0\|_{L^\infty} + C\|a\|_{L^\beta((0,T),L^\gamma)}\|u\|_{L^p((0,T),L^\infty)} + C\|f\|_{L^\beta((0,T),L^\gamma)}. \tag{1.6.23}$$

In view of this estimate, we now proceed as follows. We choose $T_0 < T$ such that

$$C\|a\|_{L^\beta((T_0,T),L^\gamma)} \leq \frac{1}{2}. \tag{1.6.24}$$

This is always possible, since $\beta < \infty$. Given $t \in (T_0, T)$, we use equation (1.6.3) on (T_0, T). It follows from (1.6.23) and (1.6.23) that

$$\|u\|_{L^p((T_0,t),L^\infty)} \leq C\|u(T_0)\|_{L^\infty} + \frac{1}{2}\|u\|_{L^p((T_0,t),L^\infty)} + C\|h\|_{L^\beta((T_0,t),L^\gamma)};$$

and thus,

$$\|u\|_{L^p((T_0,T),L^\infty)} \leq 2C\left(\|u(T_0)\|_{L^\infty} + \|h\|_{L^\beta((T_0,t),L^\gamma)}\right).$$

For the other estimate, we proceed as in Step 4 of the proof of Theorem 1.6.8. □

Finally, we show a uniqueness result when $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$.

Theorem 1.6.12. *Assume $N \geq 3$ and that Ω is of class C^2. Let $T > 0$ and $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$. If $u \in L^\infty((0,T), L^q(\Omega))$ with $q > \frac{N}{N-2}$ satisfies*

$$u(t) = \int_0^t T(t-s)a(s)u(s)\,ds,$$

for all $t \in [0,T]$, then $u(t) \equiv 0$.

Proof. We have $au \in L^\infty((0,T), L^{r_0}(\Omega))$, with $\frac{1}{r_0} = \frac{1}{q} + \frac{2}{N}$. In particular, $1 < r_0 < \infty$, so that by maximal regularity (Theorem 1.6.1) we have $u \in L^p((0,T), W^{2,r_0}(\Omega) \cap W^{1,r_0}_0(\Omega))$ for every $p < \infty$, and u satisfies

$$u_t - \triangle u = au, \tag{1.6.25}$$

in $L^{r_0}\Omega)$ for almost all $t \in (0,T)$.

We now use a duality argument. Fix $t_0 \in (0,T)$, and $\psi \in \mathcal{D}(\Omega)$. Let $a_n = \min\{n, \max\{a, -n\}\}$. We have $(a_n)_{n\geq 0} \subset C([0,T], L^{\frac{N}{2}}(\Omega)) \cap L^\infty((0,T)\times\Omega)$. Moreover, $a_n \to a$ in $C([0,T], L^{\frac{N}{2}}(\Omega))$ as $n \to \infty$. Indeed,

$|a| \le p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$. We now argue by contradiction. Otherwise, there exist $\varepsilon > 0$, $t \in [0,T]$ and a sequence $(t_n)_{n\ge 0} \in [0,T]$ such that $t_n \to t$ and $\|a(t_n,\cdot) - a(t,\cdot)\|_{L^{\frac{N}{2}}} \ge \varepsilon$. On the other hand, by possibly extracting a subsequence, we may assume that $u(t_n) \to u(t)$ and $v(t_n) \to v(t)$ in $L^q(\Omega)$ and almost everywhere, and that there exists $\varphi \in L^q(\Omega)$ such that $|u(t_n)| + |v(t_n)| \le \varphi$ almost everywhere. It follows easily that $a(t_n) \to a(t)$ almost everywhere and that $|a(t_n)| \le C|\varphi|^{p-1} \in L^{\frac{N}{2}}(\Omega)$. By dominated convergence, we deduce $a(t_n) \to a(t)$ in $L^{\frac{N}{2}}(\Omega)$, which is absurd.

Let v_n be the solution of

$$\begin{cases} -(v_n)_t - \triangle v_n = a_n v_n, & \text{in} \quad (0,t_0)\times\Omega, \\ v_n = 0 \quad \text{in} \quad (0,t_0)\times\partial\Omega, \\ v_n(t_0) = \psi \quad \text{in} \quad \Omega. \end{cases}$$

We now multiply the equation (1.6.25) by v_n and integrate on $(0,t_0)\times\Omega$. We obtain

$$\Big[\int_\Omega u v_n\Big]_0^{t_0} = \int_0^{t_0}\int_\Omega (u(v_n)_t + u_t v_n) = \int_0^{t_0}\int_\Omega (u(-\triangle v_n - a_n v_n) + v_n(\triangle u + au)) = \int_0^{t_0}\int_\Omega (a - a_n) u v_n.$$

Therefore,

$$\int_\Omega u(t_0)\psi = \int_0^{t_0}\int_\Omega (a - a_n) u v_n.$$

Hence

$$\Big|\int_\Omega u(t_0)\psi\Big| \le t_0 \|a - a_n\|_{C([0,t_0],L^{\frac{N}{2}})} \|u\|_{L^\infty(0,t_0),L^q)} \|v_n\|_{L^\infty((0,t_0),L^\theta)}, \tag{1.6.26}$$

with $\dfrac{1}{\theta} = 1 - \dfrac{1}{q} - \dfrac{2}{N} > 0$. In particular, we have $\theta < \infty$. We claim that for every $2 \le r < \infty$ there exists a constant C (C depends on r) such that

$$\sup_{n\ge 0} \|v_n\|_{L^\infty((0,t_0),L^r)} \le C\|\psi\|_{L^r}. \tag{1.6.27}$$

Assuming the claim, we let $n \to \infty$ in (1.6.26) and we obtain

$$\int_\Omega u(t_0)\psi = 0.$$

Since $t_0 \in (0,T)$ and $\psi \in \mathcal{D}(\Omega)$ are arbitrary, we deduce that $u \equiv 0$. □

Proof of Claim (1.6.27). We use the same method as in Brezis and Kato [22]. It is convenient to introduce $w_n(t) = v_n(t_0 - t)$ so that w_n satisfies

$$\begin{cases} (w_n)_t - \triangle w_n = b_n w_n \quad \text{in} \quad (0,t_0)\times\Omega, \\ w_n = 0 \quad \text{in} \quad (0,t_0)\times\partial\Omega, \\ w_n(0) = \psi \quad \text{in} \quad \Omega, \end{cases} \tag{1.6.28}$$

with $b_n(s) = a_n(t_0 - s)$. We multiply the equation (1.6.28) by $|w_n|^{r-2}w_n$ to obtain

$$\frac{1}{r}\frac{d}{dt}\int_\Omega |w_n(t)|^r + \frac{4(r-1)}{r^2}\int_\Omega |\nabla |w_n|^{\frac{r}{2}}|^2 \le \int_\Omega |b_n|\,|w_n|^r \le \int_\Omega |b|\,|w_n|^r, \tag{1.6.29}$$

where $b(t) = a(t_0 - t)$ for $0 \le t \le t_0$. Given $j \ge 0$ to be chosen large enough, we write $b = b - b_j + b_j$, and we estimate

$$\begin{aligned}\int_\Omega |b|\,|w_n|^r &\le \int_\Omega |b-b_j|\,|w_n|^r + \int_\Omega |b_j|\,|w_n|^r \\ &\le \|b-b_j\|_{L^{\frac{N}{2}}} \|w_n\|^r_{L^{\frac{Nr}{N-2}}} + j\|w_n\|^r_{L^r} \\ &\le C\|b-b_j\|_{L^{\frac{N}{2}}} \int_\Omega |\nabla |w_n|^{\frac{r}{2}}|^2 + j\|w_n\|^r_{L^r},\end{aligned} \tag{1.6.30}$$

where the last inequality follows from Sobolev's inequality. We now choose j large enough (independent of n) so that

$$C\|b-b_j\|_{L^{\frac{N}{2}}} \le \frac{4(r-1)}{r^2}.$$

(Recall that $b_j \underset{j\to\infty}{\longrightarrow} b$ in $C([0,t_0], L^{\frac{N}{2}}(\Omega))$. It is here that we use the assumption $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$; $a \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$ would not be sufficient.) It now follows from (1.6.29) and (1.6.30) that

$$\frac{1}{r}\frac{d}{dt}\int_\Omega |w_n(t)|^r \le j\|w_n\|^r_{L^r},$$

from which we deduce $\|w_n(t)\|^r_{L^r} \le \|\psi\|^r_{L^r} e^{jrt}$. □

Remark 1.6.13. The conclusion of Theorem 1.6.12 fails if $q = \dfrac{N}{N-2}$. To construct such an example, we use the technique of Ni and Sacks [82]. Let ψ be as in Remark 3.9.11 and let v be the solution of (3.9.1) with the initial condition $v(0) = \psi$. Set $u = v - \psi$ and

$$a = \begin{cases} \dfrac{v^p - \psi^p}{v - \psi} & \text{if} \quad v \ne \psi, \\ v^{p-1} & \text{if} \quad v = \psi. \end{cases}$$

u satisfies

$$u(t) = \int_0^t T(t-s)a(s)u(s)\,ds,$$

for all $t \in [0,T]$, but $u \ne 0$.

1.6.3. Schrödinger's equation. Throughout this section, we consider the group of isometries $(T(t))_{t\in\mathbb{R}}$ associated with Schrödinger's equation, and we assume $\Omega = \mathbb{R}^N$. We use the notation of Sections 1.4.3 and 1.4.4, and in particular the notion of admissible pair (see Definition 1.4.53). We begin with a Strichartz' estimate in the nonhomogeneous case (see Strichartz [93] and Yajima [101]). We also give the proof of Theorem 1.4.54.

Theorem 1.6.14. *Let I be an interval of $\mathbb{R}$ (bounded or not), let $J = \overline{I}$ and let $t_0 \in I$. Let (γ, ρ) be an admissible pair, and let $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$. Then, for every admissible pair (q,r), the function*

$$t \mapsto \Phi_f(t) = \int_{t_0}^t T(t-s)f(s)\,ds, \text{ for } t \in I,$$

belongs to $L^q(I, L^r(\mathbb{R}^N)) \cap C(J, L^2(\mathbb{R}^N))$. Furthermore, there exists a constant C, depending only on γ and q such that

$$\|\Phi_f\|_{L^q(I,L^r)} \le C\|f\|_{L^{\gamma'}(I,L^{\rho'})},$$

for every $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$.

Remark 1.6.15. Note that the definition of Φ_f makes sense. Indeed, $L^{\rho'}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$, and so $f \in L^1(I', H^{-1}(\mathbb{R}^N))$, for every bounded interval $I' \subset I$. In particular, we have $\Phi_f \in C(I', H^{-1}(\mathbb{R}^N))$. Evidently, Theorem 1.6.14 gives an estimate of the solution of the nonhomogeneous Schrödinger equation

$$\begin{cases} iu_t + \triangle u + f = 0, \\ u(0) = 0, \end{cases}$$

in terms of f and φ.

Proof of Theorem 1.6.14. We divide the proof into six steps. For convenience, we assume that $I = [0, T)$, for some $T \in (0, \infty)$ and that $t_0 = 0$, the proof being the same in the general case. It is convenient to define, in the same way as Φ, the operators Ψ and Θ_t (where $t \in (0, T)$ is a parameter) by

$$\Psi_f(s) = \int_s^T T(s-t)f(t)\,dt, \forall s \in [0, T),$$

and

$$\Theta_{t,f}(s) = \int_0^t T(s-\sigma)f(\sigma)\,d\sigma, \forall s \in [0, T).$$

It is clear that both Ψ and Θ_t map continuously $L^1_{\text{loc}}([0, T); H^{-1}(\mathbb{R}^N))$ to $C([0, T), H^{-1}(\mathbb{R}^N))$.

Step 1. For every admissible pair (q, r), $\Phi \in \mathcal{L}(L^{q'}(0, T; L^{r'}(\mathbb{R}^N)), L^q(0, T; L^r(\mathbb{R}^N)))$, with a norm depending only on q. By density, it is sufficient to consider the case $f \in C_{\text{c}}([0, T), L^{r'}(\mathbb{R}^N))$. In this case, it follows easily from Theorem 1.4.49 that $\Phi_f \in C([0, T), L^r(\mathbb{R}^N))$, and that

$$\|\Phi_f(t)\|_{L^r} \le \int_0^t |t-s|^{-N\left(\frac{1}{2}-\frac{1}{r}\right)} \|f(s)\|_{L^{r'}}\,ds \le \int_0^T |t-s|^{\frac{-2}{q}} \|f(s)\|_{L^{r'}}\,ds.$$

It follows from the classical Riesz' potential inequalities (see Corollary A.5.17) that

$$\|\Phi_f\|_{L^q(0,T;L^r)} \le C\|f\|_{L^{q'}(0,T;L^{r'})},$$

where C depends only on q.

Step 2. By the same argument, one shows that both Ψ and Θ_t are continuous from $L^{q'}(0, T; L^{r'}(\mathbb{R}^N))$ to $L^q(0, T; L^r(\mathbb{R}^N))$, with norms depending only on q.

Step 3. For every admissible pair (q, r), $\Phi \in \mathcal{L}(L^{q'}(0, T; L^{r'}(\mathbb{R}^N)), C([0, T], L^2(\mathbb{R}^N)))$, and its norm depends only on q. By density, it is sufficient to consider the case $f \in C_{\text{c}}([0, T), L^{r'}(\mathbb{R}^N)) \cap C_{\text{c}}([0, T), L^2(\mathbb{R}^N))$. It follows that $\Phi_f \in C([0, T), L^2(\mathbb{R}^N))$; and so,

$$\begin{aligned} \|\Phi_f(t)\|^2_{L^2} &= \Big(\int_0^t T(t-s)f(s)\,ds, \int_0^t T(t-\sigma)f(\sigma)\,d\sigma\Big)_{L^2} \\ &= \int_0^t \int_0^t (T(t-s)f(s), T(t-\sigma)f(\sigma))_{L^2}\,d\sigma\,ds \\ &= \int_0^t \int_0^t (f(s), T(s-\sigma)f(\sigma))_{L^2}\,d\sigma\,ds = \int_0^t (f(s), \Theta_{t,f}(s))_{L^2}\,ds, \end{aligned}$$

where we used the property $T(t)^* = T(-t)$ (see Corollary 1.3.33). Applying Hölder's inequality in space, then in time, and applying Step 2, it follows that

$$\|\Phi_f(t)\|^2_{L^2} \le \|f\|_{L^{q'}(0,T;L^{r'})} \|\Theta_{t,f}\|_{L^q(0,T;L^r)} \le C(q)\|f\|^2_{L^{q'}(0,T;L^{r'})}.$$

Hence the result, since t is arbitrary.

Step 4. By the same argument, one shows that both Ψ and Θ_t are continuous from $L^{q'}(0,T;L^{r'}(\mathbb{R}^N))$ to $C([0,T],L^2(\mathbb{R}^N))$, with norms depending only on q.

Step 5. For every admissible pair (q,r), $\Phi \in \mathcal{L}(L^1(0,T;L^2(\mathbb{R}^N)), L^q(0,T;L^r(\mathbb{R}^N)))$, and its norm depends only on q. Let $f \in L^1(0,T;L^2(\mathbb{R}^N))$ and consider $\varphi \in C_{\rm c}([0,T),\mathcal{D}(\mathbb{R}^N))$. We have

$$\begin{aligned}\int_0^T (\Phi_f(t),\varphi(t))_{L^2}\,dt &= \int_0^T\int_0^t (T(t-s)f(s),\varphi(t))_{L^2}\,ds\,dt\\ &= \int_0^T\int_s^T (f(s),T(s-t)\varphi(t))_{L^2}\,dt\,ds\\ &= \int_0^T (f(s),\Psi_\varphi(s))_{L^2}\,ds;\end{aligned}$$

and so, by Cauchy-Schwartz' inequality and Step 4,

$$\begin{aligned}|\int_0^T (\Phi_f(t),\varphi(t))_{L^2}\,dt| &\le \|f\|_{L^1(0,T;L^2)}\|\Psi_\varphi\|_{L^\infty(0,T;L^2)}\\ &\le C(q)\|f\|_{L^1(0,T;L^2)}\|\varphi\|_{L^{q'}(0,T;L^{r'})}.\end{aligned} \tag{1.6.31}$$

On the other hand, one verifies easily that for every $g \in L^q(0,T;L^r(\mathbb{R}^N))$, one has

$$\|g\|_{L^q(0,T;L^r(\mathbb{R}^n))} = \sup\{\int_0^T (g(t),\varphi(t))_{L^2}\,dt;\ \varphi \in C_{\rm c}([0,T),\mathcal{D}(\mathbb{R}^n)),\ \|\varphi\|_{L^{q'}(0,T;L^{r'}(\mathbb{R}^n))} = 1\}.$$

The result follows from (1.6.31), and the above relation applied with $g = \Phi_f$.

Step 6. Conclusion. Let (γ,ρ) be an admissible pair. It follows from steps 1 and 3 that Φ is continuous from $L^{\gamma'}(0,T;L^{\rho'}(\mathbb{R}^N))$ to $L^\infty(0,T;L^2(\mathbb{R}^N))$ and from $L^{\gamma'}(0,T;L^{\rho'}(\mathbb{R}^N))$ to $L^\gamma(0,T;L^\rho(\mathbb{R}^N))$. Consider an admissible pair (q,r) for which $2 \le q \le \rho$, and let $\theta \in [0,1]$ be such that

$$\frac{1}{r} = \frac{\theta}{\rho} + \frac{1-\theta}{2},\ \text{and}\ \frac{1}{q} = \frac{\theta}{\gamma} + \frac{1-\theta}{\infty}.$$

By applying Hölder's inequality in space, then in time, we obtain

$$\|\Phi_f\|_{L^q(0,T;L^r)} \le \|\Phi_f\|^\theta_{L^\gamma(0,T;L^\rho)}\|\Phi_f\|^{1-\theta}_{L^\infty(0,T;L^2)} \le C\|f\|_{L^{\gamma'}(0,T;L^{\rho'})},$$

where C depends only on γ and q. Therefore, Φ maps continuously $L^{\gamma'}(0,T;L^{\rho'}(\mathbb{R}^N))$ to $L^q(0,T;L^r(\mathbb{R}^N))$.

Let now (q,r) be an admissible pair for which $\rho < r$ and let $\mu \in [0,1]$ be such that

$$\frac{1}{\gamma'} = \frac{\mu}{1} + \frac{1-\mu}{q'}\ \text{and}\ \frac{1}{\rho'} = \frac{\mu}{2} + \frac{1-\mu}{r'}.$$

By steps 1 and 5, Φ is continuous from $L^{q'}(0,T;L^{r'}(\mathbb{R}^N))$ to $L^q(0,T;L^r(\mathbb{R}^N))$ and from $L^1(0,T;L^2(\mathbb{R}^N))$ to $L^q(0,T;L^r(\mathbb{R}^N))$. By Interpolation, it follows that Φ is continuous $L^\sigma(0,T;L^\delta(\mathbb{R}^N)) \to L^q(0,T;L^r(\mathbb{R}^N))$ for every pair (σ,δ) such that, for some $\theta \in [0,1]$,

$$\frac{1}{\sigma} = \frac{\theta}{1} + \frac{1-\theta}{q'}\ \text{and}\ \frac{1}{\delta} = \frac{\theta}{2} + \frac{1-\theta}{r'}.$$

(see Theorem A.5.12.) The result follows by choosing $\theta = \mu$. □

Proof of Theorem 1.4.54. The proof is parallel to the proof of Theorem 1.6.14, and we describe only the main steps. Let

$$\Lambda_f(t) = \int_{-\infty}^{+\infty} T(t-s)f(s)\,ds, \text{ and } \Gamma_f = \int_{-\infty}^{+\infty} T(-t)f(t)\,dt.$$

One shows (see proof of Theorem 1.6.14, Step 1) that

$$\|\Lambda_f\|_{L^q(0,T;L^r)} \le C(q)\|f\|_{L^{q'}(0,T;L^{r'})},$$

for every admissible pair (q,r). It follows (see (see proof of Theorem 1.6.14, Step 3) that

$$\|\Gamma_f\|_{L^2} \le C(q)\|f\|_{L^{q'}(0,T,L^{r'})},$$

from which one obtains easily that

$$|\int_{-\infty}^{+\infty} (T(t)\varphi,\psi(t))_{L^2}\,dt| = (\varphi, \int_{-\infty}^{+\infty} T(-t)\psi(t)\,dt)_{L^2} \le C(q)\|\varphi\|_{L^2}\|\psi\|_{L^{q'}(0,T;L^{r'})},$$

for every $\varphi \in L^2(\mathbb{R}^N)$ and $\psi \in C_c([0,T),\mathcal{D}(\mathbb{R}^N))$. The result follows easily (see proof of Theorem 1.6.14, Step 5). □

Remark 1.6.16. One can obtain estimates of $u(t) = \int_0^t T(t-s)f(s)\,ds$ of the type obtained in Theorem 1.4.62. More precisely, if $f \in L^2([0,T],L^2(\mathbb{R}^N))$, then $u \in L^2([0,T],H^{1/2}(B))$ for every bounded open set $B \subset \mathbb{R}^N$ (see Constantin and Saut [30]). Therefore, there is locally a gain of half a derivative. As a matter of fact, if one is willing to reverse the time and space integrations, then the gain is one derivative. More precisely, if $\{Q_\alpha\}_{\alpha\in\mathbb{Z}^N}$ is a family of disjoint open cubes of size R such that $\mathbb{R} = \underset{\alpha\in\mathbb{Z}^N}{\cup}\overline{Q_\alpha}$, then (see Kenig, Ponce and Vega [70])

$$\sup_{\alpha\in\mathbb{Z}^N}\Bigl(\int_{Q_\alpha}\int_{-\infty}^{+\infty}|\nabla u(t,x)|^2 dt\,dx\Bigr)^{1/2} \le CR\sum_{\alpha\in\mathbb{Z}^N}\Bigl(\int_{Q_\alpha}\int_{-\infty}^{+\infty}|f(t,x)|^2 dt\,dx\Bigr)^{1/2}.$$

See also Ruiz and Vega [90] for related estimates.

1.6.4. The wave equation. Throughout this section, we assume $\Omega = \mathbb{R}^N$ and we consider the continuous group $(T(t))_{t\in\mathbb{R}}$ in $L^2(\mathbb{R}^N)\times H^{-1}(\mathbb{R}^N)$ associated with the wave equation (i.e. (1.4.27) with $\lambda = 0$, see Remark 1.4.68). We use the notation of Section 1.4.4. Given $T > 0$ and $f \in L^1((0,T),H^{-1}(\mathbb{R}^N))$, the solution u of the problem

$$\begin{cases} u_{tt} - \triangle u = f, \\ u(0) = u_t(0) = 0, \end{cases} \tag{1.6.32}$$

is the first component of U given by

$$U(t) = \int_0^t T(t-s)F(s)\,ds,$$

where $F(s) = (0,f(s))$. The Strichartz estimate for the solutions of (1.6.32) are best stated in the homogeneous Besov spaces. A typical result is the following (see Ginibre and Velo [57, Lemma 2.1]).

Theorem 1.6.17. *Assume $N \geq 3$, let $0 < T \leq \infty$ and set $I = (0, T)$. Let $2 \leq r, \rho < \dfrac{2(N-1)}{N-3}$ and let $2 < q, \gamma \leq \infty$ and $0 \leq s, \sigma < \dfrac{N+1}{2(N-1)}$ be defined by*

$$\frac{2}{q} = \frac{2(N-1)}{N+1} s = (N-1)\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{and} \quad \frac{2}{\gamma} = \frac{2(N-1)}{N+1} \sigma = (N-1)\left(\frac{1}{2} - \frac{1}{\rho}\right).$$

If $f \in L^{\gamma'}(I, \dot{B}^{\sigma}_{\rho',2}(\mathbb{R}^N))$, then the solution u of (1.6.32) verifies $\omega u \in L^q(I, \dot{B}^{-s}_{r,2}(\mathbb{R}^N))$, where $\omega = (-\triangle)^{1/2}$ is the pseudo-differential operator defined by $\widehat{\omega u}(\xi) = 2\pi|\xi|\widehat{u}(\xi)$. Moreover, there exists a constant C such that

$$\|\omega u\|_{L^q(I, \dot{B}^{-s}_{r,2})} \leq C \|f\|_{L^{\gamma'}(I, \dot{B}^{\sigma}_{\rho',2})}, \tag{1.6.33}$$

for all $f \in L^{\gamma'}(I, \dot{B}^{\sigma}_{\rho',2}(\mathbb{R}^N))$.

Proof. The proof is very similar to the proof of Theorem 1.6.1, by using the estimates of Remark 1.4.74. See Ginibre and Velo [57, Lemma 2.1]. □

Remark 1.6.18. Note that for Klein-Gordon's equation (i.e. (1.4.27) with $\lambda > 0$), similar estimates hold with the homogeneous Besov spaces replaced by the Besov spaces.

By applying Sobolev's inequalities in the homogeneous Besov spaces, one can deduce L^p estimates from Theorem 1.6.17. For example, we have the following result.

Corollary 1.6.19. *Assume $N \geq 3$, let $0 < T \leq \infty$ and set $I = (0, T)$. Let $2 \leq r, \rho < \dfrac{2(N-1)}{N-3}$ and let $2 < q, \gamma \leq \infty$ be defined by*

$$\frac{2}{q} = (N-1)\left(\frac{1}{2} - \frac{1}{r}\right) \quad \text{and} \quad \frac{2}{\gamma} = (N-1)\left(\frac{1}{2} - \frac{1}{\rho}\right).$$

Assume further that

$$\frac{1}{r} + \frac{1}{\rho} = \frac{N+1}{N-1}.$$

If $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$, then the solution u of (1.6.32) verifies $u \in L^q(I, L^r(\mathbb{R}^N))$. Moreover, there exists a constant C such that

$$\|\omega u\|_{L^q(I, L^r)} \leq C \|f\|_{L^{\gamma'}(I, L^{\rho'})},$$

for all $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$.

Proof. Let s and σ be as in Theorem 1.6.17. Since $r \geq 2$ and $\rho' \leq 2$, we have (see Bergh and Löfström [13]) $\dot{H}^{\sigma}_{\rho'}(\mathbb{R}^N)) \hookrightarrow \dot{B}^{\sigma}_{\rho',2}(\mathbb{R}^N))$ and $\dot{B}^{-s}_{r,2}(\mathbb{R}^N) \hookrightarrow \dot{H}^{-s}_r(\mathbb{R}^N)$. Therefore, it follows from (1.6.33) that

$$\|\omega u\|_{L^q(I, \dot{H}^{-s}_r)} \leq C \|f\|_{L^{\gamma'}(I, \dot{H}^{\sigma}_{\rho'})}.$$

Since $\omega^{-\sigma}$ commutes with $\triangle$, it follows that if u is the solution corresponding to f, then $\omega^{-\sigma} u$ is the solution corresponding to $\omega^{-\sigma} f$; and so,

$$\|\omega^{1-\sigma} u\|_{L^q(I, \dot{H}^{-s}_r)} \leq C \|\omega^{-\sigma} f\|_{L^{\gamma'}(I, \dot{H}^{\sigma}_{\rho'})}.$$

By definition of the homogeneous Sobolev spaces $\dot{H}_r^{-s}(\mathbb{R}^N)$, this implies that

$$\|u\|_{L^q(I,\dot{H}_r^{1-\sigma-s})} \le C\|f\|_{L^{\gamma'}(I,\dot{H}^0_{\rho'})}.$$

Finally, it follows from the assumptions that $s+\sigma = 1$. Hence the result, since $\dot{H}^0_r(\mathbb{R}^N) = L^r(\mathbb{R}^N)$ and $\dot{H}^0_{\rho'}(\mathbb{R}^N) = L^{\rho'}(\mathbb{R}^N)$. □

Remark 1.6.20. Note that in particular, one can choose $r = \rho = q = \gamma = \dfrac{2(N+1)}{N-1}$ in Theorem 1.6.19, so that $\|u\|_{L^{\frac{2(N+1)}{N-1}}(\mathbb{R}^{N+1})} \le C\|f\|_{L^{\frac{2(N+1)}{N+3}}(\mathbb{R}^{N+1})}$.

1.6.5. Stokes' equation. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class C^2. Let $E = \big(L^2(\Omega)\big)^N$, and let $F = \{\boldsymbol{u} \in (\mathcal{D}(\Omega))^N;\ \nabla\cdot\boldsymbol{u} = 0\}$. Let X be the closure of F in E. X is also a Hilbert space with the scalar product of E. Let $P : E \to X$ be the orthogonal projection on X. We consider the Stokes operator A defined by

$$\begin{cases} D(A) = \big(H^2(\Omega)\cap H^1_0(\Omega)\big)^N \cap X; \\ A\boldsymbol{u} = -P(\triangle \boldsymbol{u}),\ \text{for } \boldsymbol{u}\in D(A). \end{cases}$$

It follows from Theorem 1.2.49 that A is self-adjoint. Therefore, $-A$ is the generator of a semigroup of contractions in X which we denote $(T(t))_{t\ge 0}$. Given $T>0$ and $\boldsymbol{f}\in L^1((0,T),X)$, $\boldsymbol{u}$ defined by

$$\boldsymbol{u}(t) = \int_0^t T(t-s)\boldsymbol{f}(s)\,ds$$

is the weak solution of the problem

$$\begin{cases} \boldsymbol{u}_t + Au = f, \\ \boldsymbol{u}(0) = 0. \end{cases} \tag{1.6.34}$$

For this problem, there is a maximal regularity result similar to the one obtained for the heat equation.

Theorem 1.6.21. *Let $T>0$, $1<p,q<\infty$ and $\boldsymbol{f}\in L^q((0,T),L^p(\Omega)^N\cap L^1((0,T),X)$. If $\boldsymbol{u}$ is the corresponding solution of (1.6.34), then $\boldsymbol{u}\in W^{1,q}((0,T),L^p(\Omega)^N)$ and $\triangle u\in L^q((0,T),L^p(\Omega)^N)$. Moreover, there exists a constant C such that*

$$\|\boldsymbol{u}_t\|_{L^q((0,T),(L^p)^N)} + \|\triangle\boldsymbol{u}\|_{L^q((0,T),(L^p)^N)} \le \|\boldsymbol{f}\|_{L^q((0,T),(L^p)^N)},$$

for all $\boldsymbol{f}\in L^q((0,T),L^p(\Omega)^N\cap L^1((0,T),X)$.

For a proof of Theorem 1.6.21, see Coulhon and Lamberton [32].

Remark 1.6.22. A similar result holds when $\Omega = \mathbb{R}^N$, since every component u_i of $\boldsymbol{u}$ solves the heat equation (see Remark 1.4.79 and Theorem 1.6.1), and when $\Omega = \mathbb{R}^N\setminus D$, where D is a smooth bounded domain (see Giga and Sohr [53]).

1.6.6. Airy's equation. Let $X = L^2(\mathbb{R})$, and define the operator A on X by

$$\begin{cases} D(A) = H^3(\mathbb{R}); \\ Au = -u_{xxx} = -\dfrac{d^3u}{dx^3},\ \text{for } u\in D(A). \end{cases}$$

It follows from Theorem 1.2.52 that A is skew-adjoint; and so $-A$ is the generator of a group of isometries $(T(t))_{t\in\mathbb{R}}$ on X. Given $T>0$ and $f\in L^1((0,T),L^2(\mathbb{R}))$, u given by

$$u(t)=\int_0^t T(t-s)f(s)\,ds,$$

is the weak solution of the problem

$$\begin{cases} u_t+u_{xxx}=f,\\ u(0)=0.\end{cases} \tag{1.6.35}$$

We have the following Strichartz estimate for the problem (1.6.35) (See Ginibre and Tsutsumi [55]).

Theorem 1.6.23. *Let $0<T\le\infty$ and let $2\le r,\rho\le\infty$. Define $q,\gamma\in[6,\infty]$ by*

$$\frac{2}{q}=\frac{1}{3}\Big(1-\frac{2}{r}\Big)\quad \text{and}\quad \frac{2}{\gamma}=\frac{1}{3}\Big(1-\frac{2}{\rho}\Big).$$

If $f\in L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}))$, then the corresponding solution u of (1.6.35) belongs to $L^q((0,T),L^r(\mathbb{R}))\cap C((0,T),L^2(\mathbb{R}))$. Furthermore, there exists a constant C such that

$$\|u\|_{L^q((0,T),L^r)}\le C\|f\|_{L^{\gamma'}((0,T),L^{\rho'})},$$

for every $f\in L^{\gamma'}((0,T),L^{\rho'}(\mathbb{R}))$.

Proof. The proof is similar to the proof of Theorem 1.6.14, by applying the estimates of Remark 1.4.84 (see Ginibre and Tsutsumi [55]). □

Remark 1.6.24. One can obtain estimates of the type described in Remark 1.4.85 for the nonhomogeneous problem (1.6.35). See in particular Kenig, Ponce and Vega [70] and Ginibre and Tsutsumi [55].

1.7. Comments.

1.7.1. Semigroups of contractions are not the most general form of continuous semigroups. In particular, one can define C_0 semigroups $(T(t))_{t\ge0}$. They satisfy the following properties (cf. Pazy [85], Chapter 1):

(i) $T(t)\in\mathcal{L}(X)$, for all $t\ge0$;

(ii) $T(0)=I$;

(iii) $T(t+s)=T(t)T(s)$, for all $s,t\ge0$;

(iv) the function $t\mapsto T(t)x$ is continuous $[0,\infty)\to X$, for all $x\in X$.

It is easily verified that there exists constants $M\ge1$ and $\omega\ge0$ such that $\|T(t)\|_{\mathcal{L}(X)}\le Me^{\omega t}$, for all $t\ge0$ (see Pazy [85], Theorem 2.1, p.4). The generator of a C_0 semigroup is defined as for semigroups of contractions. However, it is sufficient to consider semigroups of contraction since an operator $-A$ in X is the generator of a C_0 semigroup of type (M,ω) if, and only if there exists an equivalent norm $|||\cdot|||$ on X such that $A+\omega I$ is m-accretive in $(X,|||\cdot|||)$ (see Pazy [85], Chapter 1, Lemma 5.1 and Theorems 5.2 and 5.3).

1.7.2. Baillon's theorem. In view of Theorem 1.3.34, one may ask if there are semigroups for which the solution of (1.5.2) solves (1.5.1) for every $x \in D(A)$ and every $f \in C([0,T],X)$. It turns out that under fairly general assumptions, the only case for which such a property holds is when A is bounded, as shows the following result of Baillon [5].

Theorem 1.7.1. *Let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup of contractions and let $-A$ be its generator. Assume there exists $t_0 > 0$ such that*

$$\int_0^{t_0} T(t_0 - s)f(s)\,ds \in D(A), \tag{1.7.1}$$

for every $f \in C([0,t_0],X)$. If X does not contain any subspace isomorphic to c_0, then A is bounded. In particular, if X is reflexive, then A is bounded.

In fact, Baillon's result asserts that the conclusion holds under (apparently) stronger conditions. However, it can be weakened by using the following lemma.

Lemma 1.7.2. *Let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup of contractions and let $-A$ be its generator. If there exists $t_0 > 0$ such that (1.7.1) holds for every $f \in C([0,t_0],X)$, then the following properties hold:*

(i) $\displaystyle\int_0^t T(t-s)f(s)\,ds \in D(A)$, *for every $f \in C([0,t_0],X)$ and every $t \in [0,t_0]$;*

(ii) *there exists a constant C such that $\|\int_0^t T(t-s)f(s)\,ds\|_{D(A)} \leq C\|f\|_{L^\infty((0,t_0),X)}$, for every $f \in C([0,t_0],X)$ and every $t \in [0,t_0]$.*

Proof. Define the operator $L : C([0,t_0],X) \to D(A)$ by

$$Lf = \int_0^{t_0} T(t_0 - s)f(s)\,ds.$$

If $f_n \underset{n\to\infty}{\longrightarrow} f$ in $C([0,t_0],X)$, then it follows from Lemma 1.5.3 that $Lf_n \underset{n\to\infty}{\longrightarrow} Lf$ in X. Therefore, if furthermore $Lf_n \underset{n\to\infty}{\longrightarrow} g$ in $D(A)$, then $g = Lf$. It follows that the graph of L is closed; and so, by the closed graph theorem, there exists a constant C such that

$$\|\int_0^{t_0} T(t_0 - s)f(s)\,ds\|_{D(A)} \leq C\|f\|_{L^\infty((0,t_0),X)}, \tag{1.7.2}$$

for every $f \in C([0,t_0],X)$. Given $f \in C([0,t_0],X)$ and $t \in [0,t_0]$, define $g \in C([0,t_0],X)$ by

$$g(s) = \begin{cases} f(0), & \text{if } 0 \leq s \leq t_0 - t; \\ f(s - t_0 + t), & \text{if } t_0 - t \leq s \leq t_0. \end{cases}$$

It follows easily that

$$\int_0^t T(t-s)f(s)\,ds = \int_0^{t_0} T(t_0 - s)g(s)\,ds - \int_t^{t_0} T(s)f(0)\,ds;$$

and so, by (1.7.2) and Lemma 1.3.9,

$$\|\int_0^t T(t-s)f(s)\,ds\|_{D(A)} \leq (C+3)\|f\|_{L^\infty((0,t_0),X)}.$$

The result follows, since t and f are arbitrary. $\square$

1.7.3. We give below an alternative proof of the $L^p - L^q$ estimates of Theorem 1.4.15, based on a technique of multipliers. That technique can be applied to certain nonlinear problems to which the comparison argument is not applicable. We have the following result, inspired by Fabes and Stroock's proof (see Fabes and Stroock [41] and Davies [36]) of the $L^1 - L^\infty$ estimate.

Proposition 1.7.3. *Let Ω be an arbitrary open subset of $\mathbb{R}^N$ and let $T > 0$. Let $f \in C([0,T], H^{-1}(\Omega))$ and let $u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], H^{-1}(\Omega))$ solve equation*

$$u_t - \triangle u = f, \text{ for all } 0 \le t \le T.$$

Assume that for every $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $\varphi(0) = 0$ and $\varphi' \ge 0$, one has $\langle \varphi(u(t)), f(t) \rangle_{H_0^1, H^{-1}} \le 0$ for almost all $t \in [0,T]$. Then, the following properties hold:

(i) *If $u(0) \in L^q(\Omega)$ for some $1 \le q \le \infty$, then $u(t) \in L^q(\Omega)$ for every $t \in (0,T]$, and*

$$\|u(t)\|_{L^q} \le \|u(0)\|_{L^q},$$

for all $t \in (0,T]$;

(ii) *if $u(0) \in L^1(\Omega) \cap L^\infty(\Omega)$, then $u(t) \in L^p(\Omega)$ for every $1 \le p \le \infty$ and every $t \in (0,T]$. In addition,*

$$\|u(t)\|_{L^p} \le C t^{-\frac{N}{2}(1-\frac{1}{p})} \|u(0)\|_{L^1},$$

for all $t \in (0,T]$, where the constant C depends only on p and N.

Proof. We proceed in several steps.

Step 1. Preliminary estimates. Let φ be as in the assumption. Define ϕ and ψ by

$$\phi(x) = \int_0^x \varphi(s)\, ds, \text{ for } x \in \mathbb{R}, \tag{1.7.3}$$

and

$$\psi(x) = \int_0^x \sqrt{\varphi'(s)}\, ds, \text{ for } x \in \mathbb{R}. \tag{1.7.4}$$

It follows that ϕ and ψ verify the same assumptions as φ. By taking the $H^{-1} - H_0^1$ scalar product of the equation with $\varphi(u) \in C([0,T], H_0^1(\Omega))$ and by applying Corollary A.3.65 and identity (A.4.24), one obtains

$$\frac{d}{dt} \int_\Omega \phi(u(t))\, dx + \int_\Omega |\nabla \psi(t)|^2\, dx \le 0, \tag{1.7.5}$$

for all $t \in [0,T]$, from which it follows in particular that

$$\int_\Omega \phi(u(t))\, dx \le \int_\Omega \phi(u(s))\, dx, \tag{1.7.6}$$

for $0 \le s \le t \le T$.

Step 2. Proof of property (i). If $q \le 2$, let $\varepsilon > 0$ and define

$$\varphi(x) = \frac{1}{q}\frac{qx^3 + 2\varepsilon x}{(\varepsilon + x^2)^{\frac{4-q}{2}}},$$

so that ϕ defined by (1.7.3) is given by

$$\phi(x) = \frac{1}{q}\frac{x^2}{(\varepsilon + x^2)^{\frac{2-q}{2}}}.$$

Applying now (1.7.6), letting $\varepsilon \downarrow 0$ and applying the dominated convergence theorem to the right-hand side and Fatou's lemma to the left-hand side, one obtains property (i). If $2 \le q < \infty$, let $\varepsilon > 0$ and define

$$\varphi(x) = \frac{1}{q}\frac{q|x|^{q-2}x + 2\varepsilon|x|^q x}{(\varepsilon + x^2)^{q/2}},$$

so that ϕ defined by (1.7.3) is given by

$$\phi(x) = \frac{1}{q}\frac{|x|^q}{(\varepsilon + x^2)^{\frac{q-2}{2}}}.$$

The same argument as above shows that the conclusions of property (i) hold. Finally, if $q = \infty$, apply the previous result for finite q and make $q \uparrow \infty$.

Step 5. Proof of property (ii). We recall that there exists A, depending only on N, such that

$$\Bigl(\int_\Omega v^2\Bigr)^{\frac{N+2}{N}} \le A\int_\Omega |\nabla v|^2 \Bigl(\int_\Omega |v|\Bigr)^{\frac{4}{N}}, \tag{1.7.7}$$

for all $v \in H_0^1(\Omega) \cap L^1(\Omega)$. This follows from Gagliardo-Nirenberg's inequality (see Theorem A.3.44 and Remark A.3.45 (ii)). Consider φ, ϕ and ψ as in step 1. It follows from inequality (1.7.7) that

$$\Bigl(\int_\Omega |\psi(u(t))|^2\Bigr)^{\frac{N+2}{N}} \le A\int_\Omega |\nabla\psi(u(t))|^2 \Bigl(\int_\Omega |\psi(u(t))|\Bigr)^{\frac{4}{N}}.$$

Therefore, inequality (1.7.5) yields

$$\frac{d}{dt}\int_\Omega \phi(u(t)) + \frac{1}{A\bigl(\int_\Omega |\psi(u(t))|\bigr)^{\frac{4}{N}}}\Bigl(\int_\Omega |\psi(u(t))|^2\Bigr)^{\frac{N+2}{N}} \le 0. \tag{1.7.8}$$

Consider now $2 \le p < \infty$. Note that by property (i), we have $u(t) \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\|u(t)\|_{L^\infty} \le \|u(0)\|_{L^\infty}$, for all $0 \le t \le T$. Set $M = \|u(0)\|_{L^\infty}$ and let φ as at step 1 be such that $\varphi(x) = |x|^{p-1}x$ for $|x| \le M$. It follows that $\phi(x) = \dfrac{1}{p}|x|^p$ for $|x| \le M$ and that $\psi(x) = \dfrac{2}{p}|x|^{\frac{p}{2}}$ for $|x| \le M$. Therefore, inequality (1.7.8) yields

$$\frac{d}{dt}\int_\Omega |u(t)|^p + \frac{4(p-1)}{Ap}\frac{1}{\bigl(\int_\Omega |u(t)|^{\frac{p}{2}}\bigr)^{\frac{4}{N}}}\Bigl(\int_\Omega |u(t)|^p\Bigr)^{1+\frac{2}{N}} \le 0,$$

for all $0 < t < T$. Note that by property (i), $\displaystyle\int_\Omega |u(t)|^{\frac{p}{2}} \le \int_\Omega |u(0)|^{\frac{p}{2}}$; and so,

$$\frac{d}{dt}\int_\Omega |u(t)|^p + \frac{4(p-1)}{Ap}\frac{1}{\bigl(\int_\Omega |u(0)|^{\frac{p}{2}}\bigr)^{\frac{4}{N}}}\Bigl(\int_\Omega |u(t)|^p\Bigr)^{1+\frac{2}{N}} \le 0.$$

Applying Theorem A.5.3, we then obtain

$$\int_\Omega |u(t)|^p \le \left(\frac{NAp}{8(p-1)}\right)^{N/2} t^{-N/2} \left(\int_\Omega |u(0)|^{\frac{p}{2}}\right)^2. \tag{1.7.9}$$

Note that since $p \ge 2$, we have $\dfrac{p}{(p-1)} \le 2$. Therefore, if we set $p = 2q$ with $1 \le q < \infty$, inequality (1.7.9) yields

$$\|u(t)\|_{L^{2q}} \le \left(\frac{NA}{4}\right)^{\frac{N}{4q}} t^{-\frac{N}{4q}} \|u(0)\|_{L^q},$$

for $0 < t \le T$. Applying the result to $u(\cdot + s)$, one obtains in fact

$$\|u(t+s)\|_{L^{2q}} \le \left(\frac{NA}{4}\right)^{\frac{N}{4q}} s^{-\frac{N}{4q}} \|u(t)\|_{L^q},$$

for $0 \le t < s \le T$. Choosing s of the form $\tau 2^{-(n+1)}$ and $q = 2^n$, we obtain

$$\|u(t + \tau 2^{-(n+1)})\|_{L^{2^{n+1}}} \le \left(\frac{NA}{4}\right)^{N2^{-(n+2)}} 2^{N(n+1)2^{-(n+2)}} \|u(t)\|_{L^{2^n}}.$$

An obvious iteration argument based on the above inequality shows that

$$\|u(\tau(2^{-1} + \cdots + 2^{-n}))\|_{L^{2^{n+1}}} \le \prod_{j=0}^{n-1} \left[\left(\frac{NA}{4\tau}\right)^{N2^{-(j+2)}} 2^{N(j+1)2^{-(j+2)}}\right] \|u(0)\|_{L^1},$$

for all $0 < \tau \le T$ and $n \ge 0$. Note that $2^{-1} + \cdots + 2^{-n} \le 1$; and so, by step 1, $\|u(\tau(2^{-1} + \cdots + 2^{-n}))\|_{L^{2^{n+1}}} \ge \|u(\tau)\|_{L^{2^{n+1}}}$. Therefore,

$$\|u(\tau)\|_{L^{2^{n+1}}} \le K_n \|u(0)\|_{L^1}, \tag{1.7.10}$$

with

$$K_n = \left(\frac{NA}{4\tau}\right)^{N\sum_{j=0}^{n-1} 2^{-(j+2)}} \left(2^{N/2}\right)^{\sum_{j=0}^{n-1} (j+1)2^{-(j+1)}}.$$

Since $\displaystyle\sum_{j=0}^{\infty} 2^{-(j+2)} = \frac{1}{2}$ and $\displaystyle\sum_{j=0}^{\infty} (j+1)2^{-(j+1)} = 2$, it follows that

$$\lim_{n\to\infty} K_n = 2^N \left(\frac{NA}{4\tau}\right)^{N/2}.$$

Therefore, inequality (1.7.10) yields

$$\|u(\tau)\|_{L^\infty} \le 2^N \left(\frac{NA}{4\tau}\right)^{N/2} \|u(0)\|_{L^1} = \left(\frac{NA}{\tau}\right)^{N/2} \|u(0)\|_{L^1},$$

for all $0 < \tau \le T$, and property (ii) follows, by applying property (i) and Hölder's inequality. □

1.8. Exercises.

Exercise 1.8.1. The object of this exercise is to show that the estimate of Lemma 1.1.16 (ii) is optimal in the sense that one can have $\|A_\lambda\|_{\mathcal{L}(X)} = 2/\lambda$ for all $\lambda > 0$. Let $X = C_0(0,\infty)$ equipped with the sup norm. Define the operator A on X by

$$\begin{cases} D(A) = \{u \in C^1([0,\infty)) \cap X;\ u' \in X\}, \\ Au = u' \text{ for all } u \in D(A). \end{cases}$$

It follows from Proposition 1.2.6 that A is m-accretive. Let $0 < \delta < M$ and define $f \in X$ as follows.

$$f(x) = \begin{cases} x/\delta \text{ if } 0 \le x \le \delta, \\ 1 \text{ if } \delta \le x \le M, \\ 1 - \dfrac{2}{\delta}(x - M) \text{ if } M \le x \le M + \delta, \\ -1 + x - M - \delta \text{ if } M + \delta \le x \le M + \delta + 1, \\ 0 \text{ if } M + \delta + 1 \le x. \end{cases}$$

Let $\lambda > 0$ and let $u = J_\lambda f$.

- Show that $u(M + \delta) \ge 1 - e^{-\frac{M+\delta}{\lambda}} - 3\dfrac{\lambda}{\delta}$.
- Show that $\|A_\lambda f\|_{\mathcal{L}(X)} \ge 2 - e^{-\frac{M+\delta}{\lambda}} - 3\dfrac{\lambda}{\delta}$.
- Show that $\|A_\lambda f\|_{\mathcal{L}(X)} = 2/\lambda$, for all $\lambda > 0$.

Exercise 1.8.2. Let A be an m-accretive operator in a Banach space X, and assume that X is reflexive. The object of this exercise is to show that $D(A)$ is dense in X. Let $x^* \in X^\star$ be such that $\langle x^*, f\rangle_{X^\star,X} = 0$ for all $f \in D(A)$. Let $x \in X$, and set $f = (I + A)^{-1}x$.

- Show that $\langle x^*, x\rangle_{X^\star,X} = \langle x^*, Af\rangle_{X^\star,X}$.
- For $\lambda > 0$, define $f_\lambda = (I + \lambda A)^{-1} f$. Show that $f_\lambda \to f$ as $\lambda \downarrow 0$, that $\|Af_\lambda\| \le \|Af\|$ and that $Af_\lambda \in D(A)$ (see Lemma 1.1.16).
- Show that $\langle x^*, Af\rangle_{X^\star,X} = 0$ and conclude.

Exercise 1.8.3. Let A be an m-accretive operator in a Banach space X, and assume that $D(A)$ is dense in X. Therefore, we may consider the operator A^* in $X^\star$ (see Brezis [17], Proposition II.16). The object of this exercise is to show that A^* is m-accretive in $X^\star$. Consider $\lambda > 0$.

- Show that $R(I + \lambda A^*) = X^\star$ (see Brezis [17], Theorem II.20). Let $x \in D(A^*)$ and $f \in X^\star$ be such that $x + \lambda A^* x = f$.
- Show that $\langle x, z\rangle_{X^\star,X} = \langle f, (I + \lambda A)^{-1} z\rangle_{X^\star,X}$ for all $z \in X$.
- Show that $\|x\|_{X^\star} \le \|f\|_{X^\star}$.
- Conclude.

Exercise 1.8.4. Let $X = L^2(0, 1)$. Define the operator A on X by

$$\begin{cases} D(A) = H^1_0(0, 1), \\ Au = u', \text{ for all } u \in D(A). \end{cases}$$

- Show that A is skew-symmetric.
- Determine $D(A^*)$ and $A^* u$ for all $u \in D(A^*)$.
- Show that A is not skew-adjoint.

Exercise 1.8.5. Let $X = L^2(0,1)$. Define the operator A on X by

$$\begin{cases} D(A) = \{u \in H^2(0,1);\ u(0) = u'(0) = u(1) = u'(1) = 0\}, \\ Au = -u'', \text{ for all } u \in D(A). \end{cases}$$

- Show that A is symmetric.
- Determine $D(A^*)$ and A^*u for all $u \in D(A^*)$.
- Show that A is not self-adjoint.

Exercise 1.8.6. Let $1 \le p < \infty$, $X = L^p(0,1)$. Consider the m-accretive operator A on X defined by

$$\begin{cases} D(A) = \{u \in W^{1,p}(0,1);\ u(0) = 0\}, \\ Au = u', \text{ for all } u \in D(A). \end{cases}$$

(See Remark 1.2.5 (ii).)

- Determine $D(A^*)$ and A^*x for all $x \in D(A^*)$.

Exercise 1.8.7. Assume $N \ge 3$. Let $0 < T < \infty$ and $a, f \in C([0,T], L^{\frac{N}{2}}(\Omega))$. The object of this exercise is to show that if $u_0 \in L^\infty(\Omega)$ and $u \in L^\infty_{\mathrm{loc}}([0,T), L^\infty(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), H^1_0(\Omega))$ satisfy (1.6.3) for all $t \in (0,T)$, then $u \in L^\infty((0,T), L^q(\Omega))$ for every $q \in [1,\infty)$.

- Show that for every $\varepsilon > 0$, there exists $a_\varepsilon \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$ such that $\|a_\varepsilon\|_{L^\infty(0,T),L^{\frac{N}{2}})} \le \varepsilon$ and $a - a_\varepsilon \in L^\infty((0,T) \times \Omega)$. (Consider for example $a_\varepsilon = a$ if $|a| \ge M$ and $a_\varepsilon = 0$ if $|a| < M$, with M large enough.)
- Given $2 \le q < \infty$, multiply the equation $u_t - \triangle u = au + f$ by $|u|^{q-2}u$, and show that
$$\frac{1}{q}\frac{d}{dt}\int_\Omega |u(t)|^q + \frac{4(q-1)}{q^2}\int_\Omega |\nabla |u|^{\frac{q}{2}}|^2 \le \int_\Omega \{|a|\,|u|^q + |f|\,|u|^{q-1}\}.$$
- Conclude.

Exercise 1.8.8. Assume $N \ge 3$. Let $0 < T < \infty$, let $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$, and let u be a smooth solution on $(0,T)$ of

$$\begin{cases} u_t - \triangle u = au, \\ u_{|\partial\Omega} = 0, \end{cases}$$

The object of this exercise is to show that if $\|u(t)\|_{L^q} \underset{t\downarrow 0}{\longrightarrow} 0$ for some $q > 1$, then $u(t) \equiv 0$.

- Show that if $a_n \to a$ in $C([0,T], L^{\frac{N}{2}}(\Omega))$, then there exists $b \in C([0,T], L^{\frac{N}{2}}(\Omega))$ and a subsequence $n_k \to \infty$ such that $|a_{n_k}(t)| \le b(t)$ a.e. in Ω for all $t \in [0,T]$. (Hint: consider a subsequence $(n_k)_{k\ge 0}$ such that $\|a_{n_{k+1}} - a_{n_k}\|_{C([0,T],L^{\frac{N}{2}})} \le 2^{-k}$ and let $b = |a_0| + \sum_{k=0}^{\infty} |a_{n_{k+1}} - a_{n_k}|$.)
- Show that, given $\tau \in (0,T)$ and $f \in C^\infty_{\mathrm{c}}((0,T) \times \Omega)$, there exists $v \in L^\infty((0,T), L^r(\Omega))$ for every $r < \infty$ which is a solution of the equation
$$\begin{cases} -v_t - \triangle v = av + f, \\ u_{|\partial\Omega} = 0, \\ v(\tau) = 0. \end{cases}$$

(Hint: approximate a by a sequence $(a_n)_{n\geq 0} \subset C_c^\infty((0,T)\times\Omega)$, apply the first question then Exercise 1.8.7 to obtain estimates independent of n, and let $n\to\infty$.)

- Conclude. (Hint: use the preceding question and apply a duality argument.)

Exercise 1.8.9. Assume $N \geq 3$. Let $0 < T < \infty$, let $a \in L^\infty((0,T), L^p(\Omega))$ for some $p \geq 1$, $p > \dfrac{N}{2}$, and let u be a smooth solution on $(0,T)$ of

$$\begin{cases} u_t - \triangle u = au, \\ u_{|\partial\Omega} = 0, \end{cases}$$

Show that if $\|u(t)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} 0$, then $u(t) \equiv 0$. (Hint: given $\tau \in (0,T)$ and $f \in C_c^\infty((0,T)\times\Omega)$, consider the solution $v \in L^\infty((0,T), L^\infty(\Omega))$ of the equation

$$\begin{cases} -v_t - \triangle v = av + f, \\ v_{|\partial\Omega} = 0, \\ v(\tau) = 0, \end{cases}$$

and apply a duality argument.)

Exercise 1.8.10. Let $0 < \rho < R < \infty$, and set $\Omega = B(0,R)$ and $\omega = B(0,\rho)$. Let $u(t)$ be the solution of

$$\begin{cases} u_t - \triangle u = 0 \quad \text{in} \quad (0,\infty)\times\Omega, \\ u = 0 \quad \text{in} \quad (0,\infty)\times\partial\Omega, \\ u(0,x) = 1_\omega. \end{cases}$$

The object of this exercise is to show that there exist $c_1, c_2 > 0$ independent of R and ρ such that

$$u(t,x) \geq c_1 t^{-\frac{N}{2}} e^{-\frac{c_2}{R^2}t} e^{-\frac{R^2}{t}} R^{-1}\rho^N (R - |x|), \tag{1.8.1}$$

for all $t > 0$ and $x \in \Omega$.

- Let λ_1 be the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$ and let φ_1 be the corresponding eigenvector such that $\varphi_1(0) = 1$. Show that $\varphi_1(x) \geq c_1 R^{-1}(R - |x|)$ and $\lambda_1 = c_2 R^{-2}$.
- Let $v(t,x) = e^{-\lambda_1 t}\varphi_1(x) z(t,x)$, where $z(t)$ is the solution of

$$\begin{cases} z_t - \triangle z = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \\ z(0,x) = 1_\omega. \end{cases}$$

  Show that $u(t,x) \geq v(t,x)$ for $(t,x) \in (0,\infty)\times\Omega$ and conclude.

Exercise 1.8.11. The object of this exercise is to show that if $(T(t))_{t\geq 0}$ is the heat semigroup in a connected open set $\Omega \subset \mathbb{R}^N$ and if $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ and $u_0 \not\equiv 0$, then $T(t)u_0 > 0$ in Ω for all $t > 0$.

- Let $x_0, x \in \Omega$ and let $\gamma \in C^1([0,1],\Omega)$ be such that $\gamma(0) = x_0$ and $\gamma(1) = x$. Let ℓ be the length of the arc γ, let

$$\rho = \min_{0\leq t\leq 1} \operatorname{dist}(\gamma(t), \partial\Omega),$$

and set $\mu = \dfrac{\rho}{4}$. Show that there exists a constant $K(\ell, \mu, N)$ such that if $u_0 \geq 1$ on $B(x_0, \mu)$, then

$$T(t)u_0(x) \geq e^{-\frac{K}{t}} \quad \text{on} \quad B(x, \mu), \tag{1.8.2}$$

for all $0 < t \leq 1$. (Hint: apply j times the estimate (1.8.1), with $j = \left[\dfrac{\ell}{\mu}\right] + 1$).

- Conclude.

Exercise 1.8.12. The object of this exercise is to show that if $(T(t))_{t\geq 0}$ is the heat semigroup in a bounded, smooth, connected domain of $\mathbb{R}^N$ then there exists a constant K depending on Ω such that for all $u_0 \in M(\Omega)$, $u_0 \geq 0$,

$$T(t)u_0(x) \geq e^{-\frac{K}{t}} \Bigl(\int u_0 \delta \Bigr) \delta(x),$$

in Ω for all $0 < t \leq 1$. Here $\delta(x)$ is the distance of x to $\partial\Omega$.

- Show that there exists $\rho > 0$ such that for every $x \in \partial\Omega$ the ball of radius ρ and center $x - \rho\vec{n}(x)$ is contained in Ω. Here, $\vec{n}(x)$ is the outward unit normal vector at x. Show that there exists ℓ such that given any $x_0, x \in \Omega$ such that $\delta(x_0) \geq \rho$, $\delta(x) \geq \rho$, there exists a function $\gamma \in C^1([0,1], \Omega)$ with $\gamma(0) = x_0$ and $\gamma(1) = x$,

$$\rho \leq \min_{0\leq t\leq 1} \operatorname{dist}(\gamma(t), \partial\Omega),$$

and the length of γ is $\leq \ell$.

- Fix any $x_0 \in \Omega$ such that $\delta(x_0) \geq \rho$, let $\mu = \dfrac{\rho}{4}$ and set $\psi = 1_{B(x_0,\mu)}$. Show that there exists a constant K_0 depending on Ω such that

$$T(t)\psi(x) \geq e^{-\frac{K_0}{t}}, \tag{1.8.3}$$

for all $0 < t \leq 1$ and for all $x \in \Omega$ such that $\delta(x) \geq 3\mu$. (Hint: apply the estimate (1.8.2).) Show that there exists a constant K_1 depending on Ω such that

$$T(t)\psi \geq e^{-\frac{K_1}{t}} \delta, \tag{1.8.4}$$

for all $0 < t \leq 1$. (Hint: apply the estimates (1.8.3) and (1.8.1))

- Let $\overline{x} \in B(x_0, \mu)$ and let $\delta_{\overline{x}}$ be the Dirac mass at $\overline{x}$. Show that

$$T(t)\delta_{\overline{x}} \geq \frac{1}{2}(4\pi t)^{-\frac{N}{2}} e^{-\frac{\mu^2}{4t}} \psi, \tag{1.8.5}$$

for all $0 < t \leq \dfrac{9\mu^2}{2N}$. (Hint: show that if $t \leq \dfrac{9\mu^2}{2N}$, then $T(t)\delta_{\overline{x}}(x) \geq (4\pi t)^{-\frac{N}{2}} (e^{-\frac{|x-\overline{x}|^2}{4t}} - e^{-\frac{9\mu^2}{4t}})$.) Show that there exists a constant K_2 depending on Ω such that

$$T(t)\delta_{\overline{x}} \geq e^{-\frac{K_2}{t}} \delta, \tag{1.8.6}$$

for all $0 < t \leq 1$. (Hint: combine the estimates (1.8.5) and (1.8.4).)

- Show that for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$,

$$T(t)\varphi \geq e^{-\frac{K_2}{t}} \Bigl(\int_\Omega \varphi\delta \Bigr) \psi, \tag{1.8.7}$$

for $0 < t \leq 1$. (Hint: apply the identity $T(t)\varphi(\overline{x}) = \int_\Omega \varphi T(t)\delta_{\overline{x}}$ and the estimate (1.8.6).)

- Conclude. (Hint: establish the result for $u_0 \in \mathcal{D}(\Omega)$ by combining the estimates (1.8.7) and (1.8.4).)

Exercise 1.8.13. Let $0 < \rho < R < \infty$, and set $\Omega = B(0, R)$ and $\omega = B(0, \rho)$.

- Let u be the solution of

$$\begin{cases} -\triangle u = 0 & \text{in} \quad \Omega \setminus \overline{\omega}, \\ u = 0 & \text{in} \quad \partial\Omega, \\ u = 1 & \text{in} \quad \partial\omega. \end{cases}$$

Show that

$$u(x) = \begin{cases} \dfrac{|x|^{-N+2} - R^{-N+2}}{\rho^{-N+2} - R^{-N+2}} & \text{if} \quad N \neq 2, \\ \dfrac{\log R - \log |x|}{\log R - \log \rho} & \text{if} \quad N = 2. \end{cases}$$

- Let v be the solution of

$$\begin{cases} -\triangle v = 1_\omega & \text{in} \quad \Omega, \\ v = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

Show that

$$v(x) = \begin{cases} \dfrac{\rho^N}{N(N-2)}(\rho^{-N+2} - R^{-N+2}) + \dfrac{\rho^2 - |x|^2}{2N} & \text{if} \quad |x| \leq \rho, \\ \dfrac{\rho^N}{N(N-2)}(|x|^{-N+2} - R^{-N+2}) & \text{if} \quad \rho \leq |x| \leq R, \end{cases}$$

if $N \neq 3$ and

$$v(x) = \begin{cases} \dfrac{\rho^2}{2}(\log R - \log \rho) + \dfrac{\rho^2 - |x|^2}{4} & \text{if} \quad |x| \leq \rho, \\ \dfrac{\rho^2}{2}(\log R - \log |x|) & \text{if} \quad \rho \leq |x| \leq R, \end{cases}$$

if $N = 2$.

Exercise 1.8.14. The object of this exercise is to show that if $u \in H_0^1(\Omega) \cap C(\Omega)$ verifies

$$-\triangle u = f \quad \text{in} \quad \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a connected domain and $f \in H1-(\Omega)$, $f \geq 0$, $f \not\equiv 0$, then $u > 0$ in Ω.

- Let $x_0 \in \Omega$. If $u(x_0) = 0$, show that $u > 0$ on B, where B is any ball centered at x_0 and contained in Ω (apply the first part of Exercise 1.8.13).
- Conclude.

Exercise 1.8.15. The object of this exercise is to show that if $u \in H_0^1(\Omega) \cap C(\Omega)$ verifies

$$-\triangle u = f \quad \text{in} \quad \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smoth, connected domain and $f \in H^{-1}(\Omega)$, $f \geq 0$, $f \not\equiv 0$, then $u(x) \geq C\delta(x)$ in Ω. Here $C > 0$ and $\delta(x)$ is the distance of x to $\partial\Omega$.

- Show that there exist $\eta > 0$ and $\varepsilon > 0$ such that $u(x) \ge \varepsilon$ for all $x \in \Omega$ such that $\delta(x) \ge \eta/2$ (apply the first part of Exercise 1.8.12 and Exercise 1.8.14).
- Conclude (apply the first part of Exercise 1.8.13).

Exercise 1.8.16. The object of this exercise is to construct $f \in L^2((0,T), L^2(\Omega))$, where $\Omega \subset \mathbb{R}^2$ is a smooth domain, such that the solution u of

$$\begin{cases} u_t - \triangle u = f, \\ u_{|\partial\Omega} = 0, \\ u(0) = 0, \end{cases} \tag{1.8.8}$$

belongs to $L^p((0,T), L^\infty(\Omega))$ for every $p < \infty$, but does not belong to $L^\infty((0,T) \times \Omega)$.

- Given $f \in L^2((0,T), L^2(\Omega))$, show that $u \in L^p((0,T), L^\infty(\Omega))$ for every $p < \infty$.
- Let $\varphi \in H_0^1(\Omega) \setminus L^\infty(\Omega)$, and let $v(t) = T(t)\varphi$, where $(T(t))_{t\ge 0}$ is the heat semigroup. Define $u \in C([0,2], L^2(\Omega))$ by
$$u(t) = \begin{cases} tv(1-t) & \text{if } 0 \le t \le 1, \\ v(t-1) & \text{if } 1 \le t \le 2. \end{cases}$$
Show that u is the solution of (1.8.8) for some $f \in L^2((0,2), L^2(\Omega))$, and that $u \notin L^\infty((0,2) \times \Omega)$.

Exercise 1.8.17. The object of this exercise is to construct $a, f \in C([0,T], L^{\frac{N}{2}}(\Omega))$, where Ω is the unit ball of $\mathbb{R}^N$, $N \ge 3$, such that the solution u of (1.6.2) with $u_0 = 0$ does not belong to $L^\infty(\Omega)$ for any $t \in [0,T]$.

- Consider a decreasing function $\theta \in C^2((0,1))$ such that $\theta(r) = -\log r$ for r small and $\theta(r) = r^{2-N} - 1$ for $1 - r$ small. Define $\varphi : \Omega \to \mathbb{R}$ by $\varphi(x) = \theta(|x|)$. Show that $\varphi \in H_0^1(\Omega)$ and that
$$-\triangle\varphi = a\varphi,$$
for some $a \in L^{\frac{N}{2}}(\Omega)$.
- For a as above, construct $f \in C([0,T], L^{\frac{N}{2}}(\Omega))$ and a solution u of (1.6.2) with $u_0 = 0$ such that $u(t) \notin L^\infty(\Omega)$ for any $t \in [0,T]$.

Exercise 1.8.18. The object of this exercise is to show that if $\Omega \subset \mathbb{R}^N$ is a bounded domain and if u is the solution of the equation

$$\begin{cases} u_t - \triangle u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ u(0) = 1, \end{cases}$$

then $u_t \le 0$ in Ω and $u_t \not\equiv 0$ for all $t > 0$.

- Show that u is smooth in $(0,\infty) \times \Omega$.
- Show that $\displaystyle\int_\Omega u\varphi_1 = e^{-\lambda_1 t}$ and $\displaystyle\int_\Omega u_t\varphi_1 = -\lambda_1 e^{-\lambda_1 t}$ (and in particular $u_t \not\equiv 0$ for all $t > 0$) where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$ and φ_1 is the first eigenvector, normalized so that $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^1} = 1$.

- Let ζ_ε be the solution of
$$\begin{cases} -\varepsilon\triangle\zeta_\varepsilon + \zeta_\varepsilon = 1 \quad \text{in } \Omega,\\ \zeta_\varepsilon = 0 \quad \text{in } \partial\Omega.\end{cases}$$
Show that $\zeta_\varepsilon \rightharpoonup 1$ in $L^2(\Omega)$ as $\varepsilon \downarrow 0$.

- Let u^ε be the solution of
$$\begin{cases} u^\varepsilon_t - \triangle u^\varepsilon = 0 \quad \text{in } \Omega,\\ u^\varepsilon = 0 \quad \text{in } \partial\Omega,\\ u^\varepsilon(0) = \zeta_\varepsilon.\end{cases}$$
Show that $u^\varepsilon_t(t) \le 0$ for all $t \ge 0$ and conclude.

1.9. Open Problems.

Open Problem 1.9.1. Let $a \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$. Let $u \in L^\infty((0,T), L^q(\Omega))$ for some q possibly very large (but finite) verify
$$\begin{cases} u_t - \triangle u = au,\\ u_{|\partial\Omega} = 0,\\ u(0,x) = u_0(x) \in L^\infty(\Omega).\end{cases}$$
Does u belong to $L^p_{\text{loc}}((0,T), L^p(\Omega))$ for every finite p?

Open Problem 1.9.2. A related question is: can one replace the assumption $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$ by $a \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$ in Theorem 1.6.12? The problem is open even under the additional assumption $u \in C^\infty_{\text{c}}((0,T) \times \Omega)$.

Open Problem 1.9.3. Assume $N \ge 3$. Let $a \in C([0,T], L^{\frac{N}{2}}(\Omega))$ and $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ satisfy the equation
$$\begin{cases} u_t - \triangle u = au,\\ u_{\partial\Omega} = 0.\end{cases}$$
If $\|u(t)\|_{L^1} \underset{t\downarrow 0}{\longrightarrow} 0$, does one have $u \equiv 0$? Note that this is the case if one replaces $L^1(\Omega)$ by $L^{1+\varepsilon}(\Omega)$ or $L^{\frac{N}{2}}(\Omega))$ by $L^{\frac{N}{2}+\varepsilon}(\Omega))$ for some $\varepsilon > 0$, see Exercises 1.8.8 and 1.8.9.

Open Problem 1.9.4. Assume $N \ge 3$, and let $a \in L^{\frac{N}{2}}(\Omega)$ and $u \in L^{\frac{N}{N-2}}(\Omega)$ solve the equation
$$\begin{cases} -\triangle u = au,\\ u_{\partial\Omega} = 0.\end{cases}$$
Let s be the best constant in Sobolev's inequality $s\|u\|_{L^{\frac{2N}{N-2}}} \le \|\nabla u\|_{L^2}$. If $\|a\|_{L^{\frac{N}{2}}} < s^2$, does one have $u \equiv 0$? (Note that formally, the result would follow by multiplying the equation by u and applying Hölder's and Sobolev's inequality.)

Chapter 2. Abstract semilinear problems: global and local existence.

In this chapter, we consider abstract initial value problems of the form

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), \\ u(0) = u_0. \end{cases} \tag{2.1}$$

Throughout this chapter, A is a densely defined linear m-accretive operator in a Banach space X with the norm $\|\cdot\|$, F is a nonlinear mapping, and u_0 is a given initial value. We will consider various situations. For example, the easiest case is when $F : X \to X$ is globally Lipschitz, and then (2.1) has a solution defined for all times $t \geq 0$. Another case is when $F : D(A) \to D(A)$ is globally Lipschitz (for the graph norm), and then again (2.1) has a solution defined for all times $t \geq 0$. Next, we consider the case where F is not globally Lipschitz, but only Lipschitz continuous on every bounded set; in this case, we establish that (2.1) has a local solution defined on a maximal time interval $[0, T_{\rm m})$ and in addition, if $T_{\rm m} < \infty$, then $u(t)$ blows up as $t \uparrow T_{\rm m}$.

2.1. The case $F : X \to X$ is globally Lipschitz. Assume $F : X \to X$ is globally Lipschitz in the sense that there exists a constant L such that

$$\|F(v) - F(u)\| \leq L\|v - u\|,$$

for all $u, v \in X$. The main result of this section is the following.

Theorem 2.1.1. *Given any $u_0 \in X$, there exists a unique global weak solution u of (2.1) in the sense that $u \in C([0,\infty), X)$ and*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\, ds, \tag{2.1.1}$$

for all $t \geq 0$.

In addition, there is continuous dependence of u with respect to u_0:

$$\|v(t) - u(t)\| \leq e^{Lt}\|v_0 - u_0\|, \tag{2.1.2}$$

for all $t \geq 0$, where v is the solution of (2.1.1) with the initial value v_0.

Moreover, if $u_0 \in D(A)$ then u is Lipschitz continuous on bounded sets of $[0,\infty)$; i.e. for every $T < \infty$, there exists a constant M_T such that

$$\|u(t_2) - u(t_1)\| \leq M_T|t_2 - t_1|, \tag{2.1.3}$$

for all $0 \leq t_1, t_2 \leq T$.

Corollary 2.1.2. *Assume that X is reflexive and that $u_0 \in D(A)$. Then (2.1) has a unique global, classical solution, i.e.*

$$u \in C^1([0,\infty), X) \cap C([0,\infty), D(A)), \tag{2.1.4}$$

where $D(A)$ is equipped with the graph norm.

Proof of Corollary 2.1.2. We already know by Theorem 2.1.1 that the solution u of (2.1.1) is Lipschitz continuous on bounded sets, hence $F(u)$ is also Lipschitz continuous on bounded sets. The conclusion (2.1.4) follows from Corollary 1.5.8. □

Proof of Theorem 2.1.1. The proof proceeds in four steps.

Step 1. Uniqueness. Assume u and v are two solutions of (2.1.1). Then

$$\|u(t) - v(t)\| \le L \int_0^t \|u(s) - v(s)\|\, ds,$$

thus by Gronwall's inequality, $\|u(t) - v(t)\| \le \|u(0) - v(0)\| e^{Lt} = 0$.

Step 2. Existence. This is proved by using the contraction mapping principle in the space

$$E = \{u \in C([0,\infty), X);\ \sup_{t\ge 0} e^{-kt}\|u(t)\| < \infty\},$$

where $k > 0$ is to be chosen. E equipped with the norm

$$\|u\|_E = \sup_{t\ge 0} e^{-kt}\|u(t)\|,$$

is a Banach space. Given $u \in E$, set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\, ds,$$

for all $t \ge 0$. We first claim that $\Phi(u) \in E$. It is clear that $\Phi(u) \in C([0,\infty), X)$. Next, we have

$$\|\Phi(u)(t)\| \le \|u_0\| + \int_0^t \|F(u(s))\|\, ds.$$

But $\|F(u(s))\| \le L\|u(s)\| + \|F(0)\|$; and so,

$$\|\Phi(u)(t)\| \le \|u_0\| + t\|F(0)\| + L\|u\|_E \int_0^t e^{ks}\, ds = \|u_0\| + t\|F(0)\| + L\,\frac{e^{kt}-1}{k}\,\|u\|_E.$$

Therefore, $\Phi(u) \in E$ and

$$\|\Phi(u)\|_E \le \|u_0\| + \frac{1}{ek}\|F(0)\| + \frac{L}{k}\|u\|_E.$$

We claim that Φ is a contraction on E provided $k > L$. Indeed, we have

$$\|\Phi(u)(t) - \Phi(v)(t)\| \le L \int_0^t \|u(s) - v(s)\|\, ds \le L\|u-v\|_E \int_0^t e^{ks}\, ds = L\,\frac{e^{kt}-1}{k}\,\|u-v\|_E.$$

Thus,

$$\|\Phi(u) - \Phi(v)\|_E \le \frac{L}{k}\|u - v\|_E.$$

Choosing any $k > L$, we conclude that Φ has a fixed point $u \in E$, which is a solution of equation (2.1.1).

Step 3. Continuous dependence. Assume that u and v are two solutions of (2.1.1) associated to the initial values u_0 and v_0, respectively. Then,

$$\|u(t) - v(t)\| \le \|u_0 - v_0\| + L \int_0^t \|u(s) - v(s)\|\, ds,$$

and (2.1.2) follows from Gronwall's inequality.

Step 4. Lipschitz continuity when $u_0 \in D(A)$. Let $h > 0$. By Corollary 1.5.18 we know that $u(t+h)$ is the weak solution of (2.1) with the initial value $u(h)$. By (2.1.2), we have

$$\|u(t+h) - u(t)\| \le \|u(h) - u(0)\| e^{Lt}, \tag{2.1.5}$$

for all $t \ge 0$. On the other hand, we have

$$u(h) = T(h)u_0 + \int_0^h T(h-s)F(u(s))\, ds;$$

and so,

$$\|u(h) - u_0\| \le \|T(h)u_0 - u_0\| + h \sup_{0<s<h} \|F(u(s))\| \le h\|Au_0\| + h \sup_{0<s<h} \|F(u(s))\|, \tag{2.1.6}$$

by Proposition 1.3.4 (i). On the other hand,

$$\|u(t)\| \le \|u_0\| + \int_0^t \|F(u(s))\|\, ds \le \|u_0\| + t\|F(0)\| + L \int_0^t \|u(s)\|\, ds.$$

By Gronwall's inequality, this implies

$$\|u(t)\| \le (\|u_0\| + t\|F(0)\|) e^{Lt};$$

and so,

$$\sup_{0<s<h} \|F(u(s))\| \le \|F(0)\| + Le^{Lh}(\|u_0\| + h\|F(0)\|).$$

(2.1.3) follows from (2.1.5), (2.1.6) and the above inequality. □

Remark 2.1.3. Instead of applying the contraction mapping principle in E, one could work in $C([0,T], X)$ equipped with its usual norm, and then Φ is a contraction provided $LT < 1$. Fix any such T, then (2.1.1) has a solution of $[0,T]$, and by iteration (2.1.1) has a global solution.

Remark 2.1.4. It is essential to assume in Corollary 2.1.2 that X is reflexive. Here is an example showing that if X is not reflexive, then the weak solution of (2.1.1) needs not be a classical solution even if $u_0 \in D(A)$.

Let $X = C_0(\mathbb{R})$ and let A be defined by

$$\begin{cases} D(A) = \{u \in C^1(\mathbb{R}) \cap X;\, u' \in X\}, \\ Au = u' \text{ for } u \in D(A). \end{cases}$$

Recall that $T(t)\varphi(x) = \varphi(x-t)$ (see Proposition 1.4.1). Let $F(u) = u^+$. Clearly, $F : X \to X$ is globally Lipschitz. We claim that in this case the weak solution of (2.1.1) is given by

$$u(t,x) = e^t u_0^+(x-t) - u_0^-(x-t).$$

Indeed, $F(u(s)) = e^s u_0^+(x-s)$, and $T(t-s)F(u(s)) = e^s u_0^+(x-t)$; and so,

$$\begin{aligned} T(t)u_0 + \int_0^t T(t-s)F(u(s))\, ds &= u_0(x-t) + \int_0^t e^s u_0^+(x-t)\, ds \\ &= u_0(x-t) + (e^t - 1)u_0^+(x-t) \\ &= -u_0^-(x-t) + e^t u_0^+(x-t) = u(t). \end{aligned}$$

Choosing for example $u_0(x) = e^{-x^2} \sin x \in D(A)$, it follows that $u(t) \notin D(A)$ for $t > 0$ since the function $x \mapsto u(t,x)$ is not C^1.

Remark 2.1.5. Throughout this section, we have assumed that F is independent of t. It is very easy to extend the above results to the case where F also depends on t, under various assumptions. For example, assume that $F(t,u) : [0,\infty) \times X \to X$ is continuous and that for every $T < \infty$ there exists L_T such that

$$\|F(t,v) - F(t,u)\| \le L_T \|v - u\|,$$

for all $t \in [0,T]$ and all $u, v \in X$. Then, given any $u_0 \in X$, the problem

$$\begin{cases} \dfrac{du}{dt} + Au = F(t,u), \\ u(0) = u_0, \end{cases}$$

has a unique global weak solution. For every $T > 0$, we consider the space

$$E_T = \{u \in C([0,T], X);\ \sup_{0 \le t \le T} e^{-kt} \|u(t)\| < \infty\},$$

with $k > L_T$, and then the map Φ defined by

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s) F(s, u(s))\, ds,$$

has a unique fixed point in E_T.

2.2. The case $F : X \to X$ is globally Lipschitz and C^1. Assume $F : X \to X$ is globally Lipschitz and that $F \in C^1(X,X)$. The main result of this section is the following.

Theorem 2.2.1. *Given $u_0 \in D(A)$, there exists a unique global, classical solution u of (2.1), i.e. $u \in C^1([0,\infty), X) \cap C([0,\infty), D(A))$.*

Remark 2.2.2. In contrast with Corollary 2.1.2, we do not assume here that X is reflexive, but instead we assume that $F \in C^1(X,X)$.

Proof of Theorem 2.2.1. The idea is the following. Consider the problem (2.1), and formally differentiate it with respect to t. Thus we have

$$\frac{d}{dt}\left(\frac{du}{dt}\right) + A\frac{du}{dt} = F'(u) \cdot \frac{du}{dt};$$

hence $v = \dfrac{du}{dt}$ satisfies

$$\begin{cases} \dfrac{dv}{dt} + Av = F'(u) \cdot v, \\ v(0) = F(u_0) - Au_0. \end{cases} \tag{2.2.1}$$

So far, we do not know whether $\dfrac{du}{dt}$ really exist, but on the other hand the existence of v satisfying (2.2.1) in the weak sense follows from Remark 2.1.5. Therefore, we define v to be the weak solution of (2.2.1), and our aim is to prove that $u \in C^1([0,\infty), X)$ with $v = \dfrac{du}{dt}$.

From (2.1.1), we have

$$\frac{u(t+h)-u(t)}{h} = T(t)\frac{T(h)-I}{h}u_0 \\ + \int_0^t T(t-s)\frac{F(u(s+h))-F(u(s))}{h}\,ds + \frac{1}{h}\int_0^h T(t+h-s)F(u(s))\,ds.$$

On the other hand, from (2.2.1) we have

$$v(t) = T(t)(F(u_0) - Au_0) + \int_0^t T(t-s)F'(u(s))\cdot v(s)\,ds.$$

It follows that

$$\Big\|\frac{u(t+h)-u(t)}{h} - v(t)\Big\| \le x_1 + x_2 + x_3,$$

with

$$x_1 = \Big\|T(t)\Big(\frac{T(h)-I}{h}u_0 + Au_0\Big)\Big\|,$$
$$x_2 = \int_0^t T(t-s)\Big\|\frac{F(u(s+h))-F(u(s))}{h} - F'(u(s))\cdot v(s)\Big\|\,ds,$$
$$x_3 = \Big\|\frac{1}{h}\int_0^h T(t+h-s)F(u(s))\,ds - T(t)F(u_0)\Big\|.$$

Since $u_0 \in D(A)$, we have $x_1 \to 0$ as $h \downarrow 0$. Next, we estimate x_2. Using the fact that $F \in C^1(X,X)$, we have for every $a \in X$

$$\|F(a+z) - F(a) - F'(a)\cdot z\| \le \varepsilon_a(\|z\|)\|z\|,$$

with $\varepsilon_a(\|z\|) \to 0$ as $\|z\| \to 0$. Moreover, this estimate is uniform for a in a compact set of X. Given $T < \infty$, it follows that

$$\|F(u(s+h)) - F(u(s)) - F'(u(s))\cdot(u(s+h)-u(s))\| \le \varepsilon(\|u(s+h)-u(s)\|)\|u(s+h)-u(s)\|,$$

for all $s \in [0,T]$ and all $h \in [0,1]$. Using Theorem 2.1.1, we obtain

$$\|F(u(s+h)) - F(u(s)) - F'(u(s))\cdot(u(s+h)-u(s))\| \le \varepsilon(h)\,h;$$

and thus

$$\Big\|\frac{F(u(s+h))-F(u(s))}{h} - F'(u(s))\cdot v(s)\Big\| \le \varepsilon(h) + \Big\|F'(u(s))\cdot\Big(\frac{u(s+h)-u(s)}{h} - v(s)\Big)\Big\|.$$

Therefore,

$$x_2 \le T\varepsilon(h) + C\int_0^t \Big\|\frac{u(s+h)-u(s)}{h} - v(s)\Big\|\,ds,$$

with $C = \sup\limits_{0\le s\le T}\|F'(u(s))\|$.

Finally,

$$x_3 \le \Big\|\frac{1}{h}\int_0^h T(h-s)F(u(s))\,ds - F(u_0)\Big\| \le \eta(h),$$

with $\eta(h) \to 0$ as $h \downarrow 0$. Set

$$\varphi_h(t) = \Big\|\frac{u(t+h)-u(t)}{h} - v(t)\Big\|.$$

Combining the above inequalities, we find

$$\varphi_h(t) \le T\varepsilon(h) + \eta(h) + C\int_0^t \varphi_h(s)\,ds.$$

By Gronwall's lemma, this implies that

$$\varphi_h(t) \le (T\varepsilon(h) + \eta(h))e^{Ct},$$

for all $t \in [0,T]$. It follows that $\varphi_h(t) \to 0$ as $h \downarrow 0$. This means that u is right differentiable for all $t \in [0,T)$ and that $\frac{d^+u}{dt} = v(t)$. Since $v \in C([0,\infty), X)$, it follows from Theorem A.1.16 that $u \in C^1([0,T), X)$. This implies that $F(u) \in C^1([0,T), X)$. Going back to equation (2.1.1), we may now assert, using Proposition 1.5.4 that $u \in C([0,T), D(A))$ and that

$$\frac{du}{dt} + Au = F(u),$$

for all $t \in [0,T)$. The result follows, since T is arbitrary. □

2.3. The case $F : D(A) \to D(A)$ is globally Lipschitz for the graph norm. In this section, we assume $F : D(A) \to D(A)$ is globally Lipschitz, i.e. that there exists a constant L such that

$$\|F(v) - F(u)\| + \|A(F(v) - F(u))\| \le L(\|v-u\| + \|Av - Au\|),$$

for all $u, v \in D(A)$. The main result is the following.

Theorem 2.3.1. *Given any $u_0 \in D(A)$, there exists a unique global, classical solution u of (2.1), i.e. $u \in C^1([0,\infty), X) \cap C([0,\infty), D(A))$.*

In addition, if $u_0 \in D(A^2)$ then $\frac{du}{dt}$ and Au are Lipschitz continuous from bounded sets of $[0,\infty)$ to X. Moreover, if X is reflexive and $u_0 \in D(A^2)$, then $u \in C^1([0,\infty), D(A)) \cap C([0,\infty), D(A^2))$.

Proof. As in Theorem 1.1.28, set $X_1 = D(A)$ and consider the operator $A_{(1)}$ defined by

$$\begin{cases} D(A_{(1)}) = \{x \in X_1;\ Ax \in X_1\}, \\ A_{(1)}x = Ax \text{ for all } x \in D(A_{(1)}). \end{cases}$$

Recall that $A_{(1)}$ is a densely defined m-accretive operator in X_1 (Theorem 1.1.28) and that the semigroup generated by $-A_{(1)}$ coincides with the restriction of $(T(t))_{t\ge 0}$ to X_1 (Proposition 1.3.16). Applying Theorem 2.1.1, there exists a unique, global weak solution in the sense that $u \in C([0,\infty), D(A))$ and u verifies (2.1.1). In particular, $F(u) \in C([0,\infty), D(A))$, and it follows from Corollary 1.5.6 that $u \in C^1([0,\infty), X)$ and that u solves (2.1) in the classical sense.

In the case $u_0 \in D(A^2)$, it follows from Theorem 2.1.1 that

$$\|u(t_2) - u(t_1)\|_{D(A)} \le M_T |t_2 - t_1|,$$

for all $T < \infty$ and $0 \le t_1, t_2 \le T$, i.e. Au is Lipschitz continuous on bounded sets of $[0,\infty)$ into X. Going back to the equation (2.1), we see that $\frac{du}{dt} = F(u) - Au$ is also Lipschitz continuous on bounded sets of $[0,\infty)$ into X.

Finally, if X is reflexive, so is X_1; and from Corollary 2.1.2 we deduce that if $u_0 \in D(A^2)$, then $u \in C^1([0,\infty), X_1) \cap C([0,\infty), D(A_{(1)}))$, i.e. $u \in C^1([0,\infty), D(A)) \cap C([0,\infty), D(A^2))$. □

2.4. The case $F : X \to X$ is Lipschitz continuous on bounded sets. Maximal interval of existence. The blow up alternative. Assume $F : X \to X$ is Lipschitz continuous on bounded sets, i.e. for every constant M, there exists L_M such that

$$\|F(v) - F(u)\| \leq L_M \|v - u\|, \tag{2.4.1}$$

for all $u, v \in X$ such that $\|u\| \leq M$ and $\|v\| \leq M$. The first result of this section is the following.

Theorem 2.4.1. *For every $u_0 \in X$, there exists $0 < T < \infty$ and a unique weak solution u of (2.1) defined on $[0, T]$, i.e. $u \in C([0, T], X)$ and (2.1.1) holds for all $t \in [0, T]$.*

Proof. Set $E = C([0, T], X)$ with its usual norm, where $T > 0$ is to be chosen later. Set

$$K = \{u \in E;\ \|u(t)\| \leq \|u_0\| + 1 \text{ for all } t \in [0, T]\},$$

so that K is a closed subset of the Banach space E. Given any $u \in K$, set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t - s)F(u(s))\, ds,$$

for all $t \in [0, T]$, so that $\Phi(u) \in E$. We claim that

a) $\|\Phi(v) - \Phi(u)\|_E \leq LT\|v - u\|_E$ for all $u, v \in K$, where $L = L_M$ with $M = \|u_0\| + 1$. This is an obvious consequence of (2.4.1).

b) $\Phi : K \to K$, provided $T(\|F(0)\| + L(\|u_0\| + 1)) \leq 1$. Indeed,

$$\|\Phi(u)(t)\| \leq \|u_0\| + \int_0^t \|F(u(s))\|\, ds.$$

On the other hand,

$$\|F(u(s)) - F(0)\| \leq L\|u(s)\| \leq L(\|u_0\| + 1),$$

since $u \in K$. Therefore,

$$\|\Phi(u)(t)\| \leq \|u_0\| + T(\|F(0)\| + L(\|u_0\| + 1)),$$

and the conclusion follows.

We now choose T small enough so that

$$T(\|F(0)\| + L(\|u_0\| + 1)) < 1, \tag{2.4.2}$$

which implies in particular that $LT < 1$. Then Φ has a unique fixed point $u \in K$. This u is a weak solution of (2.1). The uniqueness of u follows from (2.4.1) and Gronwall's inequality, as in the proof of Theorem 2.1.1.
□

Remark 2.4.2. It is tempting to iterate this construction. We first get existence on an interval $[0, T_1]$ as above. Next, we have a weak solution of (2.1) starting from the initial value $u(T_1)$, and it is defined on $[0, \delta_1]$ with

$$\delta_1(\|F(0)\| + C_1(\|u(T_1)\| + 1)) < 1,$$

where $C_1 = L_M$ with $M = \|u(T_1)\| + 1$. Gluing these two functions, and using Corollary 1.5.18, we now obtain a weak solution of (2.1) defined on $[0, T_2]$ with $T_2 = T_1 + \delta_1$. We define inductively the increasing sequence $(T_n)_{n\geq 1}$. It can very well happen that $\sup_{n\geq 1} T_n < \infty$ (see Remark 2.4.4).

Maximal interval of existence. Let $T_1 < T_2$ and let u_1 and u_2 be weak solutions of (2.1) on $[0, T_1]$ and $[0, T_2]$, respectively. By uniqueness, we know that $u_1 = u_2$ on $[0, T_1]$. Consider now the family $(u_i(t))_{i\in I}$ of all weak solutions of (2.1) defined on some interval $[0, T_i]$. Set

$$T_{\mathrm{m}} = \sup_{i\in I} T_i.$$

Note that T_{m} may be $+\infty$. We define the function $u(t)$ on $[0, T_{\mathrm{m}})$ by

$$u(t) = u_i(t), \text{ if } t \in [0, T_i], i \in I.$$

This function is well defined by the uniqueness property mentioned above. Note that $u \in C([0, T_{\mathrm{m}}), X)$ and that u verifies (2.1.1) for all $t \in [0, T_{\mathrm{m}})$. This solution is called the **maximal solution** of (2.1).

Theorem 2.4.3. *Assume (2.4.1) and let u be the maximal solution of (2.1). Then, the following alternative holds.*

Either $T_{\mathrm{m}} = +\infty$,

or $T_{\mathrm{m}} < \infty$ and $\lim_{t\uparrow T_{\mathrm{m}}} \|u(t)\| = +\infty$.

In the first case, we say that u is a **global solution***, in the second case we say that u* **blows up in finite time**.

Proof. Suppose that $T_{\mathrm{m}} < \infty$ and that there exists a sequence $t_j \uparrow T_{\mathrm{m}}$ such that $\|u(t_j)\| \leq C < \infty$. Fix any $\delta > 0$ such that

$$\delta(\|F(0)\| + L(C + 1)) < 1,$$

where $L = L_M$ with $M = C + 1$. Starting from $u(t_j)$, we have a weak solution v_j of (2.1) defined on $[0, \delta]$. Gluing together u with v_j, we obtain a weak solution of (2.1) defined on $[0, t_j + \delta]$ (see Corollary 1.5.18). For j sufficiently large, $t_j + \delta > T_{\mathrm{m}}$, and this is impossible since u is the maximal solution. □

Remark 2.4.4. For a given equation (2.1), the maximal time of existence T_{m} depends on u_0. Here are some simple examples with $X = \mathbb{R}$ and $A = 0$ showing that many situations may occur.

Example 1: $\dfrac{du}{dt} = -u^3$. Here, the solution is given by $u(t) = \dfrac{u_0}{\sqrt{1 + 2tu_0^2}}$, so that $T_{\mathrm{m}} = +\infty$ for every $u_0 \in \mathbb{R}$.

Example 2: $\dfrac{du}{dt} = u^3$. Here, the solution is given by $u(t) = \dfrac{u_0}{\sqrt{1 - 2tu_0^2}}$, so that $T_{\mathrm{m}} = \begin{cases} +\infty \text{ if } u_0 = 0, \\ \dfrac{1}{2u_0^2} < \infty \text{ if } u_0 \neq 0. \end{cases}$

Example 3: $\dfrac{du}{dt} = u^2$. Here, the solution is given by $u(t) = \dfrac{u_0}{1 - tu_0}$, so that $T_{\mathrm{m}} = \begin{cases} +\infty \text{ if } u_0 \leq 0, \\ \dfrac{1}{u_0} < \infty \text{ if } u_0 > 0. \end{cases}$

Example 4: $\dfrac{du}{dt} = u^3 - u$. Here, the solution is given by $u(t) = \dfrac{u_0}{\sqrt{u_0^2 - e^{2t}(u_0^2 - 1)}}$, so that

$$T_{\mathrm{m}} = \begin{cases} +\infty \text{ if } |u_0| \le 1, \\ \dfrac{1}{2}\log\left(\dfrac{u_0^2}{u_0^2 - 1}\right) < \infty \text{ if } |u_0| > 1. \end{cases}$$

Theorem 2.4.5. *Assume (2.4.1).*

(i) *The mapping $u_0 \mapsto T_{\mathrm{m}}(u_0)$ is lower semicontinuous.*

(ii) *The solution u depends continuously on the initial value u_0 in the sense that if $u_0^j \underset{j\to\infty}{\longrightarrow} u_0$ and if u^j is the corresponding maximal solution of (2.1) defined on the interval $[0, T_j)$, then given any $T < T_{\mathrm{m}}$, u^j is defined on $[0, T]$ for j large enough and $u^j \underset{j\to\infty}{\longrightarrow} u$ in $C([0,T], X)$. More precisely, there exists C_T such that $\|u^j(t) - u(t)\| \le C_T \|u_0^j - u_0\|$ for all $t \in [0, T]$.*

Proof. Given u_0, let $T < T_{\mathrm{m}}(u_0)$, $C = \max\limits_{0\le t\le T} \|u(t)\|$ and $M = C + 1$. Let $\delta > 0$ be small enough so that

$$\delta(\|F(0)\| + L(M+1)) < 1,$$

with $L = L_{M+1}$, and let j be large enough so that

$$\|u_0^j - u_0\| e^{LT} < 1. \tag{2.4.3}$$

We claim that $T_j = T_{\mathrm{m}}(u_0^j) > T$ and that

$$\|u^j(t) - u(t)\| \le \|u_0^j - u_0\| e^{LT}, \tag{2.4.4}$$

for all $t \in [0, T]$. Indeed, it follows from (2.4.3) and Theorem 2.4.1 (see in particular formula (2.4.2)) that $T_j > \delta$ and $\sup\limits_{0\le t\le\delta} \|u^j(t)\| \le M + 1$. It follows from (2.4.1) and Gronwall's inequality that (2.4.4) holds for all $t \in [0, \delta]$. In particular, $\|u^j(t)\| \le M$, and one can iterate this construction k times with $k = [T/\delta]$. Therefore, u^j is defined on $[0, (k+1)\delta]$, and in particular $T_j > T$, and the estimate (2.4.4) holds for all $t \in [0, T]$. The result follows. □

Theorem 2.4.6. *Assume (2.4.1). Suppose $u_0 \in D(A)$ and let u be the maximal solution of (2.1).*

(i) *u is Lipschitz continuous on compact intervals of $[0, T_{\mathrm{m}})$.*

(ii) *If X is reflexive, then u is a classical solution of (2.1) on $[0, T_{\mathrm{m}})$, i.e. $u \in C^1([0, T_{\mathrm{m}}), X) \cap C([0, T_{\mathrm{m}}), D(A))$.*

(iii) *If $F \in C^1(X, X)$, then u is a classical solution of (2.1) on $[0, T_{\mathrm{m}})$.*

We omit the proof, since it is similar to the proofs of the corresponding statements when F is globally Lipschitz.

Theorem 2.4.7. *Assume (2.4.1) and set $C(M) = \sup\{\|F(x)\|;\ \|x\| \le M\}$. If u is the maximal weak solution of (2.1), then*

$$\frac{C(\|u(t)\| + \beta)}{\beta} > \frac{1}{T_{\mathrm{m}} - t}, \tag{2.4.5}$$

for all $\beta > 0$ and all $t \in [0, T_{\rm m})$.

Proof. The proof proceeds in two steps.

Step 1. Let $t \in [0, T_{\rm m})$ and $\beta > 0$. If $0 \le s \le \dfrac{\beta}{C(\|u(t)\| + \beta)}$ is such that $t + s < T_{\rm m}$, then

$$\|u(t+s)\| \le \|u(t)\| + \beta.$$

Indeed, otherwise there exists $s_0 < \dfrac{\beta}{C(\|u(t)\| + \beta)}$ such that $t + s_0 < T_{\rm m}$, $\|u(t+s)\| \le \|u(t)\| + \beta$ for $0 \le s \le s_0$ and $\|u(t+s_0)\| = \|u(t)\| + \beta$. However, it follows from (2.1.1) that

$$\|u(t+s_0)\| \le \|u(t)\| + \int_0^{s_0} \|F(u(s))\|\, ds \le \|u(t)\| + s_0 C(\|u(t)\| + \beta) < \|u(t)\| + \beta,$$

which is absurd.

Step 2. Note that (2.4.5) is equivalent to

$$T_{\rm m} > t + \frac{\beta}{C(\|u(t)\| + \beta)}.$$

If the above inequality does not hold, there exists $\beta_0 > 0$ and $t_0 \in [0, T_{\rm m})$ such that

$$T_{\rm m} \le t + \frac{\beta_0}{C(\|u(t_0)\| + \beta_0)};$$

then by Step 1, u is bounded on $[0, T_{\rm m})$. Therefore, $F(u(\cdot)) \in L^\infty((0, T_{\rm m}), X)$ and it follows from formula (2.1.1) that $u(t)$ has a limit as $t \uparrow \infty$, which is impossible. The result follows. □

Remark 2.4.8. Observe that (2.4.5) gives a lower bound of blow up which is independent of the solution. Note that this estimate often provides an optimal rate. This is the case for the examples of Remark 2.4.4 (choose $\beta = \|u(t)\|$).

Remark 2.4.9. **The case where F is locally Lipschitz.** Assume $F : X \to X$ is locally Lipschitz, i.e. for every $x \in X$, there exists $r > 0$ such that F is Lipschitz continuous $B(x, r) \to X$. Note that this assumption is weaker than the assumption that F is Lipschitz continuous on every bounded set. (It does not even imply that F is bounded on bounded sets.) The same argument as in the proof of Theorem 2.4.1 shows that for every $u_0 \in X$ there is a unique weak solution defined on a maximal interval $[0, T_{\rm m})$. We call attention on the following.

Open problem. Assume F is locally Lipschitz, and let $u_0 \in X$ be such that $T_{\rm m} < \infty$. Does $\|u(t)\| \to +\infty$ as $t \uparrow T_{\rm m}$? (It is not even clear that $\limsup_{t \uparrow T_{\rm m}} \|u(t)\| = +\infty$.)

Note that if F is bounded on bounded sets, then the answer is yes. Indeed, the conclusion of Theorem 2.4.7 holds without assumption (2.4.1), and estimate (2.4.5) implies blow up by choosing $\beta = 1$.

Theorem 2.4.10. *Assume $F : X \to X$ is locally Lipschitz and in addition that F grows at most linearly, i.e.*

$$\|F(u)\| \le C_1 \|u\| + C_2, \tag{2.4.6}$$

for all $x \in X$. Then, for every $u_0 \in X$, $T_{\rm m}(u_0) = +\infty$.

Proof. First observe that by Remark 2.4.9, estimate (2.4.5) holds. The conclusion follows by taking $\beta = \|u(t)\| + 1$ in (2.4.5), and using assumption (2.4.6). □

Here is another situation where u is globally defined for every $u_0 \in X$.

Theorem 2.4.11. *Assume that X is a Hilbert space with scalar product $(\cdot,\cdot)$ and that $F : X \to X$ is Lipschitz continuous on bounded sets. If*

$$(F(u), u) \le C_1\|u\|^2 + C_2, \tag{2.4.7}$$

for all $u \in X$, then $T_{\rm m}(u_0) = +\infty$ for every $u_0 \in X$.

Proof. Assume first that $u_0 \in D(A)$, so that u is a classical solution of (2.1). We claim that $T_{\rm m} = +\infty$ and that

$$\|u(t)\|^2 \le (\|u_0\|^2 + 2C_2)e^{2C_1 t}, \tag{2.4.8}$$

for all $t \ge 0$. Indeed, by taking the scalar product of the equation with u, we obtain using (2.4.7)

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \le C_1\|u\|^2 + C_2,$$

from which (2.4.8) follows.

Let now $u_0 \in X$ and $u_0^j \to u_0$ with $u_0^j \in D(A)$. The corresponding solutions u^j are global and verify (2.4.8), which implies that they are uniformly bounded on bounded sets of $[0,\infty)$. By continuous dependence, we see that (2.4.8) holds for all $t \in [0, T_{\rm m})$, and thus $T_{\rm m} = +\infty$. □

2.5. The case $F : D(A) \to D(A)$ is Lipschitz continuous on bounded sets.

In this section, we assume that $F : D(A) \to D(A)$ is Lipschitz continuous on bounded sets, i.e. for every constant M, there exists L_M such that

$$\|F(v) - F(u)\| + \|A(F(v) - F(u))\| \le L_M(\|v - u\| + \|Av - Au\|), \tag{2.5.1}$$

for all $u, v \in D(A)$ such that $\|u\| + \|Au\| \le M$ and $\|v\| + \|Av\| \le M$.

Theorem 2.5.1. *Assume (2.5.1). For every $u_0 \in D(A)$, there exists a unique classical solution u of (2.1) defined on a maximal interval $[0, T_{\rm m})$, with the alternative that*

either $T_{\rm m} = +\infty$,

or $T_{\rm m} < \infty$ and $\lim\limits_{t\uparrow T_{\rm m}} \|u(t)\| + \|Au(t)\| = +\infty$.

In addition, if $u_0 \in D(A^2)$, then $\dfrac{du}{dt}$ and Au are Lipschitz continuous on compact subsets of $[0, T_{\rm m})$ into X.

Moreover, if X is reflexive and $u_0 \in D(A^2)$, then $u \in C^1([0, T_{\rm m}), D(A)) \cap C([0, T_{\rm m}), D(A^2))$.

We omit the proof, since it is similar to the proof of Theorem 2.3.1; instead of using Theorem 2.1.1, we now use the results of Section 2.4.

Remark 2.5.2. In some concrete examples, it is possible to apply several existence results, and in principle they could lead to defferent values of T_{m}. Often, one can prove that the T_{m} is the same for all methods. Suppose for example $F : X \to X$ is Lipschitz continuous on bounded sets and, given $u_0 \in D(A)$, let T_1 be the maximal time of existence. Suppose that in addition $F : D(A) \to D(A)$ is Lipschitz continuous on bounded sets and let T_2 be the corresponding maximal time of existence. Then $T_1 = T_2$.
Proof. $T_2 \le T_1$. Indeed, if $T_1 < T_2$, then $\lim_{t \uparrow T_1} \|u(t)\| = +\infty$, but on the other hand, $u \in C([0, T_2), X)$. Impossible.

$T_2 \ge T_1$. Suppose not, that $T_2 < T_1$. Then, $\lim_{t \uparrow T_2} \|u(t)\| + \|Au(t)\| = +\infty$. Fix $T_3 \in (T_2, T_1)$. It follows from Theorem 2.4.6 (i) that u is Lipschitz continuous $[0, T_3] \to X$. On the other hand, we know that $u \in C^1([0, T_2), X)$, and thus $\left\| \frac{du}{dt}(t) \right\| \le C$ for all $t \in [0, T_2)$ and some constant C. Going back to the equation, we have
$$\|Au(t)\| \le \left\| \frac{du}{dt}(t) \right\| + \|F(u(t))\| \le C',$$
for all $t \in [0, T_2)$. Impossible.

Conclusion. In many concrete problems the first question is to determine whether for a given initial value u_0 the solution is global or whether it blows up in finite time. In view of the above results, the global existence follows from *a priori* estimates on every bounded set of $[0, \infty)$. In the case where $F : X \to X$ is Lipschitz continuous on bounded sets, then we can apply Theorem 2.4.3 and it suffices to estimate $\|u(t)\|$. However, in a number of important situations, F does not map $X \to X$ but it maps $D(A) \to D(A)$. In view of Theorem 2.5.1, global existence then follows from an estimate of $\|u(t)\| + \|Au(t)\|$ on every bounded set of $[0, \infty)$. Of course, this can be a rather delicate task.

The next question is the following.

(i) If the solution is global, then it is of interest to study its asymptotic behavior as $t \to \infty$.

(ii) If finite time blow up occurs, then one wants to know how $u(t)$ behaves near blow up time.

Here is still another variant of the local existence theory.

Theorem 2.5.3. *Assume X is reflexive and $F : D(A) \to D(A)$ verifies the following properties.*

(i) *F maps bounded sets of $D(A)$ into bounded sets of $D(A)$.*

(ii) *For every M, there exists L_M such that*
$$\|F(v) - F(u)\| \le L_M \|v - u\|,$$
for all $u, v \in D(A)$ such that $\|u\|_{D(A)} \le M$ and $\|v\|_{D(A)} \le M$.

Then, for every $u_0 \in D(A)$, there exists a unique, classical solution of (2.1) defined on a maximal interval $[0, T_{\mathrm{m}})$, with the alternative that

either $T_{\mathrm{m}} = +\infty$,

or $T_{\rm m} < \infty$ *and* $\lim_{t\uparrow T_{\rm m}} \|u(t)\| + \|Au(t)\| = +\infty$.

Proof. Given $T > 0$, let $E = C([0,T], D(A))$, and set

$$K = \{u \in E;\ \|u(t)\|_{D(A)} \le \|u_0\|_{D(A)} + 1,\ \text{for } t \in [0,T]\}.$$

K is equipped with the distance induced by the norm of $C([0,T], X)$. Note that K is a complete metric space, since X (hence $D(A)$) is reflexive.

Consider Φ as in the proof of Theorem 2.1.1. $\Phi : K \to K$ is a strict contraction provided $TL_M < 1$ and $T \sup\{\|F(u)\|_{D(A)};\ \|u\|_{D(A)} \le M\} \le 1$, with $M = \|u_0\|_{D(A)} + 1$. The alternative is proved as in Theorem 2.4.3. □

2.6. Smoothing effect for self-adjoint operators in Hilbert spaces. In this section, we assume that X is a Hilbert space with the scalar product $(\cdot,\cdot)$ and that A is a self-adjoint accretive operator.

Theorem 2.6.1. *Assume $F : X \to X$ is Lipschitz continous on bounded sets. Then, for every $u_0 \in X$, the weak solution of (2.1) defined on the maximal interval $[0, T_{\rm m})$ satisfies in addition $u \in C^1((0,T_{\rm m}), X) \cap C((0,T_{\rm m}), D(A))$. In particular, u is a classical solution of (2.1) on $(0, T_{\rm m})$.*

Proof. The proof is an adaptation of the proof of Theorem 1.3.34 to the nonlinear case. We first assume $u_0 \in D(A)$, so that by Theorem 2.4.6 u is a classical solution of (2.1) on $[0, T_{\rm m})$, and we obtain estimates that are independent of $\|Au_0\|$. Taking the scalar product of the equation with u, we find

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + (Au, u) = (F(u), u);$$

and so, for every $t \in [0, T_{\rm m})$,

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t (Au(s), u(s))\, ds = \frac{1}{2}\|u_0\|^2 + \int_0^t (F(u(s)), u(s))\, ds. \tag{2.6.1}$$

Next, we take the scalar product of the equation with $t\dfrac{du}{dt}$, and we obtain

$$t\Big\|\frac{du}{dt}\Big\|^2 + t(Au, \frac{du}{dt}) = t(F(u), \frac{du}{dt}).$$

Since A is self-adjoint,

$$(Au, \frac{du}{dt}) = \frac{1}{2}\frac{d}{dt}(Au, u);$$

and so,

$$\frac{1}{2}\frac{d}{dt}(t(Au, u)) + t\Big\|\frac{du}{dt}\Big\|^2 = \frac{1}{2}(Au, u) + t(F(u), \frac{du}{dt}).$$

Since $(F(u), \dfrac{du}{dt}) \le \dfrac{1}{2}\|F(u)\|^2 + \dfrac{1}{2}\Big\|\dfrac{du}{dt}\Big\|^2$, this yields after integration on $(0,t)$, $0 < t < T_{\rm m}$,

$$t(Au(t), u(t)) + \int_0^t s\Big\|\frac{du}{dt}\Big\|^2 ds \le \int_0^t (Au(s), u(s))\, ds + \int_0^t s\|F(u(s))\|^2\, ds.$$

Using now (2.6.1) we obtain

$$t(Au(t), u(t)) + \int_0^t s\Big\|\frac{du}{dt}\Big\|^2 ds \le \frac{1}{2}\|u_0\|^2 + \int_0^t \{s\|F(u(s))\|^2 + (F(u(s), u(s))\}\, ds. \tag{2.6.2}$$

Next, given $0 < h < T_{\rm m}$, we set $u_h(t) = \dfrac{u(t+h) - u(t)}{h}$ for all $0 < t < T_{\rm m} - h$. We have by equation (2.1)

$$\frac{du_h}{dt} + Au_h = \frac{F(u(t+h)) - F(u(t))}{h}.$$

Taking the scalar product of this equation with u_h, we find

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + (Au_h, u_h) = \left(\frac{F(u(t+h)) - F(u(t))}{h}, u_h\right).$$

It follows from (2.4.1) that

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 \le C(t+h)\|u_h\|^2,$$

where $C(s) = L_{M(s)}$ with $M(s) = \sup\limits_{0\le\sigma\le s} \|u(\sigma)\|$. It follows that

$$\|u_h(t)\|^2 \le \|u_h(s)\|^2 e^{2(t-s)C(t+h)}.$$

Letting $h \downarrow 0$ and using the fact that $u \in C^1([0, T_{\rm m}), X)$, we find

$$\Big\|\frac{du}{dt}(t)\Big\|^2 \le \Big\|\frac{du}{dt}(s)\Big\|^2 e^{2(t-s)C(t)},$$

for $0 \le s \le t < T_{\rm m}$. It follows that

$$s\Big\|\frac{du}{dt}(s)\Big\|^2 \ge se^{-2tC(t)}\Big\|\frac{du}{dt}(t)\Big\|^2,$$

and (2.6.2) yields

$$t(Au(t), u(t)) + \frac{t^2}{2}e^{-2tC(t)}\Big\|\frac{du}{dt}(t)\Big\|^2 \le \frac{1}{2}\|u_0\|^2 + \int_0^t \{s\|F(u(s))\|^2 + (F(u(s), u(s))\}\, ds. \tag{2.6.3}$$

for all $0 \le t < T_{\rm m}$.

Let now $u_0 \in X$, let $u_0^j \to u_0$ with $u_0^j \in D(A)$, and let u^j be the corresponding maximal solutions of (2.1). Given any $t \in (0, T_{\rm m})$, it follows from (2.6.3) and the continuous dependence (Theorem 2.4.5) that there exists C_T such that

$$t^2\Big\|\frac{du^j}{dt}(t)\Big\|^2 \le C_T,$$

for all $t \in [0, T]$ and all j sufficiently large. Using the equation, it follows that there exists K_T such that

$$t^2\|Au^j(t)\|^2 \le K_T,$$

for all $t \in [0, T]$ and all j sufficiently large. Since $u^j(t) \to u(t)$ in X as $j \to \infty$, this implies that $u(t) \in D(A)$ for all $t \in (0, T]$. Therefore, by Theorem 2.4.6, u is a classical solution of (2.1) on $[t, T]$. The result follows, since $0 < t < T < T_{\rm m}$ are arbitrary. □

2.7. Some simple examples where global existence holds.

Example 1. Consider the equation

$$\begin{cases} u_t + u_x = g(u), \quad 0 < x < 1, t \ge 0, \\ u(t, 0) = 0, \quad t \ge 0, \\ u(0, x) = u_0(x), \quad 0 < x < 1, \end{cases} \tag{2.7.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and

$$|g(u)| \leq C|u| + C, \tag{2.7.2}$$

for all $u \in \mathbb{R}$ and some constant C. Assume $u_0 \in W^{1,p}(0,1)$ with $1 < p < \infty$ and $u_0(0) = 0$. Then, there exists a unique global, classical solution u of (2.7.1), i.e. $u \in C^1([0,\infty), L^p(0,1)) \cap C([0,\infty), W^{1,p}(0,1))$.

Fix $T > 0$ and $M > 0$, and let

$$\widetilde{g}(u) = \begin{cases} g(M) \text{ if } u > M, \\ g(u) \text{ if } -M \leq u \leq M, \\ g(-M) \text{ if } u < -M, \end{cases}$$

so that $\widetilde{g}$ is globally Lipschitz and verifies (2.7.2). Applying Corollary 2.1.2 with $X = L^p(0,1)$ and Remark 1.4.2 (iii), we obtain a unique global, classical solution $\widetilde{u}$ of (2.7.1) where g is replaced by $\widetilde{g}$. From the equation

$$\widetilde{u}(t) = T(t)u_0 + \int_0^t T(t-s)\widetilde{g}(\widetilde{u}(s))\, ds,$$

and the fact that $T(t)$ is a contraction in $L^\infty(0,1)$ (by formula (1.4.3)), it follows that

$$\|\widetilde{u}(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \int_0^t (C\|\widetilde{u}(s)\|_{L^\infty} + C)\, ds.$$

By Gronwall's inequality, we obtain

$$\|\widetilde{u}(t)\|_{L^\infty} \leq (\|u_0\|_{L^\infty} + Ct)e^{Ct},$$

for all $t \geq 0$. We choose $M = (\|u_0\|_{L^\infty} + CT)e^{CT}$, so that $\widetilde{g}(\widetilde{u}(t,x)) = g(\widetilde{u}(t,x))$ for all $x \in (0,1)$ and all $t \in [0,T]$. Therefore, $\widetilde{u}$ is a classical solution of (2.7.1) on $[0,T]$. The result follows, since T is arbitrary.

Assume in addition that $g \in C^2(\mathbb{R},\mathbb{R})$ with $g(0) = 0$, and that $u_0 \in W^{2,p}(0,1)$ with $u_0(0) = u_0'(0) = 0$. Then $u \in C([0,\infty), W^{2,p}(0,1)) \cap C^1([0,\infty), W^{1,p}(0,1))$. This follows from Theorem 2.5.1. We only have to verify that $F : D(A) \to D(A)$ is Lipschitz continuous on bounded sets. Here, $D(A) = \{u \in W^{1,p}(0,1);\, u(0) = 0\}$. Given M, we have to evaluate $\|g(u) - g(v)\|_{D(A)}$, i.e. $\|g(u) - g(v)\|_{L^p} + \|g'(u)u_x - g'(v)v_x\|_{L^p}$, with $\|u\|_{D(A)}, \|v\|_{D(A)} \leq M$. First, we have

$$\|g(u) - g(v)\|_{L^p} \leq L\|u - v\|_{L^p}.$$

Next, we write

$$\begin{aligned} \|g'(u)u_x - g'(v)v_x\|_{L^p} &\leq \|g'(u)(u_x - v_x)\|_{L^p} + \|(g'(u) - g'(v))v_x\|_{L^p} \\ &\leq L\|v - u\|_{D(A)} + \|g'(u) - g'(v)\|_{L^\infty}\|v\|_{D(A)}. \end{aligned}$$

On the other hand,

$$\|g'(u) - g'(v)\|_{L^\infty} \leq K\|u - v\|_{L^\infty} \leq K\|u - v\|_{D(A)},$$

where $K = \sup\{|g''(s)|;\, |s| \leq \|u\|_{L^\infty} + \|v\|_{L^\infty}\}$. Combining all these estimates, we see that $\|g(u) - g(v)\|_{D(A)} \leq L_M\|u - v\|_{D(A)}$.

Example 2. Consider the equation

$$\begin{cases} u_t - \triangle u = g(u), & x \in \Omega, t \geq 0, \\ u(t,x) = 0, & x \in \partial\Omega, t \geq 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{2.7.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and

$$|g(u)| \leq C|u| + C,$$

for all $u \in \mathbb{R}$ and some constant C. Here, Ω is a smooth bounded domain of $\mathbb{R}^N$. Assume $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with

$$\max\left\{1, \frac{N}{2}\right\} < p < \infty.$$

Then, there exists a unique global, classical solution u of (2.7.3), i.e. $u \in C([0,\infty), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap C^1([0,\infty), L^p(\Omega))$.

The argument is the same as in the previous example, except that instead of Remark 1.4.2 (iii) we apply Proposition 1.4.19, and also Theorem 1.4.15. The assumption $p > \dfrac{N}{2}$ implies that $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (see Appendix, Section A.3.4).

Assume in addition that $g \in C^3(\mathbb{R}, \mathbb{R})$ with $g(0) = 0$, and that $u_0 \in W^{4,p}(\Omega)$ with $u_0 = \triangle u_0 = 0$ on $\partial\Omega$. Then $u \in C^1([0,\infty), W^{2,p}(\Omega)) \cap C([0,\infty), W^{4,p}(\Omega))$. The argument is the same as in the previous example, except that we now apply the following lemma.

Lemma 2.7.1. *Assume $g \in C^3(\mathbb{R}, \mathbb{R})$ and let $\max\left\{1, \frac{N}{2}\right\} < p < \infty$. Then the map $F : u \mapsto g(u)$ maps $W^{2,p}(\Omega) \to W^{2,p}(\Omega)$ and is Lipschitz continuous on bounded sets.*

Proof. Assume $\|u\|_{W^{2,p}}, \|u\|_{W^{2,p}} \leq M$, so that in particular $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq K_M$. Clearly,

$$\|g(u) - g(v)\|_{L^p} \leq C\|u - v\|_{L^p}.$$

Next, we estimate $\|Dg(u) - Dg(v)\|_{L^p}$, where $D = \dfrac{\partial}{\partial x_j}$ for some $1 \leq j \leq N$. We write

$$\begin{aligned} \|g'(u)Du - g'(v)Dv\|_{L^p} &\leq \|g'(u)(Du - Dv)\|_{L^p} + \|(g'(u) - g'(v))Dv\|_{L^p} \\ &\leq C\|u - v\|_{L^p} + C\|u - v\|_{L^\infty}\|v\|_{W^{1,p}} \\ &\leq C\|u - v\|_{W^{2,p}}. \end{aligned}$$

Finally, we estimate $\|D^2g(u) - D^2g(v)\|_{L^p}$, where $D^2 = \dfrac{\partial^2}{\partial x_j \partial x_k}$ for some $1 \leq j, k \leq N$. We write

$$\|g'(u)D^2u + g''(u)(Du)^2 - g'(v)D^2v + g''(v)(Dv)^2\|_{L^p} \leq A + B,$$

with

$$\begin{aligned} A &= \|g'(u)D^2u - g'(v)D^2v\|_{L^p}, \\ B &= \|g''(u)(Du)^2 - g''(v)(Dv)^2\|_{L^p}. \end{aligned}$$

First, we have

$$\begin{aligned} A &\leq \|g'(u)(D^2u - D^2v)\|_{L^p} + \|(g'(u) - g'(v))D^2v\|_{L^p} \\ &\leq C\|u - v\|_{L^p} + C\|u - v\|_{L^\infty}\|v\|_{W^{2,p}} \leq C\|u - v\|_{W^{2,p}} \end{aligned}$$

Next, we have

$$B \le \|g''(u)(Du+Dv)(Du-Dv)\|_{L^p} + \|(g''(u)-g''(v))(Dv)^2\|_{L^p}$$
$$\le C(\|Du\|_{L^{2p}} + \|Dv\|_{L^{2p}})\|Du-Dv\|_{L^{2p}} + C\|u-v\|_{L^\infty}\|Dv\|_{L^{2p}}^2.$$

The result now follows from the Gagliardo-Nirenberg inequality

$$\|Du\|_{L^{2p}}^2 \le C\|u\|_{L^\infty}\|u\|_{W^{2,p}}.$$

(See (A.3.10).) □

Example 3. Consider the equation

$$\begin{cases} iu_t + \triangle u = g(u), & x\in\Omega, t\ge 0,\\ u(t,x)=0, & x\in\partial\Omega, t\ge 0,\\ u(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{2.7.4}$$

where $g:\mathbb{C}\to\mathbb{C}$ is globally Lipschitz. Here, Ω is either a smooth bounded domain of $\mathbb{R}^N$, or $\Omega=\mathbb{R}^N$ (and in that case, we also assume $g(0)=0$). For every $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$, there exists a unique global, classical solution u of (2.7.4), i.e. $u\in C^1([0,\infty),L^2(\Omega))\cap C([0,\infty),H^2(\Omega)\cap H^1_0(\Omega))$.

This is a direct consequence of Theorem 2.1.1, Corollary 2.1.2, Proposition 1.2.35 and Remark 1.2.36.

Assume in addition that $N\le 3$, $g\in C^3$ with $g(0)=0$, and $u_0\in H^4(\Omega)$ with $u_0=\triangle u_0=0$ on $\partial\Omega$. Then, $u\in C([0,\infty),H^4(\Omega))\cap C^1([0,\infty),H^2(\Omega)\cap H^1_0(\Omega))$. The argument is the same as in the preceding example. The assumption $N\le 3$ allows us to apply Lemma 2.7.1 with $p=2$.

Example 4. Consider the equation

$$\begin{cases} u_{tt} - \triangle u = g(u,u_t,\nabla u), & x\in\Omega, t\ge 0,\\ u(t,x)=0, & x\in\partial\Omega, t\ge 0,\\ u(0,x)=u_0(x), u_t(0,x)=v_0(x), & x\in\Omega, \end{cases} \tag{2.7.5}$$

where $g:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is globally Lipschitz. Here, Ω is either a smooth bounded domain of $\mathbb{R}^N$, or $\Omega=\mathbb{R}^N$ (and in that case, we also assume $g(0)=0$). Assume $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$ and $v_0\in H^1_0(\Omega)$. Then, there exists a unique global, classical solution u of (2.7.5), i.e. $u\in C^2([0,\infty),L^2(\Omega))\cap C^1([0,\infty),H^1_0(\Omega))\cap C([0,\infty),H^2(\Omega))$.

As in Section 1.4.5, we write (2.7.5) as the system

$$\begin{cases} u_t - v = 0,\\ v_t - \triangle u + u = g(u,v,\nabla u) + u,\\ u(0)=u_0, v(0)=v_0, \end{cases}$$

and we work in the space $X=H^1_0(\Omega)\times L^2(\Omega)$ with

$$F(u,v) = (0, g(u,v,\nabla u)+u).$$

It is clear that $F:X\to X$ is globally Lipschitz.

In order to get further regularity results, we have to make **further** assumptions. Here are some typical situations.

(i) Assume $N = 1$, $g \in C^2(\mathbb{R}^3, \mathbb{R})$ and $g(0, p, q) \equiv 0$. If $u_0 \in H^3(\Omega)$ with $u_0 = u_0'' = 0$ on $\partial\Omega$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $u \in C^2([0, \infty), H_0^1(\Omega)) \cap C^1([0, \infty), H^2(\Omega)) \cap C([0, \infty), H^3(\Omega))$.

(ii) Assume $N \leq 3$, $g(u, p, q) = g(u)$ depends only on u, $g \in C^2(\mathbb{R}, \mathbb{R})$ with $g(0) = 0$. If $u_0 \in H^3(\Omega)$ with $u_0 = \triangle u_0 = 0$ on $\partial\Omega$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $u \in C^2([0, \infty), H_0^1(\Omega)) \cap C^1([0, \infty), H^2(\Omega)) \cap C([0, \infty), H^3(\Omega))$.

(iii) Assume $N \leq 5$, $g(u, p, q) = g(u)$ depends only on u, $g \in C^2(\mathbb{R}, \mathbb{R})$ with $g(0) = 0$ and $g'' \in L^\infty(\mathbb{R})$. If $u_0 \in H^3(\Omega)$ with $u_0 = \triangle u_0 = 0$ on $\partial\Omega$ and $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then $u \in C^2([0, \infty), H_0^1(\Omega)) \cap C^1([0, \infty), H^2(\Omega)) \cap C([0, \infty), H^3(\Omega))$.

The proof consists in showing that under the above assumptions, F maps $D(A) \to D(A)$ and is Lipschitz continuous on bounded sets. We carry the details just for the case (ii). We show that the mapping $u \mapsto g(u)$ maps $H^2(\Omega) \cap H_0^1(\Omega) \to H_0^1(\Omega)$ and is Lipschitz continuous on bounded sets. The fact that g maps $H^2(\Omega) \cap H_0^1(\Omega) \to H_0^1(\Omega)$ (in fact $H_0^1(\Omega) \to H_0^1(\Omega)$) is proved in Corollary A.3.29. Fix M and assume $u_1, u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$ with $\|u_1\|_{H^2}, \|u_2\|_{H^2} \leq M$, so that in particular $\|u_1\|_{L^\infty}, \|u_2\|_{L^\infty} \leq K_M$. We estimate

$$\begin{aligned}\|g(u_1) - g(u_2)\|_{H^1} &\leq \|g(u_1) - g(u_2)\|_{L^2} + \|g'(u_1)Du_1 - g'(u_2)Du_2\|_{L^2} \\ &\leq C\|u_1 - u_2\|_{L^2} + \|g'(u_1)(Du_1 - Du_2)\|_{L^2} + \|(g'(u_1) - g'(u_2))Du_2\|_{L^2} \\ &\leq C\|u_1 - u_2\|_{L^2} + C\|u_1 - u_2\|_{H^1} + C\|u_1 - u_2\|_{L^\infty}\|Du_2\|_{L^2} \\ &\leq C\|u_1 - u_2\|_{H^2}.\end{aligned}$$

Chapter 3. The nonlinear heat equation.

Throughout this chapter, we assume that Ω is a smooth, bounded, connected open subset of $\mathbb{R}^N$, and we consider the equation

$$\begin{cases} u_t - \triangle u = g(u), \quad x \in \Omega, t \in [0,T], \\ u(t,x) = 0, \quad x \in \partial\Omega, t \in [0,T], \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{3.1}$$

where we assume systematically that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

3.1. A general local existence result. The main result of this section is the following.

Theorem 3.1.1. *Given $u_0 \in L^\infty(\Omega)$, there exists a unique weak solution u of (3.1), defined on a maximal time interval $[0, T_{\mathrm{m}})$, i.e. $u \in L^\infty((0,T) \times \Omega)$ for all $T < T_{\mathrm{m}}$ and*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(u(s))\, ds, \tag{3.1.1}$$

for all $t \in [0, T_{\mathrm{m}})$. Moreover, we have the **blow up** *alternative*

either $T_{\mathrm{m}} = +\infty$,

or $T_{\mathrm{m}} < \infty$ **and** $\lim\limits_{t \uparrow T_{\mathrm{m}}} \|u(t)\|_{L^\infty} = +\infty$.

In addition, u depends continuously on u_0. More precisely, the mapping $u_0 \mapsto T_{\mathrm{m}}(u_0)$ is lower semicontinuous, and for every $T < T_{\mathrm{m}}$ there exists $\varepsilon > 0$ and $C < \infty$ such that if $\|v_0 - u_0\|_{L^\infty} \le \varepsilon$, then $\|v - u\|_{L^\infty((0,T)\times\Omega)} \le C\|v_0 - u_0\|_{L^\infty(\Omega)}$, where v is the solution of (3.1.1) with the initial value v_0.

Remark 3.1.2. Note that from (3.1.1), it follows that

$$u \in C((0, T_{\mathrm{m}}) \times \overline{\Omega}), \tag{3.1.2}$$

so that in particular $u(t) \in C(\overline{\Omega})$ for all $t \in (0, T_{\mathrm{m}})$.

First, note that $T(t)u_0 \in C((0,\infty) \times \overline{\Omega})$ (Corollary 1.4.24). Next, let $T < T_{\mathrm{m}}$; we claim that

$$v(t) = \int_0^t T(t-s)h(s)\, ds, \tag{3.1.3}$$

belongs to $C([0,T] \times \overline{\Omega})$ whenever $h \in L^p((0,T) \times \overline{\Omega})$ with $p > 1 + \dfrac{N}{2}$. Indeed, we have

$$\begin{aligned} \|v(t)\|_{L^\infty(\Omega)} &\le \int_0^t \|T(t-s)h(s)\|_{L^\infty(\Omega)}\, ds \\ &\le C \int_0^t (t-s)^{-\frac{N}{2p}} \|h(s)\|_{L^p(\Omega)}\, ds \\ &\le C\|h\|_{L^p((0,T)\times\Omega)} \Bigl(\int_0^t (t-s)^{-\frac{N}{2(p-1)}}\, ds \Bigr)^{\frac{p-1}{p}}, \end{aligned}$$

by Theorem 1.4.15. It follows from the standard linear theory that if h is smooth, then so is v; and so the result follows from the above estimate. Finally, we observe that $h(s) = g(u(s))$ belongs to $L^p((0,T) \times \Omega)$ for every $p < \infty$. Note also that, since $(T(t))_{t\ge 0}$ is a semigroup of contractions in $L^p(\Omega)$ for $1 \le p < \infty$, we have $u \in C([0, T_{\mathrm{m}}), L^p(\Omega))$ for all $p \in [1, \infty)$.

Proof of Theorem 3.1.1. We first prove uniqueness. Suppose that u_1 and u_2 are two solutions of (3.1.1) on $[0,T]$. Then,

$$u_1(t) - u_2(t) = \int_0^t T(t-s)(g(u_1(s)) - g(u_2(s)))\,ds.$$

Taking the L^∞ norm of both sides, and using the fact that

$$\|T(t)\varphi\|_{L^\infty} \le \|\varphi\|_{L^\infty},$$

we find

$$\|u_1(t) - u_2(t)\|_{L^\infty} \le \int_0^t \|g(u_1(s)) - g(u_2(s))\|_{L^\infty}\,ds \le K \int_0^t \|u_1(s) - u_2(s)\|_{L^\infty}\,ds,$$

for all $t \in [0,T]$, with K the Lipschitz constant of g on $[-A,A]$,

$$A = \max\{\|u_1\|_{L^\infty((0,T)\times\Omega)}, \|u_2\|_{L^\infty((0,T)\times\Omega)}\}.$$

Uniqueness now follows from Gronwall's inequality.

Let $M = \|u_0\|_{L^\infty} + 1$ and let $\widetilde{g}$ be defined by

$$\widetilde{g}(u) = \begin{cases} g(M) \text{ if } u > M, \\ g(u) \text{ if } |u| \le M, \\ g(-M) \text{ if } u < -M, \end{cases}$$

so that $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz. Applying Theorem 2.1.1, for example with $X = L^2(\Omega)$, we obtain a weak, global solution $\widetilde{u} \in C([0,\infty), L^2(\Omega))$ satisfying

$$\widetilde{u}(t) = T(t)u_0 + \int_0^t T(t-s)\widetilde{g}(\widetilde{u}(s))\,ds. \tag{3.1.4}$$

Taking the L^∞ norm of both sides and applying Theorem 1.4.15, we see that

$$\|\widetilde{u}(t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + \int_0^t \|\widetilde{g}(\widetilde{u}(s))\|_{L^\infty}\,ds \le \|u_0\|_{L^\infty} + K_M t,$$

where $K_M = \|g\|_{L^\infty(-M,M)}$. Choose T small enough so that

$$K_M T \le 1.$$

Then $\|\widetilde{u}(t)\|_{L^\infty} \le M$ for all $t \in [0,T]$; and then $\widetilde{u}$ satisfies (3.1.1) on $[0,T]$. Uniqueness, as in Section 2.4, implies the existence of a solution defined on a maximal time interval $[0, T_{\mathrm{m}})$.

To establish the blow up alternative, we argue as in the proof of Theorem 2.4.3. Suppose $T_{\mathrm{m}} < \infty$, and assume that there is a sequence $t_j \uparrow T_{\mathrm{m}}$ such that $\|u(t_j)\|_{L^\infty} \le A < \infty$. Fix $\delta > 0$ such that

$$\delta K_{A+1} \le 1.$$

Starting with the initial value $u(t_j)$, we have a weak solution v_j of (3.1) defined on $[0,\delta]$. Gluing together u with v_j, we obtain a weak solution of (3.1) defined on $[0, t_j + \delta]$ (see Corollary 1.5.18). For j large enough, $t_j + \delta > T_{\mathrm{m}}$, and this is impossible since u is the maximal solution.

Finally, we prove the continuous dependence. Given $T < T_{\rm m}$, set $M_T = \|u\|_{L^\infty((0,T)\times\Omega)} + 1$. Let $\widetilde{g}$ be as above, but with $M = M_T$ and let L_T be the Lipschitz constant of $\widetilde{g}$. Let $\widetilde{u}$ be the solution of (3.1.4) and, given $v_0 \in L^\infty(\Omega)$, let $\widetilde{v}$ be the corresponding solution of (3.1.4). It follows that

$$\|\widetilde{u}(t) - \widetilde{v}(t)\|_{L^\infty} \le \|u_0 - v_0\|_{L^\infty} + L_T \int_0^t \|\widetilde{u}(s) - \widetilde{v}(s)\|_{L^\infty}\, ds;$$

and so, by Gronwall's inequality,

$$\|\widetilde{u} - \widetilde{v}\|_{L^\infty((0,T)\times\Omega)} \le e^{TL_T}\|u_0 - v_0\|_{L^\infty(\Omega)}.$$

In particular, if $\|u_0 - v_0\|_{L^\infty(\Omega)} \le \varepsilon$ with $\varepsilon = e^{-TL_T}$, we have $\|\widetilde{u} - \widetilde{v}\|_{L^\infty((0,T)\times\Omega)} \le M_T$, so that $\widetilde{v}$ is the solution of (3.1.1) on $[0,T]$ with the initial value v_0. The continuous dependence follows easily. □

Remark 3.1.3. Assume $u_0 \in L^\infty(\Omega)$, and in addition $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for some $p \in (1,\infty)$. Then, the above solution belongs to $C([0,T_{\rm m}), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap C^1([0,T_{\rm m}), L^p(\Omega))$. This is a direct consequence of Corollary 2.1.2. Note that g is not globally Lipschitz, but one can truncate g outside the range of $u(t,x)$, $x \in \Omega$, $t \in [0,T]$, for $T < T_{\rm m}$.

Remark 3.1.4. Assume that $g = g(x,u)$ also depends on x, and consider the problem

$$\begin{cases} u_t - \triangle u = g(x,u), & x \in \Omega, t \in [0,T] \\ u(t,x) = 0, & x \in \partial\Omega, t \in [0,T] \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{3.1.5}$$

If g is measurable in x and locally Lipschitz in u (i.e. $|g(x,u) - g(x,v)| \le K_A|u-v|$ for almost all $x \in \Omega$ and all $u, v \in [-A,A]$) and if $g(\cdot,0) \in L^\infty(\Omega)$, then the conclusions of Theorem 3.1.1 hold without any modification.

3.2. Smoothing effect. The weak solution u obtained above for (3.1) (or more generally for (3.1.5)) is in fact smooth for $t \in (0,T_{\rm m})$. Here is one such result.

Theorem 3.2.1. *Assume $u_0 \in L^\infty(\Omega)$. Then*

(i) $u \in C((0,T_{\rm m}), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap C^1((0,T_{\rm m}), L^p(\Omega))$ *for every* $p < \infty$.

(ii) *For every $T < T_{\rm m}$, there exists C depending on T and $\|u\|_{L^\infty((0,T)\times\Omega)}$ such that $\|u(t)\|_{H^2} \le \dfrac{C}{t}$ and $\|u(t)\|_{H^1} \le \dfrac{C}{\sqrt{t}}$ for all $t \in (0,T)$.*

Proof. First, we write

$$u(t) = v(t) + w(t).$$

where $v(t) = T(t)u_0$ and

$$w(t) = \int_0^t T(t-s)g(u(s))\, ds.$$

It follows from the analyticity of $(T(t))_{t\ge0}$ in $L^p(\Omega)$ (see Proposition 1.4.20) that $v(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $t > 0$. On the other hand, $w(t) \in L^p((0,T), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ for $T < T_{\rm m}$ and every $p < \infty$, by Theorem 1.6.1. In particular, $u(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for almost all $t \in (0, T_{\mathrm{m}})$. Conclusion (i) now follows from Remark 3.1.3.

The estimates (ii) are consequences of the arguments used in the proof of Theorem 2.6.1. □

Remark 3.2.2. Iterating this argument, one can show that if $g \in C^\infty(\mathbb{R}, \mathbb{R})$, then $u \in C^\infty((0, T_{\mathrm{m}}) \times \overline{\Omega})$. Of course, we have to assume that $g \in C^\infty(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ if g also depends on x.

3.3. The condition $ug(x, u) \le Cu^2 + C$ implies global existence for every u_0. In this section, we assume that

$$ug(x, u) \le Cu^2 + C', \tag{3.3.1}$$

for almost all $x \in \Omega$ and for all $u \in \mathbb{R}$. The main result is the following.

Theorem 3.3.1. *For every $u_0 \in L^\infty(\Omega)$, the solution u of (3.1.5) is globally defined.*

Proof. It relies heavily on the maximum principle, or its variants. We give two different proofs.

Proof 1. Multiplication by powers. For $t \in (0, T_{\mathrm{m}})$, we multiply the equation by $|u|^{p-2}u$, with $p \ge 2$, and we integrate on Ω. We find

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p + (p-1)\int_\Omega |u|^{p-2}|\nabla u|^2 = \int_\Omega |u|^{p-2}ug(x, u);$$

and so,

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \le C\int_\Omega |u|^p + C\int_\Omega |u|^{p-2}.$$

Applying the inequality $ab \le \dfrac{a^q}{q} + \dfrac{b^{q'}}{q'}$ with $q = \dfrac{p}{p-2}$, $a = |u|^{p-2}$ and $b = 1$, we find

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \le 2C\int_\Omega |u|^p + C|\Omega|.$$

It follows that

$$\int_\Omega |u(t)|^p \le \Big(\int_\Omega |u(s)|^p + Cp|\Omega|\Big)e^{2Cp(t-s)},$$

for all $0 < s < t < T_{\mathrm{m}}$. Passing to the limit as $s \downarrow 0$ (recall that $u \in C([0, T_{\mathrm{m}}), L^p(\Omega))$ for all $p < \infty$), we find

$$\int_\Omega |u(t)|^p \le \Big(\int_\Omega |u_0|^p + Cp|\Omega|\Big)e^{2Cpt},$$

for all $t \in (0, T_{\mathrm{m}})$. Raising this inequality to the power $\dfrac{1}{p}$, we finally obtain

$$\|u(t)\|_{L^p} \le \Big(\int_\Omega |u_0|^p + Cp|\Omega|\Big)^{\frac{1}{p}} e^{2Ct} \le \Big(\|u_0\|_{L^p} + (Cp|\Omega|)^{\frac{1}{p}}\Big)e^{2Ct}.$$

Letting $p \to \infty$, it follows that

$$\|u(t)\|_{L^\infty} \le (\|u_0\|_{L^\infty} + 1)e^{2Ct},$$

for all $t \in (0, T_{\mathrm{m}})$. In view of the blow up alternative in Theorem 3.1.1, this implies $T_{\mathrm{m}} = +\infty$. □

Proof 2. Super and sub-solutions. The proof relies on the following comparison principle for nonlinear heat equations.

Theorem 3.3.2. *Assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in u and assume u and $\overline{u}$ are smooth functions in $(0,T) \times \overline{\Omega}$ such that $u, \overline{u} \in L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega))$ and*

$$\begin{cases} u_t - \triangle u \le g(x,u) \text{ in } (0,T) \times \Omega, \\ \overline{u}_t - \triangle \overline{u} \ge g(x,\overline{u}) \text{ in } (0,T) \times \Omega, \\ u \le \overline{u} \text{ in } (0,T) \times \partial\Omega, \\ u(0,x) \le \overline{u}(0,x) \text{ in } \Omega. \end{cases}$$

Then $u \le \overline{u}$ in $(0,T) \times \Omega$.

Proof. Let $h \in C^1(\mathbb{R}, \mathbb{R})$ be such that $h' \ge 0$,

$$h(s) = \begin{cases} 0 \text{ for } s \le 0, \\ > 0 \text{ for } s > 0, \end{cases}$$

and $sh(s) \le CH(s)$ for every $s \in \mathbb{R}$, where $H(s) = \displaystyle\int_0^s h(\sigma)\, d\sigma$. For example, $h(s) = (s^+)^2$. We have

$$\begin{aligned} \int_\Omega h(u-\overline{u})(u-\overline{u})_t + \int_\Omega h'(u-\overline{u})|\nabla(u-\overline{u})|^2 &\le \int_\Omega (g(x,u) - g(x,\overline{u}))h(u-\overline{u}) \\ &\le L\int_\Omega |u-\overline{u}|h(u-\overline{u}) \le LC\int_\Omega H(u-\overline{u}); \end{aligned}$$

and so,

$$\frac{d}{dt}\int_\Omega H(u-\overline{u}) \le LC\int_\Omega H(u-\overline{u}),$$

where L is the Lipschitz constant of g on $[-M,M]$ with $M = \max\{\|u\|_{L^\infty((0,T)\times\Omega)}, \|\overline{u}\|_{L^\infty((0,T)\times\Omega)}\}$. Since $H(u(0,x), \overline{u}(0,x)) = 0$, we conclude by Gronwall's inequality that $H(u-\overline{u}) \le 0$ in $(0,T) \times \Omega$. Hence $u \le \overline{u}$ in $(0,T) \times \Omega$. □

Remark. The conclusion of Theorem 3.3.2 holds under weaker assumptions on Ω, u, $\overline{u}$. See Theorem 1.4.12.

Proof 2 continued. Observe that inequality (3.3.1) implies the existence of a constant D such that

$$g(x,u) \le D(u+1), \tag{3.3.2}$$

for all $u \ge 0$. Let v be the solution of the linear problem

$$\begin{cases} v_t - \triangle v = D(v+1) \text{ in } (0,\infty) \times \Omega, \\ v = 0 \text{ in } (0,\infty) \times \partial\Omega, \\ v(0,x) = |u_0(x)| \text{ in } \Omega. \end{cases}$$

By the maximum principle (apply for example Theorem 3.3.2), it follows that $v \ge 0$ in $(0,\infty) \times \Omega$; and so, it follows from (3.3.2) that

$$v_t - \triangle v \ge g(x,v).$$

Theorem 3.3.2 now implies that $u \le v$ in $(0, T_{\rm m}) \times \Omega$. Similarly, one obtains a lower bound for u in $(0, T_{\rm m}) \times \Omega$. Again by the blow up alternative, we deduce that $T_{\rm m} = +\infty$. □

Remark 3.3.3. It follows from the proof of Theorem 3.3.1 that

$$\|u(t)\|_{L^\infty} \le K e^{Ct}(1 + \|u_0\|_{L^\infty}),$$

for all $t \ge 0$, where C is the constant in (3.3.1). In fact, this estimate can be improved. Indeed, if $C < \lambda_1$, λ_1 the first eigenvalue of $-\triangle$ in $H^1_0(\Omega)$, then

$$\|u(t)\|_{L^\infty} \le K(1 + \|u_0\|_{L^\infty}).$$

(See Exercise 3.13.2.) This estimate is optimal in the sense that in general $\|u(t)\|_{L^\infty}$ may be bounded away from 0. To see this, consider $g(u) \equiv 1$. If $\varphi_1 > 0$ is the first eigenfunction of $-\triangle$ in $H^1_0(\Omega)$ with $\displaystyle\int_\Omega \varphi_1 = 1$, it follows that

$$z(t) = \int_\Omega u(t,x)\varphi_1(x)\,dx$$

verifies the differential equation $z' + \lambda_1 z = 1$, so that

$$z(t) = e^{-\lambda_1 t} z(0) + \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t});$$

and so, if $z(0) > 0$, then $z(t) \ge \alpha > 0$ for all $t > 0$. This implies that $\inf\limits_{t \ge 0} \|u(t)\|_{L^\infty} > 0$.

If now $C > \lambda_1$, then

$$\|u(t)\|_{L^\infty} \le K(1 + \|u_0\|_{L^\infty}) e^{(C - \lambda_1)t}.$$

(See Exercise 3.13.3.) This estimate is also optimal with respect to the behavior as $t \to \infty$. Indeed, take $g(u) = Cu$ and $u_0 = \varphi_1$, so that $u(t) = e^{(C-\lambda_1)t}\varphi_1$.

Finally, if $C = \lambda_1$, then

$$\|u(t)\|_{L^\infty} \le K(t + \|u_0\|_{L^\infty}).$$

(See Exercise 3.13.4.) This estimate is also optimal in the sense that in general $\|u(t)\|_{L^\infty}$ is not bounded (see Exercise 3.13.7).

Note that if $g(u) = 1$ or if $g(u) = \lambda_1 u$, then all solutions are bounded, but that if $g(u) = \max\{1, \lambda_1 u\}$, then some solutions are unbounded (see Exercises 3.13.2, 3.13.5 and 3.13.7).

3.4. Global existence for small initial values. There are two different methods for showing that if u_0 is "small", then the solution of (3.1.5) is globally defined:

- **the energy method**,
- **the comparison method**.

They yield different results in that the smallness condition involves different norms. Throughout this section, we assume that $g(x,u)$ is C^1 in u and that

$$g(x,0) = 0, \tag{3.4.1}$$

$$\lambda_1(-\triangle - g_u(x,0)) > 0. \tag{3.4.2}$$

Here, $\lambda_1(-\triangle - g_u(x,0))$ denotes the first eigenvalue of the operator $v \mapsto -\triangle v - g_u(x,0)v$ in $H_0^1(\Omega)$. If g is independent of x, then condition (3.4.2) is equivalent to

$$g'(0) < \lambda_1(-\triangle).$$

Assumptions (3.4.1) and (3.4.2) mean that 0 is a "**stable**" solution of the stationary problem $-\triangle u - g(x,u) = 0$.

We begin with the energy method.

Theorem 3.4.1. *Assume (3.4.1), (3.4.2) and*

$$|g(x,u)| \le C(1+|u|^q), \text{ for all } u \in \mathbb{R}, \tag{3.4.3}$$

with $1 < q < \dfrac{N+2}{N-2}$ *(no condition on* $q > 1$ *if* $N = 1,2$*). There exists* $\delta > 0$ *such that if* $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ *and* $\|u_0\|_{H^1} \le \delta$*, then the solution* u *of (3.1.5) is globally defined and belongs to* $L^\infty((0,\infty), L^\infty(\Omega) \cap H_0^1(\Omega))$*. With* $\delta > 0$ *possibly smaller, we even have*

$$\|u(t)\|_{L^\infty} \le Ce^{-\mu t} \text{ for } t \ge 0,$$

where μ *is any number strictly less than* $\lambda_1(-\triangle - g_u(x,0))$*, and* C *depends on* μ*.*

Let

$$G(x,u) = \int_0^u g(x,s)\,ds.$$

The energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u), \tag{3.4.4}$$

is naturally associated with (3.1.5) and plays an important role. We will need the following lemma.

Lemma 3.4.2. *Assume* $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$*, and let* u *be the solution of (3.1.5) defined on the maximal interval* $[0, T_{\rm m})$*. Then* $u \in C([0,T_{\rm m}), H_0^1(\Omega))$ *and*

$$E(u(t)) \le E(u_0), \tag{3.4.5}$$

for all $t \in [0, T_{\rm m})$*.*

Proof. We first prove that $u \in C([0,T_{\rm m}), H_0^1(\Omega))$, and by Theorem 3.2.1, it suffices to show that u is continuous at $t = 0$. For this purpose, we use the integral formula (3.1.1). Recall that $T(t)u_0 \in C([0,\infty), H_0^1(\Omega))$ (see Theorem 1.4.8 (iii)), and therefore we need only show that $\|u(t) - T(t)u_0\|_{H^1} \to 0$ as $t \downarrow 0$, i.e.

$$\Big\| \int_0^t T(t-s)g(x,u(s))\,ds \Big\|_{H^1} \underset{t\downarrow 0}{\longrightarrow} 0.$$

We have by Theorem 1.4.11 (ii)

$$\begin{aligned} \Big\| \int_0^t T(t-s)g(x,u(s))\,ds \Big\|_{H^1} &\le \int_0^t \Big(1 + \frac{1}{\sqrt{t-s}}\Big) \|g(x,u(s))\|_{L^2}\,ds \\ &\le C(t + \sqrt{t}) \underset{t\downarrow 0}{\longrightarrow} 0. \end{aligned}$$

Next, multiply equation (3.1.5) by u_t. We find

$$\int_\Omega u_t^2 + \frac{d}{dt}E(u(t)) = 0, \tag{3.4.6}$$

for all $t \in (0, T_{\rm m})$. In particular, $E(u(t)) \le E(u(s))$ for all $0 < s \le t < T_{\rm m}$, and the result follows by letting $s \downarrow 0$. Note that $E(u(s)) \to E(u_0)$ since u is continuous into $H_0^1(\Omega)$ and by dominated convergence. □

Proof of Theorem 3.4.1. The proof proceeds in six steps.

Step 1. We show that $\sup\{\|u(t)\|_{H^1};\, 0 \le t < T_{\rm m}\} < \infty$. We observe that by (3.4.1) and (3.4.3),

$$G(x, v) \le \frac{1}{2} g_u(x, 0)v^2 + \varepsilon v^2 + C|v|^{q+1}. \tag{3.4.7}$$

Here $\varepsilon > 0$ is arbitrarily small and C depends on ε. On the other hand, assumption (3.4.2) implies the existence of $\eta > 0$ such that

$$\int_\Omega \{|\nabla v|^2 - g_u(x, 0)v^2\} \ge \eta \int_\Omega \{|\nabla v|^2 + v^2\}. \tag{3.4.8}$$

It follows from the energy inequality (3.4.5), (3.4.7) with $\varepsilon = \eta$ and (3.4.8) that

$$\frac{\eta}{2} \int_\Omega |\nabla u|^2 \le E(u_0) + C \int_\Omega |u|^{q+1} \le E(u_0) + C\|u\|_{H^1}^{q+1},$$

by Sobolev's inequalities, since $q + 1 \le \dfrac{2N}{N-2}$. Therefore,

$$\frac{1}{2}\|u(t)\|_{H^1}^2 \le AE(u_0) + \frac{B}{q+1}\|u(t)\|_{H^1}^{q+1}. \tag{3.4.9}$$

It is now convenient to draw the graph of the function $f : x \mapsto \dfrac{x^2}{2} - \dfrac{B}{q+1}x^{q+1}$.

The graph of f

The maximum of f is achieved for $x = B^{-\frac{1}{q-1}}$ and is $\dfrac{q-1}{2(q+1)}B^{-\frac{2}{q-1}}$.

We now assume that

$$AE(u_0) < \frac{q-1}{2(q+1)}B^{-\frac{2}{q-1}}, \tag{3.4.10}$$

$$\|u_0\|_{H^1} \le B^{-\frac{1}{q-1}}. \tag{3.4.11}$$

Let $0 < \alpha < \beta$ be the two solutions of $f(x) = AE(u_0)$. In view of (3.4.9), we have for every $t \in [0, T_{\rm m})$ either $\|u(t)\|_{H^1} \le \alpha$ or $\|u(t)\|_{H^1} \ge \beta$. Since the function $t \mapsto \|u(t)\|_{H^1}$ is continuous and $\|u_0\|_{H^1} \le \alpha$, we conclude that $\|u(t)\|_{H^1}$ is **trapped** in $[0,\alpha]$. Finally, note that by choosing $\|u_0\|_{H^1}$ sufficiently small, we can achieve (3.4.10) and (3.4.11).

Step 2. We show that $T_\mathrm{m}=+\infty$. Let $p=\dfrac{2N}{(N-2)(q-1)}>\dfrac{N}{2}$ if $N\ge 3$ (the case $N=2$ is similar).

We have from (3.1.1) and Theorem 1.4.15,

$$\begin{aligned}\|u(t)\|_{L^\infty} &\le \|u_0\|_{L^\infty}+\int_0^t \|T(t-s)g(x,u(s))\|_{L^\infty}\,ds\\ &\le \|u_0\|_{L^\infty}+C\int_0^t (t-s)^{-\frac{N}{2p}}\|g(u(s))\|_{L^p}\,ds\\ &\le \|u_0\|_{L^\infty}+Ct^{1-\frac{N}{2p}}+C\int_0^t (t-s)^{-\frac{N}{2p}}\|u(s)\|_{L^{pq}}^q\,ds\\ &\le \|u_0\|_{L^\infty}+Ct^{1-\frac{N}{2p}}+C\int_0^t (t-s)^{-\frac{N}{2p}}\|u(s)\|_{L^{\frac{2N}{N-2}}}^{q-1}\|u(s)\|_{L^\infty}\,ds,\end{aligned}$$

for all $t\in(0,T_\mathrm{m})$. By Step 1, we have

$$\|u(t)\|_{L^\infty}\le \|u_0\|_{L^\infty}+Ct^{1-\frac{N}{2p}}+C\int_0^t (t-s)^{-\frac{N}{2p}}\|u(s)\|_{L^\infty}\,ds.$$

If $T_\mathrm{m}<\infty$, then it follows from the generalized Gronwall inequality (see proposition A.5.7) that

$$\sup\{\|u(t)\|_{L^\infty};\,0<t<T_\mathrm{m}\}<\infty.$$

Impossible. If $N=1$, then global existence follows immediately from Step 1.

Step 3. We show that $u\in L^\infty((0,\infty),L^\infty(\Omega))$. We know that $\sup\limits_{t\ge 0}\|u(t)\|_{H^1}<\infty$. If $N=1$, the L^∞ bound is immediate. Suppose $N\ge 3$ (the case $N=2$ is similar). Observe that $g(x,0)=0$, so that by assumption (3.4.3) we have

$$\Big|\frac{g(u)}{u}\Big|\le C(1+|u|^{q-1}).$$

It follows that $\dfrac{g(u)}{u}$ is bounded in $L^r(\Omega)$ with $r=\dfrac{2N}{(N-2)(q-1)}>\dfrac{N}{2}$, and then the L^∞ estimate follows from Theorem 1.6.6.

Step 4. $\bigcup\limits_{t\ge 1}\{u(t)\}$ is relatively compact in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. Since $u\in C([1,\infty),H_0^1(\Omega)\cap L^\infty(\Omega))$, we need only show that if $t_n\to\infty$, then there exists a subsequence $(t_{n_k})_{k\ge 0}$ such that $u(t_{n_k})$ is convergent in both $H_0^1(\Omega)$ and $L^\infty(\Omega)$. Set $\tau_n=t_n-1$, and note that $u(\tau_n)$ is bounded in $H_0^1(\Omega)$. It follows that there exists a subsequence, which we still denote by $(\tau_n)_{n\ge 0}$, such that $u(\tau_n)$ is convergent in $L^2(\Omega)$ (apply Rellich's theorem). Next, we write

$$u(\tau_n+t)-u(\tau_k+t)=T(t)(u(\tau_n)-u(\tau_k))+\int_0^t T(t-s)(g(x,u(\tau_n+s))-g(x,u(\tau_k+s)))\,ds.$$

It follows from Theorem 1.4.11 (ii) and Step 3 that

$$\|u(\tau_n+t)-u(\tau_k+t)\|_{H^1}\le \frac{C}{\sqrt{t}}\|u(\tau_n)-u(\tau_k)\|_{L^2}+\int_0^t \frac{C}{\sqrt{t-s}}\|u(\tau_n+s)-u(\tau_k+s)\|_{L^2}\,ds,$$

for $t\le 1$. The generalized Gronwall inequality (see proposition A.5.7) yields

$$\|u(\tau_n+t)-u(\tau_k+t)\|_{H^1}\le \frac{C'}{\sqrt{t}}\|u(\tau_n)-u(\tau_k)\|_{L^2},$$

for $t \leq 1$. Taking $t = 1$, it follows that $u(t_n)$ is a convergent sequence in $H_0^1(\Omega)$. Finally, since $u(t)$ is bounded in $L^\infty(\Omega)$ and $u(\tau_n)$ is convergent in $L^2(\Omega)$, it follows from Hölder's inequality that $u(\tau_n)$ is convergent in $L^N(\Omega)$. Arguing as above but using the $L^N \to L^\infty$ smoothing effect (Theorem 1.4.15) instead of the $L^2 \to H^1$ smoothing effect, we deduce that $u(t_n)$ is convergent in $L^\infty(\Omega)$.

Step 5. $u(t) \underset{t\to\infty}{\longrightarrow} 0$ in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. Since $u(t)$ is bounded in both $H_0^1(\Omega)$ and $L^\infty(\Omega)$, we have $E(u(t)) \geq -M > -\infty$, for all $t \geq 0$. Therefore, it follows from (3.4.6) that

$$E(u(t)) \to \ell, \tag{3.4.13}$$

as $t \to \infty$, for some $\ell \in \mathbb{R}$. Let now $t_n \to \infty$ be such that $u(t_n) \to v_0$ in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. An obvious argument (see e.g. the estimates of Step 4) shows that $u(t_n + \cdot) \to v(\cdot)$ in $C([0,1], H_0^1(\Omega) \cap L^\infty(\Omega))$, where v is the solution of (3.1.5) corresponding to the initial value v_0. We deduce from (3.4.13) that $E(v(t)) = \ell$ for all $t \in [0,1]$, which, by (3.4.6), implies that $v_t \equiv 0$. Therefore, v is independent of t and is a solution of

$$-\triangle v = g(x, v). \tag{3.4.12}$$

Note that

$$g(x,v)v \leq g_u(x,0)v^2 + \varepsilon v^2 + C|v|^{q+1} \text{ for all } v \in \mathbb{R}, \tag{3.4.14}$$

$\varepsilon > 0$ arbitrarily small and C depending on ε. Multiplying the equation (3.4.12) by v and applying (3.4.14), we find

$$\int_\Omega |\nabla v|^2\,dx \leq \int_\Omega g_u(x,0)v^2 + \varepsilon \int_\Omega |v|^2\,dx + C\int_\Omega |v|^{q+1}\,dx. \tag{3.4.15}$$

From (3.4.8) and (3.4.15) with $\varepsilon = \eta$, we obtain

$$\eta \int_\Omega |\nabla v|^2\,dx \leq C\int_\Omega |v|^{q+1}\,dx.$$

Therefore, by Sobolev's inequality,

$$\|v\|_{H^1}^2 \leq C'\|v\|_{H^1}^{q+1}.$$

It follows that either $\|v\|_{H^1} \geq (C')^{-\frac{1}{q-1}}$ or $v = 0$. Finally, observe that $\|v\|_{H^1} \leq \limsup_{t\to\infty} \|u(t)\|_{H^1} \leq \alpha$, where α is as in Step 1. Since clearly $\alpha \downarrow 0$ as $\delta \downarrow 0$, if we choose δ small enough, then $v = 0$. By Step 4, this implies that $u(t) \underset{t\to\infty}{\longrightarrow} 0$ in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$.

Step 6. Exponential decay. Since $u(t) \underset{t\to\infty}{\longrightarrow} 0$ in $L^\infty(\Omega)$, the exponential decay follows from Theorem 3.4.5 below. □

Remark 3.4.3. Note that if g verifies (3.4.1)—(3.4.3) with $q > \dfrac{N+2}{N-2}$, then in general u can blow up in finite time for $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ arbitrarily small in $H_0^1(\Omega)$ (see Exercise 3.13.6). However, without any assumption on g for u large, for every M there exists $\delta_M > 0$ such that if $\|u_0\|_{L^\infty} \leq M$ and $\|u_0\|_{H^1} \leq \delta_M$, then u is globally defined (see Exercise 3.13.10).

Remark 3.4.4. Some of the results presented in the steps of Theorem 3.4.1 hold under more general assumptions. More precisely, let $g(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz in u, and let $u_0 \in L^\infty(\Omega)$ be such that the solution u of (3.1.5) is global. Assume furthermore that $\sup_{t\geq 0}\|u(t)\|_{L^\infty} < \infty$. Then, the following properties hold.

(i) $\sup_{t\geq 1}\|u(t)\|_{H^1} < \infty$. This follows easily from the boundedness of u and $g(u)$ in $L^\infty(\Omega)$, hence in $L^2(\Omega)$, and the $L^2 \to H^1$ smoothing effect.

(ii) $\cup_{t\geq 1}\{u(t)\}$ is relatively compact in $H^1_0(\Omega)$ and in $L^\infty(\Omega)$. The proof is the same as in Step 4 of Theorem 3.4.1.

(iii) If $t_n \to \infty$, then there exists a subsequence $(n_k)_{k\geq 0}$ and a solution $v \in H^1_0(\Omega)\cap L^\infty(\Omega)$ of the elliptic equation

$$-\triangle v = g(x, v) \text{ in } \Omega,$$

such that $u(t_{n_k}) \to v$ in $H^1_0(\Omega)$ and in $L^\infty(\Omega)$. The set of all such limit points v is denoted by $\omega(u_0)$ and is called the ω-limit set of u_0; see Section 3.12. One shows that $\omega(u_0)$ is a connected, compact subset of $H^1_0(\Omega)$ and of $L^\infty(\Omega)$ (see Dafermos [35]).

We now present the comparison method.

Theorem 3.4.5. *Assume that (3.4.1) and (3.4.2) hold. There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega)$ and $\|u_0\|_{L^\infty} \leq \delta$, then the solution u of (3.1.5) is globally defined.*

With $\delta > 0$ possibly smaller, we even have

$$\|u(t)\|_{L^\infty} \leq Ce^{-\mu t} \text{ for } t \geq 0,$$

where μ is any number strictly less than $\lambda_1(-\triangle - g_u(x, 0))$, and C depends on μ.

Proof. Let $v \in H^1_0(\Omega)$ be the solution of

$$-\triangle v - g_u(x, 0)v = |g_u(x, 0)| + 1.$$

It follows from Proposition A.4.21 and Theorem A.4.13 that $v \geq 0$ and $v \in L^\infty(\Omega)$. Set

$$w = \eta(v + 1),$$

with $\eta > 0$. We claim that for η sufficiently small, w is a super-solution of (3.1.5). Indeed, fix $\varepsilon > 0$ such that

$$\varepsilon(\|v\|_{L^\infty} + 1) \leq 1.$$

Then, there exists $\alpha > 0$ such that

$$|g(x, t) - g_u(x, 0)t| \leq \varepsilon|t|, \text{ for } |t| \leq \alpha.$$

Choose η so small that $\eta(\|v\|_{L^\infty} + 1) \leq \alpha$. Then we have

$$\begin{aligned} g(x, w) &\leq g_u(x, 0)w + \varepsilon w \\ &= -\triangle w - \eta + \varepsilon\eta(v + 1) - \eta(|g_u(x, 0)| - g_u(x, 0)) \\ &\leq -\triangle w. \end{aligned}$$

Similarly, one shows that $-w$ is a sub-solution of (3.1.5). Therefore, if $\|u_0\|_{L^\infty} \le \eta$, the global existence follows from Theorem 3.3.2, and

$$\|u(t)\|_{L^\infty} \le \eta(\|v\|_{L^\infty} + 1),$$

for all $t \ge 0$.

We now prove the exponential decay. Fix any $\varepsilon > 0$ such that $\varepsilon < \lambda_1 = \lambda_1(-\triangle - g_u(x,0))$, and then there exists $\gamma > 0$ such that

$$|g(x,t) - g_u(x,0)t| \le \varepsilon|t|, \text{ for } |t| \le \gamma.$$

From the above discussion, we know that we may choose $\delta > 0$ sufficiently small so that if $\|u_0\|_{L^\infty} \le \delta$, then $\|u(t)\|_{L^\infty} \le \gamma$ for all $t \ge 0$. Therefore, we have

$$u_t - \triangle u \le g_u(x,0)u + \varepsilon|u|,$$

for all $t \ge 0$. Let $v(t)$ be the solution of the problem

$$\begin{cases} v_t - \triangle v - g_u(x,0)v - \varepsilon v = 0 \text{ in } (0,\infty) \times \Omega, \\ v = 0 \text{ in } (0,\infty) \times \partial\Omega, \\ v(0) = \|u_0\|_{L^\infty} \text{ in } \Omega. \end{cases}$$

Then

$$|u(t,x)| \le v(t,x) \le Ce^{-(\lambda_1 - \varepsilon)t},$$

by Exercise 3.13.5. □

Remark 3.4.6. In particular, if $g \in C^1(\mathbb{R},\mathbb{R})$ with $g'(0) < \lambda_1$ and $g(0) = 0$, then all solutions of (3.1) with sufficiently small initial values in $L^\infty(\Omega)$ are globally defined. This is not anymore true if $g'(0) \ge \lambda_1$. For example, if $g(u) = \lambda_1 u + u^3$, then for any $u_0 \ge 0$, $u_0 \not\equiv 0$, the solution of (3.1) blows up in finite time (see Remark 3.6.8).

3.5. Global existence near a stable equilibrium point. In this section, we assume that $g(x,u)$ is C^1 in u, and that the stationary problem $-\triangle u = g(x,u)$ has a "stable" solution w. In other words, we assume that $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ verifies

$$-\triangle w = g(x,w),$$

and that

$$\lambda_1(-\triangle - g_u(x,w)) > 0, \tag{3.5.1}$$

Given $u_0 \in L^\infty(\Omega)$ and u the solution of (3.1.5), then $\widetilde{u} = u - w$ solves the equation

$$\begin{cases} \widetilde{u}_t - \triangle \widetilde{u} = \widetilde{g}(x,\widetilde{u}), \quad x \in \Omega, t \in [0,T] \\ \widetilde{u}(t,x) = 0, \quad x \in \partial\Omega, t \in [0,T] \\ \widetilde{u}(0,x) = u_0(x) - w(x), \quad x \in \Omega, \end{cases}$$

with

$$\widetilde{g}(x,u) = g(x, w(x) + u) - g(x, w(x)).$$

In particular, 0 is a stationary solution and, since $\widetilde{g}_u(x,0) = g_u(x,w)$, it follows from (3.5.1) that

$$\lambda_1(-\triangle - \widetilde{g}_u(x,0)) > 0.$$

Therefore, we can apply the results of the preceding section and we obtain the following theorems.

Theorem 3.5.1. *Assume that g verifies (3.4.3) with $1 < q < \dfrac{N+2}{N-2}$ (no condition on $q > 1$ if $N = 1, 2$). There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ and $\|u_0 - w\|_{H^1} \le \delta$, then the solution u of (3.1.5) is globally defined and belongs to $L^\infty((0,\infty), L^\infty(\Omega) \cap H_0^1(\Omega))$.*

With $\delta > 0$ possibly smaller, we even have

$$\|u(t) - w\|_{L^\infty} \le Ce^{-\mu t} \text{ for } t \ge 0,$$

where μ is any number strictly less than $\lambda_1(-\triangle - g_u(x,w))$, and C depends on μ.

Theorem 3.5.2. *There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega)$ and $\|u_0 - w\|_{L^\infty} \le \delta$, then the solution u of (3.1.5) is globally defined.*

With $\delta > 0$ possibly smaller, we even have

$$\|u(t) - w\|_{L^\infty} \le Ce^{-\mu t} \text{ for } t \ge 0,$$

where μ is any number strictly less than $\lambda_1(-\triangle - g_u(x,w))$, and C depends on μ.

3.6. Some simple cases where blow up does occur. We begin with the simple model problem

$$\begin{cases} u_t - \triangle u = |u|^{p-1}u \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ in } (0,T) \times \partial\Omega, \\ u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{3.6.1}$$

Recall that if u_0 is "small enough", then (3.6.1) has a global solution. Here, we show that if u_0 is "big enough", then the solution does blow up in finite time.

Theorem 3.6.1. *Assume $p > 1$. Let $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with*

$$E(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \frac{1}{p+1}\int_\Omega |u_0|^{p+1}\,dx \le 0, \tag{3.6.2}$$

and $u_0 \not\equiv 0$. If u is the solution of (3.6.1), then $T_{\mathrm{m}} < \infty$.

Proof. Suppose $T_{\mathrm{m}} = +\infty$ and set

$$\varphi(t) = \int_\Omega u(t,x)^2\,dx. \tag{3.6.3}$$

We will derive a differential inequality for φ, which cannot hold for all $t \ge 0$. We have

$$\begin{aligned} \varphi'(t) &= 2\int_\Omega uu_t = 2\int_\Omega u(\triangle u + |u|^{p-1}u) \\ &= 2\int_\Omega |u|^{p+1} - 2\int_\Omega |\nabla u|^2 \\ &= -4E(u) + \frac{2(p-1)}{p+1}\int_\Omega |u|^{p+1} \\ &\ge -4E(u_0) + \frac{2(p-1)}{p+1}\int_\Omega |u|^{p+1}, \end{aligned}$$

where the last inequality follows from (3.4.6). Using the assumption (3.6.2), we are led to

$$\varphi'(t) \geq \frac{2(p-1)}{p+1} \int_\Omega |u|^{p+1} \geq \alpha \varphi(t)^{\frac{p+1}{2}},$$

with $\alpha > 0$, by Hölder's inequality. In particular, φ is nondecreasing, and since $\varphi(0) > 0$, whe have $\varphi(t) > 0$ for all $t \geq 0$. Finally,

$$\frac{d}{dt}\Big(-\frac{2}{p-1}\varphi(t)^{-\frac{p-1}{2}} \Big) = \frac{\varphi'(t)}{\varphi(t)^{\frac{p+1}{2}}} \geq \alpha;$$

and so,

$$0 \leq \frac{2}{p-1}\varphi(t)^{-\frac{p-1}{2}} \leq \frac{2}{p-1}\varphi(0)^{-\frac{p-1}{2}} - \alpha t,$$

for all $t \geq 0$. Impossible, and therefore $T_{\mathrm{m}} < +\infty$. □

Remark 3.6.2. The previous argument shows that

$$T_{\mathrm{m}} \leq \frac{2}{\alpha(p-1)\varphi(0)^{\frac{p-1}{2}}}.$$

However, the proof **does not** imply $\lim\limits_{t \uparrow T_{\mathrm{m}}} \varphi(t) = +\infty$. In fact, in some cases it can happen that $\varphi(t)$ remains bounded as $t \uparrow T_{\mathrm{m}}$ (See Section 3.12).

Remark 3.6.3. Note that in some sense (3.6.2) holds for "large" initial values. Indeed, given any $v_0 \not\equiv 0$, then $u_0 = \lambda v_0$ verifies (3.6.2) for $|\lambda|$ large enough.

The previous argument can be extended to more general nonlinearities.

Theorem 3.6.4. *Assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in u and verifies*

$$ug(x,u) \geq (2+\varepsilon)G(x,u) = (2+\varepsilon)\int_0^u g(x,t)\,dt, \tag{3.6.4}$$

for all $u \in \mathbb{R}$, with $\varepsilon > 0$. Let $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with

$$E(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx - \int_\Omega G(x,u_0)\,dx \leq 0$$

and $u_0 \not\equiv 0$. If u is the solution of (3.1.5), then $T_{\mathrm{m}} < \infty$.

Proof. Suppose by contradiction that $T_{\mathrm{m}} = +\infty$. Let φ be given by (3.6.3). We have

$$\varphi'(t) = -2\int_\Omega |\nabla u|^2 + 2\int_\Omega ug(x,u) \geq -2\int_\Omega |\nabla u|^2 + 2(2+\varepsilon)\int_\Omega G(x,u),$$

by (3.6.4). On the other hand, it follows from (3.4.6) that

$$\int_\Omega G(x,u(t))\,dx = -E(u_0) + \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \int_0^t \int_\Omega u_t(s,x)^2\,dx\,ds;$$

and so,

$$\varphi'(t) \geq \varepsilon \int_\Omega |\nabla u|^2 - (4+2\varepsilon)E(u_0) + (4+2\varepsilon)\int_0^t\int_\Omega u_t^2 \geq \varepsilon\int_\Omega |\nabla u|^2 + (4+2\varepsilon)\int_0^t\int_\Omega u_t^2. \tag{3.6.5}$$

We claim that

$$\int_0^1 \int_\Omega u_t(s,x)^2 \, dx \, ds = \delta > 0. \tag{3.6.6}$$

Otherwise, we have $u_t = 0$ for $t \in (0,1)$; and so, $u \equiv u_0$. This implies that $\varphi' = 0$ on $(0,1)$, and by (3.6.5) this gives

$$\int_\Omega |\nabla u_0|^2 \, dx = 0.$$

Hence $u_0 \equiv 0$. Impossible. Hence, we have proved (3.6.6).

It follows from (3.6.5) and (3.6.6) that

$$\varphi(t) \ge \delta(t-1), \tag{3.6.7}$$

for $t \ge 0$. We have

$$\varphi'(t) = 2\int_\Omega u u_t \le 2\|u\|_{L^2} \, \|u_t\|_{L^2};$$

and so,

$$\varphi(t) - \varphi(0) \le 2\int_0^t \sqrt{\varphi(s)} \, \|u_t\|_{L^2} \, ds \le 2\Big(\int_0^t \varphi(s) \, ds\Big)^{\frac{1}{2}} \Big(\int_0^t \int_\Omega u_t^2 \, dx \, ds\Big)^{\frac{1}{2}}.$$

Since $\varphi(t) \ge \varphi(0)$ by (3.6.5), we deduce

$$(\varphi(t) - \varphi(0))^2 \le 4\Big(\int_0^t \varphi(s) \, ds\Big) \Big(\int_0^t \int_\Omega u_t^2 \, dx \, ds\Big).$$

Set now

$$\psi(t) = \int_0^t \varphi(s) \, ds.$$

It follows from the above inequalities that

$$\psi(t)\psi''(t) \ge \Big(1 + \frac{\varepsilon}{2}\Big)(\varphi(t) - \varphi(0))^2 \ge \Big(1 + \frac{\varepsilon}{4}\Big)\varphi(t)^2,$$

for t large enough, by (3.6.7). Therefore,

$$\psi(t)\psi''(t) \ge \Big(1 + \frac{\varepsilon}{4}\Big)\psi'(t)^2,$$

for t large, which implies that $(\psi^{-\frac{\varepsilon}{4}})'' \le 0$. Since $\psi^{-\frac{\varepsilon}{4}}$ is positive and converges to 0 as $t \to \infty$, we obtain a contradiction. □

Remark 3.6.5. Assume that (3.6.4) holds only for $|u|$ large enough. Then there exists $K < \infty$ such that for every $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $E(u_0) \le -K$, the solution of (3.1.5) blows up in finite time (see Exercise 3.13.12).

In the previous Theorems 3.6.1 and 3.6.4, blow up occurs for large initial data. In the next result, blow up occurs for all initial values.

Theorem 3.6.6. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and that*

$$g(u) \ge \lambda_1 u + h(u), \tag{3.6.8}$$

for all $u \in \mathbb{R}$, where $h > 0$ is a convex function $\mathbb{R} \to \mathbb{R}$ such that

$$\int_1^\infty \frac{ds}{h(s)} < \infty, \tag{3.6.9}$$

and λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. If $u_0 \in L^\infty(\Omega)$, then the solution u of (3.1) blows up in finite time.

Example 3.6.7. Let $g(u) = \lambda e^u$, then the assumptions of Theorem 3.6.6 are satisfied if $\lambda e > \lambda_1$.

Proof of Theorem 3.6.6. Suppose $T_{\mathrm{m}} = +\infty$ and let

$$f(t) = \int_\Omega u(t,x)\varphi_1(x)\,dx,$$

where $\varphi_1 > 0$ is the first eigenvector of $-\triangle$ in $H_0^1(\Omega)$ such that $\displaystyle\int_\Omega \varphi_1 = 1$. We have

$$f'(t) = \int_\Omega u_t\varphi_1 = \int_\Omega (\triangle u + g(u))\varphi_1 = -\lambda_1\varphi(t) + \int_\Omega g(u)\varphi_1 \geq \int_\Omega h(u)\varphi_1 \geq h(f(t)),$$

by Jensen's inequality. Let

$$\Gamma(t) = \int_1^t \frac{ds}{h(s)}.$$

We have

$$\frac{d}{dt}\Gamma(f(t)) = \frac{f'(t)}{h(f(t))} \geq 1;$$

and so,

$$\Gamma(f(t)) \geq \Gamma(f(0)) + t,$$

which contradicts (3.6.9) for t large enough. □

Remark 3.6.8. Assume that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and that

$$g(u) \geq \lambda_1 u + h(u),$$

for all $u \geq 0$, where $h : (0,\infty) \to (0,\infty)$ is a convex function such that (3.6.9) holds. Then for all $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, the solution u of (3.1) blows up in finite time.

Indeed, since $g(0) \geq 0$, the maximum principle (see Theorem 3.3.2) implies that $u \geq 0$, and the above argument still holds.

Remark 3.6.9. Consider the equation

$$\begin{cases} u_t - \triangle u = \lambda_1 u + u^2 \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ in } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega. \end{cases}$$

Then, for every $u_0 \geq 0$, $u_0 \not\equiv 0$, blow up occurs in finite time. On the other hand, if $u_0 \leq 0$, then $T_{\mathrm{m}} = +\infty$.

We just prove the last claim. This amounts to show that the problem

$$\begin{cases} v_t - \triangle v = \lambda_1 v - v^2 \text{ in } (0,T) \times \Omega, \\ v = 0 \text{ in } (0,T) \times \partial\Omega, \\ v = v_0 \text{ in } \Omega, \end{cases} \tag{3.6.10}$$

has global existence for $v_0 \ge 0$. For this purpose, we consider the problem

$$\begin{cases} w_t - \triangle w = \lambda_1 w - |w|w \text{ in } (0,T) \times \Omega, \\ w = 0 \text{ in } (0,T) \times \partial\Omega, \\ w = v_0 \text{ in } \Omega. \end{cases}$$

First we note that $g(w) = \lambda_1 w - |w|w$ verifies the conditions of Theorem 3.3.1; hence w is globally defined. Furthermore, if $v_0 \ge 0$, then the maximum principle implies that $w \ge 0$; and so, w satisfies (3.6.10). □

Remark 3.6.10. Assume that g verifies (3.6.8) for $u \ge \alpha > 0$, where $h : (\alpha, \infty) \to (0, \infty)$ is a convex function such that (3.6.9) holds. Then, there exists $\beta > 0$ such that if $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$ and

$$\int_\Omega u_0(x)\varphi_1(x)\, dx > \beta,$$

then the solution of (3.1) blows up in finite time (see Exercise 3.13.11).

3.7. The study of $u_t - \triangle u = \lambda g(u)$. Consider the problem

$$\begin{cases} u_t - \triangle u = \lambda g(u) \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ in } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega. \end{cases} \tag{3.7.1}$$

Here, and throughout this section

$$\lambda > 0,$$

and $g : [0, \infty) \to [0, \infty)$ is a C^1 convex, nondecreasing function with

$$g(0) > 0,$$

and

$$\int_0^\infty \frac{ds}{g(s)} < \infty, \tag{3.7.2}$$

so that in particular

$$\lim_{s \to \infty} \frac{g(s)}{s} = +\infty. \tag{3.7.3}$$

Typical examples of such functions which occur in applications are $g(u) = e^u$ and $g(u) = (1+u)^p$, $1 < p < \infty$.

Solutions u of (3.7.1) are always assumed to be nonnegative. The initial condition u_0 is always assumed to be in $L^\infty(\Omega)$ and $u_0 \ge 0$, so that a classical solution of (3.7.1) exists on a maximal interval $(0, T_{\mathrm{m}})$.

Our first result asserts that the existence of a global, classical solution of (3.7.1) implies the existence of a solution for the corresponding stationary problem:

$$\begin{cases} -\triangle u = \lambda g(u) \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega. \end{cases} \tag{3.7.4}$$

Unfortunately, this solution need not be a classical solution (see Remark 3.7.5 and Theorem 3.7.6). Therefore, we are led to the notion of a weak solution.

Definition 3.7.1. *A weak solution of (3.7.4) is a function $u \in L^1(\Omega)$, $u \geq 0$ such that*

$$g(u)\delta \in L^1(\Omega), \tag{3.7.5}$$

where δ denotes the function distance to the boundary,

$$\delta(x) = \operatorname{dist}(x, \partial\Omega), \tag{3.7.6}$$

and

$$-\int_\Omega u \triangle \zeta = \lambda \int_\Omega g(u)\zeta, \tag{3.7.7}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. (Note that the second integral makes sense since $|\zeta(x)| \leq C\delta(x)$ for all $x \in \Omega$.)

It is clear that any classical solution of (3.7.4) is a weak solution. Our first result is the following.

Theorem 3.7.2. *If there exists a global, classical solution of (3.7.1) for some $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then there exists a weak solution of (3.7.4).*

Remark 3.7.3. Theorem 3.7.2 is quite surprising since we do not assume any bound (as $t \to \infty$) for the global solution u.

The stationary problem has been extensively investigated. See Brezis, Cazenave, Martel and Ramiandrisoa [20], Brezis and Nirenberg [24], Crandall and Rabinowitz [34], Fujita [44], Gallouet, Gallouët, Mignot and Puel [49], Gelfand [50], Joseph and Lundgren [62], Keller and Cohen [68], Keller and Keener [69], Mignot and Puel [81]. We now summarize the main results concerning (3.7.4).

Lemma 3.7.4. *There exists $0 < \lambda^* < \infty$ such that:*

(a) *For every $0 < \lambda < \lambda^*$ equation (3.7.4) has a minimal, positive classical solution $u(\lambda)$, which is the unique stable solution of (3.7.4); stability means that*

$$\lambda_1(-\triangle - \lambda g'(u(\lambda))) > 0.$$

(There may exist, for some values of $\lambda \in (0, \lambda^*)$, one or many other solutions, which are all unstable.)

(b) *The map $\lambda \mapsto u(\lambda)$ is increasing.*

(c) *For $\lambda > \lambda^*$, there is no weak solution of (3.7.4).*

(d) *For $\lambda = \lambda^*$ there is a weak solution $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u(\lambda)$ of (3.7.4).*

For the proof of Lemma 3.7.4, we refer to the above mentioned references.

Remark 3.7.5. The solution u^* is sometimes a classical solution. For example when $g(u) = e^u$ and $N \le 9$ or when $g(u) = (1+u)^p$ and

$$N < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}. \tag{3.7.8}$$

Note that (3.7.8) holds for any p if $N \le 10$; if $N \ge 11$ condition (3.7.8) says that p is strictly less than some $p(N)$ (see F. Mignot and J.-P. Puel [81]). However, there are cases where there is **no classical solution** at $\lambda = \lambda^*$. For example when Ω is the unit ball of $\mathbb{R}^N$ with $N \ge 10$ and $g(u) = e^u$; in this case $\lambda^* = 2(N-2)$ and $u^*(x) = \log\left(\frac{1}{|x|^2}\right)$. Similarly, for $g(u) = (1+u)^p$, if $N \ge 11$ and $p \ge p(N)$, then $u^*(x) = |x|^{-\frac{2}{p-1}} - 1$. See Joseph and Lundgren [62] and Brezis and Vazquez [27].

There is a converse of Theorem 3.7.2.

Theorem 3.7.6. *If there exists a weak solution w of (3.7.4), then for any $u_0 \in L^\infty(\Omega)$ with $0 \le u_0 \le w$, the solution u of (3.7.1) with $u(0) = u_0$ is global.*

Remark 3.7.7. If w is a classical solution of (3.7.4), then the existence of a global solution of (3.7.1) follows immediately from the maximum principle. On the other hand, if $w \notin L^\infty(\Omega)$, then the conclusion is far from obvious. Indeed, suppose that the solution blows up in finite time T_{m}. Clearly $u(t,x) \le w(x)$ on $(0, T_{\mathrm{m}}) \times \Omega$, but this estimate in itself does not prevent $\|u(t)\|_{L^\infty}$ from blowing up in finite time. As we will see in Section 3.12, $u(t,x)$ can converge to a blow up profile $u(T_{\mathrm{m}}, x)$, which may be finite everywhere except at one point.

Putting together Theorems 3.7.2 and 3.7.6 and Lemma 3.7.4, we can now state the following.

Corollary 3.7.8. *Consider the (classical) solution u of (3.7.1) with $u_0(x) \equiv 0$. If $\lambda \le \lambda^*$, then u is global. If $\lambda > \lambda^*$, then u blows up in finite time.*

Proof of Theorem 3.7.2. We may assume that $u_0 = 0$, so that $u \ge 0$ and $u_t \ge 0$ for all $t \ge 0$. (see Step 1 of the proof of Theorem 3.8.3)

Next, observe that by (3.7.3), there exists a constant $M > 0$ such that

$$g(s) - \lambda_1 s \ge \frac{1}{2} g(s) \quad \text{for} \quad s \ge M, \tag{3.7.9}$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. Let $\varphi \in C^2(\overline{\Omega})$ with $\varphi_{|\partial\Omega} = 0$. It follows from (3.7.1) that

$$\frac{d}{dt} \int_\Omega u(t)\varphi + \int_\Omega u(t)(-\triangle\varphi) = \int_\Omega g(u(t))\varphi. \tag{3.7.10}$$

We first claim that

$$\sup_{t \ge 0} \int_\Omega g(u)\varphi_1 \le (1+\lambda_1)M, \tag{3.7.11}$$

where M is as in (3.7.9) and φ_1 is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$ such that $\int_\Omega \varphi_1 = 1$. Indeed, taking $\varphi = \varphi_1$ in (3.7.10), we find

$$\frac{d}{dt} \int_\Omega u(t)\varphi_1 + \lambda_1 \int_\Omega u(t)\varphi_1 = \int_\Omega g(u(t))\varphi_1 \ge g\left(\int_\Omega u(t)\varphi_1\right), \tag{3.7.12}$$

by Jensen's inequality. If there exists $t_0 \geq 0$ such that $\int_\Omega u(t_0)\varphi_1 > M$, then it follows from (3.7.12) and (3.7.9) that

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \geq \frac{1}{2}g\left(\int_\Omega u(t)\varphi_1\right),$$

for $t \geq t_0$, which is absurd by (3.7.2); and so

$$\int_\Omega u(t)\varphi_1 \leq M,$$

for all $t \geq 0$. Integrating (3.7.12) on $(t, t+1)$ and since $u_t \geq 0$, we find

$$\int_\Omega g(u(t))\varphi_1 \leq \int_t^{t+1}\int_\Omega g(u)\varphi_1 \leq \int_\Omega u(t+1)\varphi_1 + \lambda_1\int_t^{t+1}\int_\Omega u\varphi_1 \leq (1+\lambda_1)M,$$

hence (3.7.11).

We next claim that there exists K such that

$$\sup_{t\geq 0}\|u(t)\|_{L^1} \leq K. \tag{3.7.13}$$

Indeed, let ζ_0 be the solution of (3.7.21). Taking $\varphi = \zeta_0$ in (3.7.10) and integrating on $(t, t+1)$, we find

$$\int_\Omega u(t) \leq \int_t^{t+1}\int_\Omega u = \int_\Omega u(t)\zeta_0 - \int_\Omega u(t+1)\zeta_0 + \int_t^{t+1}\int_\Omega g(u)\zeta_0,$$

and (3.7.13) follows by applying (3.7.11).

By monotone convergence, it follows from (3.7.13) and (3.7.11) that $u(t)$ has a limit w in $L^1(\Omega)$ and that $g(u)$ converges to $g(w)$ in $L^1(\Omega, \delta(x)dx)$, as $t \to \infty$. Let $\varphi \in C^2(\overline{\Omega})$, $\varphi_{|\partial\Omega} = 0$. Integrating (3.7.10) on $(t, t+1)$, we obtain

$$\left[\int_\Omega u\varphi\right]_t^{t+1} + \int_t^{t+1}\int_\Omega u(t)(-\triangle\varphi) = \int_t^{t+1}\int_\Omega g(u(t))\varphi.$$

Letting $t \to \infty$, we find

$$\int_\Omega w(-\triangle\varphi) = \int_\Omega g(w)\varphi.$$

Therefore, w is a weak solution of (3.7.4). □

For the proof of Theorem 3.7.6 we need four lemmas. We begin with a lemma concerning the linear Laplace equation.

Lemma 3.7.9. *Given $f \in L^1(\Omega, \delta(x)dx)$, there exists a unique $v \in L^1(\Omega)$ which is a weak solution of*

$$\begin{cases} -\triangle v = f \quad \text{in} \quad \Omega, \\ v_{|\partial\Omega} = 0, \end{cases} \tag{3.7.14}$$

in the sense that

$$-\int_\Omega v\triangle\zeta = \int_\Omega f\zeta, \tag{3.7.15}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Moreover,

$$\|v\|_{L^1} \leq C\|f\|_{L^1(\Omega,\delta(x)dx)}, \tag{3.7.16}$$

for some constant C independent of f. In addition, if $f \geq 0$ a.e. in Ω, then $v \geq 0$ a.e. in Ω.

Proof. The uniqueness is clear. Indeed, let v_1 and v_2 be two solutions of (3.7.14). Then $v = v_1 - v_2$ satisfies

$$\int_\Omega v \triangle \zeta = 0,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Given any $\varphi \in \mathcal{D}(\Omega)$ let ζ be the solution of

$$\begin{cases} \triangle \zeta = \varphi & \text{in } \Omega, \\ \zeta_{|\partial\Omega} = 0. \end{cases}$$

It follows that

$$\int_\Omega v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.

For the existence, we may assume that $f \geq 0$ (otherwise we write $f = f_+ - f_-$). Given an integer $k \geq 0$ set $f_k(x) = \min\{f(x), k\}$, so that $f_k \underset{k\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let v_k be the solution of

$$\begin{cases} -\triangle v_k = f_k & \text{in } \Omega, \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7.17}$$

The sequence $(v_k)_{k\geq 0}$ is clearly monotone nondecreasing. It is also a Cauchy sequence in $L^1(\Omega)$ since

$$\int_\Omega (v_k - v_\ell) = \int_\Omega (f_k - f_\ell)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} -\triangle \zeta_0 = 1 & \text{in } \Omega, \\ \zeta_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7.18}$$

Hence

$$\int_\Omega |v_k - v_\ell| \leq C \int_\Omega |f_k - f_\ell| \delta(x)\, dx.$$

Passing to the limit in (3.7.17) (after multiplication by ζ), we obtain (3.7.15). Finally, taking $\zeta = \zeta_0$ in (3.7.15), we obtain

$$\|v\|_{L^1} = \int_\Omega v = \int_\Omega f\zeta_0 \leq C\|f\|_{L^1(\Omega,\delta(x)dx)},$$

and (3.7.16) follows. □

Our next lemma is a variant of Kato's inequality (see [65] and Theorem A.5.20).

Lemma 3.7.10. *Let $f \in L^1(\Omega, \delta(x)dx)$, and let $u \in L^1(\Omega)$ be the weak solution of (3.7.14). Let $\Phi \in C^2(\mathbb{R})$ be concave, with Φ' bounded and $\Phi(0) = 0$. Then*

$$-\triangle \Phi(u) \geq \Phi'(u) f,$$

in the sense that

$$-\int_\Omega \Phi(u) \triangle \zeta \geq \int_\Omega \Phi'(u) f \zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ *such that* $\zeta = 0$ *on* $\partial\Omega$.

Proof. Consider $(f_n)_{n\geq 0} \subset \mathcal{D}(\Omega)$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^1(\Omega, \delta(x)dx)$. Let u_n be the solution of

$$\begin{cases} -\triangle u_n = f_n \quad \text{in} \quad \Omega, \\ u_n = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

It follows from Lemma 3.7.9 that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^1(\Omega)$. On the other hand we have

$$\triangle\Phi(u_n) = \Phi'(u_n)\triangle u_n + \Phi''(u_n)|\nabla u_n|^2 \leq \Phi'(u_n)\triangle u_n = -\Phi'(u_n)f_n.$$

Therefore,

$$-\int_\Omega \Phi(u_n)\triangle\zeta \geq \int_\Omega \Phi'(u_n)f_n\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geq 0$ such that $\zeta = 0$ on $\partial\Omega$; and so the result follows easily by letting $n \to \infty$. □

Lemma 3.7.11. *Let* $0 < \varepsilon < g(0)$ *and set*

$$h_0(t) = \int_0^t \frac{ds}{g(s)}, \quad h_\varepsilon(t) = \int_0^t \frac{ds}{g(s) - \varepsilon},$$

and

$$\Phi_\varepsilon(t) = h_\varepsilon^{-1}(h_0(t)),$$

for all $t \geq 0$. *Then*

(i) $\Phi_\varepsilon \in C^2([0,\infty))$, $\Phi_\varepsilon(0) = 0$ *and* $0 \leq \Phi_\varepsilon(t) \leq t$.

(ii) Φ_ε *is increasing, concave and* $\Phi'_\varepsilon(t) = \dfrac{g(\Phi_\varepsilon(t)) - \varepsilon}{g(t)} \leq 1$ *for all* $t \geq 0$.

(iii) $\sup\limits_{t\geq 0} \Phi_\varepsilon(t) = C_\varepsilon < \infty$, *for every* $\varepsilon \in (0, g(0))$.

Proof. Properties (i) and (iii) are clear. We have $h_\varepsilon(\Phi_\varepsilon(t)) = h_0(t)$, and thus $h'_\varepsilon(\Phi_\varepsilon(t))\Phi'_\varepsilon(t) = h'_0(t)$, which is the identity in (ii). Differentiating once more, we deduce

$$\Phi''_\varepsilon(t) = (g(\Phi_\varepsilon(t) - \varepsilon))\frac{g'(\Phi_\varepsilon(t)) - g'(t))}{g(t)^2}.$$

Since $g'(\Phi(t)) \leq g'(t)$, it follows that Φ is concave. Hence (ii). □

Lemma 3.7.12. *Let* δ *be given by (3.7.6). For every* $0 < T < \infty$, *there exists* $\varepsilon_1(T) > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_1$, *then the solution* Z *of the equation*

$$\begin{cases} Z_t - \triangle Z = -\varepsilon \quad \text{in} \quad (0,\infty)\times\Omega, \\ Z = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \\ Z(0) = \delta, \end{cases}$$

verifies $Z \geq 0$ *on* $[0,T]\times\overline{\Omega}$.

Proof. Consider the solution ζ_0 of (3.7.18). We have

$$\zeta_0 = T(t)\zeta_0 + \int_0^t T(s)1_\Omega \, ds,$$

for all $t \geq 0$. Since $T(t)\zeta_0 \geq 0$, it follows that

$$\int_0^t T(s)1_\Omega \, ds \leq \zeta_0 \leq C\delta, \tag{3.7.19}$$

for all $t \geq 0$. On the other hand, we have

$$Z(t) = T(t)\delta - \varepsilon \int_0^t T(s)1_\Omega \, ds;$$

and so,

$$Z(t) \geq T(t)\delta - \varepsilon C\delta.$$

Consider now $c_0, c_1 > 0$ such that $c_0\varphi_1 \leq \delta \leq c_1\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$, associated to the eigenvalue λ_1. We have

$$T(t)\delta \geq c_0 T(t)\varphi_1 = c_0 e^{-\lambda_1 t}\varphi_1 \geq \frac{c_0}{c_1} e^{-\lambda_1 t}\delta.$$

Therefore,

$$Z(t) \geq \left(\frac{c_0}{c_1} e^{-\lambda_1 t} - \varepsilon C \right) \delta.$$

It follows that $Z(t) \geq 0$ on $[0, T]$, provided $\varepsilon \leq \dfrac{c_0}{c_1 C} e^{-\lambda_1 T}$. □

Proof of Theorem 3.7.6. If $w \in L^\infty(\Omega)$, then by the maximum principle $u(t) \leq w$ for all $t \in [0, T_{\rm m})$; and so $T_{\rm m} = +\infty$. Therefore, we may assume

$$w \notin L^\infty(\Omega). \tag{3.7.20}$$

We proceed in five steps.

Step 1. We claim that $u(t) \leq w$ for all $t \in [0, T_{\rm m})$. This is proved using the maximum principle, but since w is not smooth, we have to be more careful. Fix $T < T_{\rm m}$. Let $h(t, x) \in \mathcal{D}((0, T) \times \Omega)$, $h \geq 0$, and let ζ be the solution of

$$\begin{cases} -\zeta_t - \triangle\zeta - \lambda g'(u)\zeta = h, \\ \zeta_{|\partial\Omega} = 0, \\ \zeta(T) = 0. \end{cases}$$

Since $g'(u) \in L^\infty((0,T)\times\Omega)$ we have in particular $\zeta \in C([0,T], C^2(\overline{\Omega}) \cap C_0(\Omega))$ and $\zeta \geq 0$. Multiplying (3.7.1) by ζ and integrating on $(0, T) \times \Omega$, we find

$$-\int_\Omega u_0\zeta(0) + \int_0^T \int_\Omega u[h + \lambda g'(u)\zeta] = \lambda \int_0^T \int_\Omega g(u)\zeta.$$

On the other hand,

$$-\int_0^T \int_\Omega w\zeta_t - \int_\Omega w\zeta(0) = 0,$$

and

$$-\int_0^T\int_\Omega w\triangle\zeta = \lambda\int_0^T\int_\Omega g(w)\zeta.$$

Therefore,

$$-\int_\Omega (u_0-w)\zeta(0) + \int_0^T\int_\Omega (u-w)h = \lambda\int_0^T\int_\Omega (g(u)-g(w)-g'(u)(u-w))\zeta.$$

By convexity,

$$g(u)-g(w)-g'(u)(u-w)\le 0.$$

Since $\zeta\ge 0$ and $u_0-w\le 0$, this yields

$$\int_0^T\int_\Omega (u-w)h\le 0.$$

Since h is arbitrary, we conclude that $u-w\le 0$.

Step 2. There exist $0<\tau<T_{\mathrm{m}}$ and $C_0, c_0>0$ such that

$$u(\tau)\le C_0\delta, \tag{3.7.21}$$

and

$$u(\tau)\le w - c_0\delta. \tag{3.7.22}$$

Set $v_0=\min\{w,1+u_0\}$. We have $v_0\ge u_0$ and $v_0\not\equiv u_0$ by (3.7.20). In particular, there exists a function $\gamma:[0,\infty)\to\mathbb{R}$ such that $\gamma(t)>0$ for $t>0$ and

$$T(t)(v_0-u_0)\ge\gamma(t)\delta, \tag{3.7.23}$$

where δ is defined by (3.7.6) (see Exercise 1.8.12). Let v be the solution of (3.7.1) with the initial value $v(0)=v_0$, and let $[0,\overline{T})$ be the maximal interval of existence of v. We have $v\ge 0$, and by Step 1, $v\le w$. Let $z(t)=u(t)+T(t)(v_0-u_0)$ for $0\le t<\overline{T}$. We have

$$\begin{cases} z_t-\triangle z=\lambda g(u)\le\lambda g(z) & \text{in} \quad (0,\overline{T})\times\Omega,\\ z=0 \quad \text{on} \quad \partial\Omega,\\ z(0)=v_0 \quad \text{in} \quad \Omega,\end{cases}$$

so that $z\le v$ by the maximum principle. Therefore,

$$u(t)\le v(t)-T(t)(v_0-u_0)\le w-T(t)(v_0-u_0)\le w-\gamma(t)\delta, \tag{3.7.24}$$

for $0\le t<\overline{T}$ by (3.7.23). Fix $0<T<\min\{\overline{T},T_{\mathrm{m}}\}$. u is bounded by some constant M on $[0,T]\times\overline{\Omega}$, so that

$$u(t)\le MT(t)1_\Omega+\lambda g(M)\int_0^t T(s)1_\Omega\,ds.$$

There exists a function $\overline{C}:(0,\infty)\to\mathbb{R}$ such that $T(t)1_\Omega\le\overline{C}(t)\delta$ for $t>0$, so that we deduce from (3.7.19) that

$$u(t)\le M\overline{C}(t)\delta+\lambda g(M)C\delta, \tag{3.7.25}$$

for $0 < t \le T$. (3.7.21) and (3.7.22) now follow from (3.7.24) and (3.7.25).

Step 3. We may assume without loss of generality that

$$u_0 \le C_0 \delta, \tag{3.7.26}$$

and

$$u_0 \le w - c_0 \delta, \tag{3.7.27}$$

where C_0, c_0 are as in Step 2. Indeed, we need only consider $u(\cdot + \tau)$ instead of $u(\cdot)$.

Step 4. Let Φ_ε be as in Lemma 3.7.11, and set $w_\varepsilon = \Phi_\varepsilon(w)$ for $0 < \varepsilon < g(0)$. Then

$$w_\varepsilon \in L^\infty(\Omega), \tag{3.7.28}$$

and

$$\int_\Omega \zeta(-\triangle w_\varepsilon) \ge \lambda \int_\Omega (g(w_\varepsilon) - \varepsilon)\zeta, \tag{3.7.29}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \ge 0$ on Ω and $\zeta_{|\partial\Omega} = 0$. Moreover, there exists $0 < \varepsilon_1 \le g(0)$ such that

$$u_0 \le w_\varepsilon - \frac{c_0}{2}\delta, \tag{3.7.30}$$

for $0 < \varepsilon < \varepsilon_1$, where c_0 is as in (3.7.27). Indeed, (3.7.28) and (3.7.29) follow from Lemmas 3.7.10 and 3.7.11. In order to prove (3.7.30), set

$$\eta = \min\{w, (C_0 + c_0)\delta\},$$

and

$$\eta_\varepsilon = \Phi_\varepsilon(\eta).$$

Here, δ is given by (3.7.6) and C_0 is as in (3.7.26). It follows from (3.7.26) and (3.7.27) that

$$u_0 \le \eta - c_0 \delta. \tag{3.7.31}$$

We claim that

$$\eta \le \eta_\varepsilon + \frac{c_0}{2}\delta, \tag{3.7.32}$$

for $\varepsilon > 0$ small enough. Note that it follows from (3.7.31) and (3.7.32) that

$$u_0 \le \eta_\varepsilon - \frac{c_0}{2}\delta,$$

and (3.7.30) follows since $\eta_\varepsilon \le w_\varepsilon$ (since Φ_ε is nondecreasing). Thus we need only prove (3.7.32). Note that

$$\eta_\varepsilon \le \eta \le M,$$

where $M = (C_0 + c_0)\|\delta\|_{L^\infty}$, and that

$$\Phi'_\varepsilon(x) \underset{\varepsilon \downarrow 0}{\longrightarrow} 1,$$

uniformly on $[0, M]$ by Lemma 3.7.11. Therefore,

$$\eta - \eta_\varepsilon \le \eta \sup_{0 \le x \le M} (1 - \Phi'_\varepsilon(x)) \le (C_0 + c_0)\delta \sup_{0 \le x \le M} (1 - \Phi'_\varepsilon(x)) \le \frac{c_0}{2}\delta,$$

for ε small enough, and (3.7.32) follows.

Step 5. Conclusion. Assume by contradiction that $T_{\rm m} < \infty$. Let $\varepsilon > 0$ be small enough so that

$$u_0 \le w_\varepsilon - \frac{c_0}{2}\delta,$$

(see Step 4), and so that the solution Z of the equation

$$\begin{cases} Z_t - \triangle Z = -\varepsilon\lambda \quad \text{in} \quad (0, T_{\rm m}) \times \Omega, \\ Z = 0 \quad \text{on} \quad \partial\Omega, \\ Z(0) = \dfrac{c_0}{2}\delta \quad \text{in} \quad \Omega, \end{cases}$$

is nonnegative on $[0, T_{\rm m}] \times \overline{\Omega}$ (see Lemma 3.7.12). Let v be the solution of

$$\begin{cases} v_t - \triangle v = \lambda(g(|v|) - \varepsilon) \quad \text{in} \quad (0, T) \times \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega, \\ v(0) = w_\varepsilon \quad \text{in} \quad \Omega. \end{cases}$$

Let $[0, S_m)$ be the maximal interval of existence of v. Set $z(t) = Z(t) + u(t)$ for $0 \le t < T_{\rm m}$. We have $z \ge u \ge 0$ and

$$\begin{cases} z_t - \triangle z = \lambda(g(u) - \varepsilon) \le \lambda(g(z) - \varepsilon) \quad \text{on} \quad (0, T_{\rm m}) \times \Omega, \\ z_{|\partial\Omega} = 0, \\ z(0) = u_0 + \dfrac{c_0}{2}\delta \le w_\varepsilon \quad \text{in} \quad \Omega. \end{cases}$$

By the maximum principle, we have $z \le v$ on $[0, \min\{T_{\rm m}, S_m\})$. In particular, $v \ge 0$ on $[0, \min\{T_{\rm m}, S_m\})$; by the maximum principle and (3.7.29), $v \le w_\varepsilon$. Since $w_\varepsilon \in L^\infty(\Omega)$, this implies that $T_{\rm m} < S_m = +\infty$. Therefore, $u \le z \le v \le w_\varepsilon$ on $[0, T_{\rm m})$, which is absurd. □

We have the following result.

Theorem 3.7.13. *Let $\lambda \in (0, \lambda^*)$, and let $\underline{u}$ be the minimal solution of (3.7.4). There exists $\delta > 0$ such that if $u_0 \in L^\infty(\Omega)$ verifies $0 \le u_0 \le \underline{u} + \delta$, then the solution u of (3.7.1) is global. Moreover, there exist $\mu > 0$ and a constant C such that*

$$\|u(t) - \underline{u}\|_{L^\infty} \le Ce^{-\mu t},$$

for all $t \ge 0$.

Proof. The proof proceeds in three steps.

Step 1. There exists $\delta > 0$ such that if $\|u_0 - \underline{u}\|_{L^\infty} \le \delta$, then there exist $\mu > 0$ and a constant C such that

$$\|u(t) - \underline{u}\|_{L^\infty} \le Ce^{-\mu t},$$

for all $t \ge 0$. This follows from Theorem 3.5.2.

Step 2. The conclusion of the theorem holds when $u_0 = 0$. Indeed, $\underline{u}$ is a super-solution and 0 is a sub-solution. Therefore, u is global and bounded; moreover, $u_t \ge 0$ (see Step 1 of the proof of Theorem 3.8.3).

It follows from Remark 3.4.4 that there exists a solution w of (3.7.4) such that $u(t) \to w$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ as $t \to \infty$. In particular, we have $w \le \underline{u}$; and so $w = \underline{u}$, since $\underline{u}$ is the minimal solution of (3.7.4). Therefore, $u(t) \to \underline{u}$ in $L^\infty(\Omega)$ as $t \to \infty$, and the result follows from Step 1.

Step 3. Conclusion. Let $v_1(t)$ be the solution of (3.7.1) with the initial value $v_1(0) = 0$ and let $v_2(t)$ be the solution of (3.7.1) with the initial value $v_2(0) = \underline{u} + \delta$ (δ as in Step 1). By the maximum principle, the solution u stays between v_1 and v_2. The conclusion follows from Steps 1 and 2. □

Remark 3.7.14. Note that if $\lambda \in (0, \lambda^*)$ then there exists $\beta > 0$ such that if $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$ verifies

$$\int_\Omega u_0 \varphi_1 > \beta,$$

then u blows up in finite time (see Remark 3.6.10).

Remark 3.7.15 Suppose $g(u) = e^u$ and $\lambda = \lambda^*$.

(i) If $N \le 9$, then the equation (3.7.4) has a unique, positive, smooth solution u^*, and in addition $\lambda_1(-\triangle - e^{u^*}) = 0$. In this case, one can show that for every $u_0 \in L^\infty(\Omega)$ such that $u_0 \le u^*$, the solution u of (3.7.1) is global. Moreover, $\|u(t) - u^*\|_{L^\infty} \to 0$ as $t \to \infty$; and if $u_0 \in L^\infty(\Omega)$, $u_0 \ge u^*$ and $u_0 \not\equiv u^*$, then u blows up in finite time (see Exercise 3.13.15).

(ii) If $N \ge 10$, then the behavior is quite different. Suppose Ω is the unit ball of $\mathbb{R}^N$. Then $\lambda^* = 2(N-2)$ and $u^*(x) = -2\log|x|$ (see Remark 3.7.5). If $u_0 \in L^\infty(\Omega)$, $u_0 \le u^*$, then the solution $u(t)$ of (3.7.1) converges to u^* as $t \uparrow \infty$, in $L^p(\Omega)$ for any $1 \le p < \infty$. However, if $u_0 \ge u^*$, $u_0 \not\equiv u^*$, then there is instantaneous blow up: there is no weak solution of (3.7.1) on any interval $(0,T)$ with $T > 0$. See Peral and Vazquez [87].

3.8. Analysis of $\|u(t)\|_{L^q}$ near blow up time. Recall that if u is a solution of (3.1) which blows up in finite time, then $\lim_{t\uparrow T_{\rm m}} \|u(t)\|_{L^\infty} = +\infty$. We warn the reader that in general $u(t,x)$ **does not** blow up as $t \uparrow T_{\rm m}$ **for every** $x \in \Omega$. In fact, it may happen that $u(t,x) \underset{t\uparrow T_{\rm m}}{\longrightarrow} \infty$ only for one point $x_0 \in \Omega$, and that $|u(t,x)|$ remains bounded for $x \ne x_0$ as $t \uparrow T_{\rm m}$. (See Section 3.12.)

Here, we prove that $\|u(t)\|_{L^q}$ blows up for q sufficiently large.

Theorem 3.8.1. *Assume $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and satisfies*

$$|g(u)| \le C(|u|^p + 1) \text{ for all } u \in \mathbb{R}, \tag{3.8.1}$$

for some $p \in (1,\infty)$. Let u be a solution of (3.1) which blows up in finite time. Then,

$$\lim_{t\uparrow T_{\rm m}} \|u(t)\|_{L^q} = +\infty$$

for all $\infty \ge q > \dfrac{N(p-1)}{2}$, $q \ge 1$. More precisely,

$$\liminf_{t\uparrow T_{\rm m}} (T_{\rm m} - t)^\delta \|u(t)\|_{L^q} > 0 \tag{3.8.2}$$

with $\delta = \dfrac{1}{p-1} - \dfrac{N}{2q}$.

Proof. We proceed in two steps.

Step 1. $\limsup_{t\uparrow T_{\rm m}} \|u(t)\|_{L^{pq}} = \infty$. We argue by contradiction and assume that

$$\limsup_{t\uparrow T_{\rm m}} \|u(t)\|_{L^{pq}} < \infty, \tag{3.8.3}$$

and we apply Theorem 1.6.4 with $f(t,x) = g(0)$ and

$$a(t,x) = \frac{g(u(t,x)) - g(0)}{u(t,x)}.$$

It follows from (3.8.1) and (3.8.3) that $a \in L^\infty((0,T_{\rm m}), L^\sigma(\Omega))$ with $\sigma = \dfrac{pq}{p-1}$. We have $\sigma \ge \dfrac{q}{p-1} > \dfrac{N}{2}$ and $\sigma \ge \dfrac{p}{p-1} > 1$; and so, $u \in L^\infty((0,T_{\rm m}), L^\infty(\Omega))$. This is impossible by the blow up alternative.

Step 2. Proof of (3.8.2). Let $0 \le t < T_{\rm m}$ and $0 \le s < T_{\rm m} - t$. We have

$$u(t+s) = T(s)u(t) + \int_0^s T(s-\sigma) g(u(t+\sigma))\, d\sigma.$$

It follows from (3.8.1) and Theorem 1.4.15 that

$$\|u(t+s)\|_{L^{qp}} \le s^{-\frac{N(p-1)}{2qp}} \|u(t)\|_{L^q} + CT_{\rm m}|\Omega|^{\frac{1}{qp}} + C\int_0^s (s-\sigma)^{-\frac{N(p-1)}{2qp}} \|u(t+\sigma)\|_{L^{qp}}^p.$$

We now apply Theorem A.5.10 with $f(t) = \|u(t)\|_{L^{pq}}$, $g(t) = \|u(t)\|_{L^q}$ and $\alpha = \dfrac{N(p-1)}{2qp}$. The result follows, since

$$\gamma = \frac{1-\alpha p}{p-1} = \frac{1}{p-1} - \frac{N}{2q}.$$

□

It is an open problem whether in general $\|u(t)\|_{L^q}$ blows up for $q = \dfrac{N(p-1)}{2}$ (see Open Problem 3.14.3). However, in many cases this conclusion holds (see Section 3.12). Here are two such cases.

Theorem 3.8.2. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and that*

$$|g(u)| \le C|u|^p, \quad G(u) \ge \alpha |u|^{p+1}, \quad ug(u) \ge (2+\varepsilon)G(u),$$

for $|u| \ge M$, with $\varepsilon > 0$ and $p = 1 + \dfrac{4}{N}$. If u is a solution of (3.1) which blows up in finite time, then

$$\lim_{t\uparrow T_{\rm m}} \|u(t)\|_{L^2} = +\infty.$$

Note that $2 = \dfrac{N(p-1)}{2}$.

Proof. We set

$$\varphi(t) = \int_\Omega u(t,x)^2\, dx,$$

for $t \in [0, T_{\mathrm{m}})$. We have

$$\begin{aligned}\varphi'(t) &= -2\int_\Omega |\nabla u|^2 + 2\int_\Omega ug(u) = -2\int_\Omega |\nabla u|^2 + 2\int_{|u|\ge M} ug(u) + 2\int_{|u|\le M} ug(u)\\ &\ge -2\int_\Omega |\nabla u|^2 + 2(2+\varepsilon)\int_{|u|\ge M} G(u) + 2\int_{|u|\le M} ug(u)\\ &= -2\int_\Omega |\nabla u|^2 + 2(2+\varepsilon)\int_\Omega G(u) + \int_{|u|\le M}(2ug(u) - 2(2+\varepsilon)G(u)).\end{aligned}$$

It follows from (3.4.6) that

$$\begin{aligned}\varphi'(t) &\ge -4E(u_0) + 2\varepsilon\int_\Omega G(u) + \int_{|u|\le M}(2ug(u) - 2(2+\varepsilon)G(u))\\ &\ge -4E(u_0) + 2\varepsilon\alpha\int_\Omega |u|^{p+1} + \int_{|u|\le M}(2ug(u) - 4G(u) - 2\varepsilon\alpha|u|^{p+1})\\ &\ge 2\varepsilon\alpha\int_\Omega |u|^{p+1} - K,\end{aligned}$$

with $K = 4E(u_0) + |\Omega| \sup\limits_{|u|\le M}\{2|ug(u)| + 4|G(u)| + 2\varepsilon\alpha|u|^{p+1}\}$.

Therefore, by integrating the above inequality, the conclusion of the theorem follows if we show that

$$\int_0^{T_{\mathrm{m}}}\int_\Omega |u(t,x)|^{p+1}\,dx\,dt = \infty. \tag{3.8.4}$$

We prove (3.8.4) by contradiction, and we assume that

$$\int_0^{T_{\mathrm{m}}}\int_\Omega |u(t,x)|^{p+1}\,dx\,dt < \infty. \tag{3.8.5}$$

Let $g(u) = au + f$, with $f = g(0)$ and $a = \dfrac{g(u) - g(0)}{u}$. We have $f \in L^\infty((0,T_{\mathrm{m}}), L^\infty(\Omega))$. Furthermore,

$$|a| \le C(1 + |u|^{p-1}),$$

so that (3.8.5) implies that $a \in L^{\frac{p+1}{p-1}}((0,T_{\mathrm{m}}), L^{\frac{p+1}{p-1}}(\Omega))$. It now follows from Theorem 1.6.11 that $u \in L^q((0,T_{\mathrm{m}}), L^q(\Omega))$ for all $q < \infty$. Therefore, $a \in L^q((0,T_{\mathrm{m}}), L^q(\Omega))$ for all $q < \infty$; and therefore $u \in L^\infty((0,T_{\mathrm{m}}), L^\infty(\Omega))$ by Remark 1.6.5. This contradicts the blow up alternative, thus proving (3.8.4). □

Under more restrictive assumptions on the initial value u_0, one can still show that $\lim\limits_{t\uparrow T_{\mathrm{m}}} \|u(t)\|_{L^q} = +\infty$ for $q = \dfrac{N(p-1)}{2} \ge 1$. Here is such a result (see also Weissler [96], Remark 3.9.16and Exercise 3.13.19).

Theorem 3.8.3. *Assume that $N \ge 3$ and that*

$$|g(u)| \le C(1 + |u|^p),$$

with $p \ge 1 + \dfrac{2}{N}$. Let $u_0 \in L^\infty(\Omega) \cap H^2(\Omega) \cap H^1_0(\Omega)$ be such that the solution u of (3.1) blows up in finite time. If $\triangle u_0 + g(u_0) \ge 0$ in Ω, then

$$\lim_{t\uparrow T_{\mathrm{m}}} \|u(t)\|_{L^q} = +\infty,$$

for $q = \dfrac{N(p-1)}{2}$.

Proof. We proceed in three steps.

Step 1. $u_t(t,x) \ge 0$, for all $t \in [0,T_{\rm m})$ and all $x \in \Omega$. Given $0 < h < T_{\rm m}$, define

$$v(t) = \frac{u(t+h) - u(t)}{h},$$

for $0 \le t < T - h$. It follows that

$$v_t - \triangle v = \frac{g(u(t+h)) - g(u(t))}{h}.$$

Multiplying the equation by $-v^-$ and integrating on Ω, we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^-(t)^2 + \int_\Omega |\nabla v^-|^2 \le C\int_\Omega v^-(t)^2,$$

for all $0 \le t < T_1 - h$, with $h < T_1 < T_{\rm m}$. Here, C is the constant of g on $[-A,A]$ where $A = \|u\|_{L^\infty((0,T_1)\times\Omega)}$. It follows that

$$\int_\Omega v^-(t)^2 \le e^{Ct}\int_\Omega v^-(0)^2.$$

Letting $h \downarrow 0$, we obtain $u_t \ge 0$.

Step 2. We show that $\limsup\limits_{t\uparrow T_{\rm m}} \|u(t)\|_{L^q} = +\infty$. Assume by contradiction that $\limsup\limits_{t\uparrow T_{\rm m}} \|u(t)\|_{L^q} < \infty$. Since $u(t,x)$ is a nondecreasing function of t for every $x \in \Omega$, it follows from the monotone convergence theorem that $u(t)$ has a limit in $L^q(\Omega)$ as $t \uparrow T_{\rm m}$; and so, $u \in C([0,T_{\rm m}], L^q(\Omega))$. Let $h \in C_{\rm c}(\mathbb{R})$ be such that $g(u) = h(u)$ for $|u| \le 1$. We now write $g(u) = au + f$ with $f = g(u)$. One verifies easily that $a, f \in C([0,T_{\rm m}], L^{\frac{N}{2}}(\Omega))$. By Exercise 1.8.7 and a bootstrap argument, we deduce that $u \in L^\infty((0,T_{\rm m})\times\Omega)$, which contradicts the blow up alternative.

Step 3. Conclusion. Since $u(t,x)$ is a nondecreasing function of t for every $x \in \Omega$, we have clearly $\limsup\limits_{t\uparrow T_{\rm m}} \|u(t)\|_{L^q} = \lim\limits_{t\uparrow T_{\rm m}} \|u(t)\|_{L^q}$, and the result follows from Step 2. □

3.9. Local existence for initial data in L^q, $q < \infty$. The bad sign. We now return to the question of local existence, for the model problem

$$\begin{cases} u_t - \triangle u = |u|^{p-1}u & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{in } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.9.1}$$

where $p > 1$. Recall (See Theorem 3.1.1) that if $u_0 \in L^\infty(\Omega)$, there is a unique weak solution defined on a maximal interval $[0,T_{\rm m})$. We now address the question of what happens if $u_0 \notin L^\infty(\Omega)$, but instead $u_0 \in L^q(\Omega)$ for some $q < \infty$. The value

$$q = \frac{N(p-1)}{2},$$

plays a critical role, and one has to distinguish two cases:

Case 1: $q \ge \dfrac{N(p-1)}{2}$.

Case 2: $q < \dfrac{N(p-1)}{2}$.

Roughly speaking, in case 1 we obtain the existence and uniqueness of a local solution for any $u_0 \in L^q(\Omega)$. In case 2, it seems that there exists no local solution in any reasonable sense for some initial conditions $u_0 \in L^q(\Omega)$ (see Weissler [97, 98] and Open Problem 3.14.11).

Our main existence and uniqueness result is the following.

Theorem 3.9.1. *Assume $q > \dfrac{N(p-1)}{2}$ (resp. $q = \dfrac{N(p-1)}{2}$) and $q \geq 1$ (resp. $q > 1$), $N \geq 1$. Given any $u_0 \in L^q(\Omega)$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^q(\Omega))$ with $u(0) = u_0$, which is a classical solution of (3.9.1) on $(0,T) \times \overline{\Omega}$.*

Moreover, we have:

(i) Smoothing effect and continuous dependence, namely

$$\|u(t) - v(t)\|_{L^q} + t^{\frac{N}{2q}} \|u(t) - v(t)\|_{L^\infty} \leq C \|u_0 - v_0\|_{L^q}, \tag{3.9.2}$$

for all $t \in (0,T]$ where $T = \min\{T(u_0), T(v_0)\}$ and C can be estimated in terms of $\|u_0\|_{L^q}$ and $\|v_0\|_{L^q}$.

(ii) $\lim\limits_{t \downarrow 0} t^{\frac{N}{2q}} \|u(t)\|_{L^\infty} = 0$.

(iii) If $u_0 \geq 0$, then $u(t) \geq 0$ for all $t \in [0, T(u_0)]$.

Furthermore, for any bounded set (resp. compact set) $\mathcal{K}$ in $L^q(\Omega)$, there is a (uniform) time $T = T(\mathcal{K})$ such that for any $u_0 \in \mathcal{K}$ the solution of (3.9.1) exists on $[0,T]$.

Many people have established uniqueness results for nonlinear evolution equations with singular initial conditions, in particular the Navier-Stokes and the Euler equations (see e.g. Kato and Fujita [66], Kato [64], Ben-Artzi [10], Weissler [97]). In all these works it is assumed that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ and **also** that $\lim\limits_{t \downarrow 0} t^\alpha \|u(t)\|_{L^\infty} = 0$ for some appropriate $\alpha > 0$. Our main point is that such an assumption is redundant. A similar observation has first been made in [18].

Remark 3.9.2. Since u is a classical solution on $(0,T) \times \overline{\Omega}$, the usual blow up alternative holds: either $T_{\text{m}} = +\infty$ or else $T_{\text{m}} < \infty$ and $\lim\limits_{t \uparrow T_{\text{m}}} \|u(t)\|_{L^\infty} = +\infty$.

Remark 3.9.3. The "doubly critical" case, $q = \dfrac{N(p-1)}{2}$ and $q = 1$, in Theorem 3.9.1 is delicate and widely open (see Remark 3.9.13 below). For example, when $N = 1$, the very simple equation

$$u_t - u_{xx} = u^3,$$

with an initial condition $u_0 \in L^1(\Omega)$, enters in this category.

It seems that for some $u_0 \in L^1(\Omega)$ there is not even a local solution. See Open Problem 3.14.7. We are, at least, able to find some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that equation (3.9.1) has no **nonnegative** solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. See Theorem 3.9.14 below.

When $q \geq p$, it makes sense to talk about weak solutions $u \in C([0,T], L^q(\Omega))$ in the integral sense, i.e.

$$u(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\, ds, \tag{3.9.3}$$

for all $t \in [0, T]$. Uniqueness holds in that class:

Theorem 3.9.4. *Assume $q > \dfrac{N(p-1)}{2}$ (resp. $q = \dfrac{N(p-1)}{2}$) and $q \geq p$ (resp. $q > p$), $N \geq 1$. Then uniqueness for (3.9.3) holds in the class $C([0,T], L^q(\Omega))$.*

Remark 3.9.5. In the "doubly critical" case $q = \dfrac{N(p-1)}{2}$ and $q = p$, i.e. $q = p = \dfrac{N}{N-2}$ with $N \geq 3$, the conclusion of Theorem 3.9.4 fails, i.e. uniqueness fails in the class $C([0,T], L^q(\Omega))$. See Ni and Sacks [82] and Remark 3.9.11 below.

Remark 3.9.6. The solution u of (3.9.1) given in Theorem 3.9.1 also satisfies (3.9.3); here, there is no restriction about q except for the assumptions of Theorem 3.9.1. This is not completely obvious since the integral on the right-hand side of (3.9.3) need not be well-defined. To establish the convergence of this integral, we rely on the smoothing effect (3.9.2). Clearly we have

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\, d\sigma, \tag{3.9.4}$$

for all $0 < s < t < T$. We let $s \downarrow 0$ in (3.9.4); to justify this passage to the limit it suffices to check that

$$\int_0^t \|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q}\, d\sigma < \infty.$$

The only difficulty is when σ is near 0. But

$$\|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q} \leq (t-\sigma)^{-\frac{N}{2}\left(\frac{q-1}{q}\right)} \|u(\sigma)\|_{L^p}^p,$$

by Theorem 1.4.15. We may always assume that $p > q$ (the case $q \geq p$ has been handled above); and so,

$$\|u(\sigma)\|_{L^p}^p \leq \|u(\sigma)\|_{L^q}^q \|u(\sigma)\|_{L^\infty}^{p-q} \leq C\sigma^{-\frac{N(p-q)}{2q}}.$$

The result follows, since $\dfrac{N(p-q)}{2q} = 1 - \dfrac{1}{q}\left(q - \dfrac{N(p-1)}{2}\right) - \dfrac{N}{2q}(q-1) < 1$.

In several places, it is convenient to view the nonlinear equation (3.9.1) as a linear problem

$$u_t - \triangle u = au,$$

and we have collected in the Appendix some useful facts about this linear heat equation with a potential.

Proof of Theorem 3.9.4. We consider separately two cases:

Case A: $q > \dfrac{N(p-1)}{2}$ and $q \geq p$,

Case B: $q = \dfrac{N(p-1)}{2}$ and $q > p$.

Case A. Let u and v be two solutions, $u, v \in C([0,T], L^q(\Omega))$. We have

$$u(t) - v(t) = \int_0^t T(t-s)\left(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\right) ds. \tag{3.9.5}$$

Thus, by the smoothing effect of $T(t) : L^{\frac{q}{p}}(\Omega) \to L^q(\Omega)$,

$$
\begin{aligned}
\|u(t) - v(t)\|_{L^q} &\le C \int_0^t (t-s)^{-\alpha} \| (|u|^{p-1} + |v|^{p-1})|u - v| \|_{L^{\frac{q}{p}}} \, ds \\
&\le C \int_0^t (t-s)^{-\alpha} (\|u\|_{L^q}^{p-1} + \|v\|_{L^q}^{p-1}) \|u - v\|_{L^q} \, ds,
\end{aligned}
$$

where $\alpha = \dfrac{N(p-1)}{2q} < 1$ since we are in case A. Let $M = \sup\limits_{0 \le t \le T} \|u(t)\|_{L^q} + \|v(t)\|_{L^q}$ and

$$
\psi(t) = \sup_{0 \le s \le t} \|u(t) - v(t)\|_{L^q},
$$

for $t \in [0,T]$. We deduce that

$$
\psi(t) \le C M^{p-1} \frac{T^{1-\alpha}}{1-\alpha} \psi(t).
$$

Hence $\psi(t) = 0$ for t sufficiently small. Repeating the same argument, we see that $\psi(t) = 0$ for $t \in [0,T]$.

Case B. Note that $q = \dfrac{N(p-1)}{2} > p$, thus $N \ge 3$. Let u, v be two solutions and let $w = u - v$. We set

$$
a(t,x) = \begin{cases} \dfrac{|u|^{p-1}u - |v|^{p-1}v}{u - v} & \text{if } u \ne v, \\ p|u|^{p-1} & \text{if } u = v, \end{cases} \tag{3.9.6}
$$

so that

$$
w(t) = \int_0^t T(t-s) a(s) w(s) \, ds.
$$

We claim that

$$
a \in C([0,T], L^{\frac{N}{2}}(\Omega)). \tag{3.9.7}
$$

We may then apply Theorem 1.6.12 to conclude that $w \equiv 0$. Note that (since we are in case B) $q > \dfrac{N}{N-2}$.□

Proof of (3.9.7). We have $|a| \le p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0,T), L^{\frac{N}{2}}(\Omega))$. We now establish (3.9.7) by contradiction. Otherwise, there exist $\varepsilon > 0$, $t \in [0,T]$ and a sequence $(t_n)_{n \ge 0} \in [0,T]$ such that $t_n \to t$ and $\|a(t_n, \cdot) - a(t, \cdot)\|_{L^{\frac{N}{2}}} \ge \varepsilon$. On the other hand, by possibly extracting a subsequence, we may assume that $u(t_n) \to u(t)$ and $v(t_n) \to v(t)$ in $L^q(\Omega)$ and almost everywhere, and that there exists $\varphi \in L^q(\Omega)$ such that $|u(t_n)| + |v(t_n)| \le \varphi$ almost everywhere. It follows easily that $a(t_n) \to a(t)$ almost everywhere and that $|a(t_n)| \le C|\varphi|^{p-1} \in L^{\frac{N}{2}}(\Omega)$. By dominated convergence, we deduce $a(t_n) \to a(t)$ in $L^{\frac{N}{2}}(\Omega)$, which is absurd. □

Proof of the existence part in Theorem 3.9.1 when $q > \dfrac{N(p-1)}{2}$ and $q \ge 1$. We first establish the existence of a solution $u \in L^\infty((0,T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^{pq}(\Omega))$. We use the contraction mapping principle in a somewhat unusual space (this idea is due to F.B. Weissler [98]). Fix $M \ge \|u_0\|_{L^q}$ and let

$$
E = L^\infty((0,T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^{pq}(\Omega)),
$$

and

$$
K = K(T) = \{u \in E;\ \|u(t)\|_{L^q} \le M + 1 \text{ and } t^\alpha \|u(t)\|_{L^{pq}} \le M + 1 \text{ for } t \in (0,T)\},
$$

with $\alpha = \dfrac{N(p-1)}{2pq} < \dfrac{1}{p} < 1$. We equip K with the distance

$$d(u,v) = \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}},$$

so that (K,d) is a nonempty complete metric space. Given $u \in K$, we set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\,ds.$$

For $u \in K$, we have

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^q} &\le \|u_0\|_{L^q} + \int_0^t \|u(s)\|_{L^{pq}}^p\,ds \\
&\le \|u_0\|_{L^q} + \Bigl(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^{pq}}\Bigr)^p \int_0^t s^{-p\alpha}\,ds \\
&\le \|u_0\|_{L^q} + \frac{T^{1-p\alpha}}{1-p\alpha}(M+1)^p.
\end{aligned}$$

Next,

$$\begin{aligned}
t^\alpha \|\Phi(u)(t)\|_{L^{pq}} &\le \|u_0\|_{L^q} + t^\alpha \int_0^t (t-s)^{-\alpha} \|u(s)\|_{L^{pq}}^p\,ds \\
&\le \|u_0\|_{L^q} + t^\alpha (M+1)^p \int_0^t (t-s)^{-\alpha} s^{-p\alpha}\,ds \\
&\le \|u_0\|_{L^q} + T^{1-p\alpha}(M+1)^p \int_0^1 (1-\sigma)^{-\alpha}\sigma^{-p\alpha}\,d\sigma.
\end{aligned}$$

Similarly, one shows that for $u, v \in K$,

$$t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^{pq}} \le CT^{1-p\alpha}(M+1)^{p-1} d(u,v).$$

It follows from the above estimates that if T is small enough (depending on M), then $\Phi : K \to K$ is a strict contraction. Thus Φ has a unique fixed point in K.

To complete the argument, it suffices to show that $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ (once $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$, it must be a classical solution on $(0,T) \times \overline{\Omega}$). Since $u \in K$ and $p\alpha < 1$, we have $|u|^{p-1}u \in L^1((0,T), L^q(\Omega))$. This implies that $u \in C([0,T], L^q(\Omega))$. (Recall that, in a general setting, if $f \in L^1((0,T), X)$ and $u(t) = \int_0^t T(t-s)f(s)\,ds$, then $u \in C([0,T], X)$.)

Next, we prove that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\text{loc}}((0,T), L^{pq}(\Omega))$. Therefore, we may apply Theorem 1.6.7 (with q replaced by pq and $\sigma = \dfrac{pq}{p-1}$) on every interval $(\varepsilon, T-\varepsilon)$, with $a = |u|^{p-1}$.

Note that the choice of T depends only on M. This establishes the last assertion in Theorem 3.9.1. □

For the proof of the existence part in Theorem 3.9.1 when $q = \dfrac{N(p-1)}{2}$ and $q > 1$, we will use the following lemma.

Lemma 3.9.7. *Given a compact set $\mathcal{K} \subset L^q(\Omega)$ and $q < r \le \infty$, there exists a function $\gamma : (0,1] \to (0,\infty)$ with*

$$\lim_{t\downarrow 0} \gamma(t) = 0,$$

such that

$$t^\alpha \|T(t)u_0\|_{L^r} \le \gamma(t),$$

for all $t \in (0,1)$ *and all* $u_0 \in \mathcal{K}$, *where* $\alpha = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$.

Proof. If $\mathcal{K}$ is reduced to a single point u_0, the result is clear. Indeed, for any $v_0 \in L^\infty(\Omega)$

$$\begin{aligned} t^\alpha \|T(t)u_0\|_{L^r} &\le t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} + t^\alpha \|T(t)v_0\|_{L^r} \\ &\le \|u_0 - v_0\|_{L^q} + Ct^\alpha \|v_0\|_{L^\infty}; \end{aligned}$$

and so

$$\limsup_{t\downarrow 0} t^\alpha \|T(t)u_0\|_{L^r} \le \|u_0 - v_0\|_{L^q}.$$

The assertion follows since v_0 is arbitrary.

In the general case, given any $\rho > 0$, there is a finite covering of $\mathcal{K}$ by balls $B(u_i, \rho)$ in $L^q(\Omega)$. Any $u_0 \in \mathcal{K}$ belongs to some $B(u_i, \rho)$, and we then write

$$\begin{aligned} t^\alpha \|T(t)u_0\|_{L^r} &\le t^\alpha \|T(t)(u_0 - u_i)\|_{L^r} + t^\alpha \|T(t)u_i\|_{L^r} \\ &\le \|u_0 - u_i\|_{L^q} + t^\alpha \|T(t)u_i\|_{L^r} \\ &\le \rho + t^\alpha \|T(t)u_i\|_{L^r}. \end{aligned}$$

The conclusion of the lemma then follows from the first assertion. □

Proof of the existence part in Theorem 3.9.1 when $q = \frac{N(p-1)}{2}$ **and** $q > 1$**.** The strategy is the same as in the case $q = \frac{N(p-1)}{2}$, with some minor technical differences. Fix any $r \in (q, pq)$, $r \ge p$, and set

$$\widetilde{E} = L^\infty((0,T), L^q(\Omega)) \cap \{u \in L^\infty_{\text{loc}}((0,T), L^r(\Omega));\ t^\alpha u \in L^\infty((0,T), L^r(\Omega))\},$$

and

$$E = L^\infty((0,T), L^q(\Omega)) \cap \{u \in L^\infty_{\text{loc}}((0,T), L^r(\Omega));\ t^\alpha u \in C_0([0,T], L^r(\Omega))\},$$

with $\alpha = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right) < \frac{1}{p} < 1$ (since $r < pq$). Here C_0 means that we consider functions which vanish at $t = 0$. Fix $M \ge \|u_0\|_{L^q}$. Given $\delta > 0$ to be chosen later, let

$$\widetilde{K} = \widetilde{K}(T) = \{u \in \widetilde{E};\ \|u(t)\|_{L^q} \le M + 1 \text{ and } t^\alpha \|u(t)\|_{L^r} \le \delta \text{ for } t \in (0,T)\},$$

and

$$K = K(T) = \widetilde{K} \cap E.$$

We equip $\widetilde{K}$ with the distance

$$d(u,v) = \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^r},$$

so that $(\widetilde{K}, d)$ and (K, d) are nonempty complete metric spaces. Consider the same mapping Φ as in the proof of the existence part when $q = \frac{N(p-1)}{2}$ and $q > 1$. Let $a = \frac{N}{2}\left(\frac{p}{r} - \frac{1}{q}\right)$. For $u \in \widetilde{K}$, we have by using the smoothing effect $L^{\frac{r}{p}} \to L^q$ (note that $r < pq$, so that $r/p < q$),

$$\begin{aligned} \|\Phi(u)(t)\|_{L^q} &\le \|u_0\|_{L^q} + \int_0^t (t-s)^{-a} \|u(s)\|_{L^r}^p \, ds \\ &\le \|u_0\|_{L^q} + \left(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^r}\right)^p \int_0^t (t-s)^{-a} s^{-p\alpha} \, ds \\ &\le \|u_0\|_{L^q} + C_1 \delta^p, \end{aligned} \tag{3.9.8}$$

since $a+p\alpha = 1$. Here the constant C_1 (and the constants C_2, C_3 below) depends only on p, q, r, N. Therefore,

$$\|\Phi(u)(t)\|_{L^q} \leq M + 1,$$

provided

$$C_1 \delta^p \leq 1. \tag{3.9.9}$$

Next, using the smoothing effect $L^{\frac{r}{p}} \to L^r$, we have

$$\begin{aligned} t^\alpha \|\Phi(u)(t)\|_{L^r} &\leq \sup_{0<t<T} t^\alpha \|T(t)u_0\|_{L^r} + t^\alpha \int_0^t (t-s)^{-\frac{N(p-1)}{2r}} \|u(s)\|_{L^r}^p \, ds \\ &\leq \sup_{0<t<T} t^\alpha \|T(t)u_0\|_{L^r} + \Big(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^r} \Big)^p t^\alpha \int_0^t (t-s)^{-\frac{N(p-1)}{2r}} s^{-p\alpha} \, ds \\ &\leq \sup_{0<t<T} t^\alpha \|T(t)u_0\|_{L^r} + C_2 \delta^p, \end{aligned} \tag{3.9.10}$$

since $p\alpha + \dfrac{N(p-1)}{2r} = \alpha + 1$. Therefore,

$$\sup_{0<t<T} t^\alpha \|\Phi(u)(t)\|_{L^r} \leq \sup_{0<t<T} t^\alpha \|T(t)u_0\|_{L^r} + \frac{\delta}{2}, \tag{3.9.11}$$

provided

$$C_2 \delta^{p-1} \leq \frac{1}{2}. \tag{3.9.12}$$

Similarly, one shows that for $u, v \in \widetilde{K}$,

$$\sup_{0<t<T} t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^r} \leq C_3 \delta^{p-1} d(u,v) \leq \frac{1}{2} d(u,v), \tag{3.9.13}$$

provided

$$C_3 \delta^{p-1} \leq \frac{1}{2}, \tag{3.9.14}$$

for some constant C_3. It follows from the above estimates that $\Phi : \widetilde{K} \to \widetilde{E}$.

We fix any $\delta > 0$ small enough so that (3.9.9), (3.9.12) and (3.9.14) are satisfied. The choice of δ depends only on N, p, q, r.

Next, we fix $T > 0$ such that

$$\sup_{0<t<T} t^\alpha \|T(t)u_0\|_{L^r} \leq \frac{\delta}{2}. \tag{3.9.15}$$

In view of Lemma 3.9.7, the choice of T depends only on the compact set $\mathcal{K} \subset L^q(\Omega)$. This establishes the last assertion in Theorem 3.9.1.

By (3.9.13),(3.9.11) and (3.9.15), $\Phi : \widetilde{K} \to \widetilde{K}$ is a strict contraction, and thus has a unique fixed point in $\widetilde{K}$.

Next, we claim that this fixed point belongs to K. For this purpose, it suffices to verify that $\Phi : K \to K$. We have to check that $\Phi(u) \in C((0,T], L^r(\Omega))$ and that $\lim\limits_{t \downarrow 0} t^\alpha \Phi(u)(t) = 0$ in $L^r(\Omega)$. Since by Lemma 3.9.7 $T(t)u_0$ satisfies the above requirements, we may always assume that $u_0 = 0$. It is clear that $\Phi(u) \in K$ when $u \in C([0,T], L^\infty(\Omega))$. Since $K \cap C([0,T], L^\infty(\Omega))$ is dense in K equipped with the metric d, the result follows from (3.9.13).

We now show that $u \in L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\rm loc}((0,T), L^r(\Omega))$. Therefore, we can apply Theorem 1.6.7 (with $\sigma = \dfrac{r}{p-1}$) on every interval $(\varepsilon, T-\varepsilon)$, with $a = |u|^{p-1}$. Indeed, $r \ge p > p-1$, so that $\sigma > 1$; $r > q = \dfrac{N(p-1)}{2}$, so that $\sigma > \dfrac{N}{2}$; and $r \ge p$, so that $r \ge \sigma'$.

Finally, we show that $u \in C([0,T], L^q(\Omega))$. Indeed, we have $u \in K$, so that in particular $u \in C((0,T], L^r(\Omega)) \subset C((0,T], L^q(\Omega))$. Therefore, it remains to show that $u(t) - T(t)u_0 \underset{t\downarrow 0}{\longrightarrow} 0$ in $L^q(\Omega)$. As in (3.9.8) we have

$$\|u(t) - T(t)u_0\|_{L^q} \le C_1 \sup_{0<s<t} (s^\alpha \|u(s)\|_{L^r})^p \underset{t\downarrow 0}{\longrightarrow} 0,$$

since $u \in E$. □

Proof of the uniqueness in Theorem 3.9.1. For every $u_0 \in L^q(\Omega)$, we denote by $U(t)u_0$ **the** solution constructed via the above contraction argument on some interval $[0, T(u_0)]$. We shall need the following lemma.

Lemma 3.9.8. *Let $u_0 \in L^\infty(\Omega)$ and consider the classical solution $\widetilde{u}$ of (3.9.1) defined on the maximal interval $[0, T_{\rm m}(u_0))$. Then $T(u_0) < T_{\rm m}(u_0)$ and $\widetilde{u}(t) = U(t)u_0$ for all $t \in [0, T(u_0)]$.*

Proof. It is clear that $\widetilde{u} \in K(\tau)$ for some $0 < \tau \le T(u_0)$ sufficiently small. By uniqueness in $K(\tau)$ we have

$$\widetilde{u}(t) = U(t)u_0, \quad \text{for} \quad 0 \le t \le \tau.$$

After time τ, both $\widetilde{u}(t)$ and $U(t)u_0$ are classical solutions. Hence the result. □

End of the proof of the uniqueness in Theorem 3.9.1. Here we use the same idea as in [18]. We give the proof only in the critical case $q = \dfrac{N(p-1)}{2}$ and $q > 1$; the other case is simpler. Let $v \in C([0,T], L^q(\Omega)) \cap L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$ be a solution of (3.9.1) with $v(0) = u_0$. Recall that v is a classical solution of (3.9.1) on $(0,T) \times \overline{\Omega}$. We are going to prove that $v(t) = U(t)u_0$ on some interval $[0, T')$. Then, $v(t) = U(t)u_0$ as long as both exist, by standard uniqueness in $L^\infty(\Omega)$.

Set

$$\mathcal{K} = v([0,T]),$$

and

$$M = \sup_{0 \le t \le T} \|v(t)\|_{L^q}.$$

Since $\mathcal{K}$ is a compact set in $L^q(\Omega)$, there is a uniform $T_1 > 0$ such that $U(t)v_0$ is well defined for all $v_0 \in \mathcal{K}$ and all $t \in [0, T_1]$. Moreover, since $U(t)v(s) \in K(T_1)$ (considered as a function of t), we have

$$\begin{aligned} &\|U(t)v(s)\|_{L^q} \le M + 1, \\ &t^\alpha \|U(t)v(s)\|_{L^r} \le \delta, \end{aligned} \tag{3.9.16}$$

for all $s \in (0,T)$ and all $t \in (0, T_1)$.

Fix any $0 < s < T$. It follows from Lemma 3.9.8 that

$$v(t+s) = U(t)v(s) \quad \text{for} \quad 0 \le t \le \min\{T-s, T_1\}. \tag{3.9.17}$$

Combining (3.9.16) and (3.9.17) we obtain

$$\|v(t+s)\|_{L^q} \le M+1,$$
$$t^\alpha \|v(t+s)\|_{L^r} \le \delta,$$

for $t+s<T$ and $t<T_1$. Passing to the limit as $s \downarrow 0$, we deduce that

$$\|v(t)\|_{L^q} \le M+1,$$
$$t^\alpha \|v(t)\|_{L^r} \le \delta,$$

for $0 < t < \min\{T, T_1\}$. Therefore, $v(t) \in \widetilde{K}(T')$ where $T' = \min\{T, T_1\}$. We may now argue as in Remark 3.9.6 to assert that

$$v(t) = T(t)u_0 + \int_0^t T(t-s)|v(s)|^{p-1}v(s)\,ds, \tag{3.9.18}$$

i.e. $v = \Phi(v)$. By (3.9.13) we deduce $v(t) = U(t)u_0$ on $[0, T']$. □

Proof of (i), (ii) and (iii) in Theorem 3.9.1 (Smoothing effect and stability). To prove (3.9.2) we consider three cases. The methods are essentially the same in all three cases with some minor technical changes.

Case a: $q > \dfrac{N(p-1)}{2}$, $q \ge p-1$ and $q \ge 1$;
Case b: $q > \dfrac{N(p-1)}{2}$ and $1 \le q < p-1$;
Case c: $q = \dfrac{N(p-1)}{2}$ and $q > 1$;

Case a: $q > \dfrac{N(p-1)}{2}$ **and** $q \ge p-1$**.** We apply Theorem 1.6.7 with a given by (3.9.6). We have $|a| \le p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0,T), L^\sigma(\Omega))$ with $\sigma = \dfrac{q}{p-1} > \dfrac{N}{2}$, $\sigma \ge 1$. By (1.6.7) we have

$$\|u(t) - v(t)\|_{L^\infty} \le Ce^{Ct\|a\|^\alpha_{L^\infty((0,T),L^\sigma)}}(t^{-\frac{N}{2q}} + 1)\|u_0 - v_0\|_{L^q},$$

with $\alpha = \dfrac{N(p-1)}{2pq}$. By construction, $u, v \in K$ where M is chosen such that $M \ge \|u_0\|_{L^q}$ and $M \ge \|u_0\|_{L^q}$; and thus, the L^∞ estimate of (3.9.2) follows.

On the other hand, we have.

$$\|u(t) - v(t)\|_{L^q} \le \|u_0 - v_0\|_{L^q} + C\int_0^t (\|u\|^{p-1}_{L^{pq}} + \|v\|^{p-1}_{L^{pq}})\|u - v\|_{L^{pq}}.$$

Since $u, v \in K$, we have

$$\|u(s)\|^{p-1}_{L^{pq}} + \|v(s)\|^{p-1}_{L^{pq}} \le \frac{C}{s^{\alpha(p-1)}}(M+1)^{p-1}.$$

Therefore,

$$\sup_{0<t<T} \|u(t) - v(t)\|_{L^q} \le \|u_0 - v_0\|_{L^q} + C(M+1)^{p-1} \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}}. \tag{3.9.19}$$

Here, we use the fact that $\alpha p < 1$. Furthermore, by using the $L^q(\Omega) \to L^{pq}(\Omega)$ smoothing effect, we have

$$\|u(t) - v(t)\|_{L^{pq}} \le t^{-\alpha}\|u_0 - v_0\|_{L^q} + CA\int_0^t (t-s)^{-\alpha}s^{-\alpha(p-1)}\|u - v\|_{L^{pq}},$$

where $A = \sup\limits_{0<s<T} s^{\alpha(p-1)}(\|u(s)\|_{L^{pq}}^{p-1} + \|v(s)\|_{L^{pq}}^{p-1})$. From the singular Gronwall lemma (Proposition A.5.7), we deduce

$$\sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}} \le C\|u_0 - v_0\|_{L^q}. \tag{3.9.20}$$

Combining (3.9.19) and (3.9.20) we obtain the L^q estimate of (3.9.2).

Case b. Here, we cannot apply Theorem 1.6.7 since $\sigma = \dfrac{q}{p-1} < 1$. However, the second part of the proof in unchanged and we deduce as above that

$$\|u(t) - v(t)\|_{L^q} + t^\alpha \|u(t) - v(t)\|_{L^{pq}} \le C\|u_0 - v_0\|_{L^q},$$

for all $t \in [0,T]$. This establishes the L^q estimate of (3.9.2).

We now turn to the proof of the L^∞ estimate of (3.9.2). First note that

$$\|u(t/2) - v(t/2)\|_{L^{pq}} \le Ct^{-\alpha}\|u_0 - v_0\|_{L^q}.$$

We now apply Theorem 1.6.7 on the interval $(t/2,t)$ with q replaced by $= pq$ and $\sigma = \dfrac{pq}{p-1} > 1$ and with a given by (3.9.6). Since $|a| \le p(|u|^{p-1} + |v|^{p-1})$, we have that $a \in L^\infty((t/2,t), L^\sigma(\Omega))$; and $\|a\|_{L^\infty((t/2,t),L^\sigma)} \le C(M+1)^{p-1} t^{-\alpha(p-1)}$. It follows that

$$\|u(t) - v(t)\|_{L^\infty} \le C\exp\Bigl(Ct(M+1)^{\frac{2\sigma(p-1)}{2\sigma-N}} (t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma-N}}\Bigr)(1 + t^{-\frac{N}{2pq}})\|u(t/2) - v(t/2)\|_{L^{pq}};$$

and so,

$$\|u(t) - v(t)\|_{L^\infty} \le C\exp\Bigl(C(M+1)^{\frac{2\sigma(p-1)}{2\sigma-N}} t^\gamma\Bigr)(1 + t^{-\frac{N}{2pq}})\|u(t/2) - v(t/2)\|_{L^{pq}},$$

with $\gamma = 1 - \dfrac{2\sigma\alpha(p-1)}{2\sigma - N}$. Since $\gamma > 0$, we obtain

$$\|u(t) - v(t)\|_{L^\infty} \le Ct^{-\frac{N}{2pq}}\|u(t/2) - v(t/2)\|_{L^{pq}}.$$

Therefore,

$$\|u(t) - v(t)\|_{L^\infty} \le Ct^{-\frac{N}{2pq}} t^{-\alpha}\|u_0 - v_0\|_{L^q} = Ct^{-\frac{N}{2q}}\|u_0 - v_0\|_{L^q},$$

which is the L^∞ estimate of (3.9.2).

Case c: $q = \dfrac{N(p-1)}{2}$ and $q > 1$. Since $u - v = T(t)(u_0 - v_0) + \Phi(u) - \Phi(v)$, it follows from (3.9.13) that

$$\sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^r} \le \sup_{0<t<T} t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} + \frac{1}{2}\sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^r},$$

with $\alpha = \dfrac{N(r-q)}{2qr}$; and so,

$$\sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^r} \le 2\sup_{0<t<T} t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} \le 2\|u_0 - v_0\|_{L^q}. \tag{3.9.21}$$

Furthermore (as in (3.9.8)) we have,

$$\begin{aligned}
\|u(t) - v(t)\|_{L^q} &\le \|u_0 - v_0\|_{L^q} + C\int_0^t (t-s)^{-a}(\|u(s)\|_{L^r}^{p-1} + \|v(s)\|_{L^r}^{p-1})\|u(s) - v(s)\|_{L^r}\, ds \\
&\|u_0 - v_0\|_{L^q} + C\delta^{p-1}\sup_{0<s<t} t^\alpha \|u(s) - v(s)\|_{L^r}\int_0^t (t-s)^{-a} s^{-\alpha p}\, ds \\
&\le C\|u_0 - v_0\|_{L^q},
\end{aligned}$$

by (3.9.21). This establishes the L^q estimate of (3.9.2).

To prove the L^∞ estimate, we apply Theorem 1.6.7 on the interval $(t/2,t)$ with $\sigma = \frac{r}{p-1} > \frac{N}{2}$, $\sigma > 1$, and with a given by (3.9.6). We have $|a| \le p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((t/2,t), L^\sigma(\Omega))$; and since $u, v \in K$, $\|a\|_{L^\infty((t/2,t),L^\sigma)} \le Ct^{-\alpha(p-1)}$. It follows that

$$\|u(t) - v(t)\|_{L^\infty} \le C \exp\Big(Ct(t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma - N}}\Big)(1 + t^{-\frac{N}{2r}})\|u(t/2) - v(t/2)\|_{L^r}.$$

But $1 - \dfrac{2\sigma\alpha(p-1)}{2\sigma - N} = 0$; and so,

$$\|u(t) - v(t)\|_{L^\infty} \le C(1 + t^{-\frac{N}{2r}})\|u(t/2) - v(t/2)\|_{L^r}. \tag{3.9.22}$$

Combining (3.9.21) and (3.9.22), we obtain

$$\|u(t) - v(t)\|_{L^\infty} \le Ct^{-\frac{N}{2r}}t^{-\alpha}\|u_0 - v_0\|_{L^q} = Ct^{-\frac{N}{2q}}\|u_0 - v_0\|_{L^q},$$

which is the desired estimate. In fact in this case the constant C in (3.9.2) is independent of $\|u_0\|_{L^q}$ and $\|u_0\|_{L^q}$.

Finally, (ii) and (iii) are clearly true when $u_0 \in L^\infty(\Omega)$, and the general case follows by continuous dependence (3.9.2). □

Remark 3.9.9. Even when $q < p$, the uniqueness property in Theorem 3.9.1 holds in a class larger than $C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. More precisely, uniqueness holds in the class $C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^p(\Omega))$. Indeed, note first that the equation makes sense in that class (since then $|u|^{p-1}u \in L^\infty_{\text{loc}}((0,T), L^1(\Omega)))$. Furthermore, if $u \in L^\infty_{\text{loc}}((0,T), L^p(\Omega))$, then $|u|^{p-1} \in L^\infty_{\text{loc}}((0,T), L^{\frac{p}{p-1}}(\Omega))$. We have $p > q = \frac{N(p-1)}{2}$, so that $\frac{p}{p-1} > \frac{N}{2}$. Therefore, it follows from Theorem 1.6.7 that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$.

Remark 3.9.10. The above results hold for more general nonlinearities with similar proofs. More precisely, one can replace $|u|^{p-1}u$ by $g(u)$ where $g : \mathbb{R} \to \mathbb{R}$ verifies $|g(x) - g(y)| \le C(1 + |x|^{p-1} + |y|^{p-1})|x - y|$.

Remark 3.9.11 (The "doubly critical" case $q = \frac{N(p-1)}{2}$ and $q = p$ in Theorem 3.9.4). As we have already mentioned in Remark 3.9.5, if $q = \frac{N(p-1)}{2}$ and $q = p$, i.e. $q = p = \frac{N}{N-2}$ $(N \ge 3)$, then the conclusion of Theorem 3.9.4 fails, i.e. uniqueness fails in the class $C([0,T], L^q(\Omega))$. This is a result of Ni and Sacks [82], and we sketch their argument. First, a simple lemma.

Lemma 3.9.12. *Let $\varphi, f \in L^1(\Omega)$ satisfy the equation*

$$\begin{cases} -\triangle\varphi = f & \text{in} \quad \Omega, \\ \varphi = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

in the sense that

$$-\int_\Omega \varphi\triangle\zeta = \int_\Omega f\zeta, \tag{3.9.23}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Then

$$\varphi = T(t)\varphi + \int_0^t T(s) f \, ds, \tag{3.9.24}$$

for all $t \geq 0$.

Proof. The conclusion is trivial if φ is smooth. In the general case, let $(f_n)_{n\geq 0} \subset \mathcal{D}(\Omega)$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^1(\Omega)$ and let φ_n be the corresponding solution of (3.9.23). Then $\varphi_n \to \varphi$ in $L^1(\Omega)$ (see e.g. Lemma 3.7.9 above) and one passes to the limit in (3.9.24). □

In the case $\Omega =$ the unit ball of $\mathbb{R}^N$, Ni and Sacks [82] have constructed a radial function $\psi \in C^2(\overline{\Omega}\backslash\{0\})$, $\psi > 0$ in Ω, $\psi = 0$ on $\partial\overline{\Omega}$, $\psi \in L^p(\Omega)$, $\lim\limits_{x\to 0} \psi(x) = +\infty$, satisfying the equation

$$\begin{cases} -\triangle\psi = \psi^p & \text{in} \quad \Omega \quad \text{with} \quad p = \dfrac{N}{N-2}, \\ \quad \psi = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

in the sense that

$$-\int_\Omega \psi \triangle \zeta = \int_\Omega \psi^p \zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. In view of Lemma 3.9.12, $v(t) \equiv \psi$ is a solution of (3.9.18) in $C([0,\infty), L^p(\Omega))$. On the other hand, the solution u of (3.9.1) (with initial condition ψ) given by Theorem 3.9.1 has a smoothing effect. Hence, the two solutions are distinct. □

Remark 3.9.13 (the "doubly critical" case $q = \dfrac{N(p-1)}{2}$ and $q = 1$ in Theorem 3.9.1). If $q = \dfrac{N(p-1)}{2}$ and $q = 1$, i.e. $p = \dfrac{N+2}{N}$ $(N \geq 1)$ Theorem 3.9.1 does not apply and we suspect that the conclusions fail. (This concerns for example the simple case $N = 1$, $p = 3$, $q = 1$.) See Open Problems 3.14.7—3.14.10. Here is some evidence suggesting that the answers to these open problems might be positive. (See also Exercise 3.13.20.)

Theorem 3.9.14. *Assume again $p = \dfrac{N+2}{N}$, $q = 1$. There is some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that for every $T > 0$ problem (3.9.1) has no nonnegative solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$.*

Here $N \geq 1$ and Ω can be arbitrary.

Proof. Fix any open ball $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$. Let $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ be such that $v(t) = T(t)u_0$ satisfies

$$\int_0^1 \int_\omega v^p(t,x) \, dx \, dt = +\infty. \tag{3.9.25}$$

(See Exercise 3.13.21. Note that $v \geq 0$ by the maximum principle.)

Assume by contradiction that for some $T > 0$ there is a nonnegative solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ of (3.9.1). We have

$$u(t+s) \geq T(t)u(s),$$

for all $t \geq 0$, $s > 0$, $t + s < T$. As $s \downarrow 0$ we find

$$u(t) \geq T(t)u_0 = v(t), \tag{3.9.26}$$

for all $t \in [0, T]$. Since u is a classical solution of (3.9.1) for $t \in (0, T)$, we may multiply (3.9.1) by $\zeta \in \mathcal{D}(\Omega)$, $\zeta \geq 0$ on Ω, $\zeta \geq 1$ on ω and we obtain

$$\frac{d}{dt}\int_\Omega u\zeta + \int_\Omega u(-\triangle\zeta) = \int_\Omega u^p\zeta \geq \int_\omega u^p.$$

Integrating on (ε, T) and letting $\varepsilon \downarrow 0$ (since $u \in C([0, T], L^1(\Omega))$), we deduce that

$$\int_0^T \int_\omega u^p < \infty,$$

which contradicts (3.9.25) and (3.9.26). $\square$

Remark 3.9.15. Baras [6] has given examples showing that uniqueness for problem (3.9.1) fails in the class $C([0, T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0, T), L^\infty(\Omega))$ for $1 \leq q < \dfrac{N(p-1)}{2}$. Here, the initial condition can be any smooth function u_0, for example $u_0 = 0$. Such a phenomenon had been observed earlier by Haraux and Weissler [59] when $\Omega = \mathbb{R}^N$.

Remark 3.9.16. Let $u_0 \in L^\infty(\Omega)$, let u be the corresponding solution of (3.9.1) and assume that $T_{\mathrm{m}} < \infty$. Suppose in addition that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and that $\triangle u_0 + |u_0|^{p-1}u_0 \geq 0$ a.e. in Ω. If $\dfrac{N(p-1)}{2} > 1$, then $\lim\limits_{t\uparrow T_{\mathrm{m}}} \|u(t)\|_{L^{\frac{N(p-1)}{2}}} = +\infty$. Indeed, suppose by contradiction that $\liminf\limits_{t\uparrow T_{\mathrm{m}}} \|u(t)\|_{L^{\frac{N(p-1)}{2}}} < \infty$ and let $(t_n)_{n\geq 0}$ be a sequence such that $t_n \uparrow T_{\mathrm{m}}$ as $n \to \infty$ and $\sup\limits_{n\geq 0} \|u(t_n)\|_{L^{\frac{N(p-1)}{2}}} < \infty$. We claim that $u(t_n)$ is contained in a compact set of $L^{\frac{N(p-1)}{2}}(\Omega)$. This is indeed the case since, by the maximum principle, $u_t(t, x) \geq 0$ on $(0, T_{\mathrm{m}}) \times \Omega$; and thus $(u(t_n))_{n\geq 0}$ is a nonincreasing sequence and has a limit in $L^{\frac{N(p-1)}{2}}(\Omega)$. Applying Theorem 3.9.1 with $u(t_n)$ as initial condition, we obtain a uniform $T > 0$. Thus $T_{\mathrm{m}} \geq t_n + T$. This is impossible as $n \to \infty$. The same conclusion holds for more general nonlinearities (see Remark 3.9.10). Note that in the case $N \geq 3$, the result follows from Theorem 3.8.3.

3.10. Initial conditions in $L^1(\Omega)$ or measures. In this section, we consider the two problems

$$\begin{cases} u_t - \triangle u = |u|^{p-1}u & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.10.1}$$

and

$$\begin{cases} u_t - \triangle u + |u|^{p-1}u = 0 & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.10.2}$$

and we shall concentrate on the case where u_0 is either in $L^1(\Omega)$ or a measure. We first recall a special case of Theorem 3.9.1.

Theorem 3.10.1. *Assume*

$$p < \frac{N+2}{N}. \tag{3.10.3}$$

Then, given any $u_0 \in L^1(\Omega)$*, there exist a time* $T = T(u_0) > 0$ *and a unique function* $u \in C([0,T], L^1(\Omega))$ *with* $u(0) = u_0$*, which is a classical solution of (3.10.1) on* $(0,T) \times \overline{\Omega}$*.*

This result can be extended to the case where the initial condition is a measure, $u_0 \in M(\Omega)$,

$$M(\Omega) = C_0(\Omega)^*,$$

and $C_0(\Omega)$ denotes the space of continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$.

Theorem 3.10.2. *Assume (3.10.3). Then, given any* $u_0 \in M(\Omega)$*, there exist a time* $T = T(u_0) > 0$ *and a unique function* u *which is a classical solution of (3.10.1) on* $(0,T) \times \overline{\Omega}$ *and which satisfies the initial condition* $u(0) = u_0$ *in the sense*

$$\lim_{t \downarrow 0} \int_\Omega u(t,x)\varphi(x)\, dx = \int_\Omega u_0 \varphi, \tag{3.10.4}$$

for every $\varphi \in C_0(\Omega)$*.*

Moreover,

$$\int_0^T \int_\Omega |u(t,x)|^p \, dx \, dt < \infty, \tag{3.10.5}$$

and

$$u(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\, ds, \tag{3.10.6}$$

for all $t \in (0,T)$*. In particular,* $u(t) - T(t)u_0 \in C([0,T], L^1(\Omega))$*.*

Proof. The proof is almost the same as the proof of Theorem 3.9.1. We first establish the existence of a solution $u \in L^\infty((0,T), L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^p(\Omega))$. Fix $M \geq \|u_0\|_{M(\Omega)}$ and let

$$E = L^\infty((0,T), L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^p(\Omega)),$$

and

$$K = K(T) = \{u \in E;\ \|u(t)\|_{L^1} \leq M+1 \text{ and } t^\alpha \|u(t)\|_{L^p} \leq M+1 \text{ for } t \in (0,T)\},$$

with $\alpha = \dfrac{N(p-1)}{2p} < \dfrac{1}{p} < 1$. We equip K with the distance

$$d(u,v) = \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^p},$$

so that (K,d) is a nonempty complete metric space. Given $u \in K$, we set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\, ds.$$

For $u \in K$, we have by (1.4.13)

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^1} &\leq \|u_0\|_{M(\Omega)} + \int_0^t \|u(s)\|_{L^p}^p \, ds \\
&\leq \|u_0\|_{M(\Omega)} + \Big(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^p}\Big)^p \int_0^t s^{-p\alpha}\, ds \\
&\leq \|u_0\|_{M(\Omega)} + \frac{T^{1-p\alpha}}{1-p\alpha}(M+1)^p.
\end{aligned}$$

Next, using again (1.4.13)

$$
\begin{aligned}
t^\alpha \|\Phi(u)(t)\|_{L^p} &\le \|u_0\|_{M(\Omega)} + t^\alpha \int_0^t (t-s)^{-\alpha} \|u(s)\|_{L^p}^p \, ds \\
&\le \|u_0\|_{M(\Omega)} + t^\alpha (M+1)^p \int_0^t (t-s)^{-\alpha} s^{-p\alpha} \, ds \\
&\le \|u_0\|_{M(\Omega)} + T^{1-p\alpha} (M+1)^p \int_0^1 (1-\sigma)^{-\alpha} \sigma^{-p\alpha} \, d\sigma.
\end{aligned}
$$

Similarly, one shows that for $u, v \in K$,

$$
t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^p} \le C T^{1-p\alpha} (M+1)^{p-1} d(u,v).
$$

It follows from the above estimates that if T is small enough (depending on M), then $\Phi : K \to K$ is a strict contraction. Thus Φ has a unique fixed point in K, which is a solution of (3.10.6).

Next, we prove that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\text{loc}}((0,T), L^p(\Omega))$. Therefore, we may apply Theorem 1.6.7 (with q replaced by p and $\sigma = \dfrac{p}{p-1}$) on every interval $(\varepsilon, T-\varepsilon)$, with $a = |u|^{p-1}$. Since $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$, it must be a classical solution on $(0,T) \times \overline{\Omega}$; and since $u \in K$ and $p\alpha < 1$, we have $|u|^{p-1}u \in L^1((0,T) \times \Omega)$. This implies that u verifies (3.10.5). This also implies (by (3.10.6)) that $u(t) - T(t)u_0 \in C([0,T], L^1(\Omega))$ and using (1.4.15), we deduce (3.10.4).

Note that the choice of T depends only on M.

We now establish uniqueness. For every $u_0 \in M(\Omega)$, we denote by $U(t)u_0$ **the** solution constructed via the above contraction argument on some interval $[0, T(u_0)]$. Let v be any solution as in the statement of the theorem. We are going to prove that $v(t) = U(t)u_0$ on some interval $[0, T')$. Then, $v(t) = U(t)u_0$ as long as both exist, by standard uniqueness in $L^\infty(\Omega)$. We observe that (3.10.4) means that $v(t) \to u_0$ in $M(\Omega)$ weak-$\star$. This implies (since v is smooth for $t > 0$) that

$$
M = \sup_{0 \le t \le T} \|v(t)\|_{M(\Omega)} < \infty.
$$

Set

$$
\mathcal{K} = v([0,T]).
$$

Since $M < \infty$,there is a uniform $T_1 > 0$ such that $U(t)v_0$ is well defined for all $v_0 \in \mathcal{K}$ and all $t \in [0, T_1]$. Moreover, since $U(t)v(s) \in K(T_1)$ (considered as a function of t), we have

$$
\begin{aligned}
&\|U(t)v(s)\|_{L^1} \le M+1, \\
&t^\alpha \|U(t)v(s)\|_{L^p} \le M+1,
\end{aligned}
\tag{3.10.7}
$$

for all $s \in (0,T)$ and all $t \in (0, T_1)$.

Fix any $0 < s < T$. It follows from Lemma 3.9.8 that

$$
v(t+s) = U(t)v(s) \quad \text{for} \quad 0 \le t \le \min\{T-s, T_1\}. \tag{3.10.8}
$$

Combining (3.10.7) and (3.10.8) we obtain

$$
\begin{aligned}
&\|v(t+s)\|_{L^1} \le M+1, \\
&t^\alpha \|v(t+s)\|_{L^p} \le M+1,
\end{aligned}
$$

for $t+s<T$ and $t<T_1$. Passing to the limit as $s\downarrow 0$, we deduce that

$$
\begin{aligned}
&\|v(t)\|_{L^1}\le M+1,\\
&t^\alpha\|v(t)\|_{L^p}\le M+1,
\end{aligned}
\tag{3.10.9}
$$

for $0<t<\min\{T,T_1\}$. Therefore, $v(t)\in K(T')$ where $T'=\min\{T,T_1\}$. Given $\varepsilon\in(0,T')$, we have

$$
v(t+\varepsilon)=T(t)v(\varepsilon)-\int_0^t T(t-s)|v(\varepsilon+s)|^{p-1}v(\varepsilon+s)\,ds,
\tag{3.10.10}
$$

for all $0\le t\le T'-\varepsilon$. We now let $\varepsilon\downarrow 0$. It follows from (3.10.9) that the integral on the right-hand side of (3.10.10) converges to

$$
\int_0^t T(t-s)|v(s)|^{p-1}v(s)\,ds,
$$

and it follows from Theorem 1.4.25 (v) (since $v(\varepsilon)\underset{\varepsilon\downarrow 0}{\longrightarrow}u_0$ in $M(\Omega)$ weak-$\star$) that $T(t)v(\varepsilon)\underset{\varepsilon\downarrow 0}{\longrightarrow}T(t)u_0$ in $C_0(\Omega)$; and so,

$$
v(t)=T(t)u_0-\int_0^t T(t-s)|v(s)|^{p-1}v(s)\,ds,
$$

which implies that $v(t)=U(t)u_0$ on $[0,T']$. □

In the "good case" (3.10.2), we have a similar result with the additional property that the solution is now global.

Theorem 3.10.3. *Assume (3.10.3). Then, given any $u_0\in M(\Omega)$, there exists a unique function u which is a classical solution of (3.10.2) on $(0,\infty)\times\overline{\Omega}$ and which satisfies the initial condition $u(0)=u_0$ in the sense (3.10.4).*

Moreover, u satisfies (3.10.5) and

$$
u(t)=T(t)u_0-\int_0^t T(t-s)|u(s)|^{p-1}u(s)\,ds,
\tag{3.10.11}
$$

for all $t>0$. In particular, $u(t)-T(t)u_0\in C([0,\infty),L^1(\Omega))$.

Local existence and uniqueness follows from the argument of the proof of Theorem 3.10.2. The solution being classical at $t>0$ is global by Theorem 3.3.1.

Condition (3.10.3) is essential in Theorem 3.10.3. We suspect that it is also essential in the "bad case" Theorem 3.10.2 (see open Problem 3.14.14). More precisely, if $u_0=\delta_0$ the Dirac mass at 0 and $0\in\Omega$, there is no solution of (3.10.2) in the weakest possible sense:

Theorem 3.10.4. *Suppose*

$$
p\ge\frac{N+2}{N}.
$$

Given any $T>0$, there is no function $u\in L^p_{\mathrm{loc}}((0,T)\times\Omega$ satisfying (3.10.2) in $\mathcal{D}'((0,T)\times\Omega)$ and such that

$$
\operatorname{ess}\lim_{t\downarrow 0}\int_\Omega u(t,x)\varphi(x)\,dx=\varphi(0),
$$

for all $\varphi\in C_c(\Omega)$.

See Brezis and Friedman [21] for the proof of Theorem 3.10.4. The method uses a statement about removable singularities for the equation (3.10.2). See also Section 3.12.

In particular the equation $u_t - u_{xx} + u^3 = 0$ in $\Omega = (-1, 1)$ has no solution with $u_0 = \delta_0$.

Remark 3.10.5. Note that the nonexistence result is purely local. No boundary condition is prescribed.

Remark 3.10.6. Consider the problem (3.10.2) with $p \geq \dfrac{N+2}{N}$. Suppose $(u_{0,j})_{j\geq 0}$ is a sequence of smooth initial data which converges weak-$\star$ to δ_0. Let $(u_j)_{j\geq 0}$ is a sequence of corresponding solutions of (3.10.2). Note that we have good estimates on $(u_j)_{j\geq 0}$: $\|u_j\|_{L^\infty((0,\infty),L^1(\Omega))} \leq C$ and $\|u_j(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2}}$ (see the proof of Theorem 3.10.8). The reader may wonder what happens to the sequence $(u_j)_{j\geq 0}$, since the limiting problem has no solution. The answer is that $u_j \underset{j\to\infty}{\longrightarrow} 0$ uniformly on $[\varepsilon, \infty] \times \overline{\Omega}$ for any $\varepsilon > 0$. See Brezis and Friedman [21]. This shows that **the initial condition may be lost in the process of passing to the limit under weak convergence of the initial conditions**.

Remark 3.10.7. It is possible to solve the problem (3.10.2) for some values of $p \geq \dfrac{N+2}{N}$ and some measures u_0 less singular than δ_0 (for example, a distribution of charges on a surface). Baras and Pierre [8] have described precise conditions on u_0.

We conclude this section with an existence result for (3.10.2) where $1 \leq p < \infty$ is arbitrary and $u_0 \in L^1(\Omega)$.

Theorem 3.10.8. *Given any $u_0 \in L^1(\Omega)$, there exists a unique function $u \in C([0,\infty), L^1(\Omega))$ with $u(0) = u_0$, which is a classical solution of (3.10.2) on $(0,\infty) \times \overline{\Omega}$.*

Moreover, u satisfies (3.10.5) and we continuous dependence and have smoothing effect, namely

$$\|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}, \tag{3.10.12}$$

and

$$t^{\frac{N}{2}} \|u(t) - v(t)\|_{L^\infty} \leq \|u_0 - v_0\|_{L^1}, \tag{3.10.13}$$

for all $t \geq 0$.

Proof. Let $(u_{0,j})_{j\geq 0} \subset \mathcal{D}(\Omega)$ with $u_{0,j} \underset{j\to\infty}{\longrightarrow} u_0$ in $L^1(\Omega)$, and let u_j be the corresponding global, smooth solutions of (3.10.2). We claim that

$$\|u_j(t) - u_k(t)\|_{L^1} + \int_0^t \int_\Omega \left| |u_j|^{p-1} u_j - |u_k|^{p-1} u_k \right| \, dx \, dt \leq \|u_{0,j} - u_{0,k}\|_{L^1}, \tag{3.10.14}$$

for all $t \geq 0$. To establish (3.10.14), we multiply the equation

$$(u_j - u_k)_t - \triangle(u_j - u_k) + |u_j|^{p-1} u_j - |u_k|^{p-1} u_k = 0,$$

by $\theta_m(u_j - u_k)$ where θ_m is a smooth approximation of the signum function, and we let $m \to \infty$.

Thus, $(u_j)_{j\geq 0}$ is a Cauchy sequence in $C([0,\infty), L^1(\Omega))$ and $(|u_j|^{p-1}u_j)_{j>0}$ is a Cauchy sequence in $L^1((0,\infty) \times \Omega)$. Hence $u_j \underset{j\to\infty}{\longrightarrow} u$ in $C([0,\infty), L^1(\Omega))$ and $|u_j|^{p-1} u_j \underset{j\to\infty}{\longrightarrow} |u|^{p-1} u$ in $L^1((0,\infty) \times \Omega)$, for some

$u \in C([0,\infty), L^1(\Omega)) \cap L^p((0,\infty)\times\Omega)$. This implies that u satisfies (3.10.11). To obtain further estimates, we use Kato's inequality (Theorem A.5.20). Setting $\varphi_j = |u_j|$, we have

$$\frac{\partial \varphi_j}{\partial t} - \triangle \varphi_j \le \left(\frac{\partial u_j}{\partial t} - \triangle u_j\right) \mathrm{sign} u_j = -|u_j|^p \le 0.$$

By the maximum principle, $\varphi_j(t) \le T(t)|u_{0,j}| \le t^{-\frac{N}{2}} \|u_{0,j}\|_{L^1}$. This implies that $u(t) \in L^\infty(\Omega)$ for all $t > 0$ and thus is a classical solution.

We now prove uniqueness and the estimates (3.10.12) and (3.10.13). Let $u(t)$ and $v(t)$ be two solutions with initial values u_0 and v_0. Applying Kato's inequality and the maximum principle as above, we obtain the **pointwise** estimate

$$|u(t) - v(t)| \le T(t)|u_0 - v_0|,$$

in Ω for all $t > 0$. The result follows. □

Remark 3.10.9. The conclusion of Theorem 3.10.8 holds if we replace the nonlinearity $|u|^{p-1}u$ by any locally Lipschitz function $f(u)$ which is monotone nondecreasing in u.

3.11. Further results.

3.11.1. The necessary (and almost sufficient) condition of Baras and Pierre for the existence of a solution on $(0,T)$. Consider the problem

$$\begin{cases} u_t - \triangle u = u^p \quad \text{in} \quad (0,T)\times\Omega, \\ u = 0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ u(0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.11.1}$$

with $u_0 \in L^\infty(\Omega)$ and $u_0 \ge 0$. It has a solution on $(0, T_{\mathrm{m}}) \times \Omega$. Let us first derive a simple condition relating u_0 and T_{m}.

Fix $T < T_{\mathrm{m}}$, let $h \in C^\infty([0,T]\times\overline{\Omega})$, $h \ge 0$, $h \equiv 0$ on a neighborhood of $[0,T]\times\partial\Omega$, $h(T,\cdot) \not\equiv 0$, and let

$$\zeta(t) = \int_t^T T(s-t)h(s)\,ds,$$

for $0 \le t \le T$, so that ζ verifies

$$\begin{cases} -\zeta_t - \triangle\zeta = h \quad \text{in} \quad (0,T)\times\Omega, \\ \zeta = 0 \quad \text{in} \quad (0,T)\times\partial\Omega, \\ \zeta(T) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

and $\zeta > 0$ on $(0,T)\times\Omega$.

Multiplying the equation (3.11.1) by ζ and integrating on $(0,T)\times\Omega$, we obtain

$$\int_\Omega \zeta(0)u_0 = \int_0^T\int_\Omega uh - \int_0^T\int_\Omega u^p\zeta. \tag{3.11.2}$$

Recall Young's inequality $|ab| \le |a|^p + C_p|b|^{p'}$, with $C_p = (p-1)p^{-p'}$. We deduce from (3.11.2)

$$\int_\Omega \zeta(0)u_0 \le C_p \int_0^T\int_\Omega \left(\frac{h}{\zeta}\right)^{p'}\zeta. \tag{3.11.3}$$

Note that u has disappeared in inequlity (3.11.3). This is a necessary condition on u_0 for the existence of a solution up to time T. (In particular, this shows that if we fix T and u_0, the problem (3.11.1) with the initial condition λu_0 does not have a solution for λ sufficiently large.)

A remarkable fact is that condition (3.11.3) is almost a sufficient condition. More precisely:

Theorem 3.11.1. *Suppose $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, satisfies*

$$\int_\Omega \zeta(0) u_0 \le C \int_0^T \int_\Omega \left(\frac{h}{\zeta}\right)^{p'} \zeta. \tag{3.11.4}$$

for some constant $C < C_p$ and every h as above. Then (3.11.1) has a classical solution on $(0,T)\times\Omega$, i.e. $T_{\mathrm{m}}(u_0) \ge T$.

Sketch of the proof. There are two main ingredients.

Step 1. A remarkable result of Baras and Pierre [8] asserts that the condition (3.11.3) is also a sufficient condition for the existence of a **weak** solution of (3.11.1) on $(0,T)\times\Omega$. Here, a weak solution of (3.11.1) on $(0,T)\times\Omega$ is ameasurable function on $(0,T)\times\Omega$, $u \ge 0$, such that for all $S < T$, $u \in L^1((0,S)\times\Omega)$, $u^p\delta \in L^1((0,S)\times\Omega)$ (where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$), and

$$\int_0^T \int_\Omega u^p \xi = -\int_0^T \int_\Omega u(\xi_t + \triangle \xi) - \int_\Omega u_0 \xi(0), \tag{3.11.5}$$

for every $\xi \in C^2([0,T]\times\overline{\Omega})$ such that $\xi = 0$ on $(0,T)\times\partial\Omega$ and $\xi(t,x) = 0$ for all $x \in \Omega$ and all t near T.

At this stage it is not clear that this u is a classical solution. In fact, it is plausible that such a u need not be a classical solution on $(0,T)$. See Open Problem 3.14.16.

Step 2. Suppose now that u_0 satisfies (3.11.4). Then by Step 1 there is a weak solution of the equation

$$\begin{cases} u_t - \triangle u = \lambda u^p & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

for some $\lambda > 1$.

We claim that for any $\varepsilon \in (0,1)$ the problem

$$\begin{cases} v_t - \triangle v = (1-\varepsilon)\lambda v^p & \text{in} \quad (0,T)\times\Omega, \\ v = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ v(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{3.11.6}$$

has a **classical** solution on $(0,T)$.

To prove this, one uses the same kind of device as in [20]. Namely, fix $A \ge \|u_0\|_{L^\infty}$ and consider the function

$$\Phi(t) = \begin{cases} t & \text{for} \quad 0 \le t \le A, \\ \left(\dfrac{\varepsilon}{A^{p-1}} + \dfrac{1-\varepsilon}{t^{p-1}}\right)^{-\frac{1}{p-1}} & \text{for} \quad t \ge A. \end{cases}$$

This function is **bounded**, monotone increasing and **concave**. Moreover,

$$\Phi'(t) \ge (1-\varepsilon)\frac{\Phi(t)^p}{t^p}, \tag{3.11.7}$$

for all $t \geq 0$. Next, observe that $w = \Phi(u)$ is a supersolution of the problem (3.11.6). Indeed, by Kato's inequality (see Lemma 3.7.10 and Theorem A.5.20) w satisfies

$$w_t - \triangle w \geq \Phi'(u)(u_t - \triangle u) = \Phi'(u)\lambda u^p \geq (1-\varepsilon)\lambda w^p,$$

by (3.11.7). (Clearly, $w(0) = u_0$.) By the maximum principle, $v \leq w$, thus v is a classical solution on $(0,T)$ (recall that w is bounded).

Finally, we choose $1 - \varepsilon = \dfrac{1}{\lambda}$. □

3.11.2. Complete blow up after $T_{\rm m}$: is there a life after death? Let $p > 1$ and given $u_0 \in L^\infty(\Omega)$ consider the solution u of the equation (3.11.1) defined on the maximal interval $[0, T_{\rm m})$. Suppose that $T_{\rm m} < \infty$.

A natural question is the following: can one extend the solution u **after** the blow up time $T_{\rm m}$ as a "weak" solution on $[0,T)$, $T_{\rm m} < T \leq \infty$? For this purpose, we propose a simple strategy: approximate the nonlinearity g by a sequence of nonlineartities $(g_n)_{n\geq 0}$ which are globally bounded (or globally lipschitz), so that the solution u_n of the problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \triangle u_n = g_n(u_n) \quad \text{in} \quad (0,\infty)\times\Omega, \\ u_n = 0 \quad \text{in} \quad (0,\infty)\times\partial\Omega, \\ u_n(0,x) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.11.8}$$

is global; then let $n \to \infty$ and study the existence of a pointwise limit of $u_n(t,x)$ for $t > T_{\rm m}$.

Baras and Cohen [7] have shown that in many situations, u blows up **completely** after $T_{\rm m}$ in the sense that for every $x \in \Omega$ and every $t > T_{\rm m}$, $u_n(t,x) \to \infty$ as $n \to \infty$. More precisely, we have the following result (see Baras and Cohen [7] and Martel [79]).

Theorem 3.11.2. *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, and consider the solution u of the equation (3.11.1) defined on the maximal interval $[0,T_{\rm m})$. Suppose that $T_{\rm m} < \infty$. Suppose furthermore that $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$ and that $\triangle u_0 + u_0^p \geq 0$ a.e. in Ω.*

Let $(g_n)_{n\geq 0}$ be any sequence of locally Lipschitz functions $[0,\infty) \to [0,\infty)$ such that

$$\begin{gathered} 0 \leq g_n(s) \leq n \quad \text{for} \quad n \geq 0, s \geq 0, \\ g_n(s) \uparrow s^p \quad \text{as} \quad n \to \infty \quad \text{for} \quad s \geq 0, \end{gathered} \tag{3.11.9}$$

and for every $n > 0$, let u_n be the (global) solution of the equation (3.11.8).

Then for every $T < T_{\rm m}$, $u_n \underset{n\to\infty}{\longrightarrow} u$ uniformly on $[0,T]\times\Omega$;

and for every $T > T_{\rm m}$

$$\frac{u_n(t,x)}{\delta(x)} \underset{n\to\infty}{\longrightarrow} +\infty, \tag{3.11.10}$$

uniformly on $[T,+\infty)\times\Omega$, where δ is the function distance to the boundary, $\delta(x) = \operatorname{dist}(x,\partial\Omega)$.

Sketch of the proof. We follow the argument of Martel [79]. We first observe that by the maximum principle (see the proof of Theorem 3.8.3),

$$u_t \geq 0, \tag{3.11.11}$$

on $(0, T_{\rm m}) \times \Omega$. For convenience, we set

$$g(s) = s^p.$$

We now proceed in four steps.

Step 1. For every $t \in (0, T_{\rm m})$, there exists $c(t) > 0$ such that $u(t) \geq u_0 + c(t)\delta$ a.e. in Ω.

Fix $\tau \in (0, T_{\rm m})$ and let $\varepsilon \in (0,1)$ be such that $\tau - \varepsilon > 0$.

We first claim that $u(\tau - \varepsilon) - u_0 \geq 0$, $u(\tau - \varepsilon) - u_0 \not\equiv 0$. Indeed, note that by (3.11.11) $u(t)$ is a nondecreasing function of t. Therefore, if $u(\tau - \varepsilon) = u_0$, it follows that u is constant on $(0, \tau - \varepsilon)$, from which we deduce easily that u_0 is a stationary solution; and so $T_{\rm m} = +\infty$, which is absurd.

Next, we claim that

$$u(\tau) \geq u_0 + T(\varepsilon)(u(\tau - \varepsilon) - u_0). \tag{3.11.12}$$

Indeed, we have

$$\begin{aligned} u(\tau) &= T(\varepsilon)u(\tau - \varepsilon) + \int_0^\varepsilon T(\varepsilon - s)g(u(\tau - \varepsilon + s))\,ds \\ &= T(\varepsilon)(u(\tau - \varepsilon) - u_0) + T(\varepsilon)u_0 + \int_0^\varepsilon T(\varepsilon - s)g(u(\tau - \varepsilon + s))\,ds. \end{aligned}$$

Since $u(t)$ is a nondecreasing function of t, we have $g(u(\tau - \varepsilon + s)) \geq g(u(s))$ and also

$$u(\varepsilon) = T(\varepsilon)u_0 + \int_0^\varepsilon T(\varepsilon - s)g(u(s))\,ds \geq u_0;$$

and so we deduce (3.11.12).

The result now follows from (3.11.12) and the inequality

$$T(t)\varphi \geq e^{-\frac{C}{t}}\|\varphi\delta\|_{L^1}\delta, \tag{3.11.13}$$

for all $\varphi \geq 0$ (see Exercise 1.8.12).

Step 2. $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^\infty((0,T)\times\Omega)$ for every $T < T_{\rm m}$. Indeed, note that $g_n \underset{n\to\infty}{\longrightarrow} g$ uniformly on bounded subsets of $[0, \infty)$. Furthermore, note that $u_n \geq 0$; and that, since $g_n \leq g$, $u_n \leq u$ on $(0, T_{\rm m}) \times \Omega$. The result now follows from a classical continuous dependence argument.

Step 3. Fix any $\tau \in (0, T_{\rm m})$ and let $T = T_{\rm m} + \tau$. Then

$$\lim_{n\to\infty} \|u_n(T)\delta\|_{L^1} = +\infty. \tag{3.11.14}$$

We argue by contradiction and we assume that

$$\varliminf_{n\to\infty} \|u_n(T)\delta\|_{L^1} < \infty. \tag{3.11.15}$$

The idea is the following. Assuming (3.11.15), we show:

(i) that u_n converges to a "weak solution" u of (3.11.1) on $[0, T)$,

(ii) using that weak solution, we construct a bounded supersolution of (3.11.1) on $[0, T_{\rm m})$, which contradicts the blow up alternative.

Proof of (i). Consider $f \in L^\infty((\tau,T)\times\Omega)$, $\zeta \in H^1_0(\Omega)$ and let ξ be the solution of the equation

$$\begin{cases} -\xi_t - \triangle \xi = f \quad \text{in} \quad (\tau,T)\times\Omega, \\ \xi = 0 \quad \text{in} \quad (\tau,T)\times\partial\Omega, \\ \xi(T) = \zeta \quad \text{in} \quad \Omega. \end{cases} \tag{3.11.16}$$

Multiplying the equation (3.11.8) by ξ and integrating on $(\tau,T)\times\Omega$, we obtain after integration by parts

$$\int_\tau^T \int_\Omega g_n(u_n)\xi = \int_\tau^T \int_\Omega u_n f + \int_\Omega u_n(T)\zeta - \int_\Omega u_n(\tau)\xi(\tau). \tag{3.11.17}$$

We first choose $\zeta = \varphi_1$ and $f \equiv 0$, so that $\xi(t) = e^{-\lambda_1(T-t)}\varphi_1$ (here, λ_1 is the first eigenvalue of $-\triangle$ in $H^1_0(\Omega)$ and φ_1 is a corresponding, positive eigenvector). We obtain

$$e^{-\lambda_1 T}\int_\tau^T \int_\Omega g_n(u_n)\varphi_1 \le \int_\Omega u_n(T)\varphi_1.$$

Since $\varphi_1 \approx \delta$, we deduce by applying (3.11.15)

$$\sup_{n\ge 0}\int_\tau^T \int_\Omega g_n(u_n)\delta < \infty. \tag{3.11.18}$$

We next take $\zeta = 0$ and $f \equiv 1$ in (3.11.17), and we obtain

$$\int_\tau^T \int_\Omega u_n = \int_\tau^T \int_\Omega g_n(u_n)\xi + \int_\Omega u_n(\tau)\xi(\tau).$$

Since $\xi(t) \le C\delta$ for some C independent of $t \in [0,T]$ (by (3.7.19)), we obtain from the above inequality and (3.11.18)

$$\sup_{n\ge 0}\int_\tau^T \int_\Omega u_n < \infty. \tag{3.11.19}$$

Note that g_n is nondecreasing in n, so that u_n is also nondecreasing in n. Therefore, it follows from the monotone convergence theorem and the estimate (3.11.19), and from Step 2, that there exists a function $\overline{u} \in L^1((0,T)\times\Omega)$ such that

$$u_n \uparrow \overline{u} \quad \text{in} \quad L^1((0,T)\times\Omega) \quad \text{and a.e. in} \quad (0,T)\times\Omega.$$

In addition,

$$\overline{u} = u \quad \text{on} \quad (0,T_{\mathrm{m}})\times\Omega.$$

Next, since $u_n \uparrow \overline{u}$ and $g_n \uparrow g$, we deduce that $g_n(u_n) \to g(\overline{u})$ a.e. in $(0,T)\times\Omega$. Therefore, we deduce from (3.11.18) and the dominated convergence theorem (and Step 2) that $g(\overline{u})\delta \in L^1((0,T)\times\Omega)$, and that

$$g_n(u_n)\delta \underset{n\to\infty}{\longrightarrow} g(u)\delta \quad \text{in} \quad L^1((0,T)\times\Omega) \quad \text{and a.e. in} \quad (0,T)\times\Omega.$$

Passing to the limit in (3.11.17), we obtain that

$$\int_\tau^T \int_\Omega g(\overline{u})\xi = \int_\tau^T \int_\Omega \overline{u} f - \int_\Omega u(\tau)\xi(\tau),$$

for all $f \in L^\infty((\tau,T)\times\Omega)$, where ξ is the solution of (3.11.16) with $\zeta = 0$. In other words, $\overline{u}$ is a **weak** solution of the equation (3.11.1) on the interval $(0,T)$ (cf. the proof of Theorem 3.11.1, Step 1).

Proof of (ii). It follows from Step 1 that there is a constant $c_0 > 0$ such that

$$u(\tau) \geq u_0 + c_0\delta. \tag{3.11.20}$$

Fix $\varepsilon > 0$ small enough so that the solution Z of the equation

$$\begin{cases} Z_t - \triangle Z = -\varepsilon \quad \text{in} \quad (0, T_{\mathrm{m}}) \times \Omega, \\ Z = 0 \quad \text{in} \quad (0, T_{\mathrm{m}}) \times \partial\Omega, \\ Z(0) = c_0\delta \quad \text{in} \quad \Omega, \end{cases} \tag{3.11.21}$$

satisfies $Z \geq 0$ on $(0, T_{\mathrm{m}}) \times \Omega$ (see Lemma 3.7.12). Set $z(t) = u(t) + Z(t) \geq u(t)$. We have

$$\begin{cases} z_t - \triangle z = u^p - \varepsilon \leq z^p - \varepsilon \leq (z^p - \varepsilon)^+ \quad \text{in} \quad (0, T_{\mathrm{m}}) \times \Omega, \\ z = 0 \quad \text{in} \quad (0, T_{\mathrm{m}}) \times \partial\Omega, \\ z(0) = u_0 + c_0\delta. \end{cases}$$

Fix $A \geq \|u(\tau)\|_{L^\infty}$, $A > \varepsilon^{\frac{1}{p}}$ and consider the function

$$\Phi(t) = \begin{cases} t \quad \text{for} \quad 0 \leq t \leq A, \\ h^{-1}\left(\dfrac{1}{p-1}(A^{1-p} - t^{1-p})\right) \quad \text{for} \quad t \geq A, \end{cases}$$

where

$$h(s) = \int_A^s \frac{d\sigma}{\sigma^p - \varepsilon}.$$

The function Φ is **bounded**, monotone increasing and **concave**. Moreover,

$$\Phi'(t) \geq \frac{(\Phi(t)^p - \varepsilon)^+}{t^p}, \tag{3.11.22}$$

for all $t \geq 0$.

Let $w = \Phi(\overline{u})$, so that w is **bounded** on $[0, T) \times \Omega$. Next, observe that by Kato's inequality (see Lemma 3.7.10 and Theorem A.5.20) the function w satisfies (in the weak sense)

$$w_t - \triangle w \geq \Phi'(\overline{u})(\overline{u}_t - \triangle\overline{u}) = \Phi'(\overline{u})\overline{u}^p \geq (w^p - \varepsilon)^+,$$

by (3.11.22). Set now

$$v(t) = w(t + \tau),$$

for $0 \leq t < T_{\mathrm{m}}$, so that v is bounded on $[0, T_{\mathrm{m}}) \times \Omega$. We have

$$v_t - \triangle v \geq (v^p - \varepsilon)^+.$$

Furthermore, $v(0) = w(\tau) = u(\tau) \geq u_0 + c_0\delta = z(0)$. Therefore, it follows from the maximum principle that $z \leq v$ on $[0, T_{\mathrm{m}})$. Since $u \leq z$ and v is bounded, this yields a contradiction with the blow up alternative.

This completes the proof of Step 3, i.e. $\lim\limits_{n\to\infty} \|u_n(t)\delta\|_{L^1} = +\infty$ for every $t \in (T_{\mathrm{m}}, 2T_{\mathrm{m}})$.

Step 4. Proof of (3.11.10). Fix any $t_0 > T_{\mathrm{m}}$. Fix any $t' < t_0$ with $T_{\mathrm{m}} < t' < 2T_{\mathrm{m}}$. By Step 3,

$$\|u_n(t')\delta\|_{L^1} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Since $u_n(t_0) \geq T(t_0 - t')u_n(t')$ (here, $T(\cdot)$ refers to the heat semigroup), it follows from the inequality (3.11.13) that

$$\frac{u_n(t_0)}{\delta} \underset{n\to\infty}{\longrightarrow} +\infty, \tag{3.11.23}$$

uniformly on Ω.

It remains to prove that the convergence in (3.11.10) is uniform in t on $[t_0, +\infty)$. We proceed as follows. Given any $K > 0$, we construct a function v such that $v \geq K\delta$ and such that v is a subsolution of the equation (3.11.8) on $[t_0, \infty)$.

Let φ_1 be a first eigenfunction of $-\triangle$ in $H_0^1(\Omega)$, normalized such that

$$\max_{\Omega} \varphi_1 = 2. \tag{3.11.24}$$

Note that there exists a constant $\gamma > 0$ such that

$$\varphi_1 \geq \gamma\delta. \tag{3.11.25}$$

Let ψ be the solution of the elliptic equation

$$\begin{cases} -\triangle\psi = (\varphi_1 - 1)^+, \\ \psi_{|\partial\Omega} = 0. \end{cases}$$

Since $(\varphi_1 - 1)^+ \not\equiv 0$, there exist two constants $0 < \alpha_0 < \alpha_1 < \infty$ such that

$$\alpha_0\varphi_1 \leq \psi \leq \alpha_1\varphi_1.$$

Let now $\ell = \dfrac{1}{\alpha_0}$ and $w = \ell\psi$. We have

$$\varphi_1 \leq w \leq \ell\alpha_1\varphi_1,$$

and

$$\begin{cases} -\triangle w = \ell(\varphi_1 - 1)^+ \leq \ell(w-1)^+, \\ w_{|\partial\Omega} = 0. \end{cases} \tag{3.11.26}$$

On the other hand,

$$s^p \geq \ell(s-c)^+,$$

with $c = \ell^{\frac{1}{p-1}}$ Fix now

$$K \geq c + 1.$$

Since $g_n \uparrow g$ as $n \to \infty$, uniformly on bounded sets, it follows that for n large enough

$$g_n(s) \geq \ell(s - c - 1)^+ \quad \text{for} \quad 0 \leq s \leq 2K\ell\alpha_1. \tag{3.11.27}$$

Setting

$$v = Kw,$$

we deduce from (3.11.24) and (3.11.26) that

$$K\varphi_1 \leq v \leq K\ell\alpha_1\varphi_1 \leq 2K\ell\alpha_1, \tag{3.11.28}$$

and that

$$\begin{cases} -\triangle v \le \ell(v-K)^+ \le \ell(v-c-1)^+, \\ v_{|\partial\Omega} = 0. \end{cases}$$

By (3.11.27) and (3.11.28), this implies that for n large enough

$$\begin{cases} -\triangle v \le g_n(v), \\ v_{|\partial\Omega} = 0. \end{cases}$$

By (3.11.23) and (3.11.28), we have $u_n(t_0) \ge v$ for n large enough; and so, v is a subsolution of the equation (3.11.8) on $[t_0, \infty)$. It follows from the maximum principle that $u_n(t) \ge v$ for all $t \ge t_0$; and by (3.11.25) and (3.11.28) we deduce

$$\frac{u_n(t)}{\delta} \ge K\gamma,$$

in Ω for all $t \ge t_0$, provided n is large enough. Since $t_0 > T_{\mathrm{m}}$ and $K \ge c+1$ are arbitrary, this completes the proof. □

Corollary 3.11.3. *Let $u_0 \in L^\infty(\Omega) \cap H^2(\Omega) \cap H^1_0(\Omega)$, $u_0 \ge 0$, and assume $\triangle u_0 + u_0^p \ge 0$ a.e. in Ω. Let u be a weak solution of the equation (3.11.1) on $[0,T)$ for some $T > 0$, in the sense of (3.11.5). Then $T \le T_{\mathrm{m}}$.*

This corollary settles the question we raised at the beginning of for the special initial conditions u_0 as in Corollary 3.11.3, i.e. for nondecreasing (in time) solutions: there is **no** way of extending the solution after T_{m}, even in the weak sense (3.11.5).

Remark 3.11.4. The reader may wonder whether any weak solution in Corollary 3.11.3 coincides with the usual solution on $[0,T)$. This need not be the case: even if $u_0 = 0$, there exist nontrivial weak solutions. (See Baras [6] and Haraux and Weissler [59].) The only property we have is that any weak solution is larger than or equal to the classical solution (see the proof of Corollary 3.11.3).

A verifier

Proof of Corollary 3.11.3. For any $n \in \mathbb{N}$, let

$$g_n(t) = \min\{t^p, n\},$$

and let u_n be the (global, classical) solution of (3.11.8). We claim that

$$u \ge u_n \quad \text{on} \quad (0,T) \times \Omega. \tag{3.11.29}$$

The conclusion of Corollary 3.11.3 now follows from Theorem 3.11.2.

We now prove claim (3.11.29). Set

$$v = (u_n - u)^+,$$

so that by Kato's inequality (see Lemma 3.7.10 and Theorem A.5.20)

$$\begin{aligned} v_t - \triangle v &\le (g_n(u_n) - g(u))\mathrm{sign}^+(u_n - u) \\ &\le (g_n(u_n) - g_n(u))\mathrm{sign}^+(u_n - u) \\ &\le L_n v, \end{aligned}$$

where L_n is the Lipschitz constant of g_n. Since $v(0) = 0$, we have $v \leq 0$ by the maximum principle. □

In Theorem 3.11.2, we made the assumption $\triangle u_0 + u_0^p \geq 0$. We now discuss whether this assumption is essential. The answer seems to depend on p.

Remark 3.11.5. If $p < \dfrac{N+2}{N-2}$ and $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then $\lim\limits_{n\to\infty} u_n(t,x) = +\infty$ for all $t > T_{\rm m}$ and $x \in \Omega$ (presumably, $\dfrac{u_n(t)}{\delta} \underset{n\to\infty}{\longrightarrow} +\infty$ uniformly on $[T,\infty) \times \Omega$, $T > T_{\rm m}$, but we have not checked it). See Baras and Cohen [7]. Incidentally, the proof is rather complicated, and it would be interesting to have a simple proof. We suspect that the same conclusion holds for $p = \dfrac{N+2}{N-2}$.

consulter Yvan. Si pas de reponse, mettre un probleme ouvert

Remark 3.11.6. When $p > \dfrac{N+2}{N-2}$, it seems that there might be life after death for some initial conditions. Under some further restrictions on p, Galaktionov and Vazquez [48] have constructed solutions of

$$u_t - \triangle u = u^p \quad \text{in} \quad (0, +\infty) \times \mathbf{R}^N,$$

which are smooth except at $t = T > 0$ and with $\lim\limits_{t\uparrow T} \|u(t)\|_{L^\infty} = +\infty$. It would be very interesting to investigate whether a similar phenomenon holds in bounded domains. Can one have a situation where $u = \lim\limits_{n\to\infty} u_n$ has a "cascade" of blow up times and/or blows up completely after some time $T > T_{\rm m}$? See Open Problems 3.14.16 and 3.14.17.

3.11.3. ???

3.12. Comments.

Even if Ω is not smooth, the conclusion of Theorem 3.1.1 hold, provided $|\Omega| < \infty$. Note that in this case, the solution u belongs to $C((0, T_{\rm m}), L^\infty(\Omega))$ and to $C([0, T_{\rm m}), L^p(\Omega))$ for any $p \in [1, \infty)$ (see Remark 3.1.2).

In the case $|\Omega| = \infty$, then we have the following result.

Theorem 3.12.?. *Given $p \in [1, \infty)$ and $u_0 \in L^p(\Omega) \cap L^\infty(\Omega)$, there exists a unique weak solution u of (3.1), defined on a maximal time interval $[0, T_{\rm m})$, i.e. $u \in L^\infty((0,T) \times \Omega) \cap L^\infty((0,T), L^p(\Omega))$ for all $T < T_{\rm m}$ and u solves (3.1.1) for all $t \in [0, T_{\rm m})$. Moreover, we have the alternative*

either $T_{\rm m} = +\infty$,

or $T_{\rm m} < \infty$ and $\lim\limits_{t\uparrow T_{\rm m}} \|u(t)\|_{L^\infty} = +\infty$.

In addition, u depends continuously on u_0. The mapping $u_0 \mapsto T_{\rm m}$ is lower semicontinuous $L^p(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$, and for every $T < T_{\rm m}$ there exists $\varepsilon > 0$ and $C < \infty$ such that if $\|v_0 - u_0\|_{L^\infty} \leq \varepsilon$, then $\|v - u\|_{L^\infty((0,T)\times\Omega)} \leq C\|v_0 - u_0\|_{L^\infty(\Omega)}$ and $\|v - u\|_{L^\infty((0,T),L^p(\Omega))} \leq C\|v_0 - u_0\|_{L^p(\Omega)}$, where v is the solution of (3.1.1) with the initial value v_0.

Note that in this case, the solution u belongs to $C((0, T_{\rm m}), L^\infty(\Omega))$ and to $C([0, T_{\rm m}), L^p(\Omega))$.

Le Theorem 3.10.3 est du a Brezis and Friedman [21]

La methode de suite de Cauchy du Theorem 3.10.8 est adaptee de Brezis and Strauss [26]

Etudier (evt dans les problemes ouverts) ce qui se passe lorsqu'une suite $(u_{0,n})_{n\geq 0}$ converge faiblement vers u_0. (Phenomenes de perte de condition initiale ???)

3.13. Exercises. Unless otherwise specified, we still assume that Ω is a smooth bounded open subset of $\mathbb{R}^N$.

Exercise 3.13.1. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz in u, and assume that

$$ug(x,u) \leq Au^2,$$

for almost all $x \in \Omega$ and $|u| \geq M$. Let $u_0 \in L^\infty(\Omega)$ and let u be the solution of (3.1.5).

- Show that $\|u(t)\|_{L^\infty} \leq \max\{\|u_0\|_{L^\infty}, M\} e^{At}$ for all $t \geq 0$. (Hint: Show that $\max\{\|u_0\|_{L^\infty}, M\} e^{At}$ is a super-solution of (3.1.5).)

Exercise 3.13.2. Under the assumptions of Exercise 3.13.1, show that if $A < \lambda_1$, with λ_1 the first eigenvalue of $-\triangle$ in $H^1_0(\Omega)$, then $\|u(t)\|_{L^\infty} \leq C \max\{\|u_0\|_{L^\infty}, M\}$ for all $t \geq 0$. (Hint: Consider the solution $\varphi \in H^1_0(\Omega)$ of $-\triangle\varphi = A(\varphi + 1)$, and show that $\max\{\|u_0\|_{L^\infty}, M\}(1+\varphi)$ is a super-solution of (3.1.5).)

Exercise 3.13.3. Let g and u be as in Exercise 3.13.1, and assume that $A > \lambda_1$, with λ_1 the first eigenvalue of $-\triangle$ in $H^1_0(\Omega)$. The object of this exercise is to show that $\|u(t)\|_{L^\infty} \leq C \max\{\|u_0\|_{L^\infty}, M\} e^{(A-\lambda_1)t}$ for all $t \geq 0$.

Let $B \geq 0$ and consider the solution v of

$$\begin{cases} v_t - \triangle v = \lambda_1 v + Be^{-(A-\lambda_1)t} \text{ in } (0,\infty)\times\Omega, \\ v = 0 \text{ in } (0,\infty)\times\partial\Omega, \\ v(0) = \|u_0\|_{L^\infty} \text{ in } \Omega. \end{cases}$$

- Show that $\|v(t)\|_{L^\infty} \leq e^{\lambda_1 t}\left(\|u_0\|_{L^\infty} + \dfrac{B}{A}\right)$. (Observe that 0 is a sub-solution of the equation and that $e^{\lambda_1 t}\left(\|u_0\|_{L^\infty} + \dfrac{B}{A}(1-e^{-At})\right)$ is a super-solution.)
- Multiply the equation by v, and show that $\|v(t)\|_{L^2} \leq |\Omega|^{\frac{1}{2}}\left(\|u_0\|_{L^\infty} + \dfrac{B}{A-\lambda_1}\right)$.
- Let $\sigma > \dfrac{N}{2}$ and $1 \leq q \leq r \leq \infty$ be such that $\dfrac{1}{\sigma} = \dfrac{1}{q} - \dfrac{1}{r}$. Show that

  $$\|v(t+s)\|_{L^r} \leq Cs^{-\frac{N}{2\sigma}}\|v(t)\|_{L^q} + \lambda_1 \int_0^s \|v(t+\tau)\|_{L^r}\,d\tau + \frac{B}{A-\lambda_1}|\Omega|^{\frac{1}{r}},$$

  and that $\|v(t+1)\|_{L^r} \leq C(\|v(t)\|_{L^q} + 1)$.
- Iterate this estimate, and show that there exists an integer m such that $\|v(t+m)\|_{L^\infty} \leq C(\|v(t)\|_{L^2} + 1)$.
- Show that $\|v(t)\|_{L^\infty} \leq C \max\{\|u_0\|_{L^\infty}, M\}$.

- Show that $e^{(A-\lambda_1)t}v(t)$ is a super-solution of (3.1.5) provided B is large enough, and conclude.

Exercise 3.13.4. Let g and u be as in Exercise 3.13.1, and assume that $A = \lambda_1$, with λ_1 the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. The object of this exercise is to show that $\|u(t)\|_{L^\infty} \le C(\|u_0\|_{L^\infty} + t)$ for all $t \ge 0$.

Consider the solution v of

$$\begin{cases} v_t - \triangle v = \lambda_1 v + B \text{ in } (0,\infty) \times \Omega, \\ v = 0 \text{ in } (0,\infty) \times \partial\Omega, \\ v(0) = \|u_0\|_{L^\infty} \text{ in } \Omega. \end{cases}$$

- Show that $\|v(t)\|_{L^\infty} \le C(\|u_0\|_{L^\infty} + t)$ for all $t \ge 0$, and that v is a super-solution of (3.1.5) provided B is large enough. (c.f. Exercise 3.13.3.)

Exercise 3.13.5. Let $a \in L^\infty(\Omega)$, let $v_0 \in L^\infty(\Omega)$ and let v be the solution of

$$\begin{cases} v_t - \triangle v + av = 0 \text{ in } (0,\infty) \times \Omega, \\ v = 0 \text{ in } (0,\infty) \times \partial\Omega, \\ v(0) = v_0 \text{ in } \Omega. \end{cases}$$

- Show that

$$\|v(t)\|_{L^\infty} \le Ce^{-\lambda_1 t}\|v_0\|_{L^\infty},$$

for all $t \ge 0$, where C is independent of v_0 and $\lambda_1 = \lambda_1(-\triangle + a)$. (Hint: Show that $\|v(t)\|_{L^2} \le e^{-\lambda_1 t}\|u_0\|_{L^2}$, and use the smoothing effect.)

Exercise 3.13.6. Assume $N \ge 3$. Let $q > \dfrac{N+2}{N-2}$ and set $g(u) = |u|^{q-1}u$. The object of this exercise is to show that for every $\varepsilon > 0$ there exists $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $\|u_0\|_{H^1} \le \varepsilon$ and such that the solution u of (3.1) blows up in finite time.

- Let $B \subset \Omega$ be a ball. Show that there exists $v_0 \in \mathcal{D}(B)$, $v_0 \ge 0$, such that the solution v of (3.1) with $\Omega = B$ blows up in finite time, say at time T.

- Define $\widetilde{v}$ in $(0,T) \times \mathbb{R}^N$ by

$$\widetilde{v}(t,x) = \begin{cases} v(t,x) \text{ if } x \in B, \\ 0 \text{ if } x \in \mathbb{R}^N \setminus B. \end{cases}$$

  Show that $\widetilde{v}_t - \triangle\widetilde{v} \le |\widetilde{v}|^{q-1}\widetilde{v}$ in $\mathcal{D}'(\mathbb{R}^N)$ for all $t \in (0,T)$.

- Given $\lambda \in (0,1)$, set

$$\widetilde{v}^\lambda(t,x) = \lambda^{-\frac{2}{q-1}}\widetilde{v}\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right).$$

  Let u^λ be the solution of (3.1) with the initial value $u_0^\lambda = \widetilde{v}^\lambda(0)$. Show that $u^\lambda(t) \ge \widetilde{v}^\lambda(t)$ in Ω for all $t \in (0, \min\{\lambda^2 T, T_{\mathrm{m}}(u_0^\lambda)\})$, and conclude.

Exercise 3.13.7. Let g and u be as in Exercise 3.13.1, and assume that

$$ug(u) \ge \lambda_1 u^2, \quad ug(u) \not\equiv \lambda_1 u^2,$$

with λ_1 the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. The object of this exercise is to show that $\|u(t)\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} \infty$ for certain u_0.

- Assuming $g(u) \geq \lambda_1 u + \varepsilon$ for $0 \leq a < u < b$, with $\varepsilon > 0$, let $\varphi_1 > 0$ be the first eigenvector of $-\triangle$ in $H_0^1(\Omega)$ such that $\|\varphi_1\|_{L^\infty} = b$.
- Set $u_0 = \varphi_1$, and show that $u \geq \varphi_1$.
- Show that $\displaystyle \frac{d}{dt}\int_\Omega u(t,x)\varphi_1(x)\,dx \geq \varepsilon \int_{\{a<u<b\}} \varphi_1$.
- Assume by contradiction that $\sup\limits_{t\geq 0}\|u(t)\|_{L^\infty} < \infty$, and show that u is bounded in $C^{0,\frac{1}{2}}(\overline{\Omega})$ (use the analyticity of the semigroup in $L^p(\Omega)$).
- Show that there exists $\delta > 0$ such that $\displaystyle \frac{d}{dt}\int_\Omega u(t,x)\varphi_1(x)\,dx \geq \delta$ and conclude.

Exercise 3.13.8. Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, and assume that there exist $a < 0 < b$ such that

$$g(a) = g(b) = 0.$$

Let $u_0 \in L^\infty(\Omega)$ and let u be the solution of (3.1). If $a \leq u_0 \leq b$, show that $T_{\mathrm{m}} = +\infty$ and that $a \leq u(t) \leq b$ for all $t \geq 0$.

Exercise 3.13.9. Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, and assume that there exist $a > 0$ such that $g(a) = 0$, and that

$$\limsup_{u\to-\infty} \frac{g(u)}{u} < \infty.$$

Let $u_0 \in L^\infty(\Omega)$ and let u be the solution of (3.1). If $u_0 \leq a$, show that $T_{\mathrm{m}} = +\infty$ and that $u(t) \leq a$ for all $t \geq 0$.

Exercise 3.13.10. Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

$$ug(u) \leq \frac{\lambda_1 - \varepsilon}{2}u^2 \text{ for } |u| \leq \alpha,$$

with $\alpha > 0$ and λ_1 the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. The object of this exercise is to show that for every constant M, there exists $\delta_M > 0$ such that if $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ verifies $\|u_0\|_{L^\infty} \leq M$ and $\|u_0\|_{H^1} \leq \delta_M$, then the solution u of (3.1) is globally defined.

Let $\delta_M > 0$ to be chosen, and set $T = \sup\{t \in [0, T_{\mathrm{m}}); \|u(s)\|_{L^\infty} \leq 8M \text{ on } [0,t]\}$. In the sequel, the constants depend on M, but not on u.

- With the argument of Step 1 of the proof of Theorem 3.4.1, show that $\|u(t)\|_{H^1} \leq 2\delta_M$ for all $t \in [0,T]$, provided δ_M is small enough. (Observe that $E(u) \leq C\|u\|_{H^1}^2$ and that $ug(u) \leq \dfrac{\lambda_1-\varepsilon}{2}u^2 + C|u|^q$ for $1 < q < \dfrac{2N}{N-2}$.)
- Show that $g(u) = fu$ with $f \in L^\infty((0,T), L^\infty(\Omega))$ (observe that $|g(u)| \leq C|u|$).
- Show that $\|u\|_{L^\infty((0,T),L^\infty)} \leq 4\|u_0\|_{L^\infty} + C\delta_M$ (apply Theorem 1.6.6).
- Show that if δ_M is small enough, then $\|u\|_{L^\infty((0,T),L^\infty)} \leq 6\|u_0\|_{L^\infty}$, and conclude.

Exercise 3.13.11. Assume that g verifies (3.6.8) for $u \geq \alpha > 0$, where $h : (\alpha, \infty) \to (0, \infty)$ is a convex function such that (3.6.9) holds. Show that there exists $\beta > 0$ such that if $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and

$$\int_\Omega u_0(x)\varphi_1(x)\,dx \geq \beta,$$

then the solution of (3.1) blows up in finite time.

Exercise 3.13.12. Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, and assume that

$$ug(u) \geq (2+\varepsilon)G(u) = (2+\varepsilon)\int_0^u g(s)\,ds,$$

for $|u| \geq M$, with $\varepsilon > 0$. Show that there exists a constant K such that if

$$E(u_0) = \frac{1}{2}\int_\Omega |\nabla u_2|^2 - \int_\Omega G(u_0) \leq -K,$$

then the solution u of (3.1) blows up in finite time.

Exercise 3.13.13. The object of this exercise is to prove the following result.

Theorem. *Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Assume that there exists $\varepsilon > 0$ and $M < \infty$ such that*

$$ug(u) \geq (2+\varepsilon)\int_0^u g(s)\,ds, \quad \text{for } |u| \geq M. \tag{3.13.1}$$

If $N \geq 3$, assume further that there exists $p < \dfrac{N+2}{N-2}$ such that

$$|g(u)| \leq C(1+|u|^p), \tag{3.13.2}$$

for all $u \in \mathbb{R}$; and if $N = 2$, assume that

$$\lim_{|u|\to\infty} e^{-\mu u^2}|g(u)| = 0, \tag{3.13.3}$$

for all $\mu > 0$ (no condition if $N = 1$). If $u_0 \in L^\infty(\Omega)$ is such that the solution u of (3.1) is global, then $\sup\limits_{t\geq 0}\|u(t)\|_{L^\infty} < \infty$.

Step 1. Show that u verifies $\sup\limits_{t\geq 0}\|u(t)\|_{L^2} < \infty$ and $\displaystyle\int_1^\infty\int_\Omega u_t^2 < \infty$.

To prove this, show that (with the notation of Theorem 3.6.4),

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \geq \varepsilon\int_\Omega |\nabla u|^2 - (4+2\varepsilon)E(u(1)) - C|\Omega| + (4+2\varepsilon)\int_1^t\int_\Omega u_t^2,$$

for $t \geq 1$. Show that

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \geq \varepsilon\lambda_1\|u(t)\|_{L^2}^2 - (4+2\varepsilon)E(u(1)) - C|\Omega|,$$

and conclude.

Step 2. Show that there exists B independent of $t \geq 1$ such that if $\|u_t(t)\|_{L^2} \leq 1$, then $\|u(t)\|_{H^1} \leq B$.

To show this, suppose that $\|u_t(t)\|_{L^2} \leq 1$ for some $t \geq 1$, and show that

$$\varepsilon\int_\Omega |\nabla u|^2 \leq \frac{d}{dt}\|u(t)\|_{L^2}^2 + (4+2\varepsilon)E(u(1)) + C|\Omega| \leq 2\|u(t)\|_{L^2}\|u_t(t)\|_{L^2} + (4+2\varepsilon)E(u(1)) + C|\Omega|,$$

and apply Step 1.

Step 3. Conclude, if $N = 1$ or $N = 2$.

Observe that by the local existence theory (see the proof of Theorem 3.1.1), there exists $\delta_1 > 0$ such that if $\|v_0\|_{L^\infty} \le B_1$, then the solution v of (3.1) with the initial value v_0 exists on $[0, \delta_1]$ and $\sup_{0\le t\le\delta_1} \|v(t)\|_{L^\infty} \le B_1 + 1$. Show that if $t \ge 1$ is such that $\|u_t(t)\|_{L^2} \le 1$, then

$$\sup_{t\le s\le t+\delta_1} \|u(s)\|_{L^\infty} \le B_1 + 1.$$

Show that there exists $T < \infty$ such that

$$\int_T^\infty \int_\Omega u_t(t,x)^2 \, dxdt < \delta_1.$$

Show that $\sup_{t\ge T+\delta_1} \|u(t)\|_{L^\infty} \le B_1 + 1$. Conclude.

Step 4. Conclude, if $N \ge 3$.

Show that for any $M > 0$, there exists $\delta_M > 0$ independent of $t \ge 0$ such that if $\|u(t)\|_{L^{\frac{2N}{N-2}}} \le M$, then $\|u(t+s)\|_{L^{\frac{2N}{N-2}}} \le M + 1$ for $0 \le s \le \delta_M$. (Use the growth assumption on g.)

Deduce that $\sup_{t\ge 0} \|u(t)\|_{L^{\frac{2N}{N-2}}} < \infty$ (cf. Step 3).

Apply Theorem 1.6.6 to conclude.

Exercise 3.13.14. Consider the operator $L = -\triangle - a(x)$ with $a \in L^\infty(\Omega)$ and $\lambda_1(L) = 0$. Let $\varphi_1 > 0$ be the first eigenfunction of L. Let $w \in H_0^1(\Omega)$ satisfy

$$Lw \le 0, \tag{3.13.4}$$

in Ω. Prove that

(-) **either** $w \le 0$ on Ω

(-) **or** $w = k\varphi_1$ for some constant $k > 0$.

Hint: Multiply (3.13.4) by w^+ and deduce that $w^+ = k\varphi_1$ for some constant $k \ge 0$.

Exercise 3.13.15. The object of this exercise is to prove the conclusions of Remark 3.7.15 (i). Suppose that $N \le 9$, and for $0 < \lambda < \lambda^*$ consider the minimal solution u_λ of (3.7.4) with $g(u) = e^u$ (see Lemma 3.7.4).

- Show that $2k \int_\Omega e^{2ku_\lambda} |\nabla u_\lambda|^2 = \lambda \int_\Omega e^{u_\lambda}(e^{2ku_\lambda} - 1)$ for all $k > 0$.
- Show that $k^2 \int_\Omega e^{2ku_\lambda} |\nabla u_\lambda|^2 \ge \lambda \int_\Omega e^{u_\lambda}(e^{ku_\lambda} - 1)^2$ for all $k > 0$ (apply the property $\lambda_1(-\triangle - \lambda e^{u_\lambda}) > 0$).
- Show that $\limsup_{\lambda\uparrow\lambda^*} \int_\Omega e^{(2k+1)u_\lambda} < \infty$ for $0 < k < 2$.
- Show that $\limsup_{\lambda<\lambda^*} \|u_\lambda\|_{L^\infty} < \infty$ (hint: use the equation to derive that u_λ is bounded in $W^{2,p}(\Omega)$ for every $1 \le p < 5$).
- Show that u_λ converges as $\lambda \uparrow \lambda^*$ to a solution $u^* \in L^\infty$ of (3.7.4) with $\lambda = \lambda^*$.
- Show that $\lambda_1(-\triangle - \lambda^* e^{u^*}) = 0$ (use the maximality of the interval $(0, \lambda^*)$).

- Let v be a smooth solution of (3.7.4) with $\lambda = \lambda^*$. Show that $v = u^*$. (Using the convexity of the exponential, show that $f = -\triangle(v - u^*) - \lambda^* e^{u^*}(v - u^*) \geq 0$. Next, recall that $\lambda_1(-\triangle - \lambda^* e^{u^*}) = 0$ and let $\varphi > 0$ be the corresponding eigenfunction. Note that $\displaystyle\int f\varphi = 0$ and conclude that $f \equiv 0$.)

- Consider now $u_0 \in L^\infty(\Omega)$ and let u be the solution of (3.7.1) with $\lambda = \lambda^*$. If $u_0 \leq u^*$, show that u is global and converges to u^* as $t \to \infty$.

- Suppose now $u_0 \geq u^*$, $u_0 \not\equiv u^*$. Let φ be the first eigenfunction of $-\triangle - \lambda^* e^{u^*}$. Show that $\displaystyle\frac{d}{dt}\int_\Omega (u - u^*)\varphi = \lambda^* \int_\Omega \Big(e^u - e^{u^*} - e^{u^*}(u - u^*)\Big)\varphi \geq \frac{\lambda^*}{2}\int_\Omega (u - u^*)^2\varphi$.

- Show that $T_{\mathrm{m}} < \infty$.

Exercise 3.13.16. Consider the equation (3.7.1) with $\lambda > 0$, where $g(u) = (1 + u)^p$ and $0 < p < 1$. The object of this exercise is to show that there exists a unique stationary solution u_λ, and that for every $u_0 \in L^\infty(\Omega)$, the solution u of (3.7.1) is global and converges exponentially to u_λ as $t \to \infty$. For convenience, set $g(u) = \lambda(1 + |u|)^p$.

- Show that there exists a stationary solution (for example, minimize $\displaystyle\int \{|\nabla u|^2 - G(u)\}$).

- Show that all stationary solutions are nonnegative.

- Show that if u is a stationary solution, then $\lambda_1(-\triangle - g'(u)) > 0$ (minimize $\displaystyle\int \{|\nabla w|^2 - g'(u)w^2\}$ on $\{w \in H^1_0(\Omega);\ \|w\|_{L^2} = 1\}$, and show that $\displaystyle\int w[g(u) - ug'(u)] = \lambda_1 \int uw$).

- Show that if u is a stationary solution, then any other stationary solution v verifies $v \leq u$ (use the property $\lambda_1(-\triangle - g'(u)) > 0$).

- Show the uniqueness of the stationary solution.

- Show that for every $u_0 \in L^\infty(\Omega)$, the solution u of (3.7.1) is global and converges exponentially to u_λ as $t \to \infty$ (use the energy to show that u is bounded).

Exercise 3.13.17. Consider the equation

$$\begin{cases} u_t - \triangle u = \lambda(A - e^{-u}) & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{in } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.13.5}$$

where $\lambda > 0$ and $A > 1$.

- Show that there exists a unique stationary solution $u_\lambda \geq 0$ and that $\lambda_1(-\triangle - \lambda e^{-u_\lambda}) > 0$.

- Show that for every $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, the solution u of (3.13.5) is global and converges exponentially to u_λ as $t \to \infty$.

Exercise 3.13.18. Consider the equation

$$\begin{cases} u_t - u_{xx} = |u|^{p-1}u & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{in } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.13.6}$$

where $p > 1$ and $\Omega = (0, 1)$.

- Show that there exists a unique positive stationary solution φ.
- Show that for every $u_0 \in L^\infty(\Omega)$, $0 \le u_0 \le \varphi$, $u_0 \not\equiv \varphi$, the solution u of (3.13.6) is global and converges exponentially to 0 as $t \to \infty$ (compute $\dfrac{d}{dt} \displaystyle\int \varphi(u - \varphi)$).
- Show that for every $u_0 \in L^\infty(\Omega)$, $u_0 \ge \varphi$, $u_0 \not\equiv \varphi$, the solution u of (3.13.6) blows up in finite time (one can use the property $u\varphi(u^{p-1} - \varphi^{p-1}) \ge \delta(\varphi(u - \varphi))^{\frac{p+1}{2}}$ for some $\delta > 0$).

Exercise 3.13.19. Let $g(u) = |u|^{p-1}u$ with $p > 1 + \dfrac{4}{N}$ and $(N - 2)p < N + 2$. Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)$ be such that $u_0 \ge 0$ and $\triangle u_0 + u_0^p \ge 0$ a.e. in Ω, and let u be the corresponding solution of (3.1). The object of this exercise is to show by an energy method that if $T_{\rm m} < \infty$, then $\|u(t)\|_{L^q} \underset{t \uparrow T_{\rm m}}{\longrightarrow} \infty$ for $q = \dfrac{N(p-1)}{2}$.

- Show that $u(t) \ge 0$ and $u_t(t) \ge 0$ a.e. in Ω for all $t \in [0, T_{\rm m})$.
- Set $E_q(t) = \dfrac{1}{p^2} \displaystyle\int_\Omega |\nabla u^{\frac{q}{2}}|^2 - \dfrac{1}{p+q-1} \displaystyle\int_\Omega u^{p+q-1}$, and show that $E_q(t) \le E_q(0)$ for all $t \in [0, T_{\rm m})$.
- Show that $\dfrac{d}{dt} \displaystyle\int_\Omega u^q = -q(q-1)E_q(t) + \dfrac{p(p+1-q)}{p+q-1} \displaystyle\int_\Omega u^{p+q-1}$.
- Show that $\|u(t)\|_{L^q} \underset{t \uparrow T_{\rm m}}{\longrightarrow} \infty$ (hint: apply Theorem 3.8.1).

Exercise 3.13.20. The object of this exercise is to show that if Ω is a smooth bounded domain of $\mathbb{R}^2$, then there exists $u_0 \in L^1(\Omega)$ such that the equation

$$\begin{cases} u_t - \triangle u = u^2, \\ u(0) = u_0, \end{cases} \tag{3.13.7}$$

does not have any solution $u \in C([0, T], L^1(\Omega)) \cap L^1_{\rm loc}((0, T), L^2(\Omega))$ for any $T > 0$.

Let $u \in C([0, T], L^1(\Omega)) \cap L^1_{\rm loc}((0, T), L^2(\Omega))$ be a solution of (3.13.7), and set $v(t) = T(t)u_0$, where $(T(t))_{t \ge 0}$ is the semigroup of the heat equation.

- Show that $u(t) \ge v(t)$. (Hint: write Duhamel's formula between $\varepsilon > 0$ and t, then let $\varepsilon \downarrow 0$.)
- Show that $\zeta u \in L^2((0, T) \times \Omega)$, for every $\zeta \in \mathcal{D}(\Omega)$, $\zeta \ge 0$. (Hint: multiply the equation by ζ and integrate on $(0, T) \times \Omega$.)
- Assuming $u_0 \ge 0$, show that $\zeta v \in L^2((0, T) \times \Omega)$.
- Conclude. (Apply Exercise 3.13.21 below.)

Exercise 3.13.21. Let $N \ge 1$ and let $\Omega \subset \mathbb{R}^N$ be an arbitrary open domain. Fix any open ball $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$. The object of this exercise is to show that there is some $u_0 \in L^1(\Omega)$, $u_0 \ge 0$ such that $v(t) = T(t)u_0$ satisfies

$$\int_0^1 \int_\omega v^{\frac{N+2}{N}}(t, x)\, dx\, dt = +\infty.$$

Argue by contradiction and suppose that for every $u_0 \in L^1(\Omega)$, $u_0 \ge 0$,

$$\int_0^1 \int_\omega v^{\frac{N+2}{N}}(t, x)\, dx\, dt < \infty.$$

- Show that $\displaystyle\int_0^1\int_\omega |v|^{\frac{N+2}{N}}(t,x)\,dx\,dt<\infty$, for every $u_0\in L^1(\Omega)$.
- Show that there is a constant C such that
$$\int_0^1\int_\omega |v|^{\frac{N+2}{N}}(t,x)\,dx\,dt\le C\|u_0\|_{L^1}^{\frac{N+2}{N}}, \tag{3.13.8}$$
for every $u_0\in L^1(\Omega)$. (Apply the closed graph theorem to the operator $u_0\mapsto v_{|(0,1)\times\omega}$.)
- Let $K(t)$ be the fundamental solution of the heat equation in $\mathbb{R}^N$, i.e. $K(t,x)=(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$. Let d be the distance of ω to $\partial\Omega$, i.e. $d=\inf\{|x-y|;\,x\in\omega,\,y\in\partial\Omega\}$. Show that for all $u_0\in\mathcal{D}(\mathbb{R}^N)$ with $\operatorname{supp}(u)\subset\omega$ and $u_0\ge0$,
$$(K(t)\star u_0)_{|\Omega}\le T(t)u_0+c\int_\Omega u_0,$$
for all $t\ge0$, where $c=\sup\limits_{t>0}(4\pi t)^{-\frac{N}{2}}e^{-\frac{d^2}{4t}}$. (Compare $u(t)=T(t)u_0$ and $v(t)=(K(t)\star u_0)_{|\Omega}-c\|u_0\|_{L^1}$.)
- Show that
$$\int_0^1\int_\omega |K(t,x)|^{\frac{N+2}{N}}(x)\,dx\,dt<\infty.$$
(Consider a sequence $(u_0^n)_{n\ge0}\in\mathcal{D}(\omega)$ such that $u_0^n\ge0$, $\|u_0^n\|_{L^1}\le1$ and $u_0^n\underset{n\to\infty}{\longrightarrow}\delta$ (= the Dirac mass at $x_0\in\omega$) in the weak* topology of measures, apply (3.13.8) to the corresponding solutions $v^n(t,x)$ and use Fatou's lemma.)
- Show by a direct calculation that $\displaystyle\int_0^1\int_\omega |K(t,x)|^{\frac{N+2}{N}}(t,x)\,dx\,dt=\infty$ and conclude.

3.14. Open problems.

Open Problem 3.14.1. What hapens if $q=\dfrac{N+2}{N-2}$ in Exercise 3.13.6?
(cf. Júlia)

Open Problem 3.14.2. What happens in Theorem 3.4.1 if $q=\dfrac{N+2}{N-2}$? (See Remark 3.4.3).
(cf. Júlia)

Open Problem 3.14.3. What happens if $q=\dfrac{N(p-1)}{2}$ in Theorem 3.8.1? Does there exist $\mu>0$ such that $\limsup\limits_{t\uparrow T_{\mathrm{m}}}|\log(T_{\mathrm{m}}-t)|^{-\mu}\|u(t)\|_{L^q}>0$?

Open Problem 3.14.4. What happens in Exercise 3.13.13 if g does not satisfy the growth assumptions (3.13.2) or (3.13.3)?

Open Problem 3.14.5. Assume $N\ge3$, and let $u\in C([0,T],L^{\frac{N}{N-2}}(\Omega))$ solve the equation
$$\begin{cases} u_t-\triangle u=|u|^{\frac{2}{N-2}}u,\\ u_{|\partial\Omega}=0,\\ u(0,x)=0,\end{cases}$$
Does one have $u\equiv0$?

Open Problem 3.14.6. Recall that in the critical case $q=\dfrac{N(p-1)}{2}$ and $q>1$ in Theorem 3.9.1, the time of existence $T(u_0)$ depends (in our proof of Theorem 3.9.1) on u_0 and not only on $\|u_0\|_{L^q}$. It would be interesting to clarify this point. In particular, is it possible to construct a sequence of initial conditions $(u_0^n)_{n\geq 0}$ which is bounded in $L^q(\Omega)$ and such that $T_{\rm m}(u_0^n) \underset{n\to\infty}{\longrightarrow} 0$?

Open Problem 3.14.7. Is there some $u_0 \in L^1(\Omega)$ for which there is no (local) solution of (3.9.1)? This means that there is no $T > 0$ and no function $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$ satisfying (3.9.1) in the sense of Theorem 3.9.1.

Open problem 3.14.8. Is there some $u_0 \in L^1(\Omega)$ for which uniqueness fails in the class $C([0,T], L^1(\Omega)) \cap L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$ for some $T > 0$?

Open problem 3.14.9. Could there be failure of the maximum principle? More precisely, is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ and a solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$ for some $T > 0$ which does not preserve the positivity?

Open problem 3.14.10. Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that problem (3.9.1) with the "truncated" initial condition

$$u_0^n = \min\{u_0, n\},$$

has a (classical) solution u^n on some maximal interval $[0, T_{\rm m}(u_0^n))$ satisfying $T_{\rm m}(u_0^n) \underset{n\to\infty}{\longrightarrow} 0$?

Alternatively, consider the "truncated" problem

$$\begin{cases} u_t^n - \triangle u^n = g^n(u^n) \quad \text{in} \quad (0,\infty)\times\Omega, \\ u^n = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \\ u^n(0) = u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where $g^n(t) = \min\{|t|^p, n\}\,{\rm sign}t$. Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that $u^n(t,x) \underset{n\to\infty}{\longrightarrow} +\infty$ for all $x \in \Omega$ and all $t > 0$?

Open Problem 3.14.11. Is there some $u_0 \in L^q(\Omega)$ for which there is no (local) solution of (3.9.1)? This means that given any $T > 0$ (as small as we please) there is no function $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\rm loc}((0,T), L^\infty(\Omega))$ satisfying (3.9.1).

Here is a suggestion how to construct such a u_0. Let Ω be the unit ball in $\mathbb{R}^N$, and let $\varphi = \varphi(r)$ with $r = |x|$, $\varphi \in C^1(\overline{\Omega})$, $\varphi > 0$ in Ω, $\varphi = 0$ on $\partial\Omega$, $\varphi'(r) < 0$ for $r \in (0,1)$, $\varphi''(0) < 0$ and $\triangle\varphi + \varphi^p \geq 0$ in Ω be such that the solution v of (3.9.1) with the initial condition $v(0) = \varphi$ blows up in finite time $T_{\rm m}$. (It is well known that such a φ exists.) By Theorem 2.4 of Friedman and McLeod [43],

$$\sup_{0\leq t<T_{\rm m}} \|v(t)\|_{L^q} < \infty \quad \text{for all} \quad 1 \leq q < \frac{N(p-1)}{2}.$$

Set

$$u_0 = \lim_{t\uparrow T_{\rm m}} v(t).$$

This u_0 belongs to $L^q(\Omega)$ for all $1 \leq q < \dfrac{N(p-1)}{2}$. We suspect that for such an initial condition u_0, there exists no local solution of (3.9.1) in any reasonable sense. That there is no nonnegative solution follows from

Baras and Cohen [7]. Indeed, suppose there is a nonnegative solution u of (3.9.1) with the initial condition $u(0) = u_0$ on $[0, T]$ for some $T > 0$. Set

$$w(t) = \begin{cases} v(t) & \text{for} \quad 0 < t \le T_\mathrm{m}, \\ u(t - T_\mathrm{m}) & \text{for} \quad T_\mathrm{m} \le t \le T_\mathrm{m} + T. \end{cases}$$

This is an integral solution of (3.9.1) in the sense of Baras-Cohen [7] and Baras-Pierre [8] which blows up at $t = T_\mathrm{m}$. From [7], one knows that the only way to continue a solution beyond blow up time is by $+\infty$ everywhere.

Open Problem 3.14.12. What happens if $N = 1$ or $N = 2$ in Theorem 3.8.3? (Note that by Remark 3.9.16, the conclusion of the theorem holds under more restrictive assumptions on g.)

Open Problem 3.14.13. Does $\|u(t)\|_{L^q}$ remain bounded as $t \uparrow T_\mathrm{m}$, for **any** $u_0 \in L^\infty(\Omega)$ with $T_\mathrm{m} < \infty$ and any q, $1 \le q < \dfrac{N(p-1)}{2}$? (The answer is positive in some cases, see Theorem 2.4 of Friedman and McLeod [43].)

Attention, suivre les developpements!

Open Problem 3.14.14. Assume $p \ge \dfrac{N+2}{N}$ and $0 \in \Omega$. Prove or disprove that given any $T > 0$, there is a function $u \in L^p_{\mathrm{loc}}((0, T) \times \Omega)$ satisfying (3.10.1) in $\mathcal{D}'((0, T) \times \Omega)$ and such that

$$\operatorname*{ess\,lim}_{t \downarrow 0} \int_\Omega u(t, x) \varphi(x)\, dx = \varphi(0),$$

for all $\varphi \in C_\mathrm{c}(\Omega)$. (We suspect that the answer is negative. One knows that there is no solution $u \ge 0$. The proof is the same as in Theorem 3.9.14.)

Open Problem 3.14.15. We know by Theorem 3.1.1 that the mapping $u_0 \mapsto T_\mathrm{m}(u_0)$ is lower semicontinuous on $L^\infty(\Omega)$. Prove or disprove that this mapping is continuous on $L^\infty(\Omega)$.

In connection with this problem, we call attention to a positive result of Baras and Cohen [7] when $g(u) = u^p$ with $p < \dfrac{N+2}{N-2}$ and $u_0 \ge 0$. A possible suggestion to construct a discontinuity of T_m when $p \ge \dfrac{N+2}{N-2}$ would be the following: consider an initial condition $u_0 \ge 0$ such that the problem

$$\begin{cases} u_t - \triangle u = u^p, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

has a "weak" solution u on $(0, T) \times \Omega$ which blows up (in the L^∞ norm) at some $T_\mathrm{m} < T$. For the existence of such a u, see Open Problem 3.14.16. By Theorem ???, $T_\mathrm{m}((1-\varepsilon)u_0) \ge T$, and thus $T_\mathrm{m}((1-\varepsilon)u_0)$ cannot converge to $T_\mathrm{m}(u_0)$.

Open Problem 3.14.16. peaking solutions

Open Problem 3.14.17. peut-on avoir une solution qui explose a T_m, qui peut etre etendue a $[0, T)$ $T_\mathrm{m} < T < \infty$, et qui explose totalement en T?

Appendix.

In this chapter, we introduce the basic tools that are necessary for the study of evolution equations. They concern functional analysis, integration theory, Sobolev spaces, elliptic equations and some inequalities. As a rule, we only give the proof of those results that do not appear frequently in the litterature in the present form, or whose proof is especially simple. We give references for the results that we do not prove.

A.1. Functional analysis. We recall here some useful theorems of functional analysis. Some of those results are quite classical and can be found in any textbook on elementary functional analysis (see for example Brezis [17], Yosida [102]).

Theorem A.1.1. (The Banach fixed point Theorem) *Let (E, d) be a complete metric space and let $f : E \to E$ be Lipschitz continuous with Lipschitz constant L. If $L < 1$, then f has a unique fixed point $x_0 \in E$.*

Theorem A.1.2. (The closed graph Theorem) *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear mapping. Then $A \in \mathcal{L}(X, Y)$ if and only if the graph of A (i.e. the set of $(x, y) \in X \times Y$ such that $y = Ax$) is a closed subspace of $X \times Y$.*

Theorem A.1.3. (The Lax-Milgram Theorem) *Let H be a Hilbert space with norm $\| \cdot \|_H$ and consider a bilinear functional $a : H \times H \to \mathbb{R}$. If there exist $C < \infty$ and $\alpha > 0$ such that*

$$\begin{cases} |a(u,v)| \le C\|u\| \, \|v\|_H, \text{ for all } (u,v) \in H \times H \text{(continuity)}, \\ |a(u,u)| \ge \alpha \|u\|_H^2, \text{ for all } u \in H \text{ (coerciveness)}, \end{cases}$$

then, for every $f \in H^\star$ (the dual space of H), the equation

$$a(u,v) = \langle f, v \rangle_{H^\star, H}, \text{ for all } v \in H,$$

has a unique solution $u \in H$.

Proposition A.1.4. *Let X and Y be Banach spaces, let E be a subset of X, and let $(A_\lambda)_{\lambda \in (-1,1)}$ be a bounded family in $\mathcal{L}(X, Y)$. If $\lim_{\lambda \to 0} A_\lambda x = 0$, for all $x \in E$, then $\lim_{\lambda \to 0} A_\lambda x = 0$, for all $x \in \overline{E}$.*

Proof. Let $x \in \overline{E}$ and let $(x_n)_{n \in \mathbb{N}} \subset E$ converge to x as $n \to \infty$. There exists $C < \infty$ such that for all $n \in \mathbb{N}$,

$$\|A_\lambda x\| \le \|A_\lambda x_n\| + C\|x - x_n\|.$$

Given $\varepsilon > 0$, we have $C\|x - x_{n_0}\| \le \varepsilon/2$, for n_0 large enough. Then for λ small enough, we have $\|A_\lambda x_{n_0}\| \le \varepsilon/2$. Hence the result. □

Proposition A.1.5. *Let X and Y be two Banach spaces such that $X \hookrightarrow Y$ (i.e. $X \subset Y$ with continuous injection), with dense embedding. Then, the following properties hold:*

(i) *$Y^\star \hookrightarrow X^\star$, where the embedding is defined by $\langle f, x \rangle_{X^\star, X} = \langle f, x \rangle_{Y^\star, Y}$, for all $x \in X$ and $f \in Y^\star$;*

(ii) *if X is reflexive, then the embedding $Y^\star \hookrightarrow X^\star$ is dense.*

Proof. (i) Consider $y' \in Y^\star$ and $x \in X \hookrightarrow Y$. Let $\Phi_{y'}(x) = \langle y', x\rangle_{Y^\star,Y}$. It is clear that $\Phi \in \mathcal{L}(Y^\star, X^\star)$. Suppose that $\Phi_{y'} = \Phi_{z'}$, for some $y', z' \in Y^\star$. Then $\langle y' - z', x\rangle_{Y^\star,Y} = 0$, for every $x \in X$. By density, it follows that $\langle y' - z', y\rangle_{Y^\star,Y} = 0$, for every $y \in Y$; and so $y' = z'$. Hence (i), Φ being the embedding.

(ii) Assume to the contrary that $\overline{Y^\star} \neq X^\star$. Then there exists $x_0 \in X^{\star\star} = X$ such that $\langle y', x_0\rangle_{X^\star,X} = 0$, for every $y' \in Y^\star$ (see Brezis [17], Corollary I.8). Let $E = \mathbb{R}x_0 \subset Y$, and let $f \in E^\star$ be defined by $f(\lambda x_0) = \lambda$, for $\lambda \in \mathbb{R}$. We have $\|f\|_{E^\star} = 1$, and by Hahn-Banach theorem (see Brezis [17], Corollary I.2) there exists $y' \in Y^\star$ such that $\|y'\|_{Y^\star} = 1$ and $\langle y', x_0\rangle_{Y^\star,Y} = 1$, which is a contradiction, since $\langle y', x_0\rangle_{Y^\star,Y} = \langle y', x_0\rangle_{X^\star,X} = 0$. □

Remark A.1.6. Reflexivity is important in property (ii). For example, if $X = \ell^1(\mathbb{N})$ and $Y = \ell^2(\mathbb{N})$, then $X \hookrightarrow Y$ with dense embedding. However, $X^\star = \ell^\infty(\mathbb{N})$ and $Y^\star = \ell^2(\mathbb{N})$, and the embedding $\ell^2(\mathbb{N}) \hookrightarrow \ell^\infty(\mathbb{N})$ is not dense.

If X is a separable Banach space, then its dual $X^\star$ needs not be separable. (For example $X = L^1(\Omega)$ is separable, but it dual $L^\infty(\Omega)$ is not). However, $X^\star$ is weak* separable, as shows the following result.

Lemma A.1.7. *Let X be a separable Banach space and let $X^\star$ be its dual. There exists a sequence $(x'_n)_{n\in\mathbb{N}} \subset X^\star$ such that for every $x' \in X^\star$, there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ with the following properties:*

(i) *$x'_{n_k} \to x'$ weak-$\star$ as $k \to \infty$.*

(ii) *$\|x'_{n_k}\|_{X^\star} \leq \|x'\|_{X^\star}$.*

(iii) *$\|x'_{n_k}\|_{X^\star} \to \|x'\|_{X^\star}$ as $k \to \infty$.*

Proof. When equipped with the weak-$\star$ topology of $X^\star$, $B' = \{x' \in X^\star;\ \|x\|_{X^\star} \leq 1\}$ is a compact metric space. In particular, B' is separable and we denote by $(y'_n)_{n\in\mathbb{N}}$ a dense sequence in B'. Let $(x'_n)_{n\in\mathbb{N}}$ be the sequence $\bigcup_{\substack{\lambda\in\mathbb{Q}\\ n\in\mathbb{N}}} \{\lambda y'_n\}$. Given $x' \in X^\star$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $y'_{n_k} \to \dfrac{x'}{\|x'\|_{X^\star}}$ weak-$\star$ as $k \to \infty$. Consider now a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k \to \|x'\|_{X^\star}$ as $k \to \infty$ and $0 < \lambda_k \leq \|x'\|_{X^\star}$. It follows that $\lambda_k y'_{n_k} \to x'$ weak-$\star$ as $k \to \infty$. Furthermore, $\|\lambda_k y'_{n_k}\|_{X^\star} \leq |\lambda_k| \|y'_{n_k}\|_{X^\star} \leq \|x'\|_{X^\star}$. Since also $\|x'\|_{X^\star} \leq \liminf_{k\to\infty} \|\lambda_k y'_{n_k}\|_{X^\star}$, the result follows. □

Lemma A.1.8. *Let $X \hookrightarrow Y$ be two Banach spaces and let $(x_n)_{n\in\mathbb{N}} \subset X$. If $x_n \rightharpoonup x$ in X, as $n \to \infty$, then $x_n \rightharpoonup x$ in Y, as $n \to \infty$.*

Proof. The embedding is continuous $X \to Y$; and so, it is also continuous $X \to Y$ for the weak topologies. The result follows. □

Lemma A.1.9. *Let $X \hookrightarrow Y$ be two Banach spaces and let $(x_n)_{n\in\mathbb{N}} \subset X$ be a bounded sequence in X such that $x_n \rightharpoonup y$ in Y, as $n \to \infty$, for some $y \in Y$. If X is reflexive, then $y \in X$ and $x_n \rightharpoonup y$ in X, as $n \to \infty$.*

Proof. Let us first prove that $y \in X$. There exists $x \in X$ and a subsequence n_k such that $x_{n_k} \rightharpoonup x$ in X, as $k \to \infty$. Therefore, by Lemma A.1.8, $x_{n_k} \rightharpoonup x$ in Y, as $k \to \infty$. It follows that $y = x \in X$.

Let us prove that $x_n \rightharpoonup y$ in X by contradiction. If not, there exists $x' \in X^\star$, $\varepsilon > 0$ and a subsequence n_k such that $|\langle x', x_{n_k} - y\rangle| \geq \varepsilon$, for every $k \in \mathbb{N}$. On the other hand, there exists $x \in X$ and a subsequence n_{k_j} such that $x_{n_{k_j}} \rightharpoonup x$ in X as $j \to \infty$. In particular, $x = y$; and so $x_{n_{k_j}} \rightharpoonup y$ in X as $j \to \infty$, which is a contradiction. □

Corollary A.1.10. *Let $X \hookrightarrow Y$ be two Banach spaces. If Y is separable and X is reflexive, then X is separable.*

Proof. Let B be the closed unit ball of X. Since $B \subset Y$ and Y is separable, it follows that B is separable for the Y norm. Therefore, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset B$ such that for every $x \in X$, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to x strongly in Y, hence weakly in X by Lemma A.1.9. Therefore, B is contained in, hence equal to the weak X closure of the set $(x_n)_{n\in\mathbb{N}}$. In particular, B is also the weak X closure of the convex hull C of the set $(x_n)_{n\in\mathbb{N}}$. Since the weak and strong closures of convex sets coincide, it follows that C is strongly X dense in B. Since the convex hull of a countable set is clearly separable, it follows that B is separable, which completes the proof. □

Remark A.1.11. Note that if X is not reflexive, then the conclusion of Corollary A.1.10 may be invalid. For example, if Ω is a bounded subset of $\mathbb{R}^N$, then $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ but $L^\infty(\Omega)$ is not separable.

Corollary A.1.12. *Let $X \hookrightarrow Y$ be two Banach spaces, let I be a bounded, open interval of $\mathbb{R}$, and let $u : \overline{I} \to Y$ be a weakly continuous function. Assume that there exists a dense subset E of I such that*

(i) $u(t) \in X$, *for all* $t \in E$,

(ii) $\sup\{\|u(t)\|_X, t \in E\} = K < \infty$.

If X is reflexive, then $u(t) \in X$ for all $t \in \overline{I}$ and $u : \overline{I} \to X$ is weakly continuous.

Proof. Let $t \in \overline{I}$ and let $(t_n)_{n\in\mathbb{N}} \subset E$ converge to t, as $n \to \infty$. Since $u(t_n) \rightharpoonup u(t)$ in Y, it follows from Lemma A.1.9 that $u(t) \in X$ and that

$$\|u(t)\|_X \leq \liminf_{n\to\infty} \|u(t_n)\|_X \leq K.$$

Let now $t \in \overline{I}$ and let $(t_n)_{n\in\mathbb{N}} \subset \overline{I}$ converge to t, as $n \to \infty$. Since $u(t_n) \rightharpoonup u(t)$ in Y and $u(t_n)$ is bounded in X, it follows from Lemma A.1.9 that $u(t_n) \rightharpoonup u(t)$ in X. Hence the result. □

The proofs of the following two lemmas are similar to the proof of Corollary A.1.12 above and are left to the reader.

Lemma A.1.13. *Let X be a uniformly convex Banach space, let I be a bounded, open interval of $\mathbb{R}$ and let $u : \overline{I} \to X$ be weakly continuous. If $t \mapsto \|u(t)\|_X$ is continuous $\overline{I} \to \mathbb{R}$, then $u \in C(\overline{I}, X)$.*

Lemma A.1.14. *Let X be a Banach space, let I be a bounded, open interval of $\mathbb{R}$ and let $u : \overline{I} \to X$ be weakly continuous. If there exists a Banach space B such that $X \hookrightarrow B$ with compact embedding, then $u \in C(\overline{I}, B)$.*

The following compactness result is very helpful for passing to the limit in certain nonlinear evolution equations. Its proof is quite simple.

Proposition A.1.15. *Let $X \hookrightarrow Y$ be two Banach spaces, let I be a bounded, open interval of $\mathbb{R}$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C(\overline{I}, Y)$. Assume that $f_n(t) \in X$, for all $(n,t) \in \mathbb{N} \times I$ and $\sup\{\|f_n(t)\|_X ; (n,t) \in \mathbb{N} \times I\} = K < \infty$, and that f_n is uniformly equicontinuous in Y (i.e. $\forall \varepsilon > 0, \exists \delta > 0, \forall n,s,t \in \mathbb{N} \times I \times I, \|f_n(t) - f_n(s)\|_Y \le \varepsilon$ if $|t-s| \le \delta$). If X is reflexive, then the following properties hold:*

(i) *there exists a function $f \in C(\overline{I}, Y)$ which is weakly continuous $\overline{I} \to X$ and a subsequence n_k such that $f_{n_k}(t) \rightharpoonup f(t)$ in X as $k \to \infty$, for all $t \in \overline{I}$;*

(ii) *if there exists a uniformly convex Banach space B such that $X \hookrightarrow B \hookrightarrow Y$ and if $(f_n)_{n\in\mathbb{N}} \subset C(\overline{I}, B)$ and $\|f_{n_k}(t)\|_B \to \|f(t)\|_B$ as $k \to \infty$, uniformly on I, then also $f \in C(\overline{I}, B)$ and $f_{n_k} \to f$ in $C(\overline{I}, B)$ as $k \to \infty$;*

(iii) *if there exists a Banach space B such that $X \hookrightarrow B \hookrightarrow Y$, where the embedding $X \hookrightarrow B$ is compact, then also $f \in C(\overline{I}, B)$ and $f_{n_k} \to f$ in $C(\overline{I}, B)$ as $k \to \infty$.*

Proof. (i) Let $(t_n)_{n\in\mathbb{N}}$ be a representation of $\mathbb{Q} \cap I$. It follows easily from the reflexivity of X and the diagonal procedure that there exists a subsequence n_k and a function $f : \mathbb{Q} \cap I \to X$ such that $f_{n_k}(t_j) \rightharpoonup f(t_j)$ in X (hence in Y) as $k \to \infty$, for all $j \in \mathbb{N}$. It follows from the uniform equicontinuity of $(f_n)_{n\in\mathbb{N}}$ and the weak lower semicontinuity of the norm that f can be extended to a function of $C(\overline{I}, Y)$. Furthermore, by Lemma A.1.9 and Corollary A.1.12, $f : \overline{I} \to X$ is weakly continuous and $\sup\{\|f(t)\|_X, t \in I\} \le K$. Consider now $t \in \overline{I}$, let $(t_j)_{j\in\mathbb{N}} \subset \mathbb{Q} \cap I$ converge to t and let $y' \in Y^\star$. We have

$$\begin{aligned} |\langle y', f_{n_k}(t) - f(t)\rangle_{Y^\star,Y}| &\le |\langle y', f_{n_k}(t) - f_{n_k}(t_j)\rangle_{Y^\star,Y}| \\ &\quad + |\langle y', f(t) - f(t_j)\rangle_{Y^\star,Y}| + |\langle y', f_{n_k}(t_j) - f(t_j)\rangle_{Y^\star,Y}|. \end{aligned}$$

Given $\varepsilon > 0$, it follows from the uniform equicontinuity that the first and second terms of the right-hand side are less than $\varepsilon/4$ for j large enough. Given such a j, the third term is less than $\varepsilon/2$ for k large enough; and so

$$|\langle x', f_{n_k}(t) - f(t)\rangle_{Y^\star,Y}| \to 0, \text{ as } k \to \infty.$$

It follows that $f_{n_k}(t) \rightharpoonup f(t)$ in Y; and so $f_{n_k}(t) \rightharpoonup f(t)$ in X, by Lemma A.1.9. Hence (i).

(ii) Note first that $f : \overline{I} \to B$ is weakly continuous. Also, $\|f\|_B : \overline{I} \to \mathbb{R}$ is continuous; and so (Lemma A.1.13) $f \in C(\overline{I}, B)$. It remains to prove that $f_{n_k} \to f$ in $C(\overline{I}, B)$. We argue by contradiction, and we assume that there exists a sequence $(t_k)_{k\in/N} \subset \overline{I}$ and $\varepsilon > 0$ such that $\|f_{n_k}(t_k) - f(t_k)\|_B \ge \varepsilon$, for every $k \in \mathbb{N}$. We may assume that $t_k \to t \in \overline{I}$, as $k \to \infty$. It follows from (i) and uniform continuity that

$f_{n_k}(t_k) \rightharpoonup f(t)$ in Y, as $k \to \infty$. Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $C(\overline{I}, B)$, we have as well $f_{n_k}(t_k) \rightharpoonup f(t)$ in B, as $k \to \infty$. Furthermore,

$$| \, \|f_{n_k}(t_k)\|_B - \|f(t)\|_B \, | \le | \, \|f_{n_k}(t_k)\|_B - \|f(t_k)\|_B \, | + | \, \|f(t_k)\|_B - \|f(t)\|_B \, |.$$

It follows that $\|f_{n_k}(t_k)\|_B \to \|f(t)\|_B$; and so, $f_{n_k}(t_k) \to f(t)$ in B, as $k \to \infty$, which is a contradiction.

(iii) It follows from Lemma A.1.14 that $(f_n)_{n\in\mathbb{N}} \subset C(\overline{I}, B)$ and $f \in C(\overline{I}, B)$. It remains to prove that $f_{n_k} \to f$ in $C(\overline{I}, B)$. We argue by contradiction, and we assume that there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset \overline{I}$ and $\varepsilon > 0$ such that $\|f_{n_k}(t_k) - f(t_k)\|_B \ge \varepsilon$, for every $k \in \mathbb{N}$. It follows from (i) and uniform continuity that $f_{n_k}(t_k) - f(t_k) \rightharpoonup 0$ in Y, as $k \to \infty$. It follows that $\|f_{n_k}(t_k) - f(t_k)\|_B \to 0$, which is a contradiction. □

The following compactness result is very helpful for passing to the limit in certain nonlinear evolution equations. Its proof is quite simple.

Theorem A.1.16. *Let X be a Banach space, let $T > 0$ and let $f \in C([0,T), X)$. Assume that f is right-differentiable for all $t \in [0,T)$. If $\dfrac{d^+ f}{dt} \in C([0,T), X)$, then $f \in C^1([0,T), X)$ and $\dfrac{df}{dt} = \dfrac{d^+ f}{dt}$.*

Proof. Set

$$g(t) = f(t) - f(0) - \int_0^t \frac{d^+ f}{dt}\, ds,$$

for all $t \in [0,T)$. It follows that $g \in C([0,T), X)$, $g(0) = 0$, g is right-differentiable for all $t \in [0,T)$ and $\dfrac{d^+ g}{dt} = 0$. Let now $\xi \in X^\star$, and set $h(t) = \langle \xi, g(t)\rangle_{X^\star, X}$. We have $h \in C([0,T))$, $h(0) = 0$, h is right-differentiable for all $t \in [0,T)$ and $\dfrac{d^+ h}{dt} = 0$. We show that $h \equiv 0$. To see this, let $\varepsilon > 0$, set $h_\varepsilon(t) = h(t) - \varepsilon t$, and let us show that $h_\varepsilon \le 0$. Otherwise, there exists $t \in [0,T)$ such that $h_\varepsilon(t) > 0$. Let $\tau = \inf\{t \in [0,T);\, h_\varepsilon(t) > 0\}$. We have $h_\varepsilon(\tau) = 0$, and there exists $t_n \downarrow \tau$ such that $h_\varepsilon(t_n) > 0$. It follows that

$$\limsup_{t \downarrow \tau} \frac{h_\varepsilon(t) - h_\varepsilon(\tau)}{t - \tau} \ge 0.$$

On the other hand, we have $\dfrac{d^+ h_\varepsilon}{dt} = -\varepsilon$, which is a contradiction. Therefore, $h_\varepsilon \le 0$. Since $\varepsilon > 0$ is arbitrary, we have $h \le 0$. Applying the same argument to $-h$, we obtain as well $h \ge 0$, hence $h \equiv 0$. Therefore, given $t \in [0,T)$, we have $\langle \xi, g(t)\rangle_{X^\star, X} = 0$ for all $\xi \in X^\star$; and so, $g(t) \equiv 0$. The result follows easily. □

We will also use some properties of the intersection and sum of Banach spaces. Consider two Banach spaces X_1 and X_2 that are subsets of a Hausdorff topological vector space $\mathcal{X}$. Let

$$X_1 \cap X_2 = \{x \in \mathcal{X};\, x \in X_1,\, x \in X_2\},$$

and

$$X_1 + X_2 = \{x \in \mathcal{X};\, \exists x_1 \in X_1,\, \exists x_2 \in X_2,\, x = x_1 + x_2\}.$$

Define

$$\|x\|_{X_1 \cap X_2} = \|x\|_{X_1} + \|x\|_{X_2}, \text{ for } x \in X_1 \cap X_2,$$

and

$$\|x\|_{X_1+X_2} = \mathrm{Inf}\{\|x_1\|_{X_1} + \|x_2\|_{X_2};\ x = x_1 + x_2\},\ \text{for } x \in X_1 + X_2.$$

We have the following result.

Proposition A.1.17. *$(X_1 \cap X_2, \| \|_{X_1 \cap X_2})$ and $(X_1 + X_2, \| \|_{X_1+X_2})$ are Banach spaces. If furthermore $X_1 \cap X_2$ is a dense subset of both X_1 and X_2, then the following properties hold:*

(i) *$(X_1 \cap X_2)^\star = X_1^\star + X_2^\star$ and $(X_1 + X_2)^\star = X_1^\star \cap X_2^\star$;*

(ii) *$\langle f, x_1 + x_2 \rangle_{X_1^\star \cap X_2^\star, X_1+X_2} = \langle f, x_1 \rangle_{X_1^\star, X_1} + \langle f, x_2 \rangle_{X_2^\star, X_2}$, for all $f \in X_1^\star \cap X_2^\star$ and $(x_1, x_2) \in X_1 \times X_2$;*

(iii) *$\langle f_1 + f_2, x \rangle_{X_1^\star + X_2^\star, X_1 \cap X_2} = \langle f_1, x \rangle_{X_1^\star, X_1} + \langle f_2, x \rangle_{X_2^\star, X_2}$, for all $(f_1, f_2) \in X_1^\star \times X_2^\star$ and $x \in X_1 \cap X_2$;*

(iv) *if X_1 and X_2 are reflexive, then $X_1 \cap X_2$ and $X_1 + X_2$ are reflexive.*

Proof. The first properties follow from Bergh and Löfström [13] (Lemma 2.3.1 and Theorem 2.7.1), as well as properties (ii) and (iii) (proof of Theorem 2.7.1). Finally, it remains to prove Property (iv). By (i), it is sufficient to show that $X_1 \cap X_2$ is reflexive. By applying Eberlein-Šmulian's theorem, we need to show that every bounded sequence $(x_n)_{n \in \mathbb{N}} \subset X_1 \cap X_2$ has a weakly convergent subsequence. Since x_n is bounded in both X_1 and X_2, there exists $x \in X_1 \cap X_2$ and a subsequence, which we still denote by $(x_n)_{n \in \mathbb{N}}$, such that $x_n \rightharpoonup x$, in X_1 and in X_2. Given $(f_1, f_2) \in X_1^\star \times X_2^\star$, we have

$$\langle f_1, x_n \rangle_{X_1^\star, X_1} + \langle f_2, x_n \rangle_{X_2^\star, X_2} \underset{n \to \infty}{\longrightarrow} \langle f_1, x \rangle_{X_1^\star, X_1} + \langle f_2, x \rangle_{X_2^\star, X_2}.$$

By property (iii), this implies that $x_n \rightharpoonup x$ in $X_1 \cap X_2$. □

Remark A.1.18. It is clear that the definition of the spaces $X_1 \cap X_2$ and $X_1 + X_2$ as well as their properties described in Proposition A.1.17 are independent of the Hausdorff space $\mathcal{X}$. It follows that an element of $X_1 + X_2$ is equal to zero if and only if it is equal to zero in some Hausdorff space containing $X_1 \cup X_2$. In particular, if X_1 and X_2 are spaces of distributions on some open set $\Omega \subset \mathbb{R}^N$, then an element of $X_1 + X_2$ is equal to zero if and only if it is equal to zero in $\mathcal{D}'(\Omega)$.

Finally, we recall below the main properties of the exponential of a linear continuous operator. Consider a Banach space and a linear continuous operator $A \in \mathcal{L}(X, X)$. We recall that e^A (the exponential of A) is the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

It is clear that the series is normally convergent in $\mathcal{L}(X)$ and that $\|e^A\| \le e^{\|A\|}$. Furthermore, if A and B commute then $e^{A+B} = e^A e^B$. In addition, the function $t \mapsto e^{tA}$ is in $C^\infty(\mathbb{R}, \mathcal{L}(X))$ and

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA} A,$$

for all $t \in \mathbb{R}$.

Proposition A.1.19. *For every $x \in X$, there exists a unique solution $u \in C(\mathbb{R}, X)$ of the following problem:*

$$\begin{cases} \dfrac{du}{dt} = Au(t), \text{ for all } t \in \mathbb{R} \\ u(0) = x. \end{cases}$$

This solution is given by $u(t) = e^{tA}x$, for $t \in \mathbb{R}$

Proof. It is clear that $e^{tA}x$ is a solution of the problem. Let v be another solution and let $w(t) = e^{-tA}v(t)$. It follows that

$$\frac{dw}{dt} = e^{-tA}Av(t) - Ae^{-tA}v(t) = 0, \text{ for every } t \in \mathbb{R}.$$

Therefore, $w(t) \equiv w(0) = x$; and so $v \equiv e^{tA}x$. □

A.2. Vector integration. Vector integration is essential in the study of evolution equation. Even though most existence and regularity results are stated in terms of continuous functions, weaker regularity classes often appear in intermediate steps.

We present here a few basic results on vector integration that are essential in the theory of evolution equations. Throughout this section, X is a Banach space with the norm $\| \ \|$ and I is an open interval of $\mathbb{R}$ (bounded or unbounded) equipped with the Lebesgue measure. We will use the basic theorems of real valued integration (Fatou's lemma, the monotone convergence theorem, the dominated convergence theorem, Egorov's theorem in particular). We will also use the main result of the Lebesgue's points theory, which we recall below (see Dunford and Schwartz [39], Theorem III.12.8, p. 217, Rudin [89], Theorem 8.8, p. 158).

Theorem A.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally integrable and let*

$$F_h(t) = \frac{1}{h}\int_t^{t+h} f(s)\,ds, \text{ for all } t \in \mathbb{R} \text{ and all } h \neq 0.$$

There exists a set E of measure 0, such that $\lim_{h\to 0} F_h(t) = f(t)$, for all $t \in \mathbb{R} \setminus E$. The set $\mathbb{R} \setminus E$ is called the set of Lebesgue's points of f. In particular, the function

$$t \mapsto \int_0^t f(s)\,ds$$

is differentiable almost everywhere and its derivative is equal to f almost everywhere.

Finally, we will use the following well known property.

Theorem A.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $1 \le p \le \infty$. If there exists $\varphi, \psi \in L^p(\mathbb{R})$ such that*

$$|f(t) - f(s)| \le \left| \int_s^t \varphi(\sigma)\,d\sigma \right| + |t-s|\,|\psi(s)|,$$

for almost all $s, t \in \mathbb{R}$, then the following properties hold:

(i) *f is differentiable almost everywhere.*

(ii) *$f' \in L^p(\mathbb{R})$ and $|f'| \le |\varphi| + |\psi|$ almost everywhere.*

(iii) $f(t) = f(0) + \int_0^t f'(s)\,ds$ *for all* $t \in \mathbb{R}$.

Proof. Observe that $f \in L^p_{\text{loc}}(\mathbb{R})$. We show that there exists $g \in L^p(\mathbb{R})$ such that

$$\int_{\mathbb{R}} g(t)\theta(t)\,dt = -\int_{\mathbb{R}} f(t)\theta'(t)\,dt, \tag{A.2.1}$$

for all $\theta \in C^1_c(\mathbb{R})$. For almost all $h > 0$, define $f_h(t) = \dfrac{f(t+h) - f(t)}{h}$. It follows that

$$|f_h(t)| \le \frac{1}{h}\int_t^{t+h} |\varphi(\sigma)|\,d\sigma + |\psi(t)|.$$

In particular, f_h is bounded in $L^p(\mathbb{R})$ (see Proposition A.2.22 below). If $p > 1$, it follows that there exist a sequence $h_n \downarrow 0$ and $g \in L^p(\mathbb{R})$ such that $f_{h_n} \to g$ as $n \to \infty$, in $L^p(\mathbb{R})$ weak (weak-$\star$ if $p = \infty$). In particular,

$$\int_{\mathbb{R}} f_{h_n}(t)\theta(t)\,dt \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} g(t)\theta(t)\,dt, \tag{A.2.2}$$

for all $\theta \in C^1_c(\mathbb{R})$. If $p = 1$, it follows from Proposition A.2.22 below that

$$\frac{1}{h}\int_{\cdot}^{\cdot+h} \varphi(\sigma)\,d\sigma \underset{h\downarrow 0}{\longrightarrow} \varphi(\cdot)$$

in $L^1(\mathbb{R})$. In particular, there exists a sequence $h_n \downarrow 0$ and $\gamma \in L^1(\mathbb{R})$ such that

$$\Big|\frac{1}{h_n}\int_t^{t+h_n} \varphi(\sigma)\,d\sigma\Big| \le \gamma(t),$$

almost everywhere. Therefore, $|f_{h_n}(t)| \le \gamma(t) + |\psi(t)|$ almost everywhere and it follows from the deep Dunford-Pettis theorem (see Dunford and Pettis [38]) that there exists $g \in L^1(\mathbb{R})$ such that (A.2.2) holds. Now

$$\int_{\mathbb{R}} f_{h_n}(t)\theta(t)\,dt = -\int_{\mathbb{R}} f(t)\frac{\theta(t-h_n) - \theta(t)}{-h_n}\,dt \underset{n\to\infty}{\longrightarrow} -\int_{\mathbb{R}} f(t)\theta'(t)\,dt.$$

(A.2.1) follows from (A.2.2) and the above identity. Next, let $(g_n)_{n\in\mathbb{N}} \subset C^1_c(\mathbb{R})$ be such that $g_n \to g$ in $L^p(\mathbb{R})$ (in $L^\infty(\mathbb{R})$ weak-$\star$ if $p = \infty$). For every $\theta \in C^1_c(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t g(s)\,ds\theta'(t)\,dt &= \lim_{n\to\infty}\int_{\mathbb{R}}\int_0^t g_n(s)\,ds\theta'(t)\,dt \\ &= -\lim_{n\to\infty}\int_{\mathbb{R}} g_n(t)\theta(t)\,dt = -\int_{\mathbb{R}} g(t)\theta(t)\,dt.\end{aligned}$$

Therefore, it follows from (A.2.1) that

$$\int_{\mathbb{R}}\Big(f(t) - \int_0^t g(s)\,ds\Big)\,\theta'(t)\,dt = 0,$$

for all $\theta \in C^1_c(\mathbb{R})$. It follows from Lemma A.2.26 below that there exists $a \in \mathbb{R}$ such that

$$f(t) = a + \int_0^t g(s)\,ds,$$

for all $t \in \mathbb{R}$. The result now follows from Theorem A.2.1. □

Theorem A.2.2 is not anymore valid for functions f with values in a Banach space X. However, it holds when X is a reflexive Banach space. (See Theorem A.2.27 below.)

A.2.1. Measurable functions.

Definition A.2.3. *A function $f : I \to X$ is measurable if there exists a set $N \subset I$ of measure* 0 *and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ such that $\lim\limits_{n\to\infty} f_n(t) = f(t)$, for all $t \in I \setminus N$.*

Remark A.2.4. It follows easily from Definition A.2.3 that if $f : I \to X$ is measurable, then $\|f\| : I \to \mathbb{R}$ is also measurable. Many properties of vector valued measurable functions follow either immediately from the definition or else from the properties of real valued measurable functions applied to $\|f - f_n\|$. In particular, one can show easily the following results.

(i) If $f : I \to X$ is measurable and if Y is a Banach space such that $X \hookrightarrow Y$, then $f : I \to Y$ is measurable.

(ii) If a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $I \to X$ converges almost everywhere (in the X topology) to a function $f : I \to X$, then f is measurable.

(iii) If $f : I \to X$ and $\varphi : I \to \mathbb{R}$ are measurable, then $f\varphi : I \to X$ is measurable. In particular, if $f : I \to X$ is measurable and if $J \subset I$ is an open interval, then $f_{|J} : J \to X$ is measurable (take $\varphi = 1_J$).

(iv) If $(x_n)_{n\in\mathbb{N}}$ is a family of elements of X and if $(\omega_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of I such that $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$, then $\sum\limits_{n=0}^{\infty} x_n 1_{\omega_n} : I \to X$ is measurable.

In Definition A.2.3 and Remark A.2.4, the strong topology of X is involved. However, in many applications, one needs to prove measurability of a function which is only the limit in the weak topology of X of a sequence of measurable functions. For that purpose, a most useful tool is the following result.

Theorem A.2.5. (Pettis' Theorem) *Consider $f : I \to X$. Then f is measurable if and only if it satisfies the following two conditions:*

(i) *f is weakly measurable (i.e. for every $x' \in X^\star$, the function $t \mapsto \langle x', f(t)\rangle$ is measurable $I \to X$);*

(ii) *there exists a set $N \subset I$ of measure* 0 *such that $f(I \setminus N)$ is separable.*

Proof. It is clear that measurability implies weak measurability; and so (i) is necessary. If f is measurable and if $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ converges to f on $I \setminus N$ with $|N| = 0$, then $f_n(I \setminus N)$ is separable; and so $f(I \setminus N)$ is also separable. Therefore (ii) is also necessary.

Let now f satisfy (i) and (ii). By possibly replacing X by the smallest closed subspace of X containing $f(I \setminus N)$, we may assume that X is separable. We first establish that for every $x \in X$, the function $t \mapsto \|f(t) - x\|$ is measurable. Indeed, for $a \geq 0$, we have

$$\{t \in I,\ \|f(t) - x\| \leq a\} = \bigcap_{x' \in S'} \{t \in I,\ |\langle x', f(t) - x\rangle| \leq a\},$$

where S' is the unit ball of $X^\star$. It follows from Lemma A.1.7 that there exists a sequence $(x'_n)_{n\in\mathbb{N}} \subset S'$such that

$$\{t \in I,\ \|f(t) - x\| \leq a\} = \bigcap_{n\in\mathbb{N}} \{t \in I,\ |\langle x'_n, f(t) - x\rangle| \leq a\}.$$

The set on the right-hand side of the above identity is clearly measurable by assumption (i); and so the function $t \mapsto \|f(t) - x\|$ is measurable.

Consider now $n \in \mathbb{N}$. The set $f(I)$ being separable, it can be covered by a countable union of balls B_j^n of center x_j^n and radius $1/n$. Consider $f_n : I \to X$ defined by $f_n = \sum_{j=0}^{\infty} x_j^n 1_{\omega_j^n}$, where $\omega_0^n = \{t \in I,\ f(t) \in B_0^n\}$ and $\omega_j^n = \{t \in I,\ f(t) \in B_j^n \setminus \cap_{k=0}^{j-1} B_k^n\}$, for $j \geq 1$. It is immediate that $\|f(t) - f_n(t)\| \leq 1/n$, for all $t \in I$. Furthermore, since the function $t \mapsto \|f(t) - x\|$ is measurable for all $x \in X$, it follows that the sets ω_j^n are measurable; and so, by Remark A.2.4, (iv), f_n is measurable. Therefore, by Remark A.2.4, (ii) f is measurable. □

Corollary A.2.6. *If $f : I \to X$ is weakly continuous (i.e. $t \mapsto \langle x', f(t)\rangle_{X^\star, X}$ is continuous for every $x' \in X^\star$), then f is measurable.*

Proof. f is clearly weakly measurable; and so by Theorem A.2.5, it is sufficient to prove that $f(I)$ is separable. It follows from the weak continuity of f that $f(I) \subset E$, where E is the weak closure of the convex hull of $f(I \cap \mathbb{Q})$. On the other hand, $E = \overline{f(I \cap \mathbb{Q})}$; and so E is separable. Hence the result. □

Corollary A.2.7. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $I \to X$ and let $f : I \to X$. If, for almost all $t \in I$, $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, then f is measurable.*

Proof. Let $x' \in X^\star$. Since $\langle x', f_n(t)\rangle \to \langle x', f(t)\rangle$ almost everywhere, it follows that the function $t \mapsto \langle x', f(t)\rangle$ is measurable; and so f is weakly measurable.

On the other hand, it follows from Theorem A.2.5 that for every $n \in \mathbb{N}$, there exists a set N_n of measure 0 such that $f_n(I \setminus N_n)$ is separable. Consider the set $N = \cup_{n=0}^{\infty} N_n$, which is also of measure 0, and let C be the convex hull of $\cup_{n=0}^{\infty} f_n(I \setminus E)$. Clearly $f(I \setminus E) \subset \tilde{C}$, where $\tilde{C}$ is the weak closure of C. Furthermore, $\tilde{C} = \overline{C}$; and so $\tilde{C}$ is separable. Hence the result, by Theorem A.2.5. □

Corollary A.2.8. *Let $X \hookrightarrow Y$ be two Banach spaces and let $u : I \to Y$ be weakly continuous. Assume that there exists a dense subset E of I such that*

(i) *$u(t) \in X$, for all $t \in E$,*

(ii) *$\sup\{\|u(t)\|_X, t \in E\} = K < \infty$.*

If X is reflexive, then $u(t) \in X$, for all $t \in I$, and $u : I \to X$ is measurable.

Proof. The result follows from Corollaries A.1.12 and A.2.6. □

Remark A.2.9. Consider two Banach spaces $X \hookrightarrow Y$, and a measurable function $f : I \to Y$. Assume that $f(t) \in X$, for almost all $t \in I$. It is natural to ask whether $f : I \to X$ is measurable. In general, the answer is negative, as shows the following example. Let $I = \Omega = (0, 1)$ and consider the function $u : I \to L^\infty(\Omega)$ given by $u(t) = 1_{(0,t)}$, for $0 < t < 1$. One verifies easily that $u \in C^{0,1/p}(\overline{I}, L^p(\Omega))$, for every $p \in [1, \infty)$. In particular, $u : I \to L^p(\Omega)$ is measurable, for every $p \in [1, \infty)$. Furthermore, $u(t) \in L^\infty(\Omega)$

for all $t \in I$. However, $u : I \to L^\infty(\Omega)$ is not measurable. To see this, observe that $\|u(t) - u(s)\|_{L^\infty} = 1$, if $t \neq s$. Therefore, $u(I)$ is a discrete subset of $L^\infty(\Omega)$; and so, given any non-countable subset A of I, $u(A) \subset L^\infty(\Omega)$ is discrete and non-countable, hence non-separable. In particular, given a subset N of I of measure 0, $u(I \setminus N)$ is not a separable subset of $L^\infty(\Omega)$. Therefore, by Theorem A.2.5, $u : I \to L^\infty(\Omega)$ is not measurable. Note that u is an elementary example of a non-measurable function. However, one can obtain measurability results under additional assumptions. This is the object of the following result.

Proposition A.2.10. *Let $X \hookrightarrow Y$ be two Banach spaces and let $f : I \to Y$ be a measurable function. If $f(t) \in X$ for almost all $t \in I$ and if X is reflexive, then $f : I \to X$ is measurable.*

Proof. By applying Theorem A.2.5 and by modifying f on a set of measure 0, we may assume that $f(I) \subset X$ and that $f(I)$ is a separable subset of Y. By replacing X by the smallest closed subspace of X containing $f(I)$, then by replacing Y by the closure of X in Y, we may assume that Y is separable and that the embedding $X \hookrightarrow Y$ is dense. By applying Lemma A.1.10, it follows that X is separable. Therefore, by applying again Theorem A.2.5, we need only check that f is weakly measurable $I \to X$. To see this, consider $x' \in X^\star$. It follows from Proposition A.1.5 that there exists $(y'_n)_{n\in\mathbb{N}} \subset Y^\star$ such that $y'_n \underset{n\to\infty}{\longrightarrow} x'$ in $X^\star$. In particular,

$$\langle y'_n, f(t)\rangle_{X^\star,X} \underset{n\to\infty}{\longrightarrow} \langle x', f(t)\rangle_{X^\star,X}, \text{ for all } t \in I.$$

On the other hand, it follows from Proposition A.1.5 that $\langle y'_n, f(t)\rangle_{X^\star,X} = \langle y'_n, f(t)\rangle_{Y^\star,Y}$. Therefore, $t \mapsto \langle y'_n, f(t)\rangle_{X^\star,X}$ is measurable; and so, $t \mapsto \langle x', f(t)\rangle_{X^\star,X}$ is measurable. Hence the result. □

A.2.2. Integrable functions.

Definition A.2.11. *A measurable function $f : I \to X$ is integrable if there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ such that $\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\, dt = 0$. Note that by Remark A.2.4, $\|f_n - f\| : I \to \mathbb{R}$ is a nonnegative measurable function, so that $\int_I \|f_n(t) - f(t)\|\, dt$ makes sense.*

Lemma A.2.12. *Let $f : I \to X$ be integrable. There exists $i(f) \in X$ such that for any sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ verifying*

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\, dt = 0,$$

one has

$$\lim_{n\to\infty} \int_I f_n(t)\, dt = i(f),$$

the above limit being for the strong topology of X.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ verify the assumption of the lemma. We have

$$\begin{aligned}\Big\| \int_I f_n(t)\, dt - \int_I f_p(t)\, dt \Big\| &\le \int_I \|f_n(t) - f_p(t)\|\, dt \\ &\le \int_I \|f_n(t) - f(t)\|\, dt + \int_I \|f_p(t) - f(t)\|\, dt.\end{aligned}$$

Therefore, $\int_I f_n(t)\,dt$ is a Cauchy sequence, that converges to some element $x \in X$. Consider another sequence $(g_n)_{n\in\mathbb{N}} \subset C_c(I, X)$. We have

$$\begin{aligned}\|\int_I g_n(t)\,dt - x\| &\le \|\int_I g_n(t) - f(t)\,dt\| + \|\int_I f(t) - f_n(t)\,dt\| + \|\int_I f_n(t)\,dt - x\| \\ &\le \int_I \|g_n(t) - f(t)\|\,dt + \int_I \|f_n(t) - f(t)\|\,dt + \|\int_I f_n(t)\,dt - x\|\end{aligned}$$

Therefore, $\int_I g_n(t)\,dt$ converges also to x, as $n \to \infty$. Hence the result, with $i(f) = x$. □

Definition A.2.13. *The element $i(f)$ constructed in Lemma A.2.12 is called the integral of f on I. We note*

$$i(f) = \int f = \int_I f = \int_I f(t)\,dt.$$

If $I = (a, b)$, we also note

$$i(f) = \int_a^b f = \int_a^b f(t)\,dt.$$

As for real-valued functions, it is convenient to note

$$\int_\alpha^\beta f(t)\,dt = -\int_\beta^\alpha f(t)\,dt,$$

if $\beta < \alpha$.

Theorem A.2.14. (Bochner's Theorem) *Let $f : I \to X$ be measurable. Then f is integrable if and only if $\|f\| : I \to \mathbb{R}$ is integrable. In addition,*

$$\|\int_I f(t)\,dt\| \le \int_I \|f(t)\|\,dt,$$

for all integrable function $f : I \to X$.

Proof. Assume that f is integrable, and consider a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ such that

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\,dt = 0.$$

We have

$$\|f\| \le \|f_n\| + \|f_n - f\|;$$

and so $\|f\|$ is integrable.

Conversely, suppose that f is measurable and that $\|f\|$ is integrable. Let $(g_n)_{n\in\mathbb{N}} \subset C_c(I, \mathbb{R})$ be a sequence such that $g_n \to \|f\|$ in $L^1(I)$ and almost everywhere, and such that $|g_n| \le g$ almost everywhere, for some $g \in L^1(I)$. Let $(f_n)_{n\in\mathbb{N}} \subset C_c(I, X)$ be a sequence such that $f_n \to f$ almost everywhere. Finally, let

$$h_n = \frac{f_n |g_n|}{\|f_n\| + 1/n}.$$

It is clear that $h_n \in C_c(I, X)$, that $\|h_n\| \le g$ almost everywhere and that $h_n \to f$ in X almost everywhere, as $n \to \infty$. It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int \|h_n(t) - f(t)\|\,dt = 0;$$

and so f is integrable. Finally,

$$\|\int f(t)\,dt\| = \lim_{n\to\infty} \|\int h_n(t)\,dt\| \le \lim_{n\to\infty} \int \|h_n(t)\|\,dt \le \int \|f(t)\|\,dt,$$

where the last inequality follows from the dominated convergence theorem. This completes the proof. □

Remark A.2.15. Theorem A.2.14 allows one to deal with vector valued integrable functions like one deals with real valued integrable functions. It suffices in general to apply the usual convergence theorems to $\|f\|$. For example, one can easily establish the following results.

(i) If $f : I \to X$ is integrable and $\varphi \in L^\infty(I)$, then $f\varphi : I \to X$ is integrable. In particular, if $f : I \to X$ is integrable and if $J \subset I$ is an open interval, then $f_{|J} : J \to X$ is integrable (take $\varphi = 1_J$).

(ii) (the dominated convergence theorem) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions $I \to X$, let $f : I \to X$ and let $g \in L^1(I)$. If

$$\begin{cases} \|f_n(t)\| \le g(t), \text{ for almost all } t \in I \text{ and all } n \in \mathbb{N}, \\ \lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t \in I, \end{cases}$$

then f is integrable and $\displaystyle\int_I f(t)\,dt = \lim_{n\to\infty} \int_I f_n(t)\,dt$.

(iii) If Y is a Banach space, if $A \in \mathcal{L}(X,Y)$, and if $f : I \to X$ is integrable, then $Af : I \to Y$ is integrable and

$$\int_I Af(t)\,dt = A\Big(\int_I f(t)\,dt\Big).$$

In particular, if $X \hookrightarrow Y$ and if $f : I \to X$ is integrable, then the integral of f in the sense of X coincides with the integral of f in the sense of Y.

Finally, we have the following important geometric property of integrable functions.

Proposition A.2.16. *Suppose $|I| < \infty$, let $K \subset X$ be a closed convex set, let $f : I \to X$ be integrable and let $y = \dfrac{1}{|I|}\displaystyle\int_I f(t)\,dt$. If $f(t) \in K$ for almost all $t \in I$, then $y \in K$.*

Proof. We argue by contradiction and we assume that $y \notin K$. It follows from Hahn-Banach's theorem (see Brezis [17], Théorème 1.7 p.7) that there exists $x' \in X^\star$ and $\varepsilon > 0$ such that $\langle x', x\rangle_{X^\star,X} \le \langle x', y\rangle_{X^\star,X} - \varepsilon$, for all $x \in K$. In particular,

$$\langle x', f(t)\rangle_{X^\star,X} \le \langle x', y\rangle_{X^\star,X} - \varepsilon,$$

for almost all $t \in I$. Integrating that above inequality and applying Remark A.2.15 (iii), we obtain

$$\langle x', y\rangle_{X^\star,X} = \frac{1}{|I|}\langle x', \int_I f(t)\,dt\rangle_{X^\star,X} = \frac{1}{|I|}\int_I \langle x', f(t)\rangle_{X^\star,X}\,dt \le \langle x', y\rangle_{X^\star,X} - \varepsilon,$$

which is a contradiction. Hence the result. □

A.2.3. The spaces $L^p(I,X)$.

Definition A.2.17. *Let $p \in [1,\infty]$. One denotes by $L^p(I,X)$ the set of (classes of) measurable functions $f : I \to X$ such that the function $t \mapsto \|f(t)\|$ belongs to $L^p(I)$. For $f \in L^p(I,X)$, one defines*

$$\|f\|_{L^p(I,X)} = \left\{ \int_I \|f(t)\|^p \, dt \right\}^{\frac{1}{p}}, \text{ if } p < \infty,$$
$$\|f\|_{L^p(I,X)} = \operatorname*{ess\,sup}_{t \in I} \|f(t)\| \text{ if } p = \infty.$$

When there is no risk of confusion, we denote $\| \, \|_{L^p(I,X)}$ by $\| \, \|_{L^p(I)}$ or $\| \, \|_{L^p}$ or $\| \, \|_p$. One denotes by $L^p_{\mathrm{loc}}(I,X)$ the set of $f : I \to X$ such that $f_{|J} \in L^p(J,X)$, for every bounded sub-interval J of I.

Remark A.2.18. The space $L^p(I,X)$ enjoys most of the properties of the space $L^p(I) = L^p(I,\mathbb{R})$, by the same proofs or by applying the classical results to the function $t \mapsto \|f(t)\|$. In particular, one obtains easily the following results.

(i) $\| \, \|_{L^p(I,X)}$ is a norm on the space $L^p(I,X)$. $L^p(I,X)$ equipped with that norm is a Banach space. If $p < \infty$, then $C^\infty_0(I,X)$ is dense in $L^p(I,X)$ (apply the classical procedure by truncation and regularization). In particular, if Y is a Banach space such that $Y \hookrightarrow X$ with dense embedding, then $C^\infty_0(I,Y)$ is dense in $L^p(I,X)$ (since $C^\infty_0(I,Y)$ is dense in $C_{\mathrm{c}}(I,X)$ for the norm of $C_{\mathrm{b}}(I,X)$).

(ii) A measurable function $f : I \to X$ belongs to $L^p(I,X)$ if and only if there exists a function $g \in L^p(I)$ such that $\|f\| \le g$ almost everywhere on I.

(iii) If $f \in L^p(I,X)$ and $\varphi \in L^q(I)$ with $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} \le 1$, then $\varphi f \in L^r(I,X)$ and

$$\|\varphi f\|_{L^r(I,X)} \le \|f\|_{L^p(I,X)} \, \|\varphi\|_{L^q(I)}.$$

In particular, if $f \in L^p(I,X)$ and if J is an open sub-interval of I, then $f_{|J} \in L^p(J,X)$.

(iv) If $f \in L^p(I,X)$ and $g \in L^q(I,X^\star)$ with $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} \le 1$, then the function h defined by $h(t) = \langle g(t), f(t)\rangle_{X^\star,X}$ is in $L^r(I)$ and we have $\|h\|_{L^r} \le \|f\|_{L^p(I,X)} \, \|g\|_{L^q(I,X^\star)}$.

(v) If $f \in L^p(I,X) \cap L^q(I,X)$ with $p < q$, then $f \in L^r(I,X)$, for every $r \in [p,q]$, and

$$\|f\|_{L^r(I,X)} \le \|f\|^{\theta}_{L^p(I,X)} \, \|f\|^{1-\theta}_{L^q(I,X)},$$

where $\dfrac{1}{r} = \dfrac{\theta}{p} + \dfrac{1-\theta}{q}$.

(vi) If I is bounded and $p \le q$, then $L^q(I,X) \hookrightarrow L^p(I,X)$ and

$$\|f\|_{L^p(I,X)} \le |I|^{\frac{q-p}{pq}} \|f\|_{L^q(I,X)},$$

for all $f \in L^q(I,X)$.

(vii) Suppose $f : I \to X$ is measurable. If $f \in L^p(J,X)$ for all $J \subset\subset I$ and if $\|f\|_{L^p(J,X)} \le C$ for some C independent of J, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le C$.

(viii) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$, then for every $f \in L^p(I,X)$ we have $Af \in L^p(I,Y)$ and

$$\|Af\|_{L^p(I,Y)} \le \|A\|_{\mathcal{L}(X,Y)} \, \|f\|_{L^p(I,X)}.$$

In particular, if $X \hookrightarrow Y$ and if $f \in L^p(I,X)$, then $f \in L^p(I,Y)$.

(ix) (The dominated convergence theorem) Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I,X)$ and let $g \in L^p(I)$. If $p < \infty$ and

$$\begin{cases} \|f_n(t)\| \le g(t), \text{ for almost all } t \in I \text{ and all } n \in \mathbb{N}, \\ \lim_{n\to\infty} f_n(t) \text{ exists for almost all } t \in I, \end{cases}$$

then $f := \lim_{n\to\infty} f_n \in L^p(I,X)$ and $\lim_{n\to\infty} f_n = f$ in $L^p(I,X)$.

(x) Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I,X)$ and let $f \in L^p(I,X)$. If $\lim_{n\to\infty} f_n = f$ in $L^p(I,X)$, then there exists $g \in L^p(I)$ and a subsequence $(n_k)_{n\in\mathbb{N}}$ such that $\|f_{n_k}(t)\| \le g(t)$ for almost all $t \in I$ and for every $k \in \mathbb{N}$.

Remark A.2.19. Duality theorems for the spaces $L^p(I,X)$ are much more difficult to obtain than for the spaces $L^p(I)$. However, if X is reflexive and if $1 < p < \infty$, then it known that $L^p(I,X)$ is reflexive and that $(L^p(I,X))^\star \approx L^{p'}(I,X^\star)$ (see Dinculeanu [37], Chapter 13, Corollary 1 of Theorem 8, p. 252). If $1 \le p < \infty$ and if X is reflexive or if $X^\star$ is separable, then $(L^p(I,X))^\star \approx L^{p'}(I,X^\star)$ (see Dinculeanu [37], Edwards [40]). Below are some special cases, in which such results are easily obtained.

(i) If X is a Hilbert space with the scalar product $(\cdot,\cdot)$, then $L^2(I,X)$ is a Hilbert space, for the scalar product

$$\langle\langle f,g\rangle\rangle = \int_I (f(t),g(t))\,dt, \text{ for } f,g \in L^2(I,X).$$

It follows that $L^2(I,X)$ is reflexive, and by Riesz' representation theorem, we have $(L^2(I,X))^\star \approx L^2(I,X)$ (or $(L^2(I,X))^\star \approx L^2(I,X^\star)$ if one does not identify $X^\star$ with X).

(ii) Let Ω be an open subset of $\mathbb{R}^N$ and let $1 \le p < \infty$. It follows easily from Fubini's Theorem and a density argument that the operator T defined on $L^p(I,L^p(\Omega))$ by $Tu(t,x) = u(t)(x)$ is an isometry from $L^p(I,L^p(\Omega))$ onto $L^p(I\times\Omega)$; and so, $L^p(I,L^p(\Omega))$ is reflexive and $(L^p(I,L^p(\Omega)))^\star \approx L^{p'}(I,L^{p'}(\Omega))$ for every $1 < p < \infty$.

(iii) The results of (ii) above are not anymore valid for $p = \infty$. For example, let $I = \Omega = (0,1)$ and consider the function $u : I \to L^\infty(\Omega)$ given by $u(t) = 1_{(0,t)}$, for $0 < t < 1$. Evidently $Tu \in L^\infty(I\times\Omega)$, but $u \notin L^\infty(I,L^\infty(\Omega))$. In fact, $u : I \to L^\infty(\Omega)$ is not even measurable, as follows from Remark A.2.9. (However, observe that $u \in C^{0,1/p}(\overline{I},L^p(\Omega))$, for every $p \in [1,\infty)$.) It follows in particular that $(L^1(I,L^1(\Omega)))^\star \not\approx L^\infty(I,L^\infty(\Omega))$ since the linear form f on $L^1(I,L^1(\Omega))$ defined by

$$f(v) = \int_I \int_\Omega v(t)u(t)\,dx\,dt$$

is continuous but cannot be written as

$$f(v) = \int_I \int_\Omega v(t)z(t)\,dx\,dt$$

for some $z \in L^\infty(I,L^\infty(\Omega))$. Indeed, the definition of f makes sense, since if $v \in L^1(I,L^1(\Omega))$, then $vu \in L^1(I,L^1(\Omega))$; and on the other hand, if z would exist, we would obtain easily that $Tz = Tu$, hence $z = u$.

Theorem A.2.20. *Let $1 \le p \le \infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(I,X)$. If there exists $f : I \to X$ such that $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, for almost all $t \in I$, then the following properties hold:*

(i) *$f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}$;*

(ii) *if $p > 1$, then $\int_I f_n(t)\varphi(t)\,dt \rightharpoonup \int_I f(t)\varphi(t)\,dt$ as $n \to \infty$, for every $\varphi \in L^{p'}(I)$.*

Proof. By Corollary A.2.7, f is measurable. If $p < \infty$, it follows from Fatou's lemma that

$$\int_I \liminf_{n\to\infty} \|f_n(t)\|^p\,dt \le \liminf_{n\to\infty} \int_I \|f_n(t)\|^p\,dt.$$

By weak lower semicontinuity of the norm, we have

$$\int_I \|f(t)\|^p\,dt \le \int_I \liminf_{n\to\infty} \|f_n(t)\|^p\,dt;$$

and so,

$$\int_I \|f(t)\|^p\,dt \le \liminf_{n\to\infty} \int_I \|f_n(t)\|^p\,dt,$$

from which (i) follows. The case $p = \infty$ follows from an obvious adaptation of this argument. Hence property (i).

We now prove (ii). Consider first $\varphi \in C_{\rm c}(I)$. Let $x' \in X^\star$ and set

$$h_n(t) = \langle x', f_n(t) - f(t)\rangle_{X^\star,X}\varphi(t),$$

for almost all $t \in I$. It follows that $h_n(t) \underset{n\to\infty}{\longrightarrow} 0$, for almost all $t \in I$ and that h_n is bounded in $L^p(I)$, as $n \to \infty$. Since h_n is supported in a compact interval, it follows easily from Lemma A.3.20 below that $h_n \to 0$ in $L^1(I)$. In particular,

$$\langle x', \int_I f_n(t)\varphi(t)\,dt\rangle_{X^\star,X} \underset{n\to\infty}{\longrightarrow} \langle x', \int_I f(t)\varphi(t)\,dt\rangle_{X^\star,X},$$

from which property (ii) follows, since x' is arbitrary. In the general case $\varphi \in L^{p'}(I)$, let $(\varphi_\ell)_{\ell\ge 0} \subset C_{\rm c}(I)$ be such that $\varphi_\ell \underset{\ell\to\infty}{\longrightarrow} \varphi$ in $L^{p'}(I)$. Given $x' \in X^\star$, we have

$$\Big|\langle x', \int_I (f_n(t) - f(t))\varphi(t)\,dt\rangle_{X^\star,X}\Big| \le \Big|\langle x', \int_I (f_n(t) - f(t))(\varphi(t) - \varphi_\ell(t)\,dt\rangle_{X^\star,X}\Big| \\ + \Big|\langle x', \int_I (f_n(t) - f(t))\varphi_\ell(t)\,dt\rangle_{X^\star,X}\Big|.$$

Given $\varepsilon > 0$, we estimate the first term on the right-hand side by $\|x'\|_{X^\star}(\|f_n\|_{L^p(I,X)} + \|f\|_{L^p(I,X)})\|\varphi_\ell - \varphi\|_{L^{p'}} \le \varepsilon/2$ if ℓ is large enough. Given such a ℓ, the second term on the right-hand side is smaller than $\varepsilon/2$ for n large enough by what precedes. Since $\varepsilon > 0$ and $x' \in X^\star$ are arbitrary, the result follows. □

Lemma A.2.21. *Let $(f_n)_{n\in\mathbb{N}} \subset L^p(I,X)$ and $f \in L^p(I,X)$, where $1 \le p \le \infty$. If $f_n \rightharpoonup f$ in $L^p(I,X)$ as $n \to \infty$, then*

$$\int_I f_n(t)\varphi(t)\,dt \rightharpoonup \int_I f(t)\varphi(t)\,dt,$$

as $n \to \infty$ for every $\varphi \in C_{\rm c}(I)$.

Proof. Without loss of generality, we may assume that $f = 0$. Consider $\varphi \in C_0^\infty(I)$ and $x' \in X^\star$ and define the linear functional F on $L^p(I, X)$ by

$$F(g) = \langle x, \int_I g(t)\varphi(t)\,dt\rangle_{X^\star, X},$$

for every $g \in L^p(I, X)$. It follows from Remark A.2.18 (iii) that F is continuous. Therefore, $F \in (L^p(I, X))^\star$, which implies that $F(f_n) \to 0$ as $n \to \infty$. Since x' is arbitrary, this implies the result. □

Proposition A.2.22. *Let $f \in L^p(\mathbb{R}, X)$ and set*

$$T_h f(t) = \frac{1}{h}\int_t^{t+h} f(s)\,ds, \text{ for } t \in \mathbb{R} \text{ and } h \neq 0. \tag{A.2.3}$$

Then $T_h f \in L^p(\mathbb{R}, X) \cap C_{\mathrm{b}}(\mathbb{R}, X)$ and T_h is a contraction on $L^p(\mathbb{R}, X)$. Furthermore, if $p < \infty$, then $\lim_{h\to 0} T_h f = f$ in $L^p(\mathbb{R}, X)$ and almost everywhere.

Proof. It follows easily from the dominated convergence theorem that $T_h f \in C(\mathbb{R}, X)$. Furthermore, applying Hölder's inequality, we obtain if $p < \infty$

$$\|T_h f(t)\|^p \le \frac{1}{h}\int_t^{t+h} \|f(s)\|^p\,dt \le \frac{1}{h}\|f\|^p_{L^p(\mathbb{R},X)}, \text{ for } t \in \mathbb{R} \text{ and } h \neq 0;$$

and so $T_h f \in C_{\mathrm{b}}(\mathbb{R}, X)$. Furthermore,

$$\begin{aligned}\int_{-\infty}^{+\infty} \|T_h f(t)\|^p\,dt &\le \frac{1}{h}\int_{-\infty}^{+\infty}\int_t^{t+h} \|f(s)\|^p\,ds\,dt \le \frac{1}{h}\int_{-\infty}^{+\infty}\int_{s-h}^{s} \|f(s)\|^p\,dt\,ds \\ &\le \int_{-\infty}^{+\infty} \|f(s)\|^p\,ds.\end{aligned}$$

Therefore, T_h is a contraction on $L^p(\mathbb{R}, X)$. The same holds in the case $p = \infty$ with an obvious modification of the argument.

Assume now $p < \infty$. It is well known that if $f \in C_{\mathrm{c}}(\mathbb{R}, X)$, then $(T_h - I)f \to 0$ in $L^p(\mathbb{R}, X)$, as $h \to 0$. By density (Remark A.2.18 (i)) and uniform boundedness of the operators T_h, it follows that for every $f \in L^p(\mathbb{R}, X)$, $(T_h - I)f \to 0$ in $L^p(\mathbb{R}, X)$, as $h \to 0$ (Proposition A.1.4).

Let now $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}, X)$ be a sequence such that $f_n(t) \to f(t)$ as $n \to \infty$, for all $t \in \mathbb{R} \setminus N$, where N is a set of measure 0 (such a sequence exists by Remark A.2.18 (i)). Given $n \in \mathbb{N}$, the function $\|f(\cdot) - f_n(\cdot)\|$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$; and so, by Theorem A.2.1, there exists a set N_n of measure 0 such that

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} \|f(s) - f_n(s)\|\,ds = \|f(t) - f_n(t)\| \text{ for all } t \in \mathbb{R} \setminus N_n.$$

Note also that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $\|T_h f_n(t) - f_n(t)\| \to 0$ as $h \to 0$. Let $E = N \cup (\bigcup_{n\in\mathbb{N}} N_n)$, so that E is a set of measure 0. Consider $\varepsilon > 0$. For $t \in \mathbb{R} \setminus E$, we have

$$\begin{aligned}\|T_h f(t) - f(t)\| \le &\|f(t) - f_n(t)\| + \frac{1}{h}\int_t^{t+h} \|f(s) - f_n(s)\|\,ds \\ &+ \|T_h f_n(t) - f_n(t)\|.\end{aligned} \tag{A.2.4}$$

For n_0 large enough, the first term in the right-hand side of (A.2.4) is less than $\varepsilon/3$. Choosing $n = n_0$ in (A.2.4), the second and third terms in the right-hand side are both less than $\varepsilon/3$ if h is small enough. Therefore, $\|T_h f(t) - f(t)\| \le \varepsilon$ if h is small enough. It follows that $T_h f \to f$ almost everywhere, as $h \to 0$. □

Corollary A.2.23. *Let $g \in L^1_{\text{loc}}(I,X)$, $t_0 \in I$ and let the function $f \in C(I,X)$ be defined by $f(t) = \int_{t_0}^t g(s)\,ds$, for $t \in I$. Then, the following properties hold:*

(i) *f is differentiable almost everywhere and $f' = g$ almost everywhere;*

(ii) *$\int_I g(t)\varphi(t)\,dt = -\int_I f(t)\varphi'(t)\,dt$ for all $\varphi \in C^1_c(I)$.*

Proof. Since properties (i) and (ii) are local, we may assume that $I = \mathbb{R}$ and $g \in L^1(I,X)$. We have

$$T_h g(t) = \frac{f(t+h) - f(t)}{h},$$

where T_h is defined by (A.2.3). Therefore, (i) follows from Proposition A.2.22.

Consider now $\varphi \in C^1_c(\mathbb{R})$. Note that $\dfrac{\varphi(t+h) - \varphi(t)}{h} \to \varphi'$ as $h \to 0$, uniformly on $\mathbf{R}$. Therefore,

$$\begin{aligned} -\int_{\mathbb{R}} f(t)\varphi'(t)\,dt &= -\lim_{h\to 0} \int_{\mathbb{R}} f(t)\frac{\varphi(t+h) - \varphi(t)}{h}\,dt \\ &= \lim_{h\to 0} \int_{\mathbb{R}} \frac{f(t-h) - f(t)}{-h}\varphi(t)\,dt \\ &= \lim_{h\to 0} \int_{\mathbb{R}} T_{-h} g(t)\varphi(t)\,dt = \int_{\mathbb{R}} g(t)\varphi(t)\,dt, \end{aligned}$$

where the last inequality is a consequence of Proposition A.2.22. Hence (ii). □

Lemma A.2.24. *If $f \in L^1_{\text{loc}}(I,X)$ is such that*

$$\int_I f(t)\varphi(t)\,dt = 0,$$

for every $\varphi \in C^\infty_c(I)$, then $f = 0$ almost everywhere.

Proof. By Pettis' theorem, there exists a set $N \subset I$ of measure 0 such that $f(I \setminus N)$ is separable. Therefore, by replacing X by its smallest closed subspace containing $f(I \setminus N)$, we may assume that X is separable. Let B' be the closed unit ball of $X^\star$ and let $(x'_n)_{n\in\mathbb{N}}$ be a weak-$\star$ dense sequence of B' (such a sequence exists by Lemma A.1.7). For every $n \in \mathbb{N}$ and every $\varphi \in C^\infty_c(I)$, we have

$$0 = \langle x'_n, \int_I f(t)\varphi(t)\,dt \rangle_{X^\star,X} = \int_I \langle x'_n, f(t)\rangle_{X^\star,X}\varphi(t)\,dt,$$

where the last identity follows from Remark A.2.15 (iii). It follows easily that there exists a set N of measure 0 such that $\langle x'_n, f(t)\rangle_{X^\star,X} = 0$ for all $t \in I \setminus N$ and all $n \in \mathbb{N}$. Hence the result. □

Lemma A.2.25. *If $f \in L^1_{\text{loc}}(I,X)$ verifies*

$$\int_I f(t)\varphi'(t)\,dt = 0,$$

for all $\varphi \in C^\infty_c(I)$, then there exists $x_0 \in X$ such that $f(t) = x_0$ for almost all $t \in I$.

Proof. Let $\theta \in C^\infty_c(I)$ be such that $\int_I \theta(t)\,dt = 1$. Let $\psi \in C^\infty_c(I)$. Consider $t_0 \in I$ such that $\theta(t) = \psi(t) = 0$ for $t \le t_0$, and let $\varphi \in C^\infty_c(I)$ be given by

$$\varphi(t) = \int_{t_0}^t \left\{ \psi(s) - \left(\int_I \psi(\sigma)\,d\sigma \right) \theta(s) \right\} ds.$$

We have $\varphi' = \psi - \left(\int_I \psi(\sigma)\, d\sigma\right)\theta$. Therefore,

$$0 = \int_I f(t)\psi(t)\, dt - x_0 \int_I \psi(\sigma)\, d\sigma,$$

where

$$x_0 = \int_I f(t)\theta(t)\, dt;$$

and so,

$$\int_I (f(t) - x_0)\psi(t)\, dt = 0,$$

for every $\psi \in C_c^\infty(I)$. The result now follows from Lemma A.2.24. □

Lemma A.2.26. *If $f, g \in L^1_{\text{loc}}(I, X)$ verify*

$$\int_I g(t)\varphi(t)\, dt = -\int_I f(t)\varphi'(t)\, dt,$$

for all $\varphi \in C_c^\infty(I)$, then given $t_0 \in I$ there exists $x_0 \in X$ such that

$$f(t) = x_0 + \int_{t_0}^t g(s)\, ds,$$

for almost all $t \in I$.

Proof. By replacing g by ξg with $\xi \in C_c^\infty(I)$, we may assume that $I = \mathbb{R}$ and that $g \in L^1(\mathbb{R}, X)$. Let $(g_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}, X)$ be such that $g_n \underset{n\to\infty}{\longrightarrow} g$ in $L^1(\mathbb{R}, X)$. For every $\varphi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} g(t)\varphi(t)\, dt &= \lim_{n\to\infty} \int_{\mathbb{R}} g_n(t)\varphi(t)\, dt \\
&= -\lim_{n\to\infty} \int_{\mathbb{R}} \left(\int_0^t g_n(s)\, ds\right) \varphi'(t)\, dt \\
&= -\int_{\mathbb{R}} \left(\int_0^t g(s)\, ds\right) \varphi'(t)\, dt,
\end{aligned}$$

by integration by parts. It follows that

$$\int_{\mathbb{R}} \left(f(t) - \int_0^t g(s)\, ds\right) \varphi'(t)\, dt = 0,$$

for every $\varphi \in C_c^\infty(I)$, and the result follows from Lemma A.2.25. □

We are now in a position to state and prove a vector valued generalization of Theorem A.2.2.

Theorem A.2.27. *Let $f : \mathbb{R} \to X$ and $1 \le p \le \infty$. If there exists $\varphi \in L^p(\mathbb{R})$ such that*

$$\|f(t) - f(s)\| \le \left| \int_s^t \varphi(\sigma)\, d\sigma \right|,$$

for all $s, t \in \mathbb{R}$ and if X is reflexive, then the following properties hold:

(i) *f is differentiable almost everywhere;*

(ii) $f' \in L^p(\mathbb{R}, X)$ *and* $\|f'\| \le |\varphi|$ *almost everywhere;*

(iii) $f(t) = f(0) + \int_0^t f'(s)\, ds$ *for all* $t \in \mathbb{R}$.

Proof. For $h \neq 0$, set

$$f_h(t) = \frac{f(t+h) - f(t)}{h}, \text{ for all } t \in \mathbb{R}.$$

It follows that

$$\|f_h(t)\| \le \left| \frac{1}{h} \int_t^{t+h} \varphi(s)\, ds \right|, \text{ for all } t \in \mathbb{R},$$

and it follows easily from Proposition A.2.22 that $\|f_h\|_{L^p(\mathbb{R},X)} \le \|\varphi\|_{L^p(\mathbb{R})}$. Since f is clearly continuous, $f(\mathbb{R})$ is separable. Therefore, by possibly replacing X by its smaller closed subspace containing $f(\mathbb{R})$, we may assume that X is both reflexive and separable; and so, that $X^\star$ is separable. Let $(x'_n)_{n\in\mathbb{N}}$ be a dense sequence in $X^\star$. For every $n \in \mathbb{N}$, the function $\psi_n(\cdot) = \langle x'_n, f(\cdot)\rangle$ verifies

$$|\psi_n(t) - \psi_n(s)| \le \|x'_n\| \, | \int_s^t \varphi(\sigma)\, d\sigma|, \text{ for all } s, t \in \mathbb{R}.$$

It follows from Theorem A.2.2 that ψ_n is differentiable on $\mathbb{R}\backslash N_n$, where N_n is a set of measure 0. Considering $N = \underset{n\in\mathbb{N}}{\cup} E_n$, we have $|N| = 0$ and

$$\lim_{h\to 0} \langle f_h(t), x'_n\rangle = \psi'_n(t), \text{ for all } n \in \mathbb{N} \text{ and } t \in \mathbb{R} \setminus N.$$

Let F be the complement of the set of Lebesgue's points of φ. We have $|F| = 0$ and it follows from Theorem A.2.1 that for every $t \in \mathbb{R} \setminus F$,

$$\|f_h(t)\| \le 2|\varphi(t)|, \text{ for } h \text{ small enough (depending on } t).$$

Take $t \in \mathbb{R} \setminus (N \cup F)$. Since X is reflexive and $\|f_h(t)\|$ is bounded, there exists a sequence $h_n \to 0$ and an element $x(t) \in X$ such that

$$\lim_{n\to\infty} f_{h_n}(t) = x(t), \text{ in } X \text{ weak.}$$

In particular, we have $\langle x'_n, x(t)\rangle = \psi'_n(t)$, for all $n \in \mathbb{N}$. Since the sequence $(x'_n)_{n\in\mathbb{N}}$ is dense in $X^\star$, $x(t)$ is independent of the sequence h_n; and so $f_h(t) \to x(t)$ in X weak, as $h \to 0$. Since f_h is bounded in $L^p(\mathbb{R}, X)$, it follows from Theorem A.2.20 that $x \in L^p(\mathbb{R}, X)$ and that for every $\theta \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} f_h(t)\theta(t)\, dt \underset{h\to 0}{\longrightarrow} \int_{\mathbb{R}} f(t)\theta(t)\, dt.$$

Since

$$\int_{\mathbb{R}} f_h(t)\theta(t)\, dt = -\int_{\mathbb{R}} f(t)\frac{\theta(t-h) - \theta(t)}{-h}\, dt \rightharpoonup -\int_{\mathbb{R}} f(t)\theta'(t)\, dt,$$

as $h \downarrow 0$, we obtain

$$\int_{\mathbb{R}} x(t)\theta(t)\, dt = -\int_{\mathbb{R}} f(t)\theta'(t)\, dt.$$

Since θ is arbitrary, it follows from Lemma A.2.26 that there exists $x_0 \in X$ such that

$$f(t) = x_0 + \int_0^t x(s)\, ds.$$

The result now follows from Corollary A.2.23. □

A.2.4. The Sobolev spaces $W^{m,p}(I,X)$. We begin with the case $m=1$.

Definition A.2.28. *Let $1 \le p \le \infty$. We say that $f \in W^{1,p}(I,X)$ if $f \in L^p(I,X)$ and if there exists $g \in L^p(I,X)$ such that*

$$\int_I g(t)\varphi(t)\,dt = -\int_I f(t)\varphi'(t)\,dt,$$

for all $\varphi \in C_c^1(I)$. By Lemma A.2.24 g is unique, and we set $f' = \dfrac{df}{dt} = g$. For $f \in W^{1,p}(I,X)$, we set

$$\|f\|_{W^{1,p}(I,X)} = \|f\|_{L^p(I,X)} + \|f'\|_{L^p(I,X)}.$$

When there is no risk of confusion, we denote $\|\ \|_{W^{1,p}(I,X)}$ by $\|\ \|_{W^{1,p}(I)}$ or $\|\ \|_{W^{1,p}}$.

Remark A.2.29. The space $W^{1,p}(I,X)$ enjoys most properties of the space $W^{1,p}(I) = W^{1,p}(I,\mathbb{R})$, with essentially the same proofs. In particular, one obtains easily the following results.

(i) $\|\ \|_{W^{1,p}(I,X)}$ is a norm on the space $W^{1,p}(I,X)$. The space $W^{1,p}(I,X)$ equipped with the norm $\|\ \|_{W^{1,p}(I,X)}$ is a Banach space.

(ii) If $f \in W^{1,p}(I,X)$ and if J is an open sub-interval of I, then $f_{|J} \in W^{1,p}(J,X)$.

(iii) If $f \in W^{1,p}(I,X) \cap W^{1,q}(I,X)$ with $p<q$, then for every $r \in [p,q]$ we have $f \in W^{1,r}(I,X)$.

(iv) If I is bounded and $p \le q$, then $W^{1,q}(I,X) \hookrightarrow W^{1,p}(I,X)$.

(v) Suppose $f \in L^p(I,X)$. If $f \in W^{1,p}(J,X)$ for all $J \subset\subset I$ and if $\|f'\|_{L^p(J,X)} \le C$ for some C independent of J, then $f \in W^{1,p}(I,X)$ and $\|f'\|_{L^p(I,X)} \le C$.

(vi) If Y is a Banach space and if $A \in \mathcal{L}(X,Y)$ then for every $f \in W^{1,p}(I,X)$, $Af \in W^{1,p}(I,Y)$ and

$$\|Af\|_{W^{1,p}(I,Y)} \le \|A\|_{\mathcal{L}(X,Y)}\ \|f\|_{W^{1,p}(I,X)}.$$

In particular, if $X \hookrightarrow Y$ and if $f \in W^{1,p}(I,X)$, then $f \in W^{1,p}(I,Y)$ (take A to be the embedding).

(vii) If $p<\infty$, then $C_0^\infty(\mathbb{R},X)$ is dense in $W^{1,p}(\mathbb{R},X)$. (This follows from the classical truncation and regularization procedure.)

(viii) If $(f_n)_{n\ge 1} \subset W^{1,p}(I,X)$ is such that $f_n \to f$ and $f_n' \to g$ in $L^p(I,X)$ as $n\to\infty$ for some $f,g \in L^p(I,X)$, then $f \in W^{1,p}(I,X)$ and $f'=g$.

Theorem A.2.30. *Let $1 \le p \le \infty$ and $f \in L^p(I,X)$. The following properties are equivalent:*

(i) *$f \in W^{1,p}(I,X)$;*

(ii) *there exists $g \in L^p(I,X)$ such that for almost all $s,t \in I$ we have $f(t) = f(s) + \displaystyle\int_s^t g(\sigma)\,d\sigma$;*

In addition, if f satisfies these properties, then one can take $g=f'$ in property (ii).

Proof. (i)⇒(ii) follows from Lemma A.2.26 and (ii)⇒(i) follows from Corollary A.2.23. □

Corollary A.2.31. *$W^{1,p}(I,X) \hookrightarrow C_{\mathrm{b,u}}(\overline{I},X)$. In particular, $W^{1,p}(I,X) \hookrightarrow L^\infty(I,X)$.*

Proof. Let $f \in W^{1,p}(I,X)$. It follows from Theorem A.2.30 that

$$\|f(t) - f(s)\| \le \int_s^t \|f'(\sigma)\|\, d\sigma,$$

for almost all $s,t \in I$. Since X is complete, one can modify f on a set of measure 0 so that the above inequality holds for all $s,t \in I$ (see the proof of Theorem A.2.37). Hence uniform continuity. Furthermore, if we set $h = \|f\|$, we have $|h(t) - h(s)| \le \|f(t) - f(s)\|$; and so, by Theorem A.2.30 and the above inequality, $h \in W^{1,p}(I) \hookrightarrow L^\infty(I)$. By Theorem A.3.34 below, $W^{1,p}(I) \hookrightarrow L^\infty(I)$, which completes the proof. □

Remark A.2.32. Note that the inclusion $W^{1,p}(I,X) \subset C_{\mathrm{b,u}}(\overline{I},X)$ is modulo modification of the functions on a set of measure 0. In other words, this means that for every $u \in W^{1,p}(I,X)$, there exists $v \in C_{\mathrm{b,u}}(\overline{I},X)$ such that $v = u$ almost everywhere.

Corollary A.2.33 *Let $I = (a,b)$, with $-\infty \le a < b \le \infty$ and let and let Y be a Banach space such that $X \hookrightarrow Y$. There exists a linear mapping A that maps functions defined almost everywhere $I \to Y$ to functions defined almost everywhere $\mathbb{R} \to Y$ and that has the following properties:*

(i) *$Af(t) = f(t)$, for almost all $t \in I$ all f defined almost everywhere $I \to Y$;*

(ii) *Af is supported in $(-a-1, b+1)$;*

(iii) *$A \in \mathcal{L}(W^{1,p}(I,Y), W^{1,p}(\mathbb{R},Y))$, for every $1 \le p \le \infty$;*

(iv) *$A \in \mathcal{L}(L^p(I,X), L^p(\mathbb{R},X))$, for every $1 \le p \le \infty$.*

Proof. Suppose $I = (0,1)$. Given f defined almost everywhere $I \to Y$, define $\widetilde{f}$ for almost all $t \in (-1,2)$ by

$$\widetilde{f}(t) = \begin{cases} f(-t), \text{ if } -1 < t < 0; \\ f(t), \text{ if } 0 < t < 1; \\ f(2-t), \text{ if } 1 < t < 2. \end{cases}$$

Evidently, $\widetilde{f}(t) = f(t)$ for almost all $t \in I$. In addition, one verifies easily that the mapping $f \mapsto \widetilde{f}$ is continuous $L^p(I,X) \to L^p((-1,2),X)$ for all $1 \le p \le \infty$. Consider $1 \le p \le \infty$ and let $f \in W^{1,p}(I,Y)$. In particular, $\widetilde{f} \in L^p((-1,2),Y)$. Furthermore, it follows easily from Theorem A.2.30 that $\widetilde{f} \in W^{1,p}((-1,2),Y)$ and that

$$(\widetilde{f})'(t) = \begin{cases} -f'(-t), \text{ if } -1 < t < 0; \\ f'(t), \text{ if } 0 < t < 1; \\ -f'(2-t), \text{ if } 1 < t < 2. \end{cases}$$

Therefore, the mapping $f \mapsto \widetilde{f}$ is continuous $W^{1,p}(I,X) \to W^{1,p}((-1,2),X)$ for all $1 \le p \le \infty$. Finally, consider $a \in \mathcal{D}(-1,2)$ such that $a \equiv 1$ on I. Given f defined almost everywhere $I \to Y$, define Af for almost all $t \in (-1,2)$ by

$$Af(t) = \begin{cases} 0, \text{ if } t < -1; \\ a(t)\widetilde{f}(t), \text{ if } -1 < t < 2; \\ 0, \text{ if } 2 < t. \end{cases}$$

It follows easily from what precedes that properties (i) through (iv) are satisfied. The case $I = (a, b)$ with $-\infty < a < b < \infty$ is treated by the same method, and the cases $I = (a, +\infty)$ and $I = (-\infty, b)$ follow from an obvious modification. □

Corollary A.2.34. *If $p < \infty$, then $C_{\mathrm{c}}^{\infty}(\overline{I}, X)$ is dense in $W^{1,p}(I, X)$. Moreover, if Y is a Banach space such that $Y \hookrightarrow X$ with dense embedding, then $C_{\mathrm{c}}^{\infty}(\overline{I}, Y)$ is dense in $W^{1,p}(I, X)$.*

Proof. Applying Corollary A.2.33, it suffices to consider the case $I = \mathbb{R}$. Density of $C_{\mathrm{c}}^{\infty}(\mathbb{R}, X)$ follows from Remark A.2.29 (vii). Finally, density of $C_{\mathrm{c}}^{\infty}(\mathbb{R}, Y)$ follows from the density of $C_{\mathrm{c}}^{\infty}(\mathbb{R}, Y)$ in $C_c^1(\mathbb{R}, X)$ for the norm of $C_b^1(\mathbb{R}, X)$. □

Corollary A.2.35. *If $p > 1$, then $W^{1,p}(I, X) \hookrightarrow C^{0,\alpha}(\overline{I}, X)$, with $\alpha = \dfrac{p-1}{p}$. Furthermore,*

$$\|f(t) - f(s)\| \le |t - s|^{\alpha} \|f'\|_{L^p},$$

for all $f \in W^{1,p}(I, X)$ and $s, t \in I$.

Proof. It follows from Theorem A.2.30 and Hölder's inequality that

$$\|f(t+h) - f(t)\| \le h^{\frac{1}{p'}} \left(\int_t^{t+h} \|f'(s)\|^p \, ds \right)^{\frac{1}{p}} \le h^{\frac{1}{p'}} \|f'\|_{L^p}.$$

Hence the result, by Corollary A.2.31. □

Corollary A.2.36. *If $[a, b] \subset I$ and $p < \infty$, then*

$$\lim_{h \to 0} \frac{f(\cdot + h) - f(\cdot)}{h} = f', \text{ in } L^p((a, b), X),$$

for every $f \in W^{1,p}(I, X)$.

Proof. By Corollary A.2.33, we may assume $I = [a, b] = \mathbb{R}$. The result now follows from Theorem A.2.30 and Proposition A.2.22. □

Theorem A.2.37. *Assume X is reflexive and let $f \in L^p(I, X)$. Then $f \in W^{1,p}(I, X)$ if and only if there exists $\varphi \in L^p(I)$ and a set N of measure 0 such that*

$$\|f(t) - f(s)\| \le \left| \int_s^t \varphi(\sigma) \, d\sigma \right|, \text{ for all } t, s \in I \setminus N. \tag{A.2.5}$$

In addition,

$$\|f'\|_{L^p(I,X)} \le \|\varphi\|_{L^p(I)}, \tag{A.2.6}$$

whenever f and φ verify (A.2.5).

Proof. It follows from Theorem A.2.30 that (A.2.5) is necessary, with for example $\varphi = \|f'\|$. Conversely, assume that (A.2.5) holds. We first modify f on the set N in such a way that (A.2.5) holds for all $s, t \in I$.

To do this, consider $t \in N$ and let $(t_n)_{n\in\mathbb{N}} \subset I \setminus N$ be such that $t_n \to t$, as $n \to \infty$. It follows from (A.2.5) that $f(t_n)$ is a Cauchy sequence in X. Let x_t be its limit. It is clear, again by (A.2.5) that x_t is independent of the sequence (t_n). We set $f(t) = x_t$, for $t \in N$. We may pass to the limit in (A.2.5); and so (A.2.5) holds for every $s, t \in I$. In particular, f is continuous on $\overline{I}$; and so, it is not difficult to extend f to a function of $L^p(\mathbb{R}, X)$ having the same properties. Therefore, we may assume that $I = \mathbb{R}$ and that (A.2.5) holds for all $s, t \in \mathbb{R}$. The result now follows from Theorems A.2.27 and A.2.30. □

As an immediate consequence of Theorem A.2.37, we have the following result, which is very useful.

Corollary A.2.38. *Assume that X is reflexive. If $f : I \to X$ is Lipschitz continuous and bounded, then $f \in W^{1,\infty}(I, X)$ and $\|f'\|_{L^\infty(I,X)} \le L$, where L is the Lipschitz constant of f.*

Corollary A.2.39. *Let $1 \le p \le \infty$, let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of $W^{1,p}(I, X)$ and let $f : I \to X$ be such that $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, for almost all $t \in I$. If X is reflexive and if $p > 1$, then $f \in W^{1,p}(I, X)$ and*

$$\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)},$$

and

$$\|f'\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n'\|_{L^p(I,X)}.$$

In particular,

$$\|f\|_{W^{1,p}(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{W^{1,p}(I,X)}.$$

In addition,

$$\int_I f_n(t)\varphi(t)\,dt \rightharpoonup \int_I f(t)\varphi(t)\,dt, \tag{A.2.7}$$

and

$$\int_I f_n'(t)\varphi(t)\,dt \rightharpoonup \int_I f'(t)\varphi(t)\,dt, \tag{A.2.8}$$

as $n \to \infty$, for all $\varphi \in C_c(I)$.

Proof. It follows from Theorem A.2.20 that $f \in L^p(I, X)$,

$$\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)},$$

and that (A.2.7) holds. Let N be a set of measure 0 such that $f_n(t) \rightharpoonup f(t)$ in X as $n \to \infty$, for all $t \in I \setminus N$. We have

$$\|f(t) - f(s)\| \le \liminf_{n\to\infty} \|f_n(t) - f_n(s)\| \le \liminf_{n\to\infty} \int_s^t \varphi_n(\sigma)\,d\sigma, \tag{A.2.9}$$

for all $s, t \in I \setminus N$, where $\varphi_n = \|f_n'\|$. Since φ_n is bounded in $L^p(I)$, there exists a subsequence n_k and a function $\varphi \in L^p(I)$ such that $\varphi_{n_k} \to \varphi$ in $L^p(I)$ weak* as $k \to \infty$, and

$$\liminf_{k\to\infty} \|\varphi_{n_k}\|_{L^p(I)} = \liminf_{n\to\infty} \|\varphi_n\|_{L^p(I)}.$$

In particular, we have

$$\|\varphi\|_{L^p(I)} \le \liminf_{n\to\infty} \|f_n'\|_{L^p(I,X)},$$

$$\lim_{k\to\infty} \int_s^t \varphi_{n_k}(\sigma)\,d\sigma = \int_s^t \varphi(\sigma)\,d\sigma, \text{ for all } s, t \in I.$$

Applying (A.2.9), we obtain

$$\|f(t) - f(s)\| \le \int_s^t \varphi(\sigma)\, d\sigma, \text{ for all } s, t \in I \setminus N.$$

We deduce from Theorem A.2.37 that $f \in W^{1,p}(I,X)$ and $\|f'\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n'\|_{L^p(I,X)}$. Finally, we prove formula (A.2.8). Let $\varphi \in C_c^1(I)$. We have

$$\int_I f_n'(t)\varphi(t)\, dt = -\int_I f_n(t)\varphi'(t)\, dt \underset{n\to\infty}{\longrightarrow} -\int_I f(t)\varphi'(t)\, dt = \int_I f'(t)\varphi(t)\, dt,$$

by (A.2.7). This proves (A.2.8) for $\varphi \in C_c^1(I)$. The general case follows from a density argument (see the proof of Theorem A.2.20 (ii)). □

Corollary A.2.40. *Let $1 \le p \le \infty$ and $f \in L^p(I,X)$. Assume that there exists K such that for all $J \subset\subset I$ and all $|h| < \operatorname{dist}(\overline{J}, \partial I)$, $\|f(\cdot + h) - f(\cdot)\|_{L^p(J,X)} \le K|h|$. If $p > 1$ and X is reflexive, then $f \in W^{1,p}(I,X)$ and $\|f'\|_{L^p(I,X)} \le K$.*

Proof. Let $J \subset\subset I$, $|h| < \operatorname{dist}(\overline{J}, \partial I)$, $h \neq 0$ and set

$$f_h(t) = \frac{1}{h}\int_t^{t+h} f(s)\, ds,$$

for $t \in J$. It follows from Theorem A.2.30 that $f_h \in W^{1,p}(J,X)$ and

$$f_h'(t) = \frac{f(t+h) - f(t)}{h},$$

for a.a. $t \in J$. In particular, $\|f_h'\|_{L^p(J,X)} \le K$. Since $f_h \to f$ a.e. on J (see Corollary A.2.33 and Proposition A.2.22), we deduce from Corollary A.2.39 that $f \in W^{1,p}(J,X)$ and $\|f'\|_{L^p(J,X)} \le K$. The result now follows from Remark A.2.29 (v). □

Corollary A.2.41 *Consider two Banach spaces $X \hookrightarrow Y$ and $1 < p, q \le \infty$. Let $(f_n)_{n\ge 0}$ be a bounded sequence in $L^q(I,Y)$ and let $f : I \to Y$ be such that $f_n(t) \rightharpoonup f(t)$ in Y as $n \to \infty$, for a.a. $t \in I$. If $(f_n)_{n\ge 0}$ is bounded in $L^p(I,X)$ and if X is reflexive, then $f \in L^p(I,X)$ and $\|f\|_{L^p(I,X)} \le \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}$.*

Proof. It follows from Theorem A.2.20 (i) that $f \in L^q(I,Y)$. Given $k \in \mathbb{N}$, let $I_k = I \cap (-k,k)$. Fix $t_0 \in I$ and consider k large enough so that $t_0 \in I_k$. Set

$$u_n(t) = \int_{t_0}^t f_n(s)\, ds, \quad u(t) = \int_{t_0}^t f(s)\, ds.$$

It follows from Theorem A.2.20 (ii) that $u_n(t) \rightharpoonup u(t)$ in Y as $n \to \infty$, for a.a. $t \in I$. On the other hand,

$$\|u_n(t)\|_X \le |t_0 - t|\, \|f_n\|_{L^p(I,X)},$$

so that by Lemma A.1.9, $u_n(t) \in X$ for a.a. $t \in I_k$ and $u_n(t) \rightharpoonup u(t)$ in X as $n \to \infty$, for a.a. $t \in I$. Since u_n is bounded in $W^{1,p}(I_k,X)$, we deduce from Corollary A.2.39 that $u \in L^p(I_k,X)$ and that

$$\|u'\|_{L^p(I_k,X)} \le \liminf_{n\to\infty} \|u_n'\|_{L^p(I_k,X)}.$$

Finally, we have $u' = f$ in Y. Applying formula (A.2.8) in Y then in X, we obtain that that $u' = f$ in Y. In particular, $f \in L^p(I_k, X)$ and

$$\|f\|_{L^p(I_k,X)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(I_k,X)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(I,X)}.$$

Since k is arbitrary, the result follows from Fatou's lemma. □

Remark A.2.42. Note that when X is not reflexive, the conclusions of Theorem A.2.37 may be invalid. This can be seen easily on the following example. Let $\theta = 1_{[1,2]} \in L^\infty(\mathbb{R})$ and consider the function $\psi(t) = \theta(\cdot + t)$. It is clear that $\psi \in C(\mathbb{R}, L^1(\mathbb{R}))$. Let now

$$g(t) = \int_0^t \psi(s)\, ds, \text{ for } t \in \mathbb{R}.$$

Note that we have also

$$g(t,x) = \int_x^{x+t} \theta(s)\, ds, \text{ for all } t, x \in \mathbb{R}.$$

It is clear that $g \in C^1(I, L^1(\mathbb{R}))$ and that

$$\|g(t) - g(s)\|_{L^\infty(\mathbb{R})} \leq |t-s| = \int_s^t \varphi(\sigma)\, d\sigma, \tag{A.2.10}$$

where $\varphi \equiv 1$. Let now $\Omega = (0,1)$ and let $X = C(\overline{\Omega})$, equipped with the L^∞-norm. If we set $f = g_{|\Omega}$, we have $f \in C^1(\mathbb{R}, L^1(\Omega))$ and $f' = \psi_{|\Omega}$, and it follows from (A.2.10) that

$$\|f(t) - f(s)\|_X \leq \int_s^t \varphi(\sigma)\, d\sigma.$$

Therefore f satisfies (A.2.5) with $\varphi \in L^\infty$, but we claim that $f \notin W^{1,1}((0,1), X)$. Indeed, if f were in $W^{1,1}((0,1), X)$, then the derivatives of f in the senses of X and $L^1(\Omega)$ would coincide, since $X \hookrightarrow L^1(\Omega)$. Therefore, we would have $f' = \psi_{|\Omega}$, which is absurd since $\psi_{|\Omega} \notin X$, for $0 < t < 2$.

Remark A.2.43. Let us observe that if X is not reflexive or if $p = 1$, the conclusions of Corollary A.2.39 may also be invalid.

Indeed, with the notation of the preceding remark, consider a sequence $(\theta_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ such that

$$\sup_{n\in\mathbb{N}} \|\theta_n\|_{L^\infty(\mathbb{R})} < \infty, \text{ and } \lim_{n\to\infty} \theta_n = \theta \text{ almost everywhere.}$$

Let $\psi_n(t) = \theta_n(\cdot + t)$, $g_n(t) = \displaystyle\int_0^t \psi_n(s)\, ds$ and $f_n = (g_n)_{|\Omega}$. We have $f_n \in C^1(\mathbb{R}, X)$ and it is not difficult to check that f_n is bounded in $W^{1,\infty}(\mathbb{R}, X)$. In addition, f_n converges to f in $C(\mathbb{R}, X)$, but we know that $f \notin W^{1,1}((0,1), X)$.

Now if $p = 1$, consider $x \in X$, $x \neq 0$ and assume for example that $I = [0,2]$. Let $f_n : I \to X$ be defined by

$$f_n(t) = \begin{cases} x, \text{ if } t \leq 1, \\ (1 - n(t-1))x, \text{ if } 1 \leq t \leq 1 + \dfrac{1}{n}, \\ 0, \text{ if } 1 + \dfrac{1}{n} \leq t. \end{cases}$$

One verifies easily that f_n is bounded in $W^{1,1}(I,X)$ and that $f_n(t)$ has a limit $f(t)$ for all $t \neq 1$, where

$$f(t) = \begin{cases} x, & \text{if } t < 1, \\ 0, & \text{if } 1 < t. \end{cases}$$

Therefore, $f \notin C(I,X)$, and so $f \notin W^{1,1}(I,X)$.

Remark A.2.44. The mapping $f \mapsto (f, f')$ identifies the space $W^{1,p}(I,X)$ with a closed subspace of $L^p(I,X) \times L^p(I,X)$. Therefore, if $L^p(I,X)$ is reflexive, then $W^{1,p}(I,X)$ is also reflexive.

The compactness properties of the spaces $W^{1,p}(I,X)$ are rather delicate. One of the first results in that direction is due to Aubin [4]. For more recent results, see for example Brezis and Browder [19] and Simon [91]. Below are two quite useful compactness results.

Theorem A.2.45. *Consider three Banach spaces $X \hookrightarrow B \hookrightarrow Y$, where the embedding $X \hookrightarrow B$ is compact. Let $1 \le p, r \le \infty$ (with $r > 1$ if $p = \infty$) and let E be a bounded subset of $W^{1,r}(I,Y)$. If E is also bounded in $L^p(I,X)$ (i.e. every $f \in E$ belongs to $L^p(I,X)$ and $\sup\{\|f\|_{L^p(I,X)}, f \in E\} < \infty$) and if I is bounded, then E is a relatively compact subset of $L^p(I,B)$ (of $C(\overline{I},B)$ if $p = \infty$).*

Proof. Observe first that if $p = \infty$ and $f \in E$, then by Corollary A.2.31 $f : \overline{I} \to Y$ is continuous and $f : I \to X$ is bounded; and so (Lemma A.1.14) $f \in C(\overline{I}, B)$. Therefore, we only have to prove compactness in $L^p(I,X)$.

The proof proceeds in two steps.

Step 1. Let us first show that E is a relatively compact subset of $L^p(I,Y)$.

We may assume without loss of generality that $I = (0,T)$, for some $T > 0$. Observe that there exists $M < \infty$ such that

$$\sup_{f \in E} \|f\|_{L^\infty(I,Y)} + \sup_{f \in E} \|f'\|_{L^r(I,Y)} + \sup_{f \in E} \|f\|_{L^p(I,X)} \le M \tag{A.2.11}$$

For $0 < \varepsilon < T/2$, we define the set $E_\varepsilon = \{(T_\varepsilon f)_{|[0,T/2]}, f \in E\}$, where T_ε is defined by (A.2.3). For $f \in E$, we have

$$T_\varepsilon f(t) - f(t) = \frac{1}{\varepsilon} \int_0^\varepsilon (\tau_s f - f)(t)\, ds, \text{ for every } t \in [0, T/2],$$

where $\tau_s f = f(\cdot + s)$. It follows easily (see the proof of Proposition A.2.22) that

$$\|T_\varepsilon f - f\|_{L^p((0,T/2),Y)} \le \sup_{0<s<\varepsilon} \|\tau_s f - f\|_{L^p((0,T/2),Y)}. \tag{A.2.12}$$

For every $t \in \mathbb{R}$, we have

$$\|\tau_s f(t) - f(t)\|_Y \le \int_t^{t+s} \|f'(\sigma)\|_Y \, d\sigma. \tag{A.2.13}$$

Therefore,

$$\begin{aligned} \|\tau_s f - f\|_{L^1((0,T/2),Y)} &\le \int_0^{T/2} \int_t^{t+s} \|f'(\sigma)\| \, d\sigma \, dt \\ &\le s \|f'\|_{L^1((0,T),Y)} \le s T^{\frac{1}{r'}} M, \end{aligned}$$

where the last inequality follows from Hölder's inequality; and so, if $p < \infty$,

$$\begin{aligned}\|\tau_s f - f\|_{L^p((0,T/2),Y)} &\le \|\tau_s f - f\|_{L^\infty((0,T/2),Y)}^{1-\frac{1}{p}} \|\tau_s f - f\|_{L^1((0,T/2),Y)}^{\frac{1}{p}} \\ &\le 2MT^{\frac{1}{pr'}} s^{\frac{1}{p}},\end{aligned}$$

by (A.2.11). If $p = \infty$, it follows from (A.2.13) and Hölder's inequality that $\|\tau_s f - f\|_{L^p((0,T/2),Y)} \le s^{\frac{1}{r'}} M$. Therefore, there exists $C < \infty$ such that

$$\|\tau_s f - f\|_{L^p((0,T/2),Y)} \le C \min(s^{\frac{1}{r'}}, s^{\frac{1}{p}}). \qquad (A.2.14)$$

It follows from (A.2.12) and (A.2.14) that for every $\delta > 0$, there exists $\varepsilon > 0$ such that

$$\sup_{f \in F} \operatorname{dist}(f, E_\varepsilon) \le \delta, \qquad (A.2.15)$$

where the distance is in $L^p((0,T/2),Y)$. Given $\varepsilon > 0$ and $t \in [0, T/2]$, we have

$$\|T_\varepsilon f(t)\|_X \le \varepsilon^{\frac{1}{p'}-1} \|f\|_{L^p(I,X)} \le M\varepsilon^{-\frac{1}{p}};$$

and so

$$\sup_{f \in E_\varepsilon} \sup_{t \in [0,T/2]} \|f(t)\|_X \le M\varepsilon^{-\frac{1}{p}}. \qquad (A.2.16)$$

In adition, for every $t, t' \in [0, T/2]$,

$$\|T_\varepsilon f(t) - T_\varepsilon f(t')\|_Y = \|\frac{1}{\varepsilon}\int_t^{t+\varepsilon} (\tau_{t'-t} f - f)\| \le \frac{1}{\varepsilon}\|\tau_{t'-t} f - f\|_{L^1((0,T/2),Y)}.$$

When (A.2.14) is applied, it follows that E_ε is uniformly equicontinuous in Y. Therefore, taking in account (A.2.16) and the compactness of the embedding $X \hookrightarrow Y$, we may apply Ascoli's theorem; and so E_ε is relatively compact in $L^p((0,T/2),Y)$. By (A.2.15), E is also relatively compact in $L^p((0,T/2),Y)$, as the uniform limit of the sets E_ε. When t is changed to $T - t$, it follows as well that E is relatively compact in $L^p((T/2,T),Y)$, hence in $L^p(I,Y)$.

Step 2. From the relative compactness in $L^p(I,Y)$, we deduce the relative compactness in $L^p(I,B)$. For this, we need the standard inequality:

$$\forall \eta > 0, \exists C(\eta) < \infty, \forall z \in X, \|z\|_B \le \eta\|z\|_X + C(\eta)\|z\|_Y. \qquad (A.2.17)$$

To see this, let $\eta > 0$ and consider $B_n = \{z \in B, \|z\|_B < \eta + n\|z\|_Y\}$. $(B_n)_{n\in\mathbb{N}}$ is an increasing sequence of open subsets of B, and its union covers B. Since the unit ball U of X is relatively compact in B, there exists n_0 such that $U \subset B_{n_0}$. Therefore,

$$\|z\|_B \le \eta + n_0\|z\|_Y, \forall z \in U,$$

and (A.2.17) follows from homogeneity. Since E is relatively compact in $L^p(I,Y)$, for every $\delta > 0$, there exists a finite subset $\{f_j\}_{j\in J}$ of E such that

$$\sup_{f\in E} \inf_{j\in J} \|f - f_j\|_{L^p(I,Y)} \le \delta.$$

It follows easily from (A.2.11) and (A.2.17) that for every $f \in E$, $j \in J$ and $\eta > 0$,

$$\sup_{f \in E} \inf_{j \in J} \|f - f_j\|_{L^p(I,B)} \le \eta M + C(\eta)\delta.$$

Therefore, given $\varepsilon > 0$, and chosing $\eta = \varepsilon/2M$, $\delta = \varepsilon/2C(\eta)$, we get

$$\sup_{f \in E} \inf_{j \in J} \|f - f_j\|_{L^p(I,B)} \le \varepsilon.$$

Thus E is relatively compact in $L^p(I, Y)$. □

Proposition A.2.46. *Let $X \hookrightarrow Y$ be two Banach spaces and let f_n be a bounded sequence of $L^\infty(I,X) \cap W^{1,r}(I,Y)$, for some $r > 1$. If X is reflexive and if I is bounded, then the following properties hold:*

(i) *there exists $f \in L^\infty(I,X)$, $f : \overline{I} \to X$ being weakly continuous, and a subsequence n_k such that $f_{n_k}(t) \rightharpoonup f(t)$ in X as $k \to \infty$, for every $t \in \overline{I}$. In particular,*

$$\int_I f_n(t)\varphi(t)\,dt \rightharpoonup \int_I f(t)\varphi(t)\,dt$$

as $n \to \infty$ for every $\varphi \in C_c(I)$;

(ii) *if Y is reflexive, then also $f \in W^{1,r}(I,Y)$;*

(iii) *if there exists a uniformly convex Banach space B such that $X \hookrightarrow B \hookrightarrow Y$ and if $(f_n)_{n\in\mathbb{N}} \subset C(\overline{I},B)$ and $\|f_{n_k}(t)\|_B \to \|f(t)\|_B$ as $k \to \infty$, uniformly on I, then also $f \in C(\overline{I},B)$ and $f_{n_k} \to f$ in $C(\overline{I},B)$ as $k \to \infty$.*

Proof. By Corollary A.2.35, properties (i) and (iii) follow from Proposition A.1.5. Property (ii) follows from Corollary A.2.39. □

Remark A.2.47. One can define higher order vector valued Sobolev spaces as follows. For $1 \le p \le \infty$, one defines

$$W^{2,p}(I,X) = \{f \in W^{1,p}(I,X), \frac{df}{dt} \in W^{1,p}(I,X)\},$$

with the correspinding norm. For $f \in W^{1,p}(I,X)$, one defines

$$\frac{d^2 f}{dt^2} = \frac{d}{dt}\frac{df}{dt}.$$

It is clear that

$$\int_I \frac{d^2 f}{dt^2}\varphi(t)\,dt = \int_I f(t)\frac{d^2\varphi}{dt^2}(t)\,dt = -\int_I \frac{df}{dt}\frac{d\varphi}{dt}(t)\,dt,$$

for all $\varphi \in C^2_c(I)$, and it follows from Corollaries A.2.31 and A.2.35 that $W^{2,1}(I,X) \hookrightarrow C^1(\overline{I},X)$ and that $W^{2,p}(I,X) \hookrightarrow C^{1,\alpha}(\overline{I},X)$ with $\alpha = \dfrac{p-1}{p}$, if $p > 1$. More generally, one defines by induction on m

$$W^{m,p}(I,X) = \{f \in W^{m-1,p}(I,X), \frac{d^f}{dt} \in W^{m-1,p}(I,X)\},$$

with the correspinding norm. For $f \in W^{m,p}(I,X)$, one defines

$$\frac{d^m f}{dt^m} = \frac{d}{dt}\frac{d^m f}{dt^m}.$$

It is clear that

$$\int_I \frac{d^m f}{dt^m}\varphi(t)\,dt = (-1)^m \int_I f(t)\frac{d^m\varphi}{dt^m}(t)\,dt = (-1)^{m-j}\int_I \frac{d^j f}{dt^j}\frac{d^{m-j}\varphi}{dt^{m-j}}(t)\,dt,$$

for $1 \le j \le m$ and for all $\varphi \in C_c^m(I)$, and it follows from Corollaries A.2.31 and A.2.35 that $W^{m,1}(I,X) \hookrightarrow C^{m-1}(\overline{I},X)$ and that $W^{m,p}(I,X) \hookrightarrow C^{m-1,\alpha}(\overline{I},X)$ with $\alpha = \dfrac{p-1}{p}$, if $p > 1$.

We know that if $p < \infty$, then $C_c^\infty(\overline{I},X)$ is dense in $L^p(I,X)$ and in $W^{1,p}(I,X)$. However, it is sometimes useful to have density of smooth functions in spaces of the type $L^p(I,X) \cap W^{1,q}(I,Y)$. This is the object of the following result.

Proposition A.2.48. *Let $1 \le p,q < \infty$ and let $X \hookrightarrow Y$ be two Banach spaces. Then $C_c^\infty(\overline{I},X)$ is dense in $L^p(I,X) \cap W^{1,q}(I,Y)$. Moreover, if Z is a Banach space such that $Z \hookrightarrow X$ with dense embedding, then $C_c^\infty(\overline{I},Z)$ is dense in $L^p(I,X) \cap W^{1,q}(I,Y)$.*

Proof. By Corollary A.2.33, we may assume that $I = \mathbb{R}$. The first statement now follows from the standard procedure of truncation an regularization by convolution with a sequence of mollifiers, and the second statement from density of $C_c^\infty(\mathbb{R},Z)$ in $C_c^1(\mathbb{R},X)$ for the norm of $C_b^1(\mathbb{R},X)$. □

Remark A.2.49. Let Ω be an open subset of $\mathbb{R}^N$, and let $1 \le p < \infty$. Consider $u \in W^{1,1}(I,L^p(\Omega))$. Then, $\dfrac{du}{dt} \in L^1(I,L^p(\Omega))$. In particular, $\dfrac{du}{dt} \in L^1_{\mathrm{loc}}(I\times\Omega) \subset \mathcal{D}'(I\times\Omega)$. On the other hand, u can be considered as a function of $L^1_{\mathrm{loc}}(I\times\Omega)$. In particular, $\dfrac{\partial u}{\partial t} \in \mathcal{D}'(I\times\Omega)$. One verifies easily that $\dfrac{\partial u}{\partial t} = \dfrac{du}{dt}$ in $\mathcal{D}'(I\times\Omega)$. Therefore, for functions u defined on $I\times\Omega$, we will in general identify $\dfrac{\partial u}{\partial t}$ and $\dfrac{du}{dt}$.

A.3. Sobolev spaces. Sobolev spaces have become an essential tool in the study of partial differential equations. We recall below the most useful and significant results of the theory. A general reference for Sobolev spaces is Adams [1].

A.3.1. Definitions. Throughout Section A.3, Ω is an open subset of $\mathbb{R}^N$. We consider only *real-valued* functions, and we refer to Section A.3.7 for the case of *complex-valued* functions. We recall that $\mathcal{D}(\Omega)$ is equipped with the topology induced by the family of seminorms $d_{K,m}$ where K is a compact subset of Ω and $m \in \mathbb{N}$, defined by

$$d_{K,m}(\varphi) = \sup_{x\in K}\sum_{|\alpha|=m}|D^\alpha\varphi(x)|, \text{ for all } \varphi\in\mathcal{D}(\Omega).$$

The set of distributions on Ω, $\mathcal{D}'(\Omega)$, is the dual space of $\mathcal{D}(\Omega)$. If $T \in \mathcal{D}'(\Omega)$ and if $\alpha \in \mathbb{N}^N$ is a multi-index, one defines the distribution

$$D^\alpha T = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}T \in \mathcal{D}'(\Omega)$$

by

$$\langle D^\alpha T,\varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle, \text{ for all } \varphi\in\mathcal{D}(\Omega).$$

A function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f,\varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx, \text{ for all } \varphi\in\mathcal{D}(\Omega).$$

It is well known that if $T_f = T_g$, then $f = g$ almost everywhere. A distribution $T \in \mathcal{D}'(\Omega)$ is said to belong to $L^p(\Omega)$ if there exists $f \in L^p(\Omega)$ such that $T = T_f$. In this case, f is unique.

For $m \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \le m\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm $\| \ \|_{W^{m,p}} = \| \ \|_{W^{m,p}(\Omega)}$ defined by

$$\|u\|_{W^{m,p}} = \sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_{L^p(\Omega)}.$$

For $1 \le p < \infty$, one defines the closed subset $W_0^{m,p}(\Omega)$ of $W^{m,p}(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. One defines $W^{m,p}_{\text{loc}}(\Omega)$ as the set of $u \in L^1_{\text{loc}}(\Omega)$ such that $u_{|\Omega'} \in W^{m,p}(\Omega')$, for every $\Omega' \subset\subset \Omega$. When $p = 2$, one sets $W^{m,p}_{\text{loc}}(\Omega) = H^m_{\text{loc}}(\Omega)$, $W^{m,p}(\Omega) = H^m(\Omega)$ and $W_0^{m,p}(\Omega) = H_0^m(\Omega)$ and one rather equips $H^m(\Omega)$ with the equivalent norm

$$\|u\|_{H^m(\Omega)} = \|u\|_{H^m} = \left(\sum_{0 \le |\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

$H^m(\Omega)$ (hence $H_0^m(\Omega)$) is then a Hilbert space with the scalar product

$$(u, v)_{H^m} = \sum_{0 \le |\alpha| \le m} \int_\Omega D^\alpha u(x) D^\alpha v(x) \, dx.$$

Note that when $\Omega = \mathbb{R}^N$, H^m can be equivalently defined in terms of the Fourier transform. The following result is an immediate consequence of Plancherel's formula.

Proposition A.3.1. *For every $m \in \mathbb{N}$, the following properties hold:*

(i) $H^m(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1 + |\xi|^2)^{m/2} \widehat{u}(\xi) \in L^2(\mathbb{R}^N)\}$;

(ii) $\|u\|_{H^m} \approx \|(1 + |\xi|^2)^{m/2} \widehat{u}(\xi)\|_{L^2}$.

In the statement of Sobolev's embedding theorems, we will need the following spaces of continuous functions. $C(\overline{\Omega})$ is the space of continuous functions $\overline{\Omega} \to \mathbb{R}$. $C_{\text{b}}(\Omega)$ is the Banach space of continuous and bounded functions $\Omega \to \mathbb{R}$, equipped with the L^∞ norm. Given a nonnegative integer m, $C^m_b(\Omega)$ is the Banach space of functions $u \in C_{\text{b}}(\Omega)$ such that $D^\alpha u \in C_{\text{b}}(\Omega)$, for all $\alpha \in \mathbb{N}^N$ such that $|\alpha| \le m$, equipped with the norm of $W^{m,\infty}(\Omega)$. $C_{\text{b,u}}(\overline{\Omega})$ is the Banach space of uniformly continuous and bounded functions $\overline{\Omega} \to \mathbb{R}$, equipped with the L^∞ norm. $C^m_{\text{b,u}}(\overline{\Omega})$ is the Banach space of functions $u \in C_{\text{b,u}}(\overline{\Omega})$ such that $D^\alpha u \in C_{\text{b,u}}(\overline{\Omega})$, for every multi-index α such that $|\alpha| \le m$. $C^m_{\text{b,u}}(\overline{\Omega})$ is equipped with the norm of $W^{m,\infty}(\Omega)$. $C^{m,\alpha}(\overline{\Omega})$ for $0 \le \alpha \le 1$, is the Banach space of functions $u \in C^m_{\text{b,u}}(\overline{\Omega})$ such that

$$\|u\|_{C^{m,\alpha}} = \|u\|_{W^{m,\infty}} + \sup \left\{ \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha};\ x, y \in \Omega,\ |\beta| = m \right\} < \infty.$$

Finally, $C_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $L^\infty(\Omega)$.

Remark A.3.2. Note that one alway have the following inclusions: $C_0(\Omega) \subset C_{\text{b,u}}(\overline{\Omega})$, $C_{\text{b,u}}(\overline{\Omega}) \subset C_{\text{b}}(\Omega)$, $C_{\text{b,u}}(\overline{\Omega}) \subset C(\overline{\Omega})$. Furthermore, $C_0(\Omega) \ne C_{\text{b,u}}(\overline{\Omega}) \ne C_{\text{b}}(\Omega)$, but if Ω is bounded, then $C_{\text{b,u}}(\overline{\Omega}) = C(\overline{\Omega})$.

We will need the notion of regularity of the domain Ω. Given $x \in \mathbb{R}^N$, $z \in S^{N-1}$, $\theta \in (0, \pi/2)$ and $\delta > 0$, the cone with vertex x, direction z, polar angle θ and heigth δ is the set

$$C(x, z, \theta, \delta) = \{y \in \mathbb{R}^N;\ \exists \lambda > 0, |y - (x + \lambda z)| < \lambda \sin(\theta)\} \cap \{y \in \mathbb{R}^N;\ |y - x| < \delta\}.$$

We now make the following definitions.

Definition A.3.3. *Let Ω be an open subset of $\mathbb{R}^N$.*

(i) *We say that $x \in \partial\Omega$ has the exterior (respectively interior) cone property, if there exists $z \in S^{N-1}$, $\theta \in (0, \pi/2)$ and $\delta > 0$ such that $C(x, z, \theta, \delta) \cap \overline{\Omega} = \emptyset$ (respectively $C(x, z, \theta, \delta) \subset \Omega$).*

(ii) *We say that Ω has the cone property, if there exists $\theta \in (0, \pi/2)$ and $\delta > 0$ such that for every $x \in \Omega$, there exists $z_x \in S^{N-1}$ such that $C(x, z_x, \theta, \delta) \subset \Omega$.*

(iii) *We say that Ω has a Lipschitz continuous boundary if for any $x \in \partial\Omega$ there exists a neighbourhood U_x of x in $\mathbb{R}^N$ and a Lipschitz continuous function $\phi_x : \mathbb{R}^{N-1} \to \mathbb{R}$ such that for some system of cartesian coordinates $(\xi_{x,1} \cdots, \xi_{x,N})$, the set $\Omega \cap U_x$ is represented by the equation*

$$\xi_{x,N} < \phi_x(\xi_{x,1} \cdots, \xi_{x,N-1}).$$

If the Lipschitz constants of ϕ_x are bounded independently of $x \in \partial\Omega$ and if there exists $\delta > 0$ such that U_x contains the ball of center x and radius δ, Ω is said to have a uniformly Lipschitz boundary.

(iv) *Given a positive integer k, we say that Ω has a C^k boundary if for any $x \in \partial\Omega$ there exists a neighbourhood U_x of x in $\mathbb{R}^N$ and a one-to-one mapping Φ_x from U_x onto the unit ball B of $\mathbb{R}^N$ such that*

$$\begin{cases} \Phi_x(\Omega \cap U_x) \subset \mathbb{R}^N_+ = \{(x_1, \cdots, x_N);\ x_N > 0\}, \\ \Phi_x(\partial\Omega \cap U_x) \subset \partial\mathbb{R}^N_+ = \{(x_1, \cdots, x_N);\ x_N = 0\}, \\ \Phi_x \text{ and } \Phi_x^{-1} \text{ are } C^k. \end{cases}$$

If the C^k norms of Φ_x and Φ_x^{-1} are bounded independently of $x \in \partial\Omega$ and if there exists $\delta > 0$ such that U_x contains the ball of center x and radius δ, Ω is said to have a uniformly C^k boundary.

Remark A.3.4. Here a few simple observations concerning regularity of domains.

(i) One verifies easily that if the domain Ω has a bounded (hence compact) boundary, then the local and uniform regularity properties are equivalent.

(ii) It is not difficult to verify that a domain Ω with a C^1 (respectively uniformly C^1) boundary has a Lipschitz (respectively uniformly Lipschitz) boundary. One verifies as well that a domain with a uniformly Lipschitz boundary posesses the cone property, and that every $x \in \partial\Omega$ has both the interior and exterior cone property.

(iii) Note also that the regularity properties are neither stable by intersection nor by union. For example, the subsets Ω_1 and Ω_2 of $\mathbb{R}^2$ defined by $\Omega_1 = \{(x, y) \in \mathbb{R}^2, |x - 1|^2 + |y|^2 > 1\}$ and $\Omega_2 = \{(x, y) \in \mathbb{R}^2, |x - 2|^2 + |y|^2 < 2\}$ both have a uniformly C^m boundary for every integer m. However, $\Omega_1 \cup \Omega_2 = \mathbb{R}^N \setminus \{0\}$ does not have a Lipschitz boundary, and $\Omega_1 \cap \Omega_2$ does not even have the cone property.

A.3.2. Basic properties of the space $W^{m,p}(\Omega)$. We begin with the following well known result (see for example Adams [1], Theorem 3.5.).

Proposition A.3.5. *If $1 < p < \infty$, then the spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are uniformly convex Banach spaces.*

For every function u defined almost everywhere on Ω, let us define the function $\overline{u}$ almost everywhere on $\mathbb{R}^N$ by

$$\overline{u} = \begin{cases} u, & \text{on } \Omega, \\ 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{A.3.1}$$

For u as above, $z \in \mathbb{R}^N \setminus \{0\}$, $h \in \mathbb{R} \setminus \{0\}$ and $i \in \{1, \cdots, N\}$, let us set

$$\tau_z u = \overline{u}(\cdot + z)_{|\Omega}, \tag{A.3.2}$$

$$\partial_i^h u = \left(\frac{\overline{u}(\cdot + he_i) - \overline{u}(\cdot)}{h} \right)_{|\Omega}, \tag{A.3.3}$$

where e_i is the vector of $\mathbb{R}^N$ whose components are equal to 0 except the i^{th} one which is equal to 1. We have the following characterization of $W^{1,p}(\Omega)$ (see Brezis [17], Proposition IX.3).

Proposition A.3.6. *let $1 < p \leq \infty$ and let $u \in L^p(\Omega)$. Then the following properties are equivalent:*

(i) $u \in W^{1,p}(\Omega)$;

(ii) *there exists C such that for every $\varphi \in \mathcal{D}(\Omega)$ and $1 \leq i \leq N$, $|\int_\Omega u \partial_i \varphi| \leq C \|\varphi\|_{L^{p'}}$;*

(iii) *there exists C such that for every $\omega \subset\subset \Omega$ and every $z \in \mathbb{R}^N \setminus \{0\}$ with $|z| \leq dist(\omega, \mathbb{R}^N \setminus \Omega)$ one has $\|\tau_z u - u\|_{L^p(\omega)} \leq C|z|$;*

(iv) *there exists C such that for every $\omega \subset\subset \Omega$, every $1 \leq i \leq N$ and every $h \in \mathbb{R} \setminus \{0\}$ satisfying $|h| \leq dist(\omega, \mathbb{R}^N \setminus \Omega)$ one has $\|\partial_i^h u\|_{L^p(\omega)} \leq C$.*

Furthermore, if u satisfies these properties, then one can take $C = \|\nabla u\|_{L^p(\Omega)}$ in (ii), (iii) and (iv).

Remark A.3.7. If $p = 1$, then (i)$\Rightarrow$(ii) $\Leftrightarrow$(iii)$\Leftrightarrow$(iv). The functions satisfying (ii) (or (iii), or (iv)) are the functions with bounded variation (i.e. the functions of L^1 whose all derivatives of order 1 in the sense of distributions are bounded measures). See see Brezis [17], Remarque 6 p. 153.

Lemma A.3.8. *Let $1 \leq p \leq \infty$. If $u \in W^{1,p}(\Omega)$, then for every $\omega \subset\subset \Omega$ we have $\partial_i^h u \to \partial_i u$ in $L^q(\omega)$ as $h \downarrow 0$, for every $i \in \{1, \cdots, N\}$ and every $1 \leq q \leq p$ such that $q < \infty$, where ∂_i^h is defined by (A.3.3).*

Proof. Consider $\omega \subset\subset \Omega$, and let $\varphi \in \mathcal{D}(\Omega)$ be such that $\varphi \equiv 1$ on a neighborhood of ω. One verifies easily that $v = \varphi u \in W_0^{1,q}(\Omega)$ (apply for example Corollary A.3.33 below). In particular, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $v_n \to v$ in $W^{1,q}(\Omega)$. For every $n \in \mathbb{N}$, we have $(\partial_i^h - \partial_i)v_n \to 0$ in $L^q(\omega)$, as $h \downarrow 0$. It follows easily from Proposition A.3.6 (or Remark A.3.7 if $p = 1$) and Proposition A.1.4 that $(\partial_i^h - \partial_i)v \to 0$ in $L^q(\omega)$, as $h \downarrow 0$. The result follows, since $u \equiv v$ on a neighborhood of ω. □

Remark A.3.9. Here are some simple consequences of the above results.

(i) Assume Ω is connected. Then it follows from Proposition A.3.6 ((i)$\Leftrightarrow$(iii)) that $W^{1,\infty}(\Omega)$ is the set of functions u such that there exists a constant C for which $|u(x) - u(y)| \leq \mathrm{d}_\Omega(x, y)$ for almost all $x, y \in \Omega$, where d_Ω is the geodesic distance (i.e. $\mathrm{d}_\Omega(x, y)$ is the infimum of the length of polygonal lines in Ω joining x to y). In particular, if Ω has a uniformly Lipschitz boundary, d_Ω is comparable with the usual distance in $\mathbb{R}^N$; and so $W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega})$. If $\partial\Omega$ is not Lipschitz, functions of $W^{1,\infty}(\Omega)$ are not necessarily Lipschitz continuous, as shows the following example. Let B be the ball of $\mathbb{R}^2$ of center 0 and radius 2, and let $\Omega = B \setminus [-1, 1] \times \{0\}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be supported in (-1,1) and verify $\varphi(0) = 1$. Define the function $u \in C^\infty(\Omega)$ by

$$u(x, y) = \begin{cases} \varphi(x), \text{ on } \Omega \cap \{y > 0\}, \\ 0, \text{ on } \Omega \cap \{y \leq 0\}. \end{cases}$$

Then $u \in W^{1,\infty}(\Omega)$, but u is not Lipschitz continuous. However, note that for any domain Ω, Lipschitz continuous and bounded functions are in $W^{1,\infty}(\Omega)$.

(ii) It follows from Proposition A.3.6 ((i)$\Rightarrow$(iii)) and Remark A.3.7 above that if $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, then $((u_n)_{|\omega})_{n\in\mathbb{N}}$ is a relatively compact subset of $L^1(\omega)$, for every $\omega \subset\subset \Omega$. In particular, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging almost everywhere in ω. Therefore, one constructs easily a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging almost everywhere in Ω.

(iii) It follows immediately from the definition that if $u \in C^m(\Omega)$, then $u \in W^{m,\infty}_{\mathrm{loc}}(\Omega)$, and the classical derivatives of u up to order m coincide vith the distributional derivatives. If furthermore all classical derivatives of u up to order m belong to $L^p(\Omega)$ for some $1 \leq p \leq \infty$, then $u \in W^{m,p}(\Omega)$.

Corollary A.3.10. *Let $m \geq 1$ and $1 < p \leq \infty$. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence of $W^{m,p}(\Omega)$, then there exist $u \in W^{m,p}(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ almost everywhere as $k \to \infty$ and*

$$\|u\|_{W^{m,p}} \leq \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}.$$

If $p < \infty$, then also $u_{n_k} \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{m,p}_0(\Omega)$, then $u \in W^{m,p}_0(\Omega)$.

Proof. Consider a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|u_{n_k}\|_{W^{m,p}} = \liminf_{n\to\infty} \|u_n\|_{W^{1,p}}.$$

There exist $u \in L^p(\Omega)$ and a subsequence, which we still denote by $(n_k)_{k\in\mathbb{N}}$, such that $u_{n_k} \to u$ in L^p weak (in L^∞ weak*, if $p = \infty$). By Remark A.3.9 (ii), we may also assume that $u_{n_k} \to u$ almost everywhere. Let α be a multi-index, $|\alpha| \leq m$. From the weak (or weak*) convergence in L^p, it follows that $u_{n_k} \to u$ in $\mathcal{D}'(\Omega)$; and so $D^\alpha u_{n_k} \to D^\alpha u$ in $\mathcal{D}'(\Omega)$. Since $D^\alpha u_{n_k}$ is bounded in L^p, it follows that $D^\alpha u \in L^p$ and that for some subsequence, which we still denote by n_k, $D^\alpha u_{n_k} \to D^\alpha u$ in L^p weak (or weak*). Thus, $u \in W^{m,p}(\Omega)$ and

$$\|u\|_{W^{m,p}} \leq \lim_{k\to\infty} \|u_{n_k}\|_{W^{m,p}} = \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}.$$

The other properties follow from the reflexivity of $W^{m,p}$ when $1 < p < \infty$. □

Corollary A.3.11. *Let $m \geq 0$, let $1 < p \leq \infty$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $W^{m,p}(\Omega)$. If there exits $u : \Omega \to \mathbb{R}$ such that $u_n \to u$ almost everywhere as $n \to \infty$, then $u \in W^{m,p}(\Omega)$ and*

$$\|u\|_{W^{m,p}} \leq \liminf_{n\to\infty} \|u_n\|_{W^{m,p}}.$$

If $p < \infty$, then also $u_n \rightharpoonup u$ in $W^{m,p}$. If $p < \infty$ and $(u_n)_{n\in\mathbb{N}} \subset W_0^{m,p}(\Omega)$, then $u \in W_0^{m,p}(\Omega)$.

Proof. The result follows immediately from Corollary A.3.10. □

Theorem A.3.12. *Let $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $F(0) = 0$, and let $p \in [1, \infty]$. If $u \in W^{1,p}(\Omega)$, then $F(u) \in W^{1,p}(\Omega)$ and $\nabla F(u) = F'(u)\nabla u$ almost everywhere on Ω. Moreover, if $p < \infty$ then the mapping $u \mapsto F(u)$ is continuous from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)$.*

Remark A.3.13. Theorem A.3.12 is due to Marcus and Mizel [75,76,77]. See also Bourdaud and Meyer [14] for the case where $W^{1,p}(\Omega)$ is replaced by $W^{s,p}(\Omega)$. Note that the formula $\nabla F(u) = F'(u)\nabla u$ almost everywhere makes sense. Indeed, it follows from Step 2 of the proof that if $f = g$ almost everywhere, then $f(u)\nabla u = g(u)\nabla u$ almost everywhere. Note that it is important that $p < \infty$ in order that the mapping $u \mapsto F(u)$ be continuous. For example, the mapping $u \to u^+$ is not continuous $W^{1,\infty}(\Omega) \to W^{1,\infty}(\Omega)$. To see this, take $\Omega = (0, 1)$. Let $u(x) = x$, for $x \in \Omega$ and $u_n(x) = x - 1/n$, for $x \in \Omega$. We have $\|u_n - u\|_{W^{1,\infty}} = 1/n$, but $\|\nabla(u_n^+ - u^+)\|_{L^\infty} = 1$ for every n. On the other hand, one shows easily that the mapping $u \mapsto F(u)$ is continuous from $W^{1,\infty}(\Omega)$ strong to $W^{1,\infty}(\Omega)$ weak-$\star$.

Proof of Theorem A.3.12. We proceed in four steps.

Step 1. If we assume in addition that $F \in C^1(\mathbb{R})$, then $F(u) \in W^{1,p}(\Omega)$ for every $u \in W^{1,p}(\Omega)$, and $\nabla F(u) = F'(u)\nabla u$ almost everywhere on Ω. This is well known. The idea of the proof is to approximate u by a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega) \cap C^\infty(\Omega)$ (see for example Brezis [17], Proposition IX.5).

Step 2. Let $1 \leq p \leq \infty$ and let $f \in W^{1,p}(\Omega)$. If $A \subset \mathbb{R}$ is a set of measure 0, then $\nabla f = 0$ almost everywhere on the set $\{x \in \Omega;\ u(x) \in A\}$. We follow the proof of Almgren and Lieb [3]. Let $U \subset \mathbb{R}$ be an open set with finite measure and let

$$M(t) = \int_0^t 1_U(s)\, ds.$$

We claim that

$$\int_\Omega M(f)\nabla \cdot \phi\, dx = -\int_\Omega 1_{\{f\in U\}}\nabla f \cdot \phi\, dx, \tag{A.3.4}$$

for every $\phi \in \mathcal{D}(\Omega)^N$. To prove this, consider a sequence $0 \leq g_1 \leq \cdots \leq g_j \leq \cdots \leq 1$ of continuous functions such that $g_j \uparrow 1_U$ almost everywhere and set

$$N_j(t) = \int_0^t g_j(s)\, ds.$$

It follows from Step 1 that $\nabla N_j(f) = N_j'(f)\nabla f = g_j(f)\nabla f$ almost everywhere. Therefore,

$$\int_\Omega N_j(f)\nabla \cdot \phi\, dx = -\int_\Omega g_j(f)\nabla f \cdot \phi\, dx. \tag{A.3.5}$$

It follows from the dominated convergence theorem that $N_j(f) \to M(f)$ almost everywhere. Since $|N_j(f)| \leq |M(f)|$ almost everywhere, it follows by dominated convergence that the left-hand side of (A.3.5) converges to the left-hand side of (A.3.4) as $j \to \infty$. As well, since $g_j \to 1_U$ almost everywhere and $0 \leq g_j \leq 1$, it follows by dominated convergence that the right-hand side of (A.3.5) converges to the right-hand side of (A.3.4) as $j \to \infty$. Hence (A.3.4). Let now $U_1 \supset U_2 \supset \cdots$ be a decreasing sequence of open subsets of $\mathbb{R}$ such that $A \subset U_j$ and $|U_j| \to 0$ as $j \to \infty$, and set $E = \bigcap_{j\in\mathbb{N}} U_j \supset A$. We apply formula (A.3.4) with $U = U_j$. It follows that

$$\int_\Omega M_j(f)\nabla\cdot\phi\,dx = -\int_\Omega 1_{\{f\in U_j\}}\nabla f\cdot\phi\,dx, \tag{A.3.6}$$

for every $\phi \in \mathcal{D}(\Omega)^N$, where

$$M_j(t) = \int_0^t 1_{U_j}(s)\,ds.$$

It follows from the dominated convergence theorem that $M_j(f) \to 0$ almost everywhere. Since $|M_j(t)| \leq |t|$ almost everywhere, it follows by dominated convergence that the left-hand side of (A.3.6) converges to zero. As well, since $1_{U_j} \to 1_E$ almost everywhere and $0 \leq 1_{U_j} \leq 1$, it follows by dominated convergence that the right-hand side of (A.3.6) converges to

$$-\int_\Omega 1_{\{f\in E\}}\nabla f\cdot\phi\,dx.$$

It follows that

$$\int_\Omega 1_{\{f\in E\}}\nabla f\cdot\phi\,dx = 0.$$

Since ϕ is arbitrary, this implies that $1_{\{f\in E\}}\nabla f = 0$ almost everywhere, which is the desired result, since $A \subset E$.

Step 3. We show that $F(u) \in W^{1,p}(\Omega)$ and $\nabla F(u) = \psi$ almost everywhere on Ω, where $\psi = F'(u)\nabla u$. Since $F(u) \in L^p(\Omega)$, we need only show that $\nabla F(u) \in L^p(\Omega)$ and that $\nabla F(u) = \psi$. To see this, define

$$F_n(t) = n\int_0^{\frac{1}{n}} (F(t+s) - F(s))\,ds,$$

for $n \in \mathbb{N}$. It follows easily that $F_n(t) \leq L|t|$ and that $F_n \to F$ uniformly as $n \to \infty$. Furthermore, $F_n \in C^1(\mathbb{R})$ and

$$F_n'(t) = n\int_0^{\frac{1}{n}} F'(t+s)\,ds,$$

which implies that $|F_n'(t)| \leq L$ and that $F_n'(t) \to F'(t)$ as $n \to \infty$ for every $t \in \mathcal{L}$, where $\mathcal{L} \in \mathbb{R}$ is such that $|\mathbb{R}\setminus\mathcal{L}| = 0$. It now follows from Step 1 that $F_n(u) \in W^{1,p}(\Omega)$ and that $\nabla F_n(u) = F_n'(u)\nabla u$. If $u(x) \in \mathcal{L}$, then $F_n'(u(x)) \to F'(u(x))$ as $n \to \infty$; and so, $\nabla F_n(u) \to F'(u)\nabla u$ almost everywhere on the set $\{x \in \Omega;\ u(x) \in \mathcal{L}\}$. Since $|\mathbb{R}\setminus\mathcal{L}| = 0$, it follows from Step 2 that $\nabla F_n(u) = 0$ almost everywhere on the set $\{x \in \Omega;\ u(x) \notin \mathcal{L}\}$. It follows that $\nabla F_n(u) \to \psi$ almost everywhere on Ω. Since $|\nabla F_n(u)| \leq L|\nabla u| \in L^p(\Omega)$, it follows that $\nabla F_n(u) \to \psi$ in $L^p(\Omega)$ (in $L^\infty(\Omega)$ weak-$\star$ if $p = \infty$). Since $F_n(u) \to F(u)$ in $L^p(\Omega)$, hence in $\mathcal{D}'(\Omega)$, it follows that $\nabla F(u) = \psi \in L^p(\Omega)$.

Step 4. Continuity. Note that the mapping $u \mapsto F(u)$ is continuous $L^p(\Omega) \to L^p(\Omega)$. Therefore, we need only show that if $u_n \to u$ in $W^{1,p}(\Omega)$, then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\nabla F(u_{n_k}) \to \nabla F(u)$

as $k \to \infty$ in $L^p(\Omega)$. Suppose first that $p > 1$. It follows from Step 3 that $\nabla F(u_n)$ is bounded in $L^p(\Omega)$. Therefore, if $p > 1$, then there exists $z \in L^p(\Omega)$ and a subsequence, which we still denote by $(u_n)_{n\in\mathbb{N}}$ such that $\nabla F(u_n) \to z$ in $L^p(\Omega)$ weak. In particular, $\nabla F(u_n) \to z$ in $\mathcal{D}'(\Omega)$, which implies that $z = \nabla F(u)$. Since $L^p(\Omega)$ is uniformly convex, it remains to show that $\|\nabla F(u_n)\|_{L^p} \to \|\nabla F(u)\|_{L^p}$. Let $f = 1_E$ where E is a measurable subset of $\mathbb{R}$. Set now $g = f - 1/2$, so that $|g| = 1/2$. We have

$$\|g(u_n)\nabla u_n\|_{L^p} = \frac{1}{2}\|\nabla u_n\|_{L^p} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\|\nabla u\|_{L^p} = \|g(u)\nabla u\|_{L^p};$$

Therefore, if we set

$$G(t) = \int_0^t g(s)\, s,$$

then it follows from what precedes that $G(u_n) \to G(u)$ in $W^{1,p}(\Omega)$, as $n \to \infty$ (note that G verifies the assumptions of the theorem); and so,

$$\|f(u_n)\nabla u_n\|_{L^p} \underset{n\to\infty}{\longrightarrow} \|f(u)\nabla u\|_{L^p}.$$

It easily follows that the above property holds when $f = \sum_{j=1}^{m} 1_{E_j}$ where the E_j's are disjoint measurable subsets of $\mathbb{R}$ and $m < \infty$. Let now $\varepsilon > 0$. Since $F' \in L^\infty(\mathbb{R})$, we can write $F' = f + h$ where f is as above and $\|h\|_{L^\infty} \le \varepsilon$. It follows that

$$|\, \|\nabla F(u_n)\|_{L^p} - \|f(u_n)\nabla u_n\|_{L^p}| \le \varepsilon \|\nabla u_n\|_{L^p},$$

and

$$|\, \|\nabla F(u)\|_{L^p} - \|f(u)\nabla u\|_{L^p}| \le \varepsilon \|\nabla u\|_{L^p};$$

and so,

$$\limsup_{n\to\infty} |\, \|\nabla F(u_n)\|_{L^p} - \|\nabla F(u)\|_{L^p}| \le \varepsilon \limsup_{n\to\infty} (\|\nabla u_n\|_{L^p} + \|\nabla u\|_{L^p}).$$

Since ε is arbitrary, the result follows. The case $p = 1$ is more complicated, and we refer the reader to Marcus and Mizel [77]. □

Corollary A.3.14. *Let $1 \le p, q, r \le \infty$ and $\alpha > 0$ be such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}$ and let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $F(0) = 0$ and*

$$|F(y) - F(x)| \le L(|y|^\alpha + |x|^\alpha)|y - x|,$$

for all $x, y \in \mathbb{R}$. Then, for every $u \in W^{1,q}(\Omega) \cap L^p(\Omega)$, we have $F(u) \in W^{1,r}(\Omega)$ and $\|F(u)\|_{W^{1,r}} \le 2L\|u\|_{L^p}^\alpha \|u\|_{W^{1,q}}$. Furthermore, $|\nabla F(u)| \le 2L|u|^\alpha |\nabla u|$ almost everywhere on Ω.

Proof. Consider the function F_n defined by

$$F_n(x) = \begin{cases} F(n), & \text{if } n < x, \\ F(x), & \text{if } -n \le x \le n, \\ F(-n), & \text{if } -n < x. \end{cases}$$

It follows that F_n is globally Lipschitz. The result now follows rather easily by applying Theorem A.3.12 then passing to the limit as $n \to \infty$ (apply Dunford-Pettis' theorem to pass to the limit if $r = 1$). □

Corollary A.3.15. *Let $p \in [1, \infty]$. If $u \in W^{1,p}(\Omega)$, then $u^+, u^-, |u| \in W^{1,p}(\Omega)$ and*

$$\nabla u^+ = \begin{cases} \nabla u, \text{ if } u > 0, \\ 0, \text{ if } u \le 0; \end{cases} \quad \nabla u^- = \begin{cases} -\nabla u, \text{ if } u < 0, \\ 0, \text{ if } u \ge 0; \end{cases} \quad \nabla |u| = \begin{cases} \nabla u, \text{ if } u > 0, \\ 0, \text{ if } u = 0, \\ -\nabla u, \text{ if } u < 0; \end{cases}$$

almost everywhere in Ω. In particular, $|\nabla |u|\,| = |\nabla u|$ almost everywhere. If $p < \infty$, then the mappings $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ are continuous on $W^{1,p}(\Omega)$.

Corollary A.3.16. *Let $p \in [1, \infty]$. If $u, v \in W^{1,p}(\Omega)$, then $\max(u, v) \in W^{1,p}(\Omega)$ and $\min(u, v) \in W^{1,p}(\Omega)$.*

Proof. $\max(u, v) = u + (v - u)^+$ and $\min(u, v) = u - (u - v)^+$; and so, the result follows from Corollary A.3.15. □

Corollary A.3.17. *Let $p \in [1, \infty]$. Consider $M \in W^{1,p}_{\text{loc}}(\Omega)$ such that $\nabla M \in L^p(\Omega)$. If $M^- \in L^p(\Omega)$, then $(u - M)^+ \in W^{1,p}(\Omega)$ for every $u \in W^{1,p}(\Omega)$, and*

$$\nabla (u - M)^+ = \begin{cases} \nabla u - \nabla M, \text{ if } u > M; \\ 0, \text{ if } u \le M; \end{cases}$$

almost everywhere. Moreover, if $p < \infty$, then the mapping $u \mapsto (u - M)^+$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. In particular, these results apply to the case where M is a nonnegative constant.

Proof. We have $(u - M) \in W^{1,p}_{\text{loc}}(\Omega)$. When Corollary A.3.15 is applied, it follows easily that $(u - M)^+ \in W^{1,p}_{\text{loc}}(\Omega)$ and that

$$\nabla (u - M)^+ = \begin{cases} \nabla u - \nabla M, \text{ if } u > M; \\ 0, \text{ if } u \le M; \end{cases}$$

almost everywhere. In particular, $|\nabla (u - M)^+| \le |\nabla u| + |\nabla M| \in L^p(\Omega)$. Since $(u - M)^+ \le |u| + M^- \in L^p(\Omega)$ it follows that $(u - M) \in W^{1,p}(\Omega)$. Continuity is proved by the technique of proof of Theorem A.3.12. □

Remark A.3.18. These properties are specific to the case $m = 1$. For example, consider $\Omega = (-1, 1)$, $F(x) = |x|$ and $u(x) = \sin(\pi x)$. Then $u \in C^\infty(\overline{\Omega})$ but $F(u) \notin W^{2,1}(\Omega)$.

Proposition A.3.19. *Let $1 \le p \le \infty$. If $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\nabla(uv) = u \nabla v + v \nabla u$ almost everywhere in Ω.*

Proof. See Brezis [17], Proposition IX.4, p.155. □

Finally, we recall below a quite useful result concerning L^p spaces.

Lemma A.3.20. *Let $1 \le p \le \infty$, let $u : \Omega \to \mathbb{R}$ and let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence of $L^p(\Omega)$ such that $u_n \to u$ almost everywhere as $n \to \infty$. If $p > 1$, then $u \in L^p(\Omega)$ and $u_n \to u$ as $n \to \infty$ in $L^q(\Omega')$,* *for every $\Omega' \subset \Omega$ of finite measure and every $q \in [1,p)$. In particular, $u_n \to u$ as $n \to \infty$, in $L^p(\Omega)$ weak if $p < \infty$, and in $L^\infty(\Omega)$ weak* if $p = \infty$.*

Proof. By extending the functions by 0 outside Ω, we may assume that $\Omega = \mathbb{R}^N$. Observe that by Fatou's lemma, we have $u \in L^p(\mathbb{R}^N)$. Let $\Omega' \subset \mathbb{R}^N$ have a finite measure and let $q \in [1,p)$. Consider $\varepsilon > 0$. By Egorov's theorem, there exists a measurable subset E of Ω' such that $u_n \to u$ uniformly on $\Omega' \setminus E$ and

$$|E|^{\frac{p-q}{p}} \sup_{n\geq 0} \left(\int_{\mathbb{R}^N} |u_n - u|^p \right)^{\frac{q}{p}} \leq \varepsilon/2.$$

Let n_0 be large enough so that $|u_n - u|^q \leq \varepsilon/2|\Omega'|$ on $\Omega' \setminus E$, for $n \geq n_0$. It follows that

$$\begin{aligned}\int_{\Omega'} |u_n - u|^q &= \int_E |u_n - u|^q + \int_{\Omega'\setminus E} |u_n - u|^q \\ &\leq |E|^{\frac{p-q}{p}} \left(\int_E |u_n - u|^p \right)^{\frac{q}{p}} + |\Omega' \setminus E| \sup_{\Omega'\setminus E} |u_n - u|^q \\ &\leq \varepsilon.\end{aligned}$$

Hence the result, since ε is arbitrary. □

A.3.3. Basic properties of the space $W_0^{m,p}(\Omega)$. Basically, $W_0^{m,p}(\Omega)$ is the set of functions of $W^{m,p}(\Omega)$ that "vanish on $\partial\Omega$". In this section, we give some characterizations of $W_0^{m,p}(\Omega)$. The case $\Omega = \mathbb{R}^N$ is quite simple, as shows the following result.

Proposition A.3.21. *Let $1 \leq p, q < \infty$ and let m, j be nonnegative integers. Then, $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{m,p}(\mathbb{R}^N) \cap W^{j,q}(\mathbb{R}^N)$. In particular, $W_0^{m,p}(\mathbb{R}^N) = W^{m,p}(\mathbb{R}^N)$.*

Proof. By the standard procedure of truncation and regularization. □

Remark A.3.22. Note that if $\overline{\Omega} \neq \mathbb{R}^N$, then $W_0^{m,p}(\Omega)$ is a strict subset of $W^{m,p}(\Omega)$.

Proposition A.3.23. *Let $1 \leq p < \infty$. For every $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, the following properties hold:*

(i) *if $u = 0$ on $\partial\Omega$, then $u \in W_0^{1,p}(\Omega)$;*

(ii) *if $u \in W_0^{1,p}(\Omega)$ and if Ω has a C^1 boundary, then $u = 0$ on $\partial\Omega$.*

Proof. See Brezis [17], Theorem IX.17 and Remark 20.

Remark A.3.24. The smoothness assumption on Ω is essential in property (ii). For example, asume $N \geq 2$, let $\Omega = \mathbb{R}^N \setminus \{0\}$ and consider $u \in \mathcal{D}(\mathbb{R}^N)$ such that $u(0) = 1$. Then it is easily verified that $u \in H_0^1(\Omega)$, but $u \equiv 1$ on $\partial\Omega$.

Proposition A.3.25. *Let $1 \leq p < \infty$, let $u \in L^p(\Omega)$ and let $\overline{u}$ be defined by (A.3.1). Then the following holds:*

(i) *if $u \in W_0^{1,p}(\Omega)$, then $\overline{u} \in W^{1,p}(\mathbb{R}^N)$;*

(ii) *if* $\overline{u} \in W^{1,p}(\mathbb{R}^N)$ *and if* Ω *has a* C^1 *boundary, then* $u \in W_0^{1,p}(\Omega)$.

Proof. See Brezis [17], Theorem IX.18 and Remark 21.

Remark A.3.26. Property (ii) is not anymore valid without some smoothness assumption on Ω, as shows the following simple example. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be such that $\varphi(0) = 1$. Let $\Omega = (-\infty, 0) \cup (0, \infty)$ and let $u = \varphi_{|\Omega}$ almost everywhere. Obviously, $\overline{u} = \varphi$ almost everywhere. However, $\overline{u} \in W^{1,p}(\mathbb{R})$ but $u \notin W_0^{1,p}(\Omega)$.

Corollary A.3.27. *Let* $1 \leq p, q < \infty$. *If* $u \in W_0^{1,p}(\Omega) \cap W^{1,q}(\Omega)$ *and if* Ω *has a* C^1 *boundary, then* $u \in W_0^{1,q}(\Omega)$.

Proof. Let $\overline{u}$ be defined by (A.3.1). It follows from Proposition A.3.25 (i) that $\overline{u} \in W^{1,p}(\mathbb{R}^N)$. We clearly have $\overline{u} \in L^q(\mathbb{R}^N)$, and since

$$\nabla \overline{u} = \begin{cases} \nabla u & \text{on} \quad \Omega, \\ 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases}$$

we have $\nabla \overline{u} \in L^q(\mathbb{R}^N)$. Therefore $\overline{u} \in W^{1,q}(\mathbb{R}^N)$; and so, $u \in W_0^{1,q}(\Omega)$ by Proposition A.3.25 (ii). □

Proposition A.3.28. *Let* $1 \leq p < \infty$ *and* $u \in W^{1,p}(\Omega)$. *If there exists* $\Omega' \subset\subset \Omega$ *such that* $u = 0$ *almost everywhere on* $\Omega \setminus \Omega'$, *then* $u \in W_0^{1,p}(\Omega)$.

Proof. See Brezis [17], Lemma IX.5.

Corollary A.3.29. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz continuous function such that* $F(0) = 0$, *and let* $p \in [1, \infty)$. *Then the mapping* $u \mapsto F(u)$ *is continuous from* $W_0^{1,p}(\Omega)$ *to* $W_0^{1,p}(\Omega)$.

Proof. By assumption, there exists $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $\varphi_n \to u$ in $W^{1,p}(\Omega)$. By Proposition A.3.28, we have $F(\varphi_n) \in W_0^{1,p}(\Omega)$. On the other hand, it follows from Theorem A.3.12 that $F(\varphi_n) \to F(u)$ in $W^{1,p}(\Omega)$. Hence the result. □

Corollary A.3.30. *Let* $p \in [1, \infty)$. *If* $u \in W_0^{1,p}(\Omega)$, *then* $u^+, u^-, |u| \in W_0^{1,p}(\Omega)$. *Moreover, the mappings* $u \mapsto u^+$, $u \mapsto u^-$ *and* $u \mapsto |u|$ *are continuous on* $W_0^{1,p}(\Omega)$.

Corollary A.3.31. *If* $1 \leq p < \infty$ *and* $u, v \in W_0^{1,p}(\Omega)$, *then* $\max(u, v) \in W_0^{1,p}(\Omega)$ *and* $\min(u, v) \in W_0^{1,p}(\Omega)$.

Proof. $\max(u, v) = u + (v - u)^+$ and $\min(u, v) = u - (u - v)^+$; and so, the result follows from Corollary A.3.30. □

Corollary A.3.32. *Let* p, q, r, α *and* F *be as in Corollary A.3.14. If* $q, r < \infty$, *then for every* $u \in W_0^{1,q}(\Omega) \cap L^p(\Omega)$, *we have* $F(u) \in W_0^{1,r}(\Omega)$.

Proof. The proof is similar to that of Corollary A.3.14, by applying Corollary A.3.29 instead of Theorem A.3.12. □

Corollary A.3.33. *Let $1 \le p < \infty$. If $u \in W^{1,p}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then $\varphi u \in W_0^{1,p}(\Omega)$.*

Proof. It follows from Proposition A.3.19 that $\varphi u \in W^{1,p}(\Omega)$. Since φu is supported in a compact subset of Ω, the result follows from Proposition A.3.28. $\square$

Proposition A.3.34. *Let $1 \le p < \infty$ and let $u \in W^{1,p}(\Omega)$. If there exists $v \in W_0^{1,p}(\Omega)$ such that $|u| \le |v|$ almost everywhere, then $u \in W_0^{1,p}(\Omega)$.*

Proof. It follows from Corollary A.3.30 that $|v| \in W_0^{1,p}(\Omega)$. Let $(w_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be such that $w_n \underset{n\to\infty}{\longrightarrow} |v|$ in $W^{1,p}(\Omega)$. It follows from Corollary A.3.15 that $(w_n - u^+)^+ \to (|v| - u^+)^+$ in $W^{1,p}(\Omega)$. On the other hand, $\operatorname{supp}(w_n - u^+)^+ \subset \operatorname{supp}(w_n)$, thus $(w_n - u^+)^+ \in H_0^1(\Omega)$ by Proposition A.3.28. This implies $(|v| - u^+)^+ \in W_0^{1,p}(\Omega)$. Since $|v| \ge |u| \ge u^+$, we have $(|v| - u^+)^+ \equiv |v| - u^+$; and so, $|v| - u^+ \in W_0^{1,p}(\Omega)$, from which we get $u^+ \in W_0^{1,p}(\Omega)$. One shows with the same argument that $u^- \in W_0^{1,p}(\Omega)$. Therefore, $u = u^+ - u^- \in W_0^{1,p}(\Omega)$. $\square$

Corollary A.3.35. *Let $1 \le p < \infty$ and let $M \in W_{\mathrm{loc}}^{1,p}(\Omega)$ be such that $\nabla M \in L^p(\Omega)$. If there exists $w \in W_0^{1,p}(\Omega)$ such that $M \ge w$ almost everywhere, then $(u - M)^+ \in W_0^{1,p}(\Omega)$ for every $u \in W_0^{1,p}(\Omega)$. Moreover, the mapping $u \mapsto (u - M)^+$ is continuous $W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$. In particular, the above results apply to the case where M is a nonnegative constant.*

Proof. Note that $(u-M)^+ \le |u|+M^- \le |u|+w \in W_0^{1,p}(\Omega)$. Therefore, it follows from Proposition A.3.34 that $(u - M)^+ \in W_0^{1,p}(\Omega)$. Continuity follows from Corollary A.3.17. $\square$

A.3.4. Sobolev's inequalities. We recall below the most useful embedding theorems and Sobolev's inequalities concerning Sobolev's spaces.

Remark A.3.36. It may be convenient to approximate functions of $W^{m,p}(\Omega)$ by smooth functions or to extend functions of $W^{m,p}(\Omega)$ to functions of $W^{m,p}(\mathbb{R}^N)$. This can be done as follows.

(i) If Ω has a uniformly Lipschitz boundary and if $1 \le p < \infty$, then the restriction to Ω of functions of $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{m,p}(\Omega)$ (see Adams [1], Theorem 3.18).

(ii) If $p \in [1, \infty)$, then $W^{m,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ (see Adams [1], Theorem 3.16).

(iii) If Ω has a bounded C^m boundary, then there exists an operator $E \in \mathcal{L}(W^{m,p}(\Omega), W^{m,p}(\mathbb{R}^N))$ such that $Eu_{|\Omega} = u$, for every $u \in W^{m,p}(\Omega)$ (see Adams [1], Theorem 4.26).

Theorem A.3.37. (Poincaré's inequality) *If $|\Omega|$ is finite (or if Ω is bounded in one direction) and if $1 \le p < \infty$, then there exists a constant C such that*

$$\|u\|_{L^p} \le C\|\nabla u\|_{L^p}, \tag{A.3.7}$$

for every $u \in W_0^{1,p}(\Omega)$.

Proof. By density, we only have to prove the estimate for $u \in \mathcal{D}(\mathbb{R}^N)$ with $\text{supp}(u) \subset \Omega$.

Assume first that $|\Omega| < \infty$. In the case $p = 1$, It follows from (A.3.8) below that $\|u\|_{L^{\frac{N}{N-1}}} \le C\|\nabla u\|_{L^1}$; and so $\|u\|_{L^1} \le C|\Omega|^{1/N}\|\nabla u\|_{L^1}$, which is the desired estimate. If $p > 1$, It follows from (A.3.8) that $\|u\|_{L^p} \le C\|\nabla u\|_{L^p}^a \|u\|_{L^1}^{1-a}$, with $a = \dfrac{N(p-1)}{N(p-1)+p} \in (0,1)$. Hence the result, since $\|u\|_{L^1} \le |\Omega|^{1/p'}\|u\|_{L^p}$.

Assume now that Ω is bounded in one direction. Without loss of generality, we may assume that $\Omega \subset (0,a) \times \mathbb{R}^{N-1}$, for some $a > 0$. Given $(x_2, \cdots, x_N) \in \mathbb{R}^{N-1}$, the function $x_1 \mapsto u(x_1, \cdots, x_N)$ belongs to $\mathcal{D}(0,a)$. It follows from the first step of the proof that there exists C such that $\|u(\cdot, x_2, \cdots, x_N)\|_{L^p(0,a)} \le C\|\partial_1 u(\cdot, x_2, \cdots, x_N)\|_{L^p(0,a)}$, from which we get easily $\|u\|_{L^p(\Omega)} \le C\|\partial_1 u\|_{L^p(\Omega)}$. □

Remark A.3.38. Here are some simple observations concerning Poincaré's inequality.

(i) Inequality (A.3.7) holds under the more general assumption that $\Omega \subset \omega \times \mathbb{R}^{N-k}$, where $0 \le k \le N$ and ω is an open subset of $\mathbb{R}^k$ of finite measure, the proof being the same.

(ii) The boundedness assumption on Ω is essential in Theorem A.3.37. In particular, if $\Omega = \mathbb{R}^N$, then inequality (A.3.7) does not hold. To see this, consider $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \not\equiv 0$. For $\lambda > 0$, let $\varphi_\lambda(x) = \varphi(x/\lambda)$. Then $\|\varphi_\lambda\|_{L^p} = \lambda^{\frac{N}{p}}\|\varphi\|_{L^p}$ and $\|\nabla\varphi_\lambda\|_{L^p} = \lambda^{\frac{N}{p}-1}\|\nabla\varphi\|_{L^p}$. Therefore $\dfrac{\|\varphi_\lambda\|_{L^p}}{\|\nabla\varphi_\lambda\|_{L^p}} = \lambda\dfrac{\|\varphi\|_{L^p}}{\|\nabla\varphi\|_{L^p}} \underset{\lambda\to\infty}{\longrightarrow} \infty$.

Corollary A.3.39. If Ω is as in Theorem A.3.37, then $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm to $\|u\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$.

Theorem A.3.40. (Sobolev's embedding theorem) *If Ω has the cone property, then the following properties hold:*

(i) *if $1 \le mp < N$ and $j \ge 0$, then $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$, for every $q \in [p, \dfrac{Np}{N-mp}]$;*

(ii) *if $mp = N$ and $j \ge 0$, then $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$, for every $q \in [p,\infty)$;*

(iii) *if $j \ge 0$, then $W^{j+N,1}(\Omega) \hookrightarrow C_b^j(\Omega)$. In particular, $W^{N,1}(\Omega) \hookrightarrow L^\infty(\Omega)$;*

(iv) *if $mp > N$ and $j \ge 0$, then $W^{j+m,p}(\Omega) \hookrightarrow C_b^j(\Omega)$. In particular, $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.*

If Ω has a uniformly Lipschitz boundary, then also

(v) *if $mp > N > (m-1)p$ and $j \ge 0$, then $W^{j+m,p}(\Omega) \hookrightarrow C^{j,\alpha}(\overline{\Omega})$, where $\alpha = \dfrac{mp-N}{p}$.*

Proof. See Adams [1], Theorem 5.4. □

Theorem A.3.41. (Rellich's compactness theorem) *If Ω is bounded and has a Lipschitz boundary. then the following properties hold:*

(i) *if $mp \le N$ and $j \ge 0$, then the embedding $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ is compact, for every $q \in [p, \dfrac{Np}{N-mp})$;*

(ii) *if $mp > N$ and $j \ge 0$, then the embedding $W^{j+m,p}(\Omega) \hookrightarrow C_b^j(\Omega)$ is compact;*

(iii) *if $mp > N \ge (m-1)p$ and $j \ge 0$, then the embedding $W^{j+m,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ is compact, for all $\lambda \in (0, \dfrac{mp-N}{p})$.*

Proof. See Adams [1], Theorem 6.2. Note that since Ω is bounded, its boundary is uniformly Lipschitz. □

Theorem A.3.42. *The conclusions of Theorems A.3.40 and A.3.41 remain valid without any smoothness assumption on Ω if one replaces $W^{m,p}(\Omega)$ by $W_0^{m,p}(\Omega)$ (note that Ω still needs to be bounded for the compact embedding).*

Proof. See Adams [1], Theorem 5.4, part III and Theorem 6.2., part IV. □

Remark A.3.69. If $p = N > 1$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $p \le q < \infty$, but $W^{1,p}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. However, Sobolev's embedding theorem can be improved by Trudinger's inequality. In particular, if $N = 2$, then for every $M < \infty$ there exists $\mu > 0$ and $K < \infty$ such that

$$\int_\Omega \left(e^{\mu \|u\|^2} - 1\right) \le K,$$

for every $u \in H_0^1(\Omega)$ with $\|u\|_{H^1} \le M$ (see Adams [1]).

Theorem A.3.44. (Gagliardo-Nirenberg's inequality) *Let $1 \le p, q, r \le \infty$ and let j, m be two integers, $0 \le j < m$. If*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{(1-a)}{q},$$

for some $a \in [\frac{j}{m}, 1]$ ($a < 1$ if $r > 1$ and $m - j - \frac{N}{r} = 0$), then there exists a constant $C(N, m, j, p, q, r)$ such that

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p} \le C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^r}\right)^a \|u\|_{L^q}^{1-a}, \tag{A.3.8}$$

for every $u \in \mathcal{D}(\mathbb{R}^N)$.

Proof. See Friedman [42], Theorem 9.3, for the general case. The case $a = r = 1$, $m - j = N$ is treated in Brezis [17], Chapter IX, Remark 14. □

Remark A.3.45. Here are some simple consequences of Theorem A.3.44.

(i) By density (Proposition A.3.21), inequality (A.3.8) holds for every $u \in W^{m,r}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, provided $q, r < \infty$. If $q = \infty$ and $N < mr < \infty$, then $W^{m,r}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, and again by density inequality (A.3.8) holds for every $u \in W^{m,r}(\mathbb{R}^N)$;

(ii) also by density (see Proposition A.3.58 below), inequality (A.3.8) is valid for every $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$, where Ω is any open domain of $\mathbb{R}^N$, provided $q, r < \infty$, or $q = \infty$ and $N < mr < \infty$;

(iii) It follows easily from (A.3.8) and (ii) above that for every open subset Ω of $\mathbb{R}^N$ and every integers $0 \le j \le m$, one has

$$\|u\|_{H^j(\Omega)} \le C \|u\|_{H^m(\Omega)}^{\frac{j}{m}} \|u\|_{L^2(\Omega)}^{\frac{m-j}{m}},$$

for every $u \in H_0^m(\Omega)$. More generally, if $p < \infty$, then

$$\|u\|_{W^{j,p}(\Omega)} \le C \|u\|_{W^{m,p}(\Omega)}^{\frac{j}{m}} \|u\|_{L^p(\Omega)}^{\frac{m-j}{m}}, \tag{A.3.9}$$

for every $u \in W_0^{m,p}(\Omega)$.

(iv) Let Ω be a domain having a bounded C^m boundary. It follows from Adams [1], Theorem 4.26 that if $q, r < \infty$, then there exists an operator E mapping functions defined almost everywhere in Ω to functions defined almost everywhere in $\mathbb{R}^N$ such that $E \in \mathcal{L}(W^{m,r}(\Omega), W^{m,r}(\mathbb{R}^N))$, $E \in \mathcal{L}(L^q(\Omega), L^q(\mathbb{R}^N))$, and $Eu = u$ almost everywhere in Ω. In fact, the proof of Theorem 4.26 in Adams [1] shows that the same conclusion holds with the spaces $L^q(\Omega)$ and $L^q(\mathbb{R}^N)$ replaced by the spaces $C_{\mathrm{b,u}}(\overline{\Omega})$ and $C_{\mathrm{b,u}}(\mathbb{R}^N)$, respectively. Therefore, if $u \in W^{m,r}(\Omega) \cap L^r(\Omega)$ and if $q, r < \infty$ or if $q = \infty$ and $N < mr < \infty$, then it follows from (i) above and inequality (A.3.8) that

$$\begin{aligned}\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p(\Omega)} &= \sum_{|\alpha|=j} \|D^\alpha Eu\|_{L^p(\mathbb{R}^N)} \le C\|Eu\|^a_{W^{m,r}(\mathbb{R}^N)} \|Eu\|^{1-a}_{L^q(\mathbb{R}^N)} \\ &\le C\|u\|^a_{W^{m,r}(\Omega)} \|u\|^{1-a}_{L^q(\Omega)};\end{aligned}$$

and so, the inequality

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p(\Omega)} \le C\|u\|^a_{W^{m,r}(\Omega)} \|u\|^{1-a}_{L^q(\Omega)}, \tag{A.3.10}$$

holds for every $u \in W^{m,r}(\Omega) \cap L^r(\Omega)$, provided $q, r < \infty$ or $q = \infty$ and $N < mr < \infty$.

Corollary A.3.46. *If $mp > N$, then $W_0^{m,p}(\Omega) \hookrightarrow C_0(\Omega)$.*

Proof. By definition of $C_0(\Omega)$, this follows immediately from the density of $\mathcal{D}(\Omega)$ in $W_0^{m,p}(\Omega)$ and the embedding $W^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. □

Corollary A.3.47. *If Ω has a uniformly C^1 boundary and if $mp > N$, then $W^{m,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C_0(\Omega)$.*

Before proceeding to the proof of Corollary A.3.47, we need the following characterization of $C_0(\Omega)$.

Lemma A.3.48. *$C_0(\Omega)$ is the set of $u \in C(\overline{\Omega})$ verifying the following properties:*

(i) *$u(x) = 0$, for all $x \in \partial\Omega$;*

(ii) *for all $\varepsilon > 0$, there exists $M < \infty$ such that $|u(x)| \le \varepsilon$, for all $x \in \Omega$ such that $|x| \ge M$.*

Proof. It follows easily from the definition of $C_0(\Omega)$ that every $u \in C_0(\Omega)$ belongs to $C(\overline{\Omega})$ and verifies (i) and (ii). Conversely, consider $u \in C(\overline{\Omega})$ verifying (i) and (ii), and let $\varepsilon > 0$. It follows easily from (i) and (ii) that

$$\{x \in \Omega;\ |v(x)| \ge \varepsilon/2\} \text{ is a compact subset of } \Omega. \tag{A.3.11}$$

Define $v \in C(\mathbb{R}^N)$ by

$$v(x) = \begin{cases} (u - \varepsilon/2)^+ - (u + \varepsilon/2)^-, \text{ in } \Omega; \\ 0, \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We have $v \in C_{\mathrm{c}}(\mathbb{R}^N)$ and it follows from (A.3.11) that $\mathrm{Supp}(v)$ is a compact subset of Ω. Furthermore,

$$\|u - v_{|\Omega}\|_{L^\infty} \le \varepsilon/2. \tag{A.3.12}$$

Finally, let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers, and let $v_n = \rho_n * v$. We have $v_n \in \mathcal{D}(\mathbb{R}^N)$, and $v_n \underset{n\to\infty}{\longrightarrow} v$ in $L^\infty(\mathbb{R}^N)$ (see Brezis [17], Proposition IV.21, p.70). Furthermore, for n large enough, $\mathrm{Supp}(v_n)$ is a compact subset of Ω. Choose n large enough, so that $\mathrm{Supp}(v_n)$ is a compact subset of Ω and

$$\|v - v_n\|_{L^\infty} \le \varepsilon/2, \tag{A.3.13}$$

and set $u_\varepsilon = (v_n)_{|\Omega}$. We have $u_\varepsilon \in \mathcal{D}(\Omega)$, and it follows from (A.3.12) and (A.3.13) that $\|u - u_\varepsilon\|_{L^\infty} \le \varepsilon$. Since ε is arbitrary, it follows that $u \in C_0(\Omega)$. Hence the result. □

Remark A.3.49. Here are some simple observations about Lemma A.3.48.

(i) It follows from Lemma A.3.48 that if Ω is bounded, then $C_0(\Omega) = \{u \in C(\overline{\Omega});\ u_{|\partial\Omega} = 0\}$. If Ω is unbounded, then $C_0(\Omega)$ is the set of $u \in C(\overline{\Omega})$ that vanish on $\partial\Omega$ and such that $|u(x)| \to 0$, as $|x| \to \infty$, $x \in \Omega$.

(ii) It follows immediately from Lemma A.3.48 that if $u \in C_0(\Omega)$, then also $u^+, u^- \in C_0(\Omega)$.

Corollary A.3.50 *Let $1 \le p < \infty$. If Ω has a C^1 boundary, then $W_0^{1,p}(\Omega) \cap C_{\mathrm{b,u}}(\overline{\Omega}) \subset C_0(\Omega)$.*

Proof. Consider $u \in W_0^{1,p}(\Omega) \cap C_{\mathrm{b,u}}(\overline{\Omega})$. In particular, u is uniformly continuous, and since also $u \in L^p(\Omega)$, it follows easily that $|u(x)| \to 0$, as $|x| \to \infty$, $x \in \Omega$. On the other hand, it follows from Proposition A.3.23 (ii) that $u_{|\partial\Omega} = 0$; and so, $u \in C_0(\Omega)$. □

Proof of Corollary A.3.47. The result follows from the embedding $W^{m,p}(\Omega) \hookrightarrow C_{\mathrm{b,u}}(\overline{\Omega})$ (see Theorem A.3.40 (v)) and from Corollary A.3.50. □

A.3.5. The Sobolev spaces $W^{-m,q}(\Omega)$.

Definition A.3.51. *For $1 \le p < \infty$ and $m \in \mathbb{N}$, one defines $W^{-m,p'}(\Omega)$ as the (topological) dual of $W_0^{m,p}(\Omega)$. One defines $H^{-m}(\Omega) = W^{-m,2}(\Omega)$, so that $H^{-m}(\Omega) = (H_0^m(\Omega))^\star$.*

Remark A.3.52. Here are some simple consequences of Definition A.3.51.

(i) It follows from the dense embedding $\mathcal{D}(\Omega) \hookrightarrow W_0^{m,p}(\Omega)$ that $W^{-m,p'}(\Omega)$ is a space of distributions on Ω. In particular,

$$\langle f, \varphi\rangle_{W^{-m,p'}, W_0^{m,p}} = \langle f, \varphi\rangle_{\mathcal{D}', \mathcal{D}},$$

for every $f \in W^{-m,p'}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Furthermore, it follows from the dense embedding $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$ and Proposition A.1.5 that $L^{p'}(\Omega) \hookrightarrow W^{-m,p'}(\Omega)$. If $p > 1$, then the embedding is dense by Proposition A.3.5 and Proposition A.1.5. In particular, $\mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega)$. Furthermore,

$$\langle f, \varphi\rangle_{W^{-m,p'}, W_0^{m,p}} = \int_\Omega f\varphi\, dx,$$

for every $f \in L^{p'}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ (and, by density, for every $\varphi \in W_0^{m,p}(\Omega)$). Note also that a distribution $T \in \mathcal{D}'(\Omega)$ defines (by density of $\mathcal{D}(\Omega)$ in $W_0^{m,p}(\Omega)$) an element of $W^{-m,p'}(\Omega)$, if and only if there exists a constant C such that

$$|\langle T, \varphi \rangle_{\mathcal{D}',\mathcal{D}}| \leq C \|\varphi\|_{W^{m,p}},$$

for all $\varphi \in \mathcal{D}(\Omega)$.

(ii) Assume that that $1 \leq q \leq \infty$ is such that $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$. Then $L^{q'}(\Omega) \hookrightarrow W^{-m,p'}(\Omega)$. Furthermore, if $p > 1$, then the embedding is dense. In the case $q < \infty$, the result follows from Proposition A.1.5 (observe that if $p > 1$, $\mathcal{D}(\Omega) \subset W_0^{m,p}(\Omega)$; and so, the embedding $W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ is dense). Suppose now that $q = \infty$, that is $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. It follows that the linear form $W_0^{m,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_f : u \mapsto \int_\Omega u(x) f(x)\, dx,$$

is continuous for every $f \in L^1(\Omega)$. This defines a mapping $L^1(\Omega) \to W^{-m,p'}(\Omega)$. This mapping is injective since $\mathcal{D}(\Omega) \subset W_0^{m,p}(\Omega)$; and so $L^1(\Omega) \hookrightarrow W^{-m,p'}(\Omega)$. If furthermore $p > 1$, then by (i) above, $\mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega)$; and so $L^1(\Omega) \supset \mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega)$.

(iii) Like any distribution, an element of $W^{-m,p'}(\Omega)$ can be localized. Indeed, if $T \in W^{-m,p'}(\Omega)$ and Ω' is an open subset of Ω, then one defines $T_{|\Omega'}$ as follows. Let $\varphi \in \mathcal{D}(\Omega')$ and let $\tilde{\varphi} \in \mathcal{D}(\Omega)$ be equal to φ on Ω' and to 0 on $\Omega \setminus \Omega'$. Then

$$\Psi(\varphi) = \langle \tilde{\varphi}, T \rangle_{W_0^{m,p}(\Omega), W^{-m,p'}(\Omega)}$$

defines a distribution $\Psi \in \mathcal{D}'(\Omega')$. Since $\|\tilde{\varphi}\|_{W_0^{m,p}(\Omega')} \leq \|\varphi\|_{W_0^{m,p}(\Omega)}$, it follows that $\Psi \in W^{-m,p'}(\Omega')$, and one sets $T_{|\Omega'} = \Psi$. It is clear that the operator

$$P_{\Omega'} : \begin{cases} W^{-m,p'}(\Omega) \to W^{-m,p'}(\Omega') \\ T \mapsto T_{|\Omega'} \end{cases}$$

is linear and continuous, and is consistent with the usual restriction of functions.

(iv) Even though $H_0^m(\Omega)$ is a Hilbert space, one generally does not identify $H^{-m}(\Omega)$ with $H_0^m(\Omega)$. One rather identifies $L^2(\Omega)$ with its dual, so that $H^{-m}(\Omega)$ becomes a subspace of $\mathcal{D}'(\Omega)$ containing $L^2(\Omega)$. In particular, if $u \in H_0^m(\Omega)$ and $v \in L^2(\Omega)$, then

$$\langle u, v \rangle_{H_0^m, H^{-m}} = \int_\Omega u(x) v(x)\, dx. \tag{A.3.14}$$

Taking $u = v \in H_0^m(\Omega)$ in (A.3.14), it follows that

$$\|u\|_{L^2}^2 \leq \|u\|_{H_0^m} \|u\|_{H^{-m}}, \text{ for all } u \in H_0^m(\Omega). \tag{A.3.15}$$

In addition, since by definition $\|u\|_{H^{-m}} = \sup\{\langle u, v \rangle_{H^{-m}, H_0^m};\ \|v\|_{H^m} = 1\}$, we deduce from (A.3.14) that

$$\|u\|_{H^{-m}} \leq \|u\|_{L^2}, \tag{A.3.16}$$

for all $u \in L^2(\Omega)$.

(v) Since D^α is bounded from $H_0^k(\Omega)$ to $H_0^{k-j}(\Omega)$, for every $k \ge j$ and every multi-index α of length j, it follows easily from the definition and identity (A.3.14) that D^α is a bounded operator from $H^{-m}(\Omega)$ to $H^{-m-j}(\Omega)$, for every $m \in \mathbb{N}$. Since also D^α is bounded from $H^k(\Omega)$ to $H^{k-j}(\Omega)$, for every $k \ge j$, it follows easily that if $k \le j$, then D^α is bounded from $H^k(\Omega)$ to $H^{k-j}(\Omega)$.

(vi) In particular, $\triangle$ defines a linear, continuous operator from $H^1(\Omega)$ to $H^{-1}(\Omega)$. Note that for $u \in H^1(\Omega)$, the linear form $\triangle u \in H^{-1}(\Omega)$ on $H_0^1(\Omega)$ is defined by

$$\langle \triangle u, v\rangle = -\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx, \text{ for } v \in H_0^1(\Omega). \tag{A.3.17}$$

This is clear for $v \in \mathcal{D}(\Omega)$ and follows by density for $v \in H_0^1(\Omega)$. We will see in Section A.5 that for λ not too negative, $\triangle - \lambda I$ defines an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.

Proposition A.3.53. *Let $1 \le p < \infty$. Then a distribution $T \in \mathcal{D}'(\Omega)$ belongs to $W^{-1,p'}(\Omega)$ if and only if there exists $f, g_1, \cdots, g_N \in L^{p'}(\Omega)$ such that*

$$T = f + \sum_{j=1}^N \frac{\partial g_j}{\partial x_j}, \text{ in } \mathcal{D}'(\Omega).$$

Furthermore, one can choose $f, g_1, \cdots, g_N$ such that $\|T\|_{W^{-m,p'}} = \|f\|_{L^{p'}} + \|g_1\|_{L^{p'}} + \cdots + \|g_N\|_{L^{p'}}$.

Proof. See Brezis [17], Proposition IX.20.

Remark A.3.54. It is easily verified that the decomposition of Proposition A.3.53 is not unique.

When $\Omega = \mathbb{R}^N$, one can define H^{-m} in terms of the Fourier transform. More precisely, the following result is an easy consequence of Proposition A.3.1.

Proposition A.3.55. *For every $m \in \mathbb{N}$, the following properties hold:*

(i) $H^{-m}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+|\xi|^2)^{-m/2}\widehat{u}(\xi) \in L^2(\mathbb{R}^N)\}$;

(ii) $\|u\|_{H^{-m}} \approx \|(1+|\xi|^2)^{-m/2}\widehat{u}(\xi)\|_{L^2}$.

Corollary A.3.56. *If m, j are nonnegative integers, then*

$$\|u\|_{L^2(\mathbb{R}^N)} \le \|u\|_{H^j(\mathbb{R}^N)}^{\frac{m}{j+m}} \|u\|_{H^{-m}(\mathbb{R}^N)}^{\frac{j}{j+m}},$$

for all $u \in H^j(\mathbb{R}^N)$.

Proof. We have

$$\|u\|_{L^2(\mathbb{R}^N)}^2 = \|\widehat{u}\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(1+|\xi|^2)^{j/2}\widehat{u}(\xi)|^{\frac{2m}{j+m}} |(1+|\xi|^2)^{-m/2}\widehat{u}(\xi)|^{\frac{2j}{j+m}}\,d\xi.$$

Applying Hölder's inequality, we get

$$\|u\|_{L^2(\mathbb{R}^N)}^2 \le \|(1+|\cdot|^2)^{j/2}\widehat{u}\|_{L^2(\mathbb{R}^N)}^{\frac{2m}{j+m}} \|(1+|\cdot|^2)^{-m/2}\widehat{u}\|_{L^2(\mathbb{R}^N)}^{\frac{2m}{j+m}}.$$

Hence the result, by Propositions A.3.1 and A.3.55. □

Corollary A.3.57. *If m, j are nonnegative integers, then*

$$\|u\|_{L^2(\Omega)} \le \|u\|_{H_0^j(\Omega)}^{\frac{m}{j+m}} \|u\|_{H^{-m}(\Omega)}^{\frac{j}{j+m}},$$

for all $u \in H_0^j(\Omega)$.

Proof. By density, it suffices to establish the result for $u \in \mathcal{D}(\Omega)$. Then the result follows by applying Corollary A.3.56 to $\overline{u}$ defined by (A.3.1) □

We end this section with a useful density result.

Proposition A.3.58. *Let m, j be nonnegative integers and let $1 \le p, q < \infty$. The following properties hold:*

(i) *$\mathcal{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap W_0^{j,q}(\Omega)$;*

(ii) *if $q > 1$, then $\mathcal{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap W^{-j,q'}(\Omega)$;*

(iii) *if $p, q > 1$, then $\mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega) \cap W^{-j,q'}(\Omega)$;*

(iv) *$\mathcal{D}(\Omega)$ is dense in $W_0^{m,p}(\Omega) \cap C_0(\Omega)$;*

(v) *if $p > 1$, then $\mathcal{D}(\Omega)$ is dense in $W^{-m,p'}(\Omega) \cap C_0(\Omega)$;*

Proof. Let $X = W_0^{m,p}(\Omega) \cap W_0^{j,q}(\Omega)$. It follows from Proposition A.1.17 that $X^\star = W^{-m,p'}(\Omega) + W^{-j,q'}(\Omega)$. Suppose that $f \in X^\star$ is such that $\langle f, \varphi \rangle_{X^\star, X} = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ and write $f = f_1 + f_2$ with $f_1 \in W^{-m,p'}(\Omega)$ and $f_2 \in W^{-j,q'}(\Omega)$. We have (see Proposition A.1.17)

$$\begin{aligned} \langle f, \varphi \rangle_{X^\star, X} &= \langle f_1, \varphi \rangle_{W^{-m,p'}, W_0^{m,p}} + \langle f_2, \varphi \rangle_{W^{-j,q'}, W_0^{j,q}} = \langle f_1, \varphi \rangle_{\mathcal{D}', \mathcal{D}} + \langle f_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} \\ &= \langle f_1 + f_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, \varphi \rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

It follows that $f = 0$ in $\mathcal{D}'(\Omega)$, hence in $X^\star$ (see Remark A.1.18). Therefore, $\mathcal{D}(\Omega)$ is dense in X. This proves property (i), and properties (ii) and (iii) are proved by the same argument (note that if $p > 1$, then $W_0^{m,p}(\Omega)$ is reflexive; and so, $(W^{-m,p'}(\Omega))^\star = W_0^{m,p}(\Omega)$). Properties (iv) and (v) are also proved by the same argument, since the dual of $C_0(\Omega)$ is also a space of distributions (since $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$). □

Remark A.3.59. Since $\mathcal{D}(\Omega)$ is not dense in $L^\infty(\Omega)$, it is clear that $\mathcal{D}(\Omega)$ is neither dense in $W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$ nor in $W^{-m,p'}(\Omega) \cap L^\infty(\Omega)$. However, one shows easily, by a standard truncation and regularization argument, that if $u \in W_0^{m,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some nonnegative integer m and some $p \in [1, \infty)$ (respectively, if $u \in W^{-m,p'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$), then there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^N)$ such that $\|u_n\|_{L^\infty} \le \|u\|_{L^\infty}$ and such that $u_n \rightharpoonup u$ in $W_0^{m,p}(\mathbb{R}^N)$ (respectively, in $W^{-m,p'}(\mathbb{R}^N)$) as $n \to \infty$.

A.3.6. Time-dependent functions with values in Sobolev spaces. In this section, we consider an open interval I of $\mathbb{R}$ (bounded or not) and we collect a few results concerning functions from I with values in Sobolev spaces. We begin with some compactness results.

Lemma A.3.60. *If I is an interval of $\mathbb{R}$ and if m, j are nonnegative integers, $j \geq 1$, then $L^\infty(I, H_0^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega)) \hookrightarrow C^{0,\frac{j}{j+m}}(\overline{I}, L^2(\Omega))$. Furthermore,*

$$\|f(t) - f(s)\|_{L^2}^2 \leq 2|t-s|^{\frac{j}{j+m}} \|f\|_{L^\infty(I,H_0^j)}^{\frac{m}{j+m}} \|f'\|_{L^\infty(I,H^{-m})}^{\frac{j}{j+m}},$$

for all $f \in L^\infty(I, H_0^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$ and $s, t \in I$.

Proof. Let $f \in L^\infty(I, H_0^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$ and $s, t \in I$. By Corollaries A.3.57 and A.2.35, we have

$$\begin{aligned}\|f(t) - f(s)\|_{L^2} &\leq \|f(t) - f(s)\|_{H_0^j}^{\frac{m}{j+m}} \|f(t) - f(s)\|_{H^{-m}}^{\frac{j}{j+m}} \\ &\leq 2^{\frac{m}{j+m}} |t-s| \, \|f\|_{L^\infty(I,H_0^j)}^{\frac{m}{j+m}} \|f'\|_{L^\infty(I,H^{-m})}^{\frac{j}{j+m}}.\end{aligned}$$

Hence the result. □

Proposition A.3.61. *Let I be a bounded interval of $\mathbb{R}$and let m, j be nonnegative integers, $j \geq 1$. If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence of $L^\infty(I, H_0^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$, then the following properties hold:*

(i) *There exists $f \in L^\infty(I, H_0^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $t \in \overline{I}$, $f_{n_k}(t) \rightharpoonup f(t)$ in $H_0^j(\Omega)$, as $k \to \infty$. In particular,*

$$\int_I f_{n_k}(t)\varphi(t)\, dt \rightharpoonup \int_I f(t)\varphi(t)\, dt,$$

in $H_0^j(\Omega)$, for every $\varphi \in C_c(I)$;

(ii) *if Ω is bounded, then also $f_{n_k} \to f$ in $C(\overline{I}, L^2(\Omega))$;*

(iii) *if $\|f_{n_k}(t)\|_{L^2} \to \|f(t)\|_{L^2}$ as $k \to \infty$, uniformly on I, then also $f_{n_k} \to f$ in $C(\overline{I}, L^2(\Omega))$ as $k \to \infty$;*

(iv) *if $(f_n)_{n\in\mathbb{N}} \subset C(\overline{I}, H_0^j(\Omega))$ and $\|f_{n_k}(t)\|_{H^j} \to \|f(t)\|_{H^j}$ as $k \to \infty$, uniformly on I, then also $f \in C(\overline{I}, H_0^j(\Omega))$ and $f_{n_k} \to f$ in $C(\overline{I}, H_0^j(\Omega))$ as $k \to \infty$.*

Proof. (i) follows from Proposition A.2.46 (i) and (ii) applied with $r = \infty$, $X = H_0^j(\Omega)$ and $Y = H^{-m}(\Omega)$. (ii) follows from Theorem A.3.42 and Proposition A.2.46 (iv) (or Theorem A.2.45) applied with $X = H_0^j(\Omega)$, $Y = H^{-m}(\Omega)$ and $B = L^2(\Omega)$. (iii) follows from Lemma A.3.60 and Proposition A.2.46 (iii) applied with $X = H_0^j(\Omega)$, $Y = H^{-m}(\Omega)$ and $B = L^2(\Omega)$. (iv) follows from Proposition A.2.46 (iii) applied with $X = B = H_0^j(\Omega)$ and $Y = H^{-m}(\Omega)$. □

Proposition A.3.62. *Let I be a bounded interval of $\mathbb{R}$, let m, j be nonnegative integers, $j \geq 1$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(I, H^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$. If Ω is bounded and has the cone property, then there exist $f \in L^\infty(I, H^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ in $C(\overline{I}, L^2(\Omega))$.*

Proof. The result follows from Theorem A.3.41, and Theorem A.2.45 applied with $X = H^j(\Omega)$, $Y = H^{-m}(\Omega)$ and $B = L^2(\Omega)$. Property $f \in L^\infty(I, H^j(\Omega)) \cap W^{1,\infty}(I, H^{-m}(\Omega))$ follows from Theorem A.2.20 and Corollary A.2.39. □

Lemma A.3.63. *Let $F : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $F(0) = 0$ and set*

$$G(x) = \int_0^x F(s)\, ds.$$

If $u \in C^1(\overline{I}, L^2(\Omega))$, then the function $f(t) = \int_\Omega G(u(t))\, dx$ belongs to $C^1(\overline{I})$, and

$$f'(t) = \int_\Omega F(u(t)) u_t(t)\, dx, \tag{A.3.18}$$

for all $t \in I$.

Proof. Let L denote the Lipschitz constant of F. Since $|G(x)| \le Lx^2/2$, it follows that $G(u(t)) \in L^1(\Omega)$ for all $t \in I$; and so, f is well defined. Furthermore, since $|G(y) - G(x)| \le L|y + x|\,|y - x|/2$, it follows that $f \in C(\overline{I})$. Finally, observe that

$$|G(y) - G(x) - (y - x)F(x)| \le \frac{L}{2}|y - x|^2;$$

and so,

$$\left| \frac{1}{h} \left\{ \int_\Omega G(u(t+h))\, dx - \int_\Omega G(u(t))\, dx \right\} - \int_\Omega F(u(t)) \frac{u(t+h) - u(t)}{h}\, dx \right| \le \frac{L}{2} \int_\Omega u(t) \frac{u(t+h) - u(t)}{h}\, dx,$$

for all $t \in \overline{I}$ and $h \neq 0$ such that $t + h \in \overline{I}$. The result follows by letting $h \downarrow 0$. □

Corollary A.3.64. *Let m be a positive integer and let $1 < p < \infty$. Then the following properties hold:*

(i) $L^p(I, H_0^m(\Omega)) \cap W^{1,p'}(I, H^{-m}(\Omega)) \hookrightarrow C_{\mathrm{b}}(\overline{I}, L^2(\Omega))$;

(ii) *if $u \in L^p(I, H_0^m(\Omega)) \cap W^{1,p'}(I, H^{-m}(\Omega))$, then the function $f(t) = \|u(t)\|_{L^2}^2$ belongs to $W^{1,1}(I)$ and*

$$f'(t) = \langle u_t(t), u(t) \rangle_{H^{-m}, H_0^m},$$

for almost all $t \in I$.

Proof. Let $u \in C_c^1(\overline{I}, H_0^m(\Omega))$. Applying Lemma A.3.63 with $F(x) = x$ and identity (A.3.14), we get

$$\|u(t)\|_{L^2}^2 = \|u(s)\|_{L^2}^2 + 2 \int_s^t \langle u(\sigma), u_t(\sigma) \rangle_{H_0^m, H^{-m}}\, d\sigma, \tag{A.3.19}$$

for all $s, t \in \overline{I}$. Applying Hölder's inequality in time, we obtain easily

$$\|u(t)\|_{L^2}^p \le C(\|u(s)\|_{L^2}^2 + \|u\|_{L^p(I, H_0^m)}^p + \|u_t\|_{L^p(I, H^{-m})}^p).$$

Integration in s yields

$$\|u\|_{L^\infty(I, L^2)}^p \le C(\|u\|_{L^p(I, H_0^m)}^p + \|u_t\|_{L^p(I, H^{-m})}^p),$$

and property (i) follows by density (see Proposition A.2.48). Finally, consider a function $u \in L^p(I, H_0^m(\Omega)) \cap W^{1,p'}(I, H^{-m}(\Omega))$ and let $(u_n)_{n \in \mathbb{N}} \subset C_c^1(\overline{I}, H_0^m(\Omega))$ be such that $u_n \underset{n\to\infty}{\longrightarrow}$ both in $L^p(I, H_0^m(\Omega))$ and in

$W^{1,p'}(I, H^{-m}(\Omega))$. After possibly extracting a subsequence, we may assume that there exist $f \in L^p(I)$ and $g \in L^{p'}(I)$ such that $\|u_n(t)\|_{H^m} \le f(t)$ and $\|(u_n)_t(t)\|_{H^{-m}} \le g(t)$ for almost all $t \in I$, and that $u_n(t) \underset{n\to\infty}{\longrightarrow} u(t)$ and $(u_n)_t(t) \underset{n\to\infty}{\longrightarrow} u_t(t)$ for almost all $t \in I$. Applying identity (A.3.19) to u_n, then letting $n \to \infty$ and applying property (i) and the dominated convergence theorem, it follows that (A.3.19) holds for u as well. Hence property (ii). □

Corollary A.3.65. *Let F and G be as in Lemma A.3.63, let m be a positive integer and let $1 < p < \infty$. If $u \in L^p(I, H_0^m(\Omega)) \cap W^{1,p'}(I, H^{-m}(\Omega))$, then the function $f(t) = \int_\Omega G(u(t))\, dx$ belongs to $W^{1,1}(I)$, and*

$$f'(t) = \langle u_t(t), F(u(t))\rangle_{H^{-1},H_0^1},$$

for almost all $t \in I$.

Proof. The proof is identical to the proof of property (ii) of Corollary A.3.64 above, by applying the integrated version of formula (A.3.18) instead of formula (A.3.19). □

Proposition A.3.66. *Let $1 \le p \le \infty$ and let $f \in L^1(I, L^p(\Omega))$. If $f(t) \ge 0$ almost everywhere on Ω for almost all $t \in I$, then $\int_I f(t)\, dt \ge 0$ almost everywhere on Ω.*

Proof. Since the set $\{u \in L^p(\Omega);\ u \ge 0$ almost everywhere on $\Omega\}$ is a closed convex cone, the result follows from Proposition A.2.16 if I is bounded, then from an obvious truncation argument if I is unbounded. □

Corollary A.3.67 *Let $1 < p < \infty$ and let $u \in L^p(I, H^1(\Omega)) \cap W^{1,p'}(I, H^{-1}(\Omega))$ and $v \in L^p(I, H_0^1(\Omega))$ be such that $u(t) \le v(t)$ almost everywhere on Ω for almost all $t \in I$. There exist $(u_n)_{n\in\mathbb{N}} \subset C_c^1(\overline{I}, H^1(\Omega))$ and $(v_n)_{n\in\mathbb{N}} \subset C_c^1(\overline{I}, H_0^1(\Omega))$ such that $u_n(t) \le v_n(t)$ almost everywhere on Ω for all $t \in I$ and such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^p(I, H^1(\Omega)) \cap W^{1,p'}(I, H^{-1}(\Omega))$ and $v_n \underset{n\to\infty}{\longrightarrow} v$ in $L^p(I, H_0^1(\Omega))$.*

Proof. It is sufficient to repeat the steps of the proof of Proposition A.2.48. By applying the extension operator constructed in Corollary A.2.33 with $a \ge 0$, one is reduced to the case $I = \mathbb{R}$ (note that if $a \ge 0$, then the extension operator is order preserving). As well, truncation by a nonnegative function is order preserving. Finally, it follows from Proposition A.3.66 that convolution with a sequence of nonnegative mollifiers is order preserving. Hence the result. □

Corollary A.3.68. *Let $1 < p < \infty$ and let $u \in L^p(I, H^1(\Omega)) \cap W^{1,p'}(I, H^{-1}(\Omega))$. If there exists $v \in L^p(I, H_0^1(\Omega))$ such that $u(t) \le v(t)$ almost everywhere on Ω for almost all $t \in I$ and if $u \in C(I, L^2(\Omega))$, then the function $f(t) = \int_\Omega u^+(t)^2\, dx$ belongs to $W^{1,1}(I)$, and*

$$f'(t) = 2\langle u_t(t), u^+(t)\rangle_{H^{-1},H_0^1},$$

for almost all $t \in I$.

Proof. We proceed in two steps.

Step 1. Suppose first that $u \in C^1_c(\overline{I}, H^1(\Omega))$ and $v \in C^1_c(\overline{I}, H^1_0(\Omega))$. It follows from Lemma A.3.63 that

$$\int_\Omega u^+(t)^2\,dx = \int_\Omega u^+(s)^2\,dx + \int_s^t \int_\Omega u^+(\sigma) u_t(\sigma)\,d\sigma,$$

for all $s, t \in \overline{I}$. By Corollary A.3.35 (applied with $u = 0$, $M = -u$ and $w = -v$) and identity (A.3.14), we get

$$\int_\Omega u^+(t)^2\,dx = \int_\Omega u^+(s)^2\,dx + \int_s^t \langle u^+(\sigma), u_t(\sigma)\rangle_{H^1_0, H^{-1}},$$

for all $s, t \in \overline{I}$.

Step 2. Let now u and v satisfy the assumptions of the corollary, and apply Corollary A.3.67. It follows from step 1 that

$$\int_\Omega u_n^+(t)^2\,dx = \int_\Omega u_n^+(s)^2\,dx + \int_s^t \langle u_n^+(\sigma), (u_n)_t(\sigma)\rangle_{H^1_0, H^{-1}}, \tag{A.3.20}$$

for all $s, t \in \overline{I}$. After possibly extracting a subsequence, we may assume that there exist $f \in L^p(I)$ and $g \in L^{p'}(I)$ such that $\|u_n(t)\|_{H^1} \le f(t)$ and $\|(u_n)_t(t)\|_{H^{-1}} \le g(t)$ for almost all $t \in I$, and that $u_n(t) \underset{n\to\infty}{\longrightarrow} u(t)$ in $H^1(\Omega)$ and $(u_n)_t(t) \underset{n\to\infty}{\longrightarrow} u_t(t)$ in $H^{-1}(\Omega)$ for almost all $t \in I$. Note that $u_n^+(t) \underset{n\to\infty}{\longrightarrow} u^+(t)$ in $H^1(\Omega)$ for almost all $t \in I$, by Corollary A.3.15. Since $u^+(t) \in H^1_0(\Omega)$ for almost all $t \in I$ (Corollary A.3.35), it follows that $u_n^+(t) \underset{n\to\infty}{\longrightarrow} u^+(t)$ in $H^1_0(\Omega)$ for almost all $t \in I$. Therefore, letting $n \to \infty$ in identity (A.3.20) and applying the dominated convergence theorem, we obtain

$$\int_\Omega u^+(t)^2\,dx = \int_\Omega u^+(s)^2\,dx + \int_s^t \langle u^+(\sigma), u_t(\sigma)\rangle_{H^1_0, H^{-1}}, \tag{A.3.21}$$

for almost all $s, t \in I$. Since the term on the right of (A.3.21) is a continuous function of s, t and since $u \in C(I, L^2(\Omega))$, it follows that (A.3.21) holds for all $s, t \in I$. Hence the result. □

A.3.7. The case of complex-valued functions. Throughout Section A.3, we considered real valued functions but the same theory can be developped for complex valued functions, with obvious modifications which we describe below.

One has to consider the spaces $\mathcal{D}(\Omega, \mathbb{C})$ and $L^p(\Omega, \mathbb{C})$ instead of the spaces $\mathcal{D}(\Omega, \mathbb{R})$ and $L^p(\Omega, \mathbb{R})$. In particular, a function $f \in L^1_{\text{loc}}(\Omega, \mathbb{C})$ defines a distribution $T_f \in \mathcal{D}'(\Omega, \mathbb{C})$ by the formula

$$\langle T_f, \varphi\rangle = \int_\Omega \operatorname{Re}(f(x)\overline{\varphi(x)})dx, \text{ for all } \varphi \in \mathcal{D}(\Omega, \mathbb{C}).$$

In particular, $W^{m,p}(\Omega, \mathbb{C}) \approx W^{m,p}(\Omega, \mathbb{R}) \times W^{m,p}(\Omega, \mathbb{R})$. In other words, a complex-valued function u belongs to $W^{m,p}(\Omega, \mathbb{C})$ if, and only if $\operatorname{Re}(u) \in W^{m,p}(\Omega, \mathbb{R})$ and $\operatorname{Im(u)} \in W^{m,p}(\Omega, \mathbb{R})$. As well, $W^{m,p}_0(\Omega, \mathbb{C}) \approx W^{m,p}_0(\Omega, \mathbb{R}) \times W^{m,p}_0(\Omega, \mathbb{R})$, and it follows in particular that $W^{-m,p'}(\Omega, \mathbb{C}) \approx W^{-m,p'}(\Omega, \mathbb{R}) \times W^{-m,p'}(\Omega, \mathbb{R})$. The scalar product on $H^m(\Omega, \mathbb{C})$ is defined by

$$(u, v)_{H^m} = \sum_{0\le|\alpha|\le m} \int_\Omega \operatorname{Re}(D^\alpha u(x)\overline{D^\alpha v(x)})\,dx. \tag{A.3.22}$$

Formula (A.3.14) becomes

$$\langle u, v\rangle_{H^m_0, H^{-m}} = \operatorname{Re}\left(\int_\Omega u(x)\overline{v(x)}\,dx\right), \tag{A.3.23}$$

and formula (A.3.17) becomes

$$\langle \triangle u, v\rangle = -\mathrm{Re}\left(\int_\Omega \nabla u(x)\cdot \overline{\nabla v(x)}\,dx\right),\ \text{ for } v\in H_0^1(\Omega,\mathbb{C}). \tag{A.3.24}$$

Therefore, most of the results that we established for real-valued functions still hold for complex-valued functions, and are proved by considering separately the real and imaginary parts. The only exceptions are Corollaries A.3.16, A.3.17, A.3.30, A.3.31, A.3.35, A.3.68 and Proposition A.3.66 that do not make sense anymore, and Theorem A.3.12 and Corollaries A.3.14, A.3.15, A.3.29, A.3.30 and A.3.32 which must be modified as follows.

Theorem A.3.69. *If $F:\mathbb{C}\to\mathbb{C}$ is a Lipschitz continuous function such that $F(0)=0$ and if $1\le p\le\infty$, then the following properties hold.*

(i) *$F(u)\in W^{1,p}(\Omega,\mathbb{C})$, for every $u\in W^{1,p}(\Omega,\mathbb{C})$.*

(ii) *If $|F(z_1)-F(z_2)|\le L(z_1,z_2)|z_1-z_2|$ for all $z_1,z_2\in\mathbb{C}$, where $L:\mathbb{C}\times\mathbb{C}\to[0,\infty)$ is some continous function, then $|\nabla F(u)|\le L(u,u)|\nabla u|$ a.e. for every $u\in W^{1,p}(\Omega,\mathbb{C})$.*

(iii) *If F is C^1 (considered as a function $\mathbb{R}^2\to\mathbb{R}^2$) except at a finite number of points, then $\nabla F(u)=DF(u)\nabla u$ a.e. for every $u\in W^{1,p}(\Omega,\mathbb{C})$. If moreover $p<\infty$, then the mapping $u\mapsto F(u)$ is continous $W^{1,p}(\Omega,\mathbb{C})\to W^{1,p}(\Omega,\mathbb{C})$.*

(iv) *If $p<\infty$, then in properties (i) and (iii) above, one may replace $W^{1,p}(\Omega,\mathbb{C})$ by $W_0^{1,p}(\Omega,\mathbb{C})$.*

Proof. We proceed in five steps.

Step 1. Suppose F is C^1 (considered as a function $\mathbb{R}^2\to\mathbb{R}^2$), then $F(u)\in W^{1,p}(\Omega,\mathbb{C})$ and $\nabla F(u)=DF(u)\nabla u$ a.e. for every $u\in W^{1,p}(\Omega,\mathbb{C})$. This is established as in Brezis [17], Proposition IX.5. The idea of the proof is to approximate u by a sequence $(u_n)_{n\in\mathbb{N}}\subset W^{1,p}(\Omega,\mathbb{C})\cap C^\infty(\Omega,\mathbb{C})$.

Step 2. Proof of Property (i). Consider a sequence of mollifiers $(\rho_j)_{j\in\mathbb{N}}\subset\mathcal{D}(\mathbb{R}^2)$ and set $F_j=\rho_j\star F$. It follows that $F_j \underset{j\to\infty}{\longrightarrow} F$ uniformly on $\mathbb{C}$. Moreover, we have $|F_j(z_1)-F_j(z_2)|\le L|z_1-z_2|$, where L is the Lipschitz constant of F. Given $u\in W^{1,p}(\Omega,\mathbb{C})$, it follows from Step 1 that $F_j(u)\in W^{1,p}(\Omega,\mathbb{C})$ and that

$$\nabla F_j(u)=DF_j(u)\nabla u.$$

In particular, $|\nabla F_j(u)|\le L|\nabla u|$. This implies that (up to a subsequence) $\nabla F_j(u)$ converges in L^p weak (weak-$\star$ if $p=\infty$) to some function ψ (apply Dunford-Pettis' theorem if $p=1$). Since $F_j(u)\to F(u)$ in $L^p(\Omega,\mathbb{C})$, it follows that $\psi=\nabla F(u)$; and so, $F(u)\in W^{1,p}(\Omega)$.

Step 3. Proof of Property (ii). Let F_j be as in Step 2. We have $|DF_j(z)|\le L(z,z)$; and so, $|\nabla F_j(u)|\le L(u,u)|\nabla u|$. Since $\nabla F_j(u)$ converges in L^p weak (weak-$\star$ if $p=\infty$) to $\nabla F(u)$ (see Step 2), we deduce that $|\nabla F(u)|\le L(u,u)|\nabla u|$. To see this, we need only show that if a sequence $(f_n)_{n\ge0}\subset L^p(\Omega)$ verifies $f_n \underset{n\to\infty}{\longrightarrow} f$ and $|f_n|\le g$ a.e., then $|f|\le g$ a.e. Let $\varphi\in L^{p'}(\Omega)$, $\varphi\ge0$. We have

$$\int_\Omega g\varphi\ge\int_\Omega f_n\varphi \underset{n\to\infty}{\longrightarrow}\int_\Omega f\varphi;$$

and so,

$$\int_\Omega (g-f)\varphi \geq 0,$$

which implies that $g \geq f$ a.e.

Step 4. Proof of Property (iii). Let $E = (x_i)_{1\leq i\leq k}$ be such that $F \in C^1(\mathbb{C}\setminus E, \mathbb{C})$, and let again F_j be as in Step 2. Note that $DF \in L^\infty(\mathbb{C}, \mathbb{C}^2)$, so that $F'_j = \rho_j \star F'$ (see Brezis [17], Lemme IX.1). It follows that $F'_j \to F'$ on $\mathbb{C}\setminus E$. Since $\nabla u = 0$ a.e. on $\omega = \{x \in \Omega;\, u(x) \in E\}$, we see that $DF_j(u)\nabla u$ converges to $DF(u)\nabla u$ a.e. It follows that $\nabla F(u) = DF(u)\nabla u$. Suppose now $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{1,p}(\Omega, \mathbb{C})$. We have

$$\nabla F(u_n) - \nabla F(u) = (DF(u_n) - DF(u))\nabla u + DF(u)(\nabla u_n - \nabla u).$$

Since $DF(u_n(x)) \underset{n\to\infty}{\longrightarrow} DF(u(x))$ if $x \notin \omega$ and $\nabla u = 0$ a.e. in ω, we see that $\nabla F(u_n) \underset{n\to\infty}{\longrightarrow} \nabla F(u)$ a.e. If $p < \infty$, then it follows that $\nabla F(u_n) \underset{n\to\infty}{\longrightarrow} \nabla F(u)$ in $L^p(\Omega, \mathbb{C})$ by dominated convergence.

Step 5. Proof of Property (iv). Let $u \in W^{1,p}_0(\Omega, \mathbb{C})$ and let $(u_n)_{n\geq 0} \subset \mathcal{D}(\Omega, \mathbb{C})$ be such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{1,p}(\Omega, \mathbb{C})$. Up to a subsequence, we may assume that there exists $\psi \in L^p(\Omega)$ such that $|\nabla u_n| \leq \psi$ a.e. It follows from (ii) that $|\nabla F(u_n)| \leq L\psi$ a.e., where L is the Lipschitz constant of F. We deduce as in Step 1 that $F(u_n) \underset{n\to\infty}{\longrightarrow} F(u)$ in $W^{1,p}(\Omega, \mathbb{C})$ weak; and so $F(u) \in W^{1,p}_0(\Omega, \mathbb{C})$. □

Remark A.3.70. When F does not satisfy the assumption of (iii), we do not know if the mapping $u \mapsto F(u)$ is continuous $W^{1,p}(\Omega, \mathbb{C}) \to W^{1,p}(\Omega, \mathbb{C})$. Note that the formula "$\nabla F(u) = DF(u)\nabla u$" does not hold in general, even for smooth functions u. Indeed, take for example

$$F(u) = \begin{cases} u & \text{if} \quad |u| \leq 1, \\ \dfrac{u}{|u|} & \text{if} \quad |u| \geq 1. \end{cases}$$

F is C^∞, except on the set $\{|u| = 1\}$ where DF is not defined. Taking for example $u(x) = e^{ia\cdot x}$, with $a \in \mathbb{R}^N$, we see that $F(u) = u$, so that $\nabla F(u) = iae^{ia\cdot x}$, but $DF(u)\nabla u$ is not defined a.e. What happens is that (as opposed to the real valued case) if $E \subset \mathbb{C}$ is a set of measure 0, then ∇u need not vanish a.e. on the set $\{u \in E\}$. Take for example u as above and $E = \{|z| = 1\}$.

Remark A.3.71. Corollaries A.3.15 and A.3.30 must be modified as follows. If $u \in W^{1,p}(\Omega, \mathbb{C})$, it follows that $\mathrm{Re}(u), \mathrm{Im}(u), |u| \in W^{1,p}(\Omega, \mathbb{R})$. In addition, one has almost everywhere $\nabla \mathrm{Re}(u) = \mathrm{Re}(\nabla u)$, $\nabla \mathrm{Im}(u) = \mathrm{Im}(\nabla u)$ and

$$|\nabla |u|\,|^2 = \begin{cases} 0, \text{ if } u = 0, \\ |\nabla u|^2 - \left|\mathrm{Im}\left(\dfrac{\overline{u}\nabla u}{|u|}\right)\right|^2, \text{ if } u \neq 0. \end{cases}$$

(In particular, one has $|\nabla|u|\,| \leq |\nabla u|$, but in general $|\nabla|u|\,| \not\equiv |\nabla u|$. Note that this is in contrast with the real valued case.) If $p < \infty$, then the mappings $u \mapsto \mathrm{Re}(u)$, $u \mapsto \mathrm{Im}(u)$ and $u \mapsto |u|$ are continous $W^{1,p}(\Omega, \mathbb{C}) \to W^{1,p}(\Omega, \mathbb{R})$. Moreover, if $u \in W^{1,p}_0(\Omega, \mathbb{C})$, then $\mathrm{Re}(u), \mathrm{Im}(u), |u| \in W^{1,p}_0(\Omega, \mathbb{R})$. This follows from properties (iii) and (iv) of Theorem A.3.69.

Corollary A.3.72. *Let $1 \leq p, q, r \leq \infty$ and $\alpha > 0$ be such that $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1}{q}$ and let $F : \mathbb{C} \to \mathbb{C}$ be a locally Lipschitz function such that $F(0) = 0$ and*

$$|F(z_1) - F(z_2)| \leq L(|z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{C}$. *Then, for every* $u \in W^{1,q}(\Omega, \mathbb{C}) \cap L^p(\Omega, \mathbb{C})$, *we have* $F(u) \in W^{1,r}(\Omega, \mathbb{C})$ *and* $\|F(u)\|_{W^{1,r}} \le 2L\|u\|_{L^p}^\alpha \|u\|_{W^{1,q}}$. *Furthermore,* $|\nabla F(u)| \le 2L|u|^\alpha |\nabla u|$ *almost everywhere on* Ω. *In addition, if* $q, r < \infty$ *and if* $u \in W_0^{1,q}(\Omega, \mathbb{C})$, *then* $F(u) \in W_0^{1,r}(\Omega, \mathbb{C})$.

Proof. Consider the function F_n defined by

$$F_n(z) = \begin{cases} F(z), \text{ if } |z| \le n, \\ F\left(n\dfrac{z}{|z|}\right), \text{ if } -n \le |z| \ge n. \end{cases}$$

It follows that F_n is globally Lipschitz. The result now follows rather easily by applying Theorem A.3.69 (ii) then passing to the limit as $n \to \infty$ (apply Dunford-Pettis' theorem to pass to the limit if $r = 1$). □

A.4. Elliptic equations. Throughout this section, we consider an open subset $\Omega \subset \mathbb{R}^N$. We consider *real-valued* functions, and we refer to Section A.4.6 for the case of *complex-valued* functions. We describe some existence and regularity results of solutions of some second order elliptic equations with Dirichlet boundary conditions. For that purpose, it is convenient to define $\lambda_1 = \lambda_1(\Omega) \in \mathbb{R}$ by

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^2, u \in H_0^1(\Omega), \int_\Omega |u|^2 = 1\right\}. \tag{A.4.1}$$

Remark A.4.1. It follows from (A.4.1) that $\lambda_1 \ge 0$. The property $\lambda_1 > 0$ is equivalent to Poincaré's inequality (A.3.7) for $p = 2$ and depends therefore on geometric properties of Ω. In particular, if Ω is bounded in one direction or if $|\Omega|$ is finite, then $\lambda_1 > 0$ (see Theorem A.3.37). On the other hand, if $\Omega = \mathbb{R}^N$, then $\lambda_1 = 0$ (see Remark A.3.38, (ii)).

We begin with a simple coercivity inequality.

Lemma A.4.2. *Let* λ_1 *be defined by (A.4.1). If* $\lambda > -\lambda_1$, *then*

$$\min\left\{1, \frac{\lambda+\lambda_1}{1+\lambda_1}\right\}\|u\|_{H^1}^2 \le \int_\Omega \{|\nabla u|^2 + \lambda|u|^2\}\, dx \le \max\left\{1, \frac{\lambda+\lambda_1}{1+\lambda_1}\right\}\|u\|_{H^1}^2,$$

for all $u \in H_0^1(\Omega)$. *In particular,*

$$|||u|||^2 = \int_\Omega \{|\nabla u|^2 + \lambda|u|^2\}\, dx \tag{A.4.2}$$

defines an equivalent norm on $H_0^1(\Omega)$.

Proof. Let $|||\cdot|||$ be defined by (A.4.2) (this makes sense by (A.4.1)). Assume first that $\lambda \ge 1$. In particular, $\dfrac{\lambda+\lambda_1}{1+\lambda_1} \ge 1$. Given $\varepsilon > 0$, it follows from (A.4.1) that

$$\begin{aligned}\|u\|_{H^1}^2 &= (1-\varepsilon)\int_\Omega |\nabla u|^2\, dx + \varepsilon\int_\Omega |\nabla u|^2\, dx + \int_\Omega |u|^2\, dx \\ &\ge (1-\varepsilon)\int_\Omega |\nabla u|^2\, dx + (\varepsilon\lambda_1 + 1)\int_\Omega |u|^2\, dx.\end{aligned}$$

Choosing $\varepsilon = \dfrac{\lambda - 1}{\lambda+\lambda_1}$, we see

$$\|u\|_{H^1}^2 \ge \frac{1+\lambda_1}{\lambda+\lambda_1}|||u|||^2.$$

Since obviously $|||u|||^2 \geq \|u\|^2_{H^1}$, the result follows. Assume now that $\lambda \leq 1$ (i.e. $\dfrac{\lambda+\lambda_1}{1+\lambda_1} \leq 1$). Given $\varepsilon > 0$, it follows from (A.4.1) that

$$\begin{aligned}|||u|||^2 &= (1-\varepsilon)\int_\Omega |\nabla u|^2\,dx + \varepsilon\int_\Omega |\nabla u|^2\,dx + \lambda\int_\Omega |u|^2\,dx\\ &\geq (1-\varepsilon)\int_\Omega |\nabla u|^2\,dx + (\varepsilon\lambda_1+\lambda)\int_\Omega |u|^2\,dx.\end{aligned}$$

Choosing $\varepsilon = \dfrac{1-\lambda}{\lambda+\lambda_1}$, we get

$$|||u|||^2 \geq \frac{\lambda+\lambda_1}{1+\lambda_1}\|u\|^2_{H^1}.$$

Since obviously $|||u|||^2 \leq \|u\|^2_{H^1}$, the proof is complete. □

We equip $H^{-1}(\Omega)$ with the dual norm, that is

$$\|u\|_{H^{-1}} = \sup\{\langle u,v\rangle_{H^{-1},H^1_0}, v\in H^1_0(\Omega), \|v\|_{H^1_0} = 1\},$$

and we denote by $()_{H^{-1}}$ the scalar product of $H^{-1}(\Omega)$.

A.4.1. Existence. We begin with a simple consequence of Lax-Milgram's theorem.

Lemma A.4.3. *For every $f \in H^{-1}(\Omega)$, there exists a unique solution $u \in H^1_0(\Omega)$ of equation*

$$-\triangle u + u = f, \text{ in } H^{-1}(\Omega).$$

Furthermore,

$$\|f\|_{H^{-1}} = \|u\|_{H^1}. \tag{A.4.3}$$

In particular,

$$\|u\|_{H^1} \leq \|f\|_{L^2}, \tag{A.4.4}$$

whenever $f \in L^2(\Omega)$.

Proof. By Theorem A.1.3, for every $f \in H^{-1}(\Omega)$ there exists a unique $u \in H^1_0(\Omega)$ such that

$$(u,v)_{H^1} = \langle f,v\rangle_{H^{-1},H^1_0}, \text{ for every } v\in H^1_0(\Omega). \tag{A.4.5}$$

(A.4.5) is equivalent, by density, to equation

$$\int_\Omega \nabla u\cdot\nabla v + uv = \langle f,v\rangle_{H^{-1},H^1_0}, \text{ for every } v\in\mathcal{D}(\Omega),$$

which is equivalent to

$$-\triangle u + u = f, \text{ in } H^{-1}(\Omega).$$

Furthermore, taking $v = u$ in (A.4.5) yields $\|u\|^2_{H^1} \leq \|f\|_{H^{-1}}\|u\|_{H^1}$; and so $\|u\|_{H^1} \leq \|f\|_{H^{-1}}$. In addition, it follows again from (A.4.5) that

$$|\langle f,v\rangle_{H^{-1},H^1_0}| \leq \|u\|_{H^1}\|v\|_{H^1},$$

for all $v \in H^1(\Omega)$. Therefore, $\|f\|_{H^{-1}} \le \|u\|_{H^1}$, from which (A.4.3) follows. (A.4.4) is a consequence of (A.4.3) and (A.3.16). □

Remark A.4.4. Here are some simple applications of Lemma A.4.3.

(i) It follows from Lemma A.4.3 that the differential operator $-\triangle + I$ defines an isometry from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.

(ii) It follows from (A.4.3) that for every $u, v \in H_0^1(\Omega)$, $(u,v)_{H^1} = (-\triangle u + u, -\triangle v + v)_{H^{-1}}$.

(iii) It follows from property (ii) above and (A.3.17) that

$$(-\triangle u + u, u)_{H^{-1}} = (u,v)_{H^1} = \int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega u(-\triangle v + v) = \int_\Omega u^2,$$

if $u \in H_0^1(\Omega)$ and if $v \in H_0^1(\Omega)$ solves equation $-\triangle v + v = u$.

Theorem A.4.5. *Let λ_1 be defined by (A.4.1). For every $\lambda > -\lambda_1$, the following holds:*

(i) *for every $f \in H^{-1}(\Omega)$, there exists a unique element $u \in H_0^1(\Omega)$ such that*

$$-\triangle u + \lambda u = f, \text{ in } H^{-1}(\Omega); \tag{A.4.6}$$

(ii) $|||f||| = \|u\|_{H^1(\Omega)}$ *defines on $H^{-1}(\Omega)$ an equivalent norm to the H^1 norm;*

(iii) $\lambda\|u\|_{H^{-1}} \le \|f\|_{H^{-1}}$;

(iv) *if $f \in L^2(\Omega)$, then $\triangle u \in L^2(\Omega)$, the equation makes sense in $L^2(\Omega)$ and $\lambda\|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}$.*

Proof. (i) For $u, v \in H_0^1(\Omega)$, let

$$b(u,v) = \int_\Omega \{\nabla u \cdot \nabla v + \lambda uv\}\, dx.$$

It follows easily from Lemma A.4.2 that b verifies the assumptions of Theorem A.1.3; and so, given $f \in H^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$b(u,v) = \langle f, v\rangle_{H^{-1},H_0^1}, \text{ for every } v \in H_0^1(\Omega). \tag{A.4.7}$$

We claim that (A.4.7) is equivalent to (A.4.6). Indeed, by density, (A.4.7) is equivalent to

$$b(u,v) = \langle f, v\rangle_{H^{-1},H_0^1}, \text{ for every } v \in \mathcal{D}(\Omega).$$

It folows from (A.3.14) and (A.3.15) that the above equation is equivalent to

$$-\triangle u + \lambda u = f, \text{ in } \mathcal{D}'(\Omega),$$

which is equivalent to (A.4.6), since all terms in the equation belong to $H^{-1}(\Omega)$. Hence (i).

(ii) It follows from Remark A.3.52, (vi) that for some constant C, $\|f\|_{H^{-1}} \le C\|u\|_{H^1} = |||f|||$. Taking $v = u$ in (A.4.7) yields $|||f||| = \|u\|_{H^1} \le C'\|f\|_{H^{-1}}$. Hence (ii)

(iii) Take the scalar product in $H^{-1}(\Omega)$ of (A.4.6) with u. It follows that

$$(-\triangle u+u,u)_{H^{-1}}+(\lambda-1)\|u\|_{H^{-1}}^2=(f,u)_{H^{-1}}.$$

Taking in account Remark A.4.4 (i) and (iii), we obtain

$$\lambda\|u\|_{H^{-1}}^2\le\|u\|_{L^2}^2+(\lambda-1)\|u\|_{H^{-1}}^2\le\|f\|_{H^{-1}}\|u\|_{H^{-1}}.$$

Hence (iii).

(iv) Assume $f\in L^2(\Omega)$. Then $\lambda u+f\in L^2(\Omega)$; and so $\triangle u\in L^2(\Omega)$. Furthermore, taking $v=u$ in (A.4.7) and applying (A.3.14), it follows that

$$\lambda\|u\|_{L^2}^2\le\int_\Omega fu\le\|f\|_{L^2}\|u\|_{L^2}.$$

Hence the result. □

These results can be generalized in the following way. Consider a function $a\in L^1_{\mathrm{loc}}(\Omega)$ and let

$$\sigma=N/2 \text{ if } N\ge 3 (\sigma=1 \text{ if } N=1; \sigma \text{ any number } >1 \text{ if } N=2). \tag{A.4.8}$$

Assume that there exist $a_1\in L^\sigma(\Omega)$ and $a_2\in L^\infty(\Omega)$ such that $a=a_1+a_2$ almost everywhere. In other words, assume that

$$a\in L^\sigma(\Omega)+L^\infty(\Omega). \tag{A.4.9}$$

Note that in this splitting, we may always assume that $\|u\|_{L^\sigma}$ is small. Indeed, given a nonnegative integer m, we always may write $a_1=\alpha_m+\beta_m$, where

$$\alpha_m=\begin{cases}a_1, \text{ if } |a_1|\ge m,\\ 0, \text{ if } |a_1|<m.\end{cases}$$

Clearly, $\alpha_m\in L^\sigma(\Omega)$, $\beta_m\in L^\infty(\Omega)$ and $\|\alpha_m\|_{L^\sigma}\to 0$, as $m\to\infty$.

It follows from Sobolev's embedding theorem that $H^1_0(\Omega)\hookrightarrow L^r(\Omega)$, with

$$r=\frac{2\sigma}{\sigma-1} \tag{A.4.10}$$

Therefore, there exists a constant K such that

$$\left|\int_\Omega a_1uv\,dx\right|\le K\|a_1\|_{L^\sigma}\|u\|_{H^1}\|v\|_{H^1}, \text{ for every } u,v\in H^1_0(\Omega).$$

By the preceding observation, we may assume that

$$\left|\int_\Omega a_1uv\,dx\right|\le\frac12\|u\|_{H^1}\|v\|_{H^1}, \text{ for every } u,v\in H^1_0(\Omega).$$

It follows that

$$\left|\int_\Omega auv\,dx\right|\le\frac12\|u\|_{H^1}\|v\|_{H^1}+M\|u\|_{L^2}\|v\|_{L^2}, \text{ for every } u,v\in H^1_0(\Omega), \tag{A.4.11}$$

where $M = \|a_2\|_{L^\infty}$. Note also that by Hölder's inequality, $\|a_1 u\|_{L^{r'}} \le \|a_1\|_{L^\sigma}\|u\|_{L^r}$ and $\|a_2 u\|_{L^2} \le \|a_2\|_{L^\infty}\|u\|_{L^2}$, for every $u \in H^1_0(\Omega)$. Since $H^1_0(\Omega) \hookrightarrow L^r(\Omega)$, it follows from Remark A.3.52 (ii) that $L^{r'}(\Omega) \hookrightarrow H^{-1}(\Omega)$. Therefore,

$$au \in H^{-1}(\Omega), \text{ for every } u \in H^1_0(\Omega). \tag{A.4.12}$$

Let

$$\lambda_1(-\triangle + a) = \inf\left\{\int_\Omega |\nabla u|^2 + \int_\Omega a|u|^2;\, u \in H^1_0(\Omega), \int_\Omega |u|^2 = 1\right\}. \tag{A.4.13}$$

It follows from (A.4.11) that $\lambda_1(-\triangle + a)$ is finite, but now λ_1 may be positive, negative or zero. We have the following result.

Lemma A.4.6. *Let a verify (A.4.8) and (A.4.9), and let $\lambda_1(-\triangle + a)$ be defined by (A.4.13). If $\lambda > -\lambda_1(-\triangle + a)$, then*

$$p(u) = \left(\int_\Omega \{|\nabla u|^2 + a|u|^2 + \lambda|u|^2\}\,dx\right)^{1/2} \tag{A.4.14}$$

defines on $H^1_0(\Omega)$ an equivalent norm to the H^1 norm.

Proof. It follows from (A.4.13) that (A.4.14) makes sense. Furthermore, given $u \in H^1_0(\Omega)$, it follows from (A.4.11) that

$$p(u)^2 \le \frac{3}{2}\|\nabla u\|^2_{L^2} + (M + 1 + \lambda)\|u\|^2_{L^2} \le \max\left\{\frac{3}{2}, M + 1 + \lambda\right\}\|u\|^2_{H^1}.$$

On the other hand, given $\varepsilon \in (0,1)$, it follows from (A.4.13) and (A.4.11) that for every $u \in H^1_0(\Omega)$,

$$\begin{aligned} p(u)^2 &\ge \varepsilon\int_\Omega |\nabla u|^2 + a|u|^2\,dx + ((1-\varepsilon)\lambda_1(-\triangle + a) + \lambda)\int_\Omega |u|^2\,dx \\ &\ge \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2\,dx + (\lambda + \lambda_1(-\triangle + a) - \varepsilon(M + 1 + \lambda_1(-\triangle + a)))\int_\Omega |u|^2\,dx. \end{aligned}$$

For ε small enough, we have $\lambda + \lambda_1(-\triangle + a) - \varepsilon(M + 1 + \lambda_1(-\triangle + a)) > 0$. Therefore, there exists $\eta > 0$ such that

$$p(u)^2 \ge \eta\|u\|^2_{H^1},$$

for all $u \in H^1_0(\Omega)$. This completes the proof. □

Theorem A.4.7. *Let a verify (A.4.8) and (A.4.9), and let $\lambda_1(-\triangle + a)$ be defined by (A.4.13). If $\lambda > -\lambda_1(-\triangle + a)$, then for every $f \in H^{-1}(\Omega)$, there exists a unique element $u \in H^1_0(\Omega)$ such that*

$$-\triangle u + au + \lambda u = f, \tag{A.4.15}$$

in $H^{-1}(\Omega)$. In addition, $\|u\|_{H^1} \le C\|f\|_{H^{-1}}$, for some constant C independent of f.

Proof. Note first that by (A.4.12), equation (A.4.15) makes sense. For $u, v \in H^1_0(\Omega)$, let

$$b(u,v) = \int_\Omega \{\nabla u \cdot \nabla v + auv + \lambda uv\}\,dx.$$

It follows from Lemma A.4.6 that b verifies the assumptions of Theorem A.1.3. Therefore, given $f \in H^{-1}(\Omega)$, there exists a unique $u \in H^1_0(\Omega)$ such that

$$b(u,v) = \langle f, v\rangle_{H^{-1},H^1_0}, \text{ for every } v \in H^1_0(\Omega). \tag{A.4.16}$$

It is easily verified (see the proof of Theorem A.4.5) that (A.4.16) is equivalent to (A.4.15). The result follows easily. □

A.4.2. H^m regularity. The H^m regularity is described by the following result (see Brezis [17], Theorem IX.25, Gilbarg and Trudinger [54], Theorems 8.12 and 8.13).

Theorem A.4.8. *Assume Ω has a bounded boundary of class C^2, let $f \in H^{-1}(\Omega)$ and let $\lambda \in \mathbb{R}$. If $u \in H_0^1(\Omega)$ solves (A.4.6), then the following properties hold:*

(i) *if $f \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and there exists a constant C depending only on Ω such that $\|u\|_{H^2} \leq C\|f\|_{L^2}$;*

(ii) *if furthermore $f \in H^m(\Omega)$ for some $m > 0$ and if the boundary of Ω is of class C^{m+2}, then $u \in H^{m+2}(\Omega)$ and there exists a constant C depending only on Ω and m such that $\|u\|_{H^{m+2}} \leq C\|f\|_{H^m}$;*

(iii) *in particular, if $f \in C^\infty(\overline{\Omega})$, and if Ω is bounded with boundary of class C^∞, then $u \in C^\infty(\overline{\Omega})$.*

Remark A.4.9. Smoothness is required in Theorem A.4.8 in order to apply the *method of translations* to obtain estimates of u near the boundary. However, without any regularity assumption on Ω, one can still obtain interior regularity. This is the object of the next result, and follows rather easily from the characterization of $H^m(\mathbb{R}^N)$ in terms of the Fourier transform (see also Gilbarg and Trudinger [54], Theorems 8.8 and 8.10, and Corollary 8.9).

Proposition A.4.10. *Let $f \in \mathcal{D}'(\Omega)$ and let $\lambda \in \mathbb{R}$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ solves equation (A.4.6) in $\mathcal{D}'(\Omega)$, then the following properties hold:*

(i) *if $f \in H^m_{\mathrm{loc}}(\Omega)$ for some $m \geq 0$, then $u \in H^{m+2}_{\mathrm{loc}}(\Omega)$. In addition, for every $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, there exists a constant C (depending only on m,Ω' and Ω'') such that $\|u\|_{H^{m+2}(\Omega'')} \leq C(\|f\|_{H^m(\Omega')} + \|u\|_{L^2(\Omega')})$;*

(ii) *if $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.*

Proof. We proceed in several steps.

Step 1. Let $m \in \mathbb{Z}$, $v \in \mathcal{S}'(\mathbb{R}^N)$ and $h \in H^m(\mathbb{R}^N)$ be such that $-\triangle v + v = h$, in $\mathcal{S}'(\mathbb{R}^N)$. Then, $v \in H^{m+2}(\mathbb{R}^N)$, and there exists a constant C such that $\|v\|_{H^{m+2}} \leq C\|h\|_{H^m}$. Indeed, we have $(1 + 4\pi^2|\xi|^2)\widehat{v} = \widehat{h}$ in $\mathcal{S}'(\mathbb{R}^N)$. It follows that $(1 + 4\pi^2|\xi|^2)^{\frac{m+2}{2}}\widehat{v} = (1 + 4\pi^2|\xi|^2)^{m/2}\widehat{v}$, and the result follows from Propositions A.3.1 and A.3.55.

Step 2. Consider $\omega'' \subset\subset \omega' \subset\subset \Omega$. Let $k \in \mathbb{Z}$, $u \in H^k(\omega')$ and $f \in H^{k-1}(\omega')$ solve equation (A.4.6) in $\mathcal{D}'(\omega')$. Then, $u \in H^{k+1}(\omega'')$, and there exists C such that $\|u\|_{H^{k+1}(\omega'')} \leq C(\|f\|_{H^{k-1}(\omega')} + \|u\|_{H^k(\omega')})$. To show this, consider $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\rho \equiv 1$ on ω'' and $\mathrm{Supp}(\rho) \subset \omega'$ and define $v \in \mathcal{D}'(\mathbb{R}^N)$ by

$$\langle v, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)} = \langle u, \rho\varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

It follows easily that $v \in H^k(\mathbb{R}^N)$ and that $\|v\|_{H^k(\mathbb{R}^N)} \leq C\|u\|_{H^k(\omega')}$. An easy calculation shows that v solves equation

$$-\triangle v + v = T_1 + T_2 + T_3, \tag{A.4.17}$$

in $\mathcal{D}'(\mathbb{R}^N)$, where the distributions T_1, T_2 and T_3 are defined by

$$\langle T_1, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \langle f + (1-\lambda)u, \rho\varphi\rangle_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$
$$\langle T_2, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = -\langle u, \varphi\triangle\rho\rangle_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$
$$\langle T_3, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = -\langle \nabla u, \varphi\nabla\rho\rangle_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$. It follows easily that $T_j \in H^{k-1}(\mathbb{R}^N)$ and that $\|T_j\|_{H^{k-1}(\mathbb{R}^N)} \le C(\|f\|_{H^{k-1}(\omega')} + \|u\|_{H^k(\omega')})$, for $j = 1, 2, 3$. Applying (A.4.17) and Step 1, we get $v \in H^{k+1}(\mathbb{R}^N)$ and $\|v\|_{H^{k+1}(\mathbb{R}^N)} \le C(\|f\|_{H^{k-1}(\omega')} + \|u\|_{H^k(\omega')})$. The result follows, since the restrictions of u and v to ω'' coincide.

Step 3. Conclusion. Assume that $f \in H^m_{\mathrm{loc}}(\Omega)$, for some $m \ge 0$. Consider $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Note that $u \in L^1(\Omega')$, so that in particular $u \in H^{-\ell}(\Omega')$ where $\ell \in \mathbb{N}$ is such that $2\ell > N$. Consider now a family $(\omega_j)_{0\le j\le m+1}$ of open subsets of Ω, such that $\omega_{m+\ell+1} = \Omega''$ and

$$\omega_{m+\ell+1} \subset\subset \cdots \subset\subset \omega_0 \subset\subset \Omega'$$

(one constructs easily such a family). It follows from Step 2 that $u \in H^{-\ell+1}(\omega_0)$ and that

$$\begin{aligned} \|u\|_{H^{-\ell+1}(\omega_0)} &\le C(\|f\|_{H^{-\ell-1}(\Omega')} + \|u\|_{H^{-\ell}(\Omega')}) \\ &\le C(\|f\|_{H^m(\Omega')} + \|u\|_{L^1(\Omega')}). \end{aligned} \tag{A.4.18}$$

Applying (A.4.18) and Step 1, we get $u \in H^{-\ell+2}(\omega_1)$, and

$$\begin{aligned} \|u\|_{H^{-\ell+2}(\omega_1)} &\le C(\|f\|_{H^{-\ell}(\omega_0)} + \|u\|_{H^{-\ell+1}(\omega_0)}) \\ &\le C(\|f\|_{H^m(\Omega')} + \|u\|_{L^1(\Omega')}). \end{aligned}$$

Iterating the above argument, one shows as well that $u \in H^{m+2}(\omega_{m+\ell+1}) = H^{m+2}(\Omega'')$, and that there exists C such that $\|u\|_{H^{m+2}(\Omega'')} \le C(\|f\|_{H^m(\Omega')} + \|u\|_{L^1(\Omega')})$. Hence property (i), since Ω' and Ω'' are arbitrary. Property (ii) follows from the inclusion $C^\infty(\Omega) \subset H^m_{\mathrm{loc}}(\Omega)$, for every $m \ge 0$. This completes the proof. □

A.4.3. L^p regularity and estimates.

Theorem A.4.11. *Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (A.4.6). If $f \in L^p(\Omega)$ for some $p \in [1,\infty]$, then $u \in L^p(\Omega)$ and $\lambda\|u\|_{L^p} \le \|f\|_{L^p}$.*

Proof. Consider $\varphi \in C^1(\mathbb{R},\mathbb{R})$ and assume that φ is nondecreasing and has bounded derivative, and that $\varphi(0) = 0$. It follows from (A.4.7), (A.3.14), Corollary A.3.29 and Theorem A.3.12 that

$$\int_\Omega \varphi'(u)|\nabla u|^2\,dx + \lambda\int_\Omega u\varphi(u)\,dx = \int_\Omega f\varphi(u)\,dx;$$

and so

$$\lambda\int_\Omega u\varphi(u)\,dx \le \int_\Omega f\varphi(u)\,dx.$$

Assume that $|\varphi(u)| \le |u|^{p-1}$. Then $|\varphi(u)|^{\frac{p}{p-1}} \le u\varphi(u)$; and so

$$\lambda\int_\Omega u\varphi(u)\,dx \le \|f\|_{L^p}\Big(\int_\Omega u\varphi(u)\,dx\Big)^{\frac{p-1}{p}}.$$

Since $u\varphi(u) \le C|u|^2 \in L^1(\Omega)$, it follows that

$$\lambda \Big(\int_\Omega u\varphi(u)\,dx \Big)^{\frac{1}{p}} \le \|f\|_{L^p}.$$

Assume first that $p \le 2$. Given $\varepsilon > 0$, take $\varphi(u) = u(\varepsilon + u^2)^{\frac{p-2}{2}}$. It follows from the preceding calculations that

$$\lambda \Big(\int_\Omega u^2(\varepsilon + u^2)^{\frac{p-2}{2}}\,dx \Big)^{\frac{1}{p}} \le \|f\|_{L^p}.$$

Letting $\varepsilon \downarrow 0$ and applying Fatou's Lemma yields the desired result.

Assume now $2 < p \le \infty$. We use a duality argument. Given $h \in C_c^\infty(\Omega)$, let $v \in H_0^1(\Omega)$ be the solution of (A.4.6) with f replaced by h. We have

$$\int_\Omega uh = (u, -\triangle v + \lambda v)_{H_0^1, H^{-1}} = (-\triangle u + \lambda u, v)_{H^{-1}, H_0^1} = (f, v)_{H^{-1}, H_0^1} = \int_\Omega fv.$$

Therefore,

$$\Big| \int_\Omega uh \Big| \le \|f\|_{L^p} \|v\|_{L^{p'}} \le \frac{1}{\lambda} \|f\|_{L^p} \|h\|_{L^{p'}},$$

since $p' < 2$. Since $h \in C_c^\infty(\Omega)$ is arbitrary, we deduce that $\|u\|_{L^p} \le \lambda^{-1} \|f\|_{L^p}$. □

Theorem A.4.12. *Let a verify (A.4.8) and (A.4.9), and assume that $a \ge 0$ almost everywhere. Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.15). If $f \in L^p(\Omega)$ for some $1 \le p \le \infty$, then $u \in L^p(\Omega)$ and $\lambda \|u\|_{L^p} \le \|f\|_{L^p}$.*

Proof. Taking in account that $a \ge 0$, the proof is the same as that of Theorem A.4.11. Note that in this case, we have $\lambda_1(-\triangle + a) \ge 0$, thus in particular, $\lambda > -\lambda_1(-\triangle + a)$. □

When $\lambda < 0$, one can still obtain L^∞ regularity results. This is also the case for the solutions of equation (A.4.15). More precisely, we have the following.

Theorem A.4.13. *Let a verify (A.4.8) and (A.4.9), let $\lambda_1(-\triangle + a)$ be defined by (A.4.13) and let $\lambda > -\lambda_1(-\triangle + a)$. Assume further that $a^- \in L^q(\Omega) + L^\infty(\Omega)$ for some $q > 1$, $q > N/2$. Let $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.15). If $f \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > 1$, $p > N/2$, then $u \in L^\infty(\Omega)$. Moreover, given $1 \le r < \infty$, there exists a constant C independent of f such that*

$$\|u\|_{L^\infty} \le C(\|f\|_{L^p + L^\infty} + \|u\|_{L^r}).$$

In particular, $\|u\|_{L^\infty} \le C(\|f\|_{L^p + L^\infty} + \|f\|_{H^{-1}})$.

Proof. The proof is adapted from Hartman and Stampacchia [60] (see also Brezis and Lions [23]). By homogeneity, we may assume that $\|u\|_{L^r} + \|f\|_{L^p + L^\infty} \le 1$. In particular, $f = f_1 + f_2$ with $\|f_1\|_{L^p} \le 1$ and $\|f_2\|_{L^\infty} \le 1$. We also write $a^- = g_1 + g_2$ with $g_1 \in L^q(\Omega)$ and $g_2 \in L^\infty(\Omega)$. Now, since $-u$ solves the same equation as u, with f replaced by $-f$ (which satisfies the same assumptions), it is sufficient to estimate $\|u^+\|_{L^\infty}$. Set $T = \|u^+\|_{L^\infty} \in [0, \infty]$, and assume that $T > 0$. For $t \in (0, T)$, set $v(t) = (u - t)^+$. We have

$v(t) \in H_0^1(\Omega)$, by Corollary A.3.35. Let now $\alpha(t) = \text{meas}\{x \in \Omega, u(x) > t\}$, for $t > 0$. Note that $\alpha(t)$ is always finite. In particular, since $v(t) \in L^2(\Omega)$ is supported in $\{x \in \Omega, u(x) > t\}$, we have $v(t) \in L^1(\Omega)$. Set

$$\beta(t) = \int_\Omega v(t)\, dx. \tag{A.4.19}$$

On applying Fubini's Theorem to the function $1_{\{u>t\}}(x)$, we obtain

$$\beta(t) = \int_t^\infty \alpha(s)\, ds,$$

so that $\beta \in W^{1,1}_{\text{loc}}(0,\infty)$ and

$$\beta'(t) = -\alpha(t), \tag{A.4.20}$$

for almost all $t > 0$. The idea of the proof is to obtain a differential inequality on $\beta(t)$ which implies that $\beta(t)$ must vanish for t large enough. It follows from (A.4.16) that

$$b(u, v(t)) = \langle f, v(t)\rangle_{H^{-1},H_0^1}, \text{ for every } t > 0,$$

with the notation of the proof of Theorem A.4.7. Therefore, by applying Theorem A.3.12 and the property $v(t) \in L^1(\Omega)$, we get

$$\int_\Omega \{|\nabla v(t)|^2 + a|v(t)|^2 + \lambda|v(t)|^2\}\, dx = \int_\Omega \{f - t(a+\lambda)\}v(t)\, dx.$$

Therefore, it folows from Lemma A.4.6 that

$$\|v(t)\|_{H^1}^2 \le C \int_\Omega \{f - t(a+\lambda)\}v(t)\, dx.$$

We now estimate the right-hand side of the above inequality.

$$\begin{aligned}\int_\Omega f v(t) &= \int_\Omega (f_1 + f_2)v(t) \\ &\le \|f_1\|_{L^p}\|v(t)\|_{L^{p'}} + \|f_2\|_{L^\infty}\|v(t)\|_{L^1} \\ &\le \|v(t)\|_{L^{p'}} + \|v(t)\|_{L^1}.\end{aligned}$$

Furthermore,

$$-t\lambda \int_\Omega v(t) \le t|\lambda| \int_\Omega v(t),$$

and

$$-t\int_\Omega a v(t) \le t\int_\Omega a^- v(t) \le Ct(\|v(t)\|_{L^{q'}} + \|v(t)\|_{L^1}).$$

Therefore,

$$\|v(t)\|_{H^1}^2 \le C(1+t)(\|v(t)\|_{L^{p'}} + \|v(t)\|_{L^{q'}} + \|v(t)\|_{L^1}). \tag{A.4.21}$$

Consider now $\rho \ge \max\{q', 1\}$ such that $\rho > 2p'$ and $\rho \le \dfrac{2N}{N-2}$ ($\rho \le \infty$ if $N = 1$, $\rho < \infty$ if $N = 2$). It follows from the assumptions that such a ρ exists. Furthermore, it follows from Sobolev's embedding theorem that $H_0^1(\Omega) \hookrightarrow L^\rho(\Omega)$. Next, given $1 \le \sigma \le \rho$, it follows from Hölder's inequality that

$$\|v(t)\|_{L^\sigma} \le \alpha(t)^{\frac{1}{\sigma}-\frac{1}{\rho}}\|v(t)\|_{L^\rho} \le C\alpha(t)^{\frac{1}{\sigma}-\frac{1}{\rho}}\|v(t)\|_{H^1};$$

and so, by (A.4.21),

$$\|v(t)\|_{H^1}^2 \le C(1+t)(\alpha(t)^{\frac{1}{p'}} + \alpha(t)^{\frac{1}{q'}} + \alpha(t))\alpha(t)^{-\frac{1}{\rho}}\|v(t)\|_{H^1}.$$

Therefore, by Sobolev's embedding theorem,

$$\|v(t)\|_{L^\rho} \le C(1+t)(\alpha(t)^{\frac{1}{p'}} + \alpha(t)^{\frac{1}{q'}} + \alpha(t))\alpha(t)^{-\frac{1}{\rho}}.$$

Finally, since $\|v(t)\|_{L^1} \le \alpha(t)^{1-\frac{1}{\rho}}\|v(t)\|_{L^\rho}$, we obtain

$$\beta(t) \le C(1+t)(\alpha(t)^{\frac{1}{p'}} + \alpha(t)^{\frac{1}{q'}} + \alpha(t))\alpha(t)^{1-\frac{2}{\rho}},$$

which we can write as

$$\beta(t) \le C(1+t)F(\alpha(t)),$$

with $F(s) = s^{2-\frac{2}{p}-\frac{2}{\rho}} + s^{2-\frac{2}{q}-\frac{2}{\rho}} + s^{2-\frac{2}{\rho}}$. It follows that

$$-\alpha(t) + F^{-1}\left(\frac{\beta(t)}{C(1+t)}\right) \le 0. \tag{A.4.22}$$

Setting $z(t) = \dfrac{\beta(t)}{C(1+t)}$, it follows from (A.4.20) and (A.4.22) that

$$z' + \frac{\psi(z(t))}{C(1+t)} \le 0,$$

with $\psi(s) = F^{-1}(s) + Cs$. Integrating the above differential inequality yields

$$\int_s^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(s)} \frac{d\sigma}{\psi(\sigma)},$$

for all $0 < s < t < T$. If $T \le 1$, then by definition $\|u^+\|_{L^\infty} \le 1$. Otherwise, we obtain

$$\int_1^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(1)} \frac{d\sigma}{\psi(\sigma)},$$

which implies in particular that

$$\int_1^T \frac{d\sigma}{C(1+\sigma)} \le \int_0^{z(1)} \frac{d\sigma}{\psi(\sigma)}.$$

Note that by assumption, there exists $\theta < 1$ such that $F^{-1}(s) \ge s^\theta$ for s small, so that $1/\psi$ is integrable near zero. Since $1/(1+\sigma)$ is not integrable at the origin, this implies that $T = \|u^+\|_{L^\infty} < \infty$. Moreover, $\|u^+\|_{L^\infty}$ is estimated in terms of $z(1)$, and

$$z(1) = \frac{1}{C}\int_\Omega (u-1)^+ \le \frac{1}{C}\int_{\{u>1\}} u \le \frac{1}{C}\int_{\{u>1\}} u^r \le \frac{1}{C}.$$

The result follows. □

Open problem. We do not know if, under the assumptions of Theorem A.4.13, the inequality $\|u\|_{L^\infty} \le C\|f\|_{L^p+L^\infty}$ holds.

One can improve the L^p estimates by using Sobolev's inequalities. In particular, we have the following result.

Theorem A.4.14. *Let a verify (A.4.8) and (A.4.9), and assume that $a \ge 0$ almost everywhere. Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.15). If $f \in L^p(\Omega)$ for some $p \in (1, \infty]$, then the following properties hold:*

(i) *If $p > N/2$, then $u \in L^p(\Omega) \cap L^\infty(\Omega)$, and there exists a constant C independent of f such that*

$$\|u\|_{L^r} \le C\|f\|_{L^p},$$

for all $r \in [p, \infty]$;

(ii) *if $p = N/2$ and $N \ge 3$, then $u \in L^r(\Omega)$ for all $r \in [p, \infty)$, and there exist constants $C(r)$ independent of f such that*

$$\|u\|_{L^r} \le C(r)\|f\|_{L^p},$$

for all $r \in [p, \infty)$;

(iii) *if $1 < p < N/2$ and $N \ge 3$, then $u \in L^p(\Omega) \cap L^{\frac{Np}{N-2p}}(\Omega)$, and there exists a constant C independent of f such that*

$$\|u\|_{L^r} \le C\|f\|_{L^p},$$

for all $r \in [p, \dfrac{Np}{N-2p}]$.

Proof. Property (i) follows from Theorems A.4.12 and A.4.13 and Hölder's inequality. It remains to establish properties (ii) and (iii). Note that in this case $N \ge 3$. By density (Proposition A.3.58), it is sufficient to establish these properties for $f \in \mathcal{D}(\Omega)$. In this case, we have $u \in L^1(\Omega) \cap L^\infty(\Omega)$ by Theorem A.4.11. Consider an odd, increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, such that φ' is bounded. Define

$$\psi(x) = \int_0^x \sqrt{\varphi'(s)}\, ds. \tag{A.4.23}$$

It follows that ψ is odd, nondecreasing, and that ψ' is bounded. It follows from Theorem A.3.12 and Corollary A.3.29 that $\varphi(u)$ and $\psi(u)$ belong to $H_0^1(\Omega)$, and that

$$|\nabla\psi(u)|^2 = \varphi'(u)|\nabla u|^2 = \nabla u \cdot \nabla(\varphi(u)), \tag{A.4.24}$$

almost everywhere. Applying formula (A.4.16) with $v = \varphi(u)$, it follows from (A.4.24) that

$$\int_\Omega (|\nabla(\psi(u))|^2 + \lambda u\varphi(u) + au\varphi(u))\, dx = \langle f, \varphi(u)\rangle_{H^{-1},H_0^1}.$$

In addition, $x\varphi(x) \ge 0$, and it follows from (A.4.23) and Cauchy-Schwarz inequality that $x\varphi(x) \ge |\psi(x)|^2$. Therefore, it follows from Lemma A.4.6 that there exists a constant C such that

$$\|\psi(u)\|_{H^1}^2 \le C\langle f, \varphi(u)\rangle_{H^{-1},H_0^1}.$$

Therefore, given $p \in [1, \infty]$, we have

$$\|\psi(u)\|_{H^1}^2 \le C\|f\|_{L^p}\|\varphi(u)\|_{L^{p'}}.$$

Since $N \geq 3$, we have

$$H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega). \tag{A.4.25}$$

It follows that, after possibly modifying C,

$$\|\psi(u)\|^2_{L^{\frac{2N}{N-2}}} \leq C\|f\|_{L^p}\|\varphi(u)\|_{L^{p'}}. \tag{A.4.26}$$

Consider now $1 < q < \infty$ such that $(q-1)p' \geq 1$. If $q \leq 2$, let $\varphi_\varepsilon(x) = x(\varepsilon + x^2)^{\frac{q-2}{2}}$. If $q > 2$, take $\varphi_\varepsilon(x) = x|x|^{q-2}(1+\varepsilon x^2)^{\frac{2-q}{2}}$. It follows that $|\varphi_\varepsilon(x)| \leq C|x|^{q-1}$ and that $|\varphi_\varepsilon(x)| \underset{\varepsilon\downarrow 0}{\longrightarrow} |x|^{q-1}$. One verifies easily that $|\psi_\varepsilon(x)|^2 \leq C|x|^q$ and that $|\psi_\varepsilon(x)|^2 \to \frac{4(q-1)}{q^2}|x|^q$. Applying (A.4.26), then letting $\varepsilon \downarrow 0$ and applying the dominated convergence theorem, it follows that

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \leq C\frac{q^2}{q-1}\|f\|_{L^p}\|u\|^{q-1}_{L^{(q-1)p'}}, \tag{A.4.27}$$

for all $1 < q < \infty$ such that $(q-1)p' \geq 1$. We now prove property (ii). Suppose that $N \geq 3$ and that $p = N/2$. Apply (A.4.27) with $q > N/2$. It follows that

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \leq C\frac{q^2}{q-1}\|f\|_{L^{N/2}}\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}}. \tag{A.4.28}$$

On the other hand, it follows from Hölder's inequality that

$$\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}} \leq \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|u\|^{\frac{N-2}{2q-N+2}}_{L^{N/2}}.$$

Applying Theorem A.4.12, it follows that

$$\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}} \leq \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|f\|^{\frac{N-2}{2q-N+2}}_{L^{N/2}}.$$

Substitution in (A.4.28) yields

$$\|u\|_{L^{\frac{Nq}{N-2}}} \leq C(q)\|f\|_{L^{N/2}}.$$

Property (ii) follows from the above estimate and Theorem A.4.12, since q is arbitrary. Finally, we prove property (iii). Let $q = \dfrac{(N-2)p}{N-2p}$. In particular, $\dfrac{Nq}{N-2} = (q-1)p' = \dfrac{Np}{N-2p}$, and it follows from (A.4.27) that

$$\|u\|_{L^{\frac{Np}{N-2p}}} \leq C'\|f\|_{L^p}.$$

Property (iii) follows from the above estimate and Theorem A.4.12. □

Corollary A.4.15. *Let a and λ be as in Theorem A.4.14. If $f \in H^{-1}(\Omega) \cap L^1(\Omega)$, then the following properties hold:*

(i) *if $N = 1$, then $u \in L^1(\Omega) \cap L^\infty(\Omega)$, and there exists a constant C independent of f such that*

$$\|u\|_{L^r} \leq C\|f\|_{L^1},$$

for all $r \in [1, \infty]$;

(ii) *if $N \geq 2$, then $u \in L^r(\Omega)$ for all $r \in \left[1, \dfrac{N}{N-2}\right)$ ($r \in [1,\infty)$ if $N = 2$), and there exist constants $C(r)$ independent of f such that*

$$\|u\|_{L^r} \leq C(r)\|f\|_{L^1},$$

for all $r \in \left[1, \frac{N}{N-2}\right)$ *(*$r \in [1, \infty)$ *if* $N = 2$*).*

Proof. If $N = 1$, then we have $u \in H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$. Furthermore, it follows from Lemma A.4.6 (note that $\lambda_1(-\triangle + a) \geq 0$, since $a \geq 0$) that there exists $\mu > 0$ such that

$$\mu \|u\|_{H^1}^2 \leq \langle f, u \rangle_{H^{-1}, H_0^1} \leq \|f\|_{L^1} \|u\|_{L^\infty} \leq C \|f\|_{L^1} \|u\|_{H^1}.$$

Therefore, $\mu \|u\|_{H^1} \leq C \|f\|_{L^1}$, and (i) follows.

In the case $N \geq 2$, we use a duality argument. Let u and f be as in the statement of the theorem. It follows from Theorem A.4.12 that $u \in L^1(\Omega)$ and

$$\|u\|_{L^1} \leq C \|f\|_{L^1}.$$

Let now $\theta \in [1, \infty)$ be such that $u \in L^\theta(\Omega)$. Fix $q > N/2$. Let $h \in C_c^\infty(\Omega)$, and let $\varphi \in H_0^1(\Omega)$ be the solution of the equation $-\triangle \varphi + a\varphi + \lambda \varphi = h$. It follows from Theorem A.4.14 that

$$\|\varphi\|_{L^\infty} \leq C \|h\|_{L^q}.$$

Since

$$\langle f, \varphi \rangle_{H^{-1}, H_0^1} = \langle -\triangle u + au + \lambda u, \varphi \rangle_{H^{-1}, H_0^1} = \langle u, -\triangle \varphi + a\varphi + \lambda \varphi \rangle_{H_0^1, H^{-1}} = \langle u, h \rangle_{H_0^1, H^{-1}},$$

we deduce

$$\left| \int_\Omega uh \right| \leq \|f\|_{L^1} \|\varphi\|_{L^\infty} \leq C \|f\|_{L^1} \|h\|_{L^q}.$$

Since $\varphi \in C_c^\infty(\Omega)$ is arbitrary, we obtain

$$\|u\|_{L^{q'}} \leq C \|f\|_{L^1}.$$

Since $q \in \left(\frac{N}{2}, \infty\right]$ is arbitrary, $q' \in \left[1, \frac{N}{N-2}\right)$ is arbitrary and the result follows. □

Remark A.4.16. The estimates of Theorem A.4.14 and Corollary A.4.15 are optimal in the following sense.

(i) If $N \geq 2$ and $f \in L^{\frac{N}{2}}(\Omega)$, then u is not necessarily in $L^\infty(\Omega)$. For example, let Ω be the unit ball, and let $u(x) = (-\log |x|)^\gamma$ with $\gamma > 0$. Then $u \notin L^\infty(\Omega)$. On the other hand, one verifies easily that if $0 < \gamma < \frac{1}{2}$ in the case $N = 2$ and $0 < \gamma < 1 - \frac{2}{N}$ in the case $N \geq 3$, then $u \in H_0^1(\Omega)$ and $-\triangle u + u \in L^{\frac{N}{2}}(\Omega)$.

(ii) If $N \geq 3$ and $f \in L^1(\Omega)$, then there is no estimate of the form $\|u\|_{L^{\frac{N}{N-2}}} \leq C \|f\|_{L^1}$. (Note that since $u \in H_0^1(\Omega)$, we always have $u \in L^{\frac{N}{N-2}}(\Omega)$.) One constructs easily a counter example as follows. Let Ω be the unit ball, and let $u = z\varphi$ with $\varphi \in \mathcal{D}(\Omega)$, $\varphi(0) \neq 0$, and $z(x) = |x|^{2-N}(-\log |x|)^\gamma$ with $\gamma < 0$. Then $-\triangle u + u \in L^1(\Omega)$ and $u \notin L^{\frac{N}{N-2}}(\Omega)$. By approximating u by smooth functions, one deduces that there is no estimate of the form $\|u\|_{L^{\frac{N}{N-2}}} \leq C \|f\|_{L^1}$.

(iii) If $N \geq 3$ and $1 < p < N/2$, then by arguing as above one shows the following properties. There is no estimate of the form $\|u\|_{L^q} \leq C\|f\|_{L^1}$ for $q > \dfrac{Np}{N-2p}$. Moreover, if $f \in L^p(\Omega)$, then in general $u \notin L^q(\Omega)$ for $q > \dfrac{Np}{N-2p}$, $q > \dfrac{2N}{N-2}$.

Corollary A.4.17. *Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.6). If $f \in L^p(\Omega)$ for some $p \in (N/2, \infty)$, $p \geq 1$ or if $f \in C_0(\Omega)$, then $u \in C_{\mathrm{b}}(\Omega)$.*

Proof. By density (Proposition A.3.58), the result follows from Theorem A.4.14, Corollary A.4.15 and Proposition A.4.10. □

Remark A.4.18. Under some smoothness assumptions on Ω, one can establish higher order L^p estimates. However, the proof of these estimates is considerably more delicate. In particular, one has the following results.

(i) If Ω has a bounded boundary of class C^2 (in fact, $C^{1,1}$ is enough) and if $1 < p < \infty$, then one can show that for every $\lambda > 0$ and $f \in L^p(\Omega)$, there exists a unique solution $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ of equation (A.4.6), and that

$$\|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|f\|_{L^p}),$$

for some constant C independent of f (see Gilbarg and Trudinger [54], Theorem 9.15, p.241). One shows as well that for every $f \in W^{-1,p}(\Omega)$, there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of equation (A.4.6) (see Agmon, Douglis and Nirenberg [2]).

(ii) Let $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.6). It follows from the preceding result that if in addition $f \in L^p(\Omega)$ for some $p \in (1, \infty)$, then $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Indeed, by density and the estimate of (i) above, one needs only consider the case $\varphi \in \mathcal{D}(\Omega)$. In this case, $u \in H^2(\Omega) \cap H_0^1(\Omega) \cap C_0(\Omega)$ by Theorem A.4.8 and Theorem A.4.28 below. On the other hand, equation (A.4.6) has a unique solution $v \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ by (i) above. So we need only show that $u = v$. If Ω is bounded, then both u and v are solutions in $W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ with $q = \min\{p, 2\}$, and so $u = v$ by uniqueness in $W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$. If Ω is unbounded, then we use uniqueness in $C_0(\Omega)$ (Corollary A.4.33 below); so we are reduced to show that $v \in C_0(\Omega)$. We observe that, since $\partial\Omega$ is bounded, there exists $R < \infty$ such that $\{|x| > R\} \subset \Omega$. Chose R possibly larger so that $\operatorname{supp}(f) \subset \{|x| < R\}$ and consider $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\rho \equiv 1$ on $\{|x| < R\}$. Set $v = w + z$ with $w = \rho z$ and $z = (1 - \rho)z$. We have $-\triangle w + \lambda w = g$ with $g = \rho f - v\triangle\rho - 2\nabla v \cdot \nabla\rho$. Since $v \in C^\infty(\Omega)$ (Proposition A.4.10) and $\nabla\rho$ and $\triangle\rho$ have compact support in Ω, we have $g \in \mathcal{D}(\Omega)$. Since furthermore w and g are supported in $\{|x| < R\}$, we are reduced to the case of a bounded domain, and it follows by uniqueness that $w \in W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$ for every $q < \infty$. In particular, $w \in C_0(\Omega)$ (Corollary A.3.47). Finally, $z \in L^p(\{|x| > R\}) \cap C^\infty(\{|x| \geq R\})$ verifies $-\triangle z + \lambda z = -v\triangle\rho - 2\nabla v \cdot \nabla\rho$. In particular, $-\triangle z + \lambda z = 0$ for $|x|$ large. It follows easily from Proposition A.4.10 that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $z(x_0) \geq \varepsilon$, then $z(x) \geq \varepsilon/2$ for $|x - x_0| \leq \delta$. Since $z \in L^p(\Omega)$, we deduce that $z(x) \to 0$ as $|x| \to \infty$. This implies that $z \in C_0(\Omega)$ and completes the proof.

(iii) Note that if $f \in L^p(\Omega)$ and if $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is the solution of (A.4.6), then

$$\lambda \|u\|_{L^p} \le \|f\|_{L^p}.$$

This follows from Theorem A.4.11 and a density argument, by using the estimate of (i) and the regularity property of (ii).

(iv) One has partial results in the cases $p = 1$ and $p = \infty$. In particular, if Ω is bounded and smooth enough, then for every $\lambda > 0$ and $f \in L^1(\Omega)$, there exists a unique solution $u \in W_0^{1,1}(\Omega)$, such that $\triangle u \in L^1(\Omega)$, of equation (A.4.6) (see Pazy [85], Theorem 3.10, p.218). It follows that $\lambda\|u\|_{L^1} \le \|f\|_{L^1}$. Moreover, if $f \in H^{-1}(\Omega) \cap L^1(\Omega)$ and if $u \in H_0^1(\Omega)$ is the solution of (A.4.6), then $u \in W_0^{1,1}(\Omega)$. (See the argument of (ii) and (iii)) In general, $u \notin W^{2,1}(\Omega)$. If Ω is bounded, it follows from Theorems A.4.5 and A.4.11 that for every $\lambda > 0$ and $f \in L^\infty(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, such that $\triangle u \in L^\infty(\Omega)$, of equation (A.4.6). It follows from Theorem A.4.11 that $\lambda\|u\|_{L^\infty} \le \|f\|_{L^\infty}$. In general, $u \notin W^{2,\infty}(\Omega)$, even if Ω is smooth. On the other hand, it follows from property (i) above that $u \in W_0^{1,p}(\Omega)$, for every $p < \infty$.

A.4.4. The maximum principle. Let $T \in \mathcal{D}'(\Omega)$. We recall that (by definition), we have $T \ge 0$ (respectively $T \le 0$) if and only if $\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} \ge 0$ (respectively $\langle T, \varphi\rangle \le 0$), for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi \ge 0$ on Ω. Clearly, if $T \in L^1_{\text{loc}}(\Omega)$, then $T \ge 0$ as a distribution if and only if $T \ge 0$ almost everywhere on Ω.

Lemma A.4.19. *Let $u \in H_0^1(\Omega)$. If $u \ge 0$ almost everywhere, then there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $u_n \ge 0$ and $u_n \underset{n\to\infty}{\longrightarrow} u$ in $H_0^1(\Omega)$.*

Proof. Consider $\varepsilon > 0$. It follows from the definition of $H_0^1(\Omega)$ and Corollary A.3.30 that there exists $v \in \mathcal{D}(\Omega)$ such that $\|v^+ - u\|_{H^1} \le \varepsilon/2$. By convolution of v^+ with a sequence of nonnegative mollifiers, one can construct $w \in \mathcal{D}(\Omega)$, $w \ge 0$ such that $\|v^+ - w\|_{H^1} \le \varepsilon/2$. It follows that $\|u - w\|_{H^1} \le \varepsilon$. Hence the result, since ε is arbitrary. □

Corollary A.4.20. *Consider a distribution $f \in H^{-1}(\Omega)$. Then $f \ge 0$ (respectively $f \le 0$) if and only if $\langle f, \varphi\rangle_{H^{-1},H_0^1} \ge 0$ (respectively $\langle f, \varphi\rangle_{H^{-1},H_0^1} \le 0$), for every $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$ almost everywhere on Ω.*

Proof. The result follows immediately from Lemma A.4.19. □

We have the following result (the weak maximum principle).

Proposition A.4.21. *Let a verify (A.4.8) and (A.4.9), let $\lambda_1(-\triangle + a)$ be defined by (A.4.13) and let $\lambda > -\lambda_1(-\triangle + a)$. Let $f \in H^{-1}(\Omega)$, and let $u \in H_0^1(\Omega)$ be the solution of (A.4.15). If $f \ge 0$ (respectively $f \le 0$) in $\mathcal{D}'(\Omega)$, then $u \ge 0$ almost everywhere on Ω (respectively $u \le 0$ almost everywhere in Ω).*

Proof. By considering $-u$, it is sufficient to establish the result when $f \ge 0$. Apply (A.4.16) with $v = -u^- \in H_0^1(\Omega)$. It follows that

$$b(v, v) = \langle f, v\rangle_{H^{-1},H_0^1}.$$

Since $v \leq 0$ almost everywhere on Ω, we have (Corollary A.4.20) $\langle f, v\rangle_{H^{-1},H^1_0} \leq 0$, and it follows from the coerciveness of b that $v = 0$. Hence the result. □

Corollary A.4.22. *Let a verify (A.4.8) and (A.4.9), let $\lambda_1(-\triangle + a)$ be defined by (A.4.13) and let $\lambda > -\lambda_1(-\triangle + a)$. Consider $f, g \in H^{-1}(\Omega)$ and let $u, v \in H^1_0(\Omega)$ be the corresponding solutions of (A.4.15). If $f \leq g$ in $\mathcal{D}'(\Omega)$, then $u \leq v$ almost everywhere in Ω.*

Proof. Apply Proposition A.4.21 to $v - u$. □

We will need to compare solutions of equation (A.4.15) with functions u that do not satisfy the boundary condition $u = 0$ on $\partial\Omega$ (by this, we mean that $u \notin H^1_0(\Omega)$). Let a verify (A.4.8) and (A.4.9), and let r be defined by (A.4.10). Given $u \in H^1(\Omega) \cap L^r(\Omega)$, one verifies easily (see the proof of (A.4.11)) that

$$\left| \int_\Omega auv\,dx \right| \leq C\|v\|_{H^1},$$

for all $v \in H^1_0(\Omega)$. It follows that

$$au \in H^{-1}(\Omega),$$

for every $u \in H^1(\Omega) \cap L^r(\Omega)$. In particular, we have

$$-\triangle u + au + \lambda u \in H^{-1}(\Omega),$$

for every $u \in H^1(\Omega) \cap L^r(\Omega)$ and every $\lambda \in \mathbb{R}$, and

$$\langle -\triangle u + au + \lambda u, v\rangle_{H^{-1},H^1_0} = \int_\Omega \nabla u \cdot \nabla v + auv + \lambda uv, \tag{A.4.31}$$

for all $v \in H^1_0(\Omega)$. We make the following definition.

Definition A.4.23. *Let a verify (A.4.8) and (A.4.9), and let r be defined by (A.4.10). Given $f \in H^{-1}(\Omega)$, a supersolution (respectively subsolution) of equation (A.4.15) is a function $u \in H^1(\Omega) \cap L^r(\Omega)$ such that $u^- \in H^1_0(\Omega)$ (respectively, $u^+ \in H^1_0(\Omega)$) and*

$$-\triangle u + au + \lambda u \geq f \,(\text{respectively} \leq f),$$

in $\mathcal{D}'(\Omega)$.

Remark A.4.24. The assumption $u^- \in H^1_0(\Omega)$ (respectively, $u^+ \in H^1_0(\Omega)$) is a weak formulation of the property $u \geq 0$ on $\partial\Omega$ (respectively, $u \leq 0$ on $\partial\Omega$).

We have the following characterization of supersolutions and subsolutions.

Lemma A.4.25. *Let a verify (A.4.8) and (A.4.9), and let r be defined by (A.4.10). Consider $f \in H^{-1}(\Omega)$ and $u \in H^1(\Omega) \cap L^r(\Omega)$ such that $u^- \in H^1_0(\Omega)$ (respectively, $u^+ \in H^1_0(\Omega)$). Then, u is a supersolution (respectively, a subsolution) of equation (A.4.15) if, and only if*

$$\int_\Omega \nabla u \cdot \nabla v + auv + \lambda uv \geq \langle f, v\rangle_{H^{-1},H^1_0} (\text{respectively, } \leq \langle f, v\rangle_{H^{-1},H^1_0}), \tag{A.4.32}$$

for all $v \in H^1_0(\Omega)$ such that $v \geq 0$ almost everywhere.

Proof. The result follows from (A.4.31) and Corollary A.4.20. □

Proposition A.4.26. *Let a verify (A.4.8) and (A.4.9), let $\lambda_1(-\triangle + a)$ be defined by (A.4.13) and let $\lambda > -\lambda_1(-\triangle + a)$. If $u \in H^1(\Omega)$ is a supersolution (respectively, subsolution) of (A.4.15) with $f = 0$, then $u \geq 0$ almost everywhere on Ω (respectively, $u \leq 0$ almost everywhere on Ω).*

Proof. By considering $-u$, we only have to establish the result for supersolutions. Take $v = u^-$ in formula (A.4.32). Applying Corollary A.3.15, it follows that

$$\int_\Omega |\nabla u^-|^2 + a(u^-)^2 + \lambda(u^-)^2 \leq 0;$$

and so $u^- = 0$, by (A.4.16). Hence the result. □

Corollary A.4.27. *Let a verify (A.4.8) and (A.4.9), let $\lambda_1(-\triangle + a)$ be defined by (A.4.13) and let $\lambda > -\lambda_1(-\triangle + a)$. Consider $f \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (A.4.15). If v is a supersolution of (A.4.15), then $u \leq v$ almost everywhere on Ω. Similarly, if v is a subsolution of (A.4.15), then $u \geq v$ almost everywhere on Ω.*

Proof. It is clear that $u - v$ is a subsolution of equation (A.4.15) with $f = 0$ (see Corollary 1.2.28), and the result follows from Proposition A.4.26. □

When Ω satisfies some smoothness conditions, the L^∞ regularity of Theorem A.4.11 can be improved by making use of super and subsolutions, and in fact u is continuous on $\overline{\Omega}$. More precisely, we have the following result.

Theorem A.4.28. *If $N \geq 2$, assume that every $x \in \partial\Omega$ has the exterior cone property. Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (A.4.6). If $f \in C_0(\Omega)$ or if $f \in L^p(\Omega)$ for some $p \in (N/2, \infty)$, $p > 1$, then $u \in C_0(\Omega)$.*

Proof. Let us first proceed to several reductions. First, note that by density (Proposition A.3.58) and Theorem A.4.11, it is sufficient to establish the result for $f \in \mathcal{D}(\Omega)$. Consider now $f \in \mathcal{D}(\Omega)$. It follows from Proposition A.4.10 that

$$u \in C^\infty(\Omega). \tag{A.4.33}$$

Next, let $R > 0$ be such that $\mathrm{Supp}(f) \subset \{x \in \mathbb{R}^N;\, |x| \leq R\}$, and let $M = \|f\|_{L^\infty}$. Choose K large enough, so that

$$K\lambda e^{-\sqrt{1+\frac{\lambda}{2}R^2}} \geq 2M,$$

and consider

$$v(x) = Ke^{-\sqrt{1+\frac{\lambda}{2}r^2}}, \tag{A.4.34}$$

where $r = |x|$. We have $v \in \mathcal{S}(\mathbb{R}^N)$; and so, $v_{|\Omega} \in H^1(\Omega)$. Furthermore,

$$-\triangle v + \lambda v = \left(\frac{\lambda}{2} + \frac{\lambda \left(1 + (N-1)\left(1 + \frac{\lambda}{2} r^2\right) + \left(1 + \frac{\lambda}{2} r^2\right)^{1/2}\right)}{2\left(1 + \frac{\lambda}{2} r^2\right)^{3/2}} \right) v \geq \frac{\lambda v}{2} \geq f.$$

Therefore, v is a supersolution of (A.4.6). Applying Corollary A.4.27, we get $u \leq v$ almost everywhere in Ω. Since $-v$ is a subsolution of equation (A.4.6), one obtains as well that $u \geq -v$; and so,

$$|u(x)| \to 0 \text{ as } |x| \to \infty, x \in \Omega. \tag{A.4.35}$$

We now claim that for every $x_0 \in \partial\Omega$, there exist $\alpha, \eta > 0$ such that

$$|u(x)| \leq |x - x_0|^\alpha, \text{ for every } x \in \Omega \text{ such that } |x - x_0| < \eta. \tag{A.4.36}$$

Suppose for a moment that (A.4.36) holds. Then, comparing with (A.4.33), we get $u \in C(\overline{\Omega})$ and $u_{|\partial\Omega} = 0$; and so, it follows from (A.4.35) and Lemma A.3.48 that $u \in C_0(\Omega)$, which is the desired result.

Finally, it remains to establish property (A.4.36). This follows from the concept of *barrier function.* In order to construct a *local barrier* at x_0, we need the following two lemmas.

Lemma A.4.29. *Given $k \geq 0$ and $0 < \theta < \pi/2$, there exist $\gamma > 0$ and a function $f \in C^2([0, \pi - \theta])$, such that*

(i) $1/2 \leq f(t) \leq 1$, *for all* $t \in [0, \pi - \theta]$;

(ii) $f'(0) = 0$;

(iii) $f''(t) + k \dfrac{\cos t}{\sin t} f'(t) + \gamma f(t) = 0$, *for all* $t \in (0, \pi - \theta]$.

Proof. The idea is to solve equation

$$f'' + k \frac{\cos t}{\sin t} f' + \gamma f = 0, \tag{A.4.37}$$

with the initial conditions $f(0) = 1$ and $f'(0) = 0$ (note that the singularity of $1/\sin t$ at $t = 0$ is eliminated by the condition $f'(0) = 0$) and to observe that the solution depends contunuously on γ, uniformly on compact subsets of $[0, \pi)$. The result will follow, since the solution for $\gamma = 0$ is $f(t) \equiv 1$. More precisely, consider $E = \{f \in C([0, \pi - \theta]); 1/2 \leq f \leq 1\}$, and equip E with the distance $d(f, g) = \|g - f\|_{L^\infty(0, \pi-\theta)}$. It is clear that (E, d) is a complete metric space. Given $f \in E$, let

$$Af(t) = \int_0^t \frac{1}{(\sin s)^k} \int_0^s (\sin\sigma)^k f(\sigma)\, d\sigma.$$

It follows easily that $Af \in C^2([0, \pi - \theta])$ and $f'(0) = 0$. Furthermore, consider $0 \leq \sigma \leq s \leq \pi - \theta$. If $s \leq \pi/2$, one has $\sin\sigma \leq \sin s$. If $s > \pi/2$, then $\sin s \geq \sin(\pi - \theta) = \sin\theta$; and so, $\sin\sigma \leq 1 = \dfrac{\sin\theta}{\sin\theta} \leq \dfrac{\sin s}{\sin\theta}$. Therefore, we have in both cases $\sin\sigma \leq \dfrac{\sin s}{\sin\theta}$. It follows that

$$0 \leq Af(t) \leq \frac{(\pi - \theta)^2}{2 \sin\theta^k} \|f\|_{L^\infty} \leq \frac{\pi^2}{2 \sin\varepsilon^k}.$$

In particular, if γ is small enough, we have

$$0 \le \gamma A f(t) \le 1/2, \text{ for all } f \in E \text{ and } t \in [0, \pi - \theta].$$

It follows that for γ small enough, the mapping

$$f \mapsto Tf = 1 - \gamma A f,$$

maps $E \to E$ and is a contraction of Lipschitz constant $L \le 1/2$. Applying Theorem A.1.1, it follows that T has a unique fixed point $f \in E$, which solves equation $f(t) = 1 - \gamma A f(t)$, for all $t \in [0, \pi - \theta]$. One verifies easily that f solves equation (A.4.37), which completes the proof. □

Lemma A.4.30. *Let $x_0 \in \partial\Omega$. If $N \ge 2$, assume that x_0 has the exterior cone property. For $\delta > 0$, define $\Omega_\delta = \{x \in \Omega;\ |x - x_0| < \delta\}$. Then, there exists $\delta, \alpha > 0$ and a function $h \in C(\overline{\Omega_\delta}) \cap C^2(\Omega_\delta) \cap H^1(\Omega_\delta)$ such that*

(i) $\triangle h = 0$, *in* Ω_δ;

(ii) $\frac{1}{2}|x - x_0|^\alpha \le h(x) \le |x - x_0|^\alpha$, *for all* $x \in \Omega_\delta$.

Proof. If $N = 1$, take $\alpha = \delta = 1$, and $h(x) = |x - x_0|$. It is clear that h has the desired properties.

In the case $N \ge 2$, by assumption, there exist $\theta \in (0, \pi/2)$, $z \in S^{N-1}$ and $\delta > 0$, such that $C(x_0, z, \theta, \delta) \cap \overline{\Omega} = \emptyset$ (compare Definition A.3.3). Without loss of generality, we may assume that $x_0 = 0$ and $z = (0, \cdots, 0, -1)$. In particular,

$$\Omega_\delta \subset \{x \in \mathbb{R}^N;\ 0 < |x| < \delta \text{ and } x_N > -|x|\cos\theta\}. \tag{A.4.38}$$

Given $x \in \Omega_\delta$, define $t \in [0, \pi - \theta)$ by

$$\cos t = \frac{x_N}{|x|}.$$

One has $t \in C^\infty(U) \cap W^{1,\infty}_{\text{loc}}(\Omega_\delta)$, where $U = \{x \in \Omega_\delta;\ t \ne 0\}$. In addition,

$$|\nabla t|^2 = 1/|x|^2, \text{ and } \triangle t = (N-2)\frac{\cos t}{\sin t}.$$

Given $\alpha > 0$ and $f \in C^2([0, \pi - \theta])$ such that $f'(0) = 0$, define

$$h(x) = |x|^\alpha f(t(x)).$$

We have $h \in C(\overline{\Omega_\delta})$. Furthermore, we have $\nabla h = \alpha|x|^\alpha(f(t)x + f'(t)\nabla t)$. Since $f'(0) = 0$, it follows easily that $h \in C^1(\Omega_\delta)$, and that $|\nabla h| \le C|x|^{\alpha-1} \in L^2(\Omega_\delta)$. In particular, $h \in H^1(\Omega_\delta)$. Furthermore, a tedious but easy calculation shows that $h \in C^2(\Omega_\delta)$, and that

$$\triangle h = |x|^{\alpha-2}\left(f''(t) + (N-2)\frac{\cos t}{\sin t}f'(t) + \alpha(\alpha + N - 2)f(t)\right), \text{ for all } x \in U.$$

Applying Lemma A.4.29, it follows that there exists $\alpha > 0$ such that one can choose f with $1/2 \le f \le 1$, and for which $\triangle h = 0$ in Ω_δ. This completes the proof. □

End of the proof of Theorem A.4.28. Consider $x_0 \in \partial\Omega$, and let α, δ, h be given by Lemma A.4.30. By choosing δ possibly smaller, we may assume, with the notation of Lemma A.4.30, that $\Omega_\delta \cap \mathrm{Supp}(f) = \emptyset$. In particular, we have

$$-\triangle u + \lambda u = 0, \text{ in } \mathcal{D}'(\Omega_\delta) \tag{A.4.39}$$

Let M be large enough, so that

$$M\left(\frac{\delta}{2}\right)^\alpha \geq 2\|f\|_{L^\infty}, \tag{A.4.40}$$

and let

$$H = \frac{Mh}{\lambda}. \tag{A.4.41}$$

Setting $w = u_{|\Omega_\delta}$, it follows from Lemma A.4.30 and (A.4.39) that

$$-\triangle(w - H) + \lambda(w - H) = -\lambda H \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega_\delta). \tag{A.4.42}$$

Therefore, if we show that

$$(w - H)^+ \in H_0^1(\Omega_\delta), \tag{A.4.43}$$

then it follows from Proposition A.4.26 that $w \leq H$, almost everywhere in Ω_δ. One shows similarly that $w \geq -H$, so that this proves the estimate (A.4.36) and completes the proof of the theorem. To prove (A.4.43), consider $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi \equiv 1$ on the set $\{x \in \mathbb{R}^N; |x| \leq \delta/2\}$, and $\varphi \equiv 0$ on the set $\{x \in \mathbb{R}^N; |x| \geq \delta\}$. Note that $\lambda H(x) \geq \|f\|_{L^\infty}$, on the set $\{x \in \Omega_\delta; |x| \geq \delta/2\}$ (compare (A.4.40) and (A.4.41)). On the other hand, it follows from Theorem A.4.12 that $\lambda\|u\|_{L^\infty} \leq \|f\|_{L^\infty}$; and so, $|u| \leq H$ on the set $\{x \in \Omega_\delta; |x| \geq \delta/2\}$. Therefore, $(w - H)^+ = (\varphi w - H)^+$ in Ω_δ. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ be such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $H_0^1(\Omega)$, and set $v_n = (\varphi u_n)_{|\Omega_\delta}$. It is clear that $v_n \underset{n\to\infty}{\longrightarrow} \varphi w$ in $H^1(\Omega_\delta)$; and so, $(v_n - H)^+ \underset{n\to\infty}{\longrightarrow} (\varphi w - H)^+$ in $H^1(\Omega_\delta)$. Therefore, it only remains to show that $(v_n - H)^+ \in H_0^1(\Omega_\delta)$. This follows from Proposition A.3.23, since $v_n - H = -H \leq 0$ on $\partial\Omega_\delta$. □

Remark A.4.31. If the smoothness assumption on Ω does not hold, the conclusion of Theorem A.4.28 may be invalid, even for $f \in \mathcal{D}(\Omega)$, as shows the following example. For $N \geq 2$, let $\Omega = \mathbb{R}^N \setminus \{0\}$. Define $\varphi(x) = \cosh x_1$. We have $\varphi \in C^\infty(\mathbb{R}^N)$, and $-\triangle\varphi + \varphi = 0$. Let now $\psi \in \mathcal{D}(\mathbb{R}^N)$ be such that $\psi \equiv 1$, for $|x| \leq 1$ and $\psi \equiv 0$, for $|x| \geq 2$. Set $u = \varphi\psi$. We have $u \in \mathcal{D}(\mathbb{R}^N)$, and also $u \in H_0^1(\Omega)$ (cf. Remark A.3.24). On the other hand, $-\triangle u + u = 0$ for $|x| \leq 1$ and for $|x| \geq 2$. In particular, if we set $f = -\triangle u + u$, we have $f \in \mathcal{D}(\Omega)$. Finally, $u \notin C_0(\Omega)$, since $u = 1$ on $\partial\Omega$.

Under a more restrictive smoothness assumption on Ω, one can improve the conclusion of Theorem A.4.28. More precisely, we have the following result.

Theorem A.4.32. *If $N \geq 2$ suppose that there exists $\rho > 0$ such that for every $x_0 \in \partial\Omega$ there exists $y(x_0) \in \mathbb{R}^N$ such that $|x_0 - y(x_0)| = \rho$ and such that $B(y_0, \rho) \cap \Omega = \emptyset$. (In other words, we replace the cone property by a uniform "ball" property.) Let $\lambda > 0$, $f \in H^{-1}(\Omega) \cap L^\infty(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (A.4.6). It follows that*

$$|u(x)| \leq C\|f\|_{L^\infty}\mathrm{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where C is independent of f.

Proof. We may assume without loss of generality that $|f| \leq 1$, so that $|u| \leq \lambda^{-1}$. We may also suppose $N \geq 2$, for the case $N = 1$ is immediate. We now construct a *local barrier* at every point of $\partial\Omega$. Given $c > 0$, set

$$w(x) = \begin{cases} \dfrac{1}{4}(\rho^2 - |x|^2) + c \log\left(\dfrac{|x|}{\rho}\right) & \text{if} \quad N = 2, \\ \dfrac{1}{2N}(\rho^2 - |x|^2) + c(\rho^{2-N} - |x|^{2-N}) & \text{if} \quad N \geq 3. \end{cases}$$

It follows that $-\triangle w = 1$ in $\mathbb{R}^N \setminus \{0\}$. Furthermore, we see that if c is large enough, then there exists $\rho_1 > \rho_0 > \rho$ such that $w(x) > 0$ for $\rho < |x| \leq \rho_1$ and $w(x) \geq \lambda^{-1}$ for $\rho_0 \leq |x| \leq \rho_1$. Given c as above, we observe that there exists a constant K such that $w(x) \leq K(|x| - \rho)$ for $\rho \leq |x| \leq \rho_1$.

Let now $x \in \Omega$ such that $2\text{dist}(x, \partial\Omega) < \rho_1 - \rho$, and let $x_0 \in \partial\Omega$ be such that $|x - x_0| \leq 2\text{dist}(x, \partial\Omega)$. Let $\widetilde{\Omega} = \{x \in \Omega;\ \rho < |x - y(x_0)| < \rho_1\}$ and set $v(x) = w(x - y(x_0))$ for $x \in \widetilde{\Omega}$. It follows that

$$0 \leq v(x) \leq K(|x - y(x_0)| - \rho) \leq K(|x - x_0| + |x_0 - y(x_0)| - \rho) = K|x - x_0| \leq 2K\text{dist}(x, \partial\Omega),$$

for all $x \in \widetilde{\Omega}$. On the other hand,

$$-\triangle(u - v) + \lambda(u - v) = f - (1 + \lambda v) \leq f - 1 \leq 0,$$

in $\widetilde{\Omega}$. We claim that

$$(u - v)^+ \in H_0^1(\widetilde{\Omega}). \tag{A.4.44}$$

It then follows from the maximum principle that $u(x) \leq v(x) \leq 2K\text{dist}(x, \partial\Omega)$ for a.a. $x \in \widetilde{\Omega}$. Changing u to $-u$, one obtains as well that $-u \leq v$, so that $|u(x)| \leq 2K\text{dist}(x, \partial\Omega)$ for a.a. $x \in \widetilde{\Omega}$. For $x \in \Omega$ such that $2\text{dist}(x, \partial\Omega) \geq \rho_1 - \rho$, we have $u(x) \leq \lambda^{-1} \leq 2\lambda^{-1}(\rho_1 - \rho)^{-1}\text{dist}(x, \partial\Omega)$, and the result follows.

It thus remain to establish the claim (A.4.44). One proceeds as in the proof of Theorem A.4.28. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on the set $\{|x - y(x_0)| \leq \rho_0\}$ and $\varphi \equiv 0$ on the set $\{|x - y(x_0)| \geq \rho_1\}$. Since $u \leq \lambda^{-1} \leq v$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \geq \rho_0\}$ and $\varphi u - v = u - v$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \leq \rho_0\}$, we see that $(u - v)^+ = (\varphi u - v)^+$ in $\widetilde{\Omega}$. Let now $(u_n)_{n \geq 0} \subset C_c^\infty(\Omega)$ be such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $H^1(\Omega)$. We see that $(\varphi u_n - v)^+ \underset{n\to\infty}{\longrightarrow} (\varphi u - v)^+ = (u - v)^+$ in $H^1(\widetilde{\Omega})$. Thus, we need only verify that $(\varphi u_n - v)^+ \in H_0^1(\widetilde{\Omega})$. This is immediate, though, because $\varphi u_n = 0$ and $v \geq 0$ on $\partial\widetilde{\Omega}$. □

Corollary A.4.33. *If $N \geq 2$, assume that every $x \in \partial\Omega$ has the exterior cone property. For every $\lambda > 0$ and for every $f \in C_0(\Omega)$, there exists a unique solution $u \in C_0(\Omega)$ of equation*

$$-\triangle u + \lambda u = f, \text{ in } \mathcal{D}'(\Omega).$$

In addition, the following properties hold:

(i) $\triangle u \in C_0(\Omega)$;

(ii) $u \in H^2_{\text{loc}}(\Omega)$;

(iii) $\lambda\|u\|_{L^\infty} \leq \|f\|_{L^\infty}$.

Proof. We proceed in three steps.

Step 1. Uniqueness. Consider two solutions u, v and let $w = v - u$. We have $w \in C_0(\Omega)$, and

$$-\triangle w + \lambda w = 0, \text{ in } \mathcal{D}'(\Omega).$$

Given $\varepsilon > 0$, note that $(w-\varepsilon)^+$ has a compact support in Ω. Let $\Omega' \subset\subset \Omega$ be such that $\mathrm{Supp}((w-\varepsilon)^+) \subset \Omega'$. Since $w \in H^1(\Omega')$ by Proposition A.4.10, it follows from Proposition A.3.28 that $(w-\varepsilon)^+ \in H_0^1(\Omega')$. Applying (A.3.17), we get

$$\int_{\Omega'} \nabla w \cdot \nabla((w-\varepsilon)^+) + \lambda w(w-\varepsilon)^+ = 0.$$

Therefore, by Corollary A.3.17,

$$\int_{\Omega'} |\nabla((w-\varepsilon)^+)|^2 + \lambda|(w-\varepsilon)^+|^2 = -\lambda\varepsilon \int_{\Omega'} (w-\varepsilon)^+ \le 0;$$

and so, $w \le \varepsilon$ in Ω', hence in Ω. Since ε is arbitrary, we get $w \le 0$. Changing w to $-w$, it follows as well that $w \ge 0$. Therefore, $w = 0$, which proves uniqueness.

Step 2. Existence. Consider $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $C_0(\Omega)$, and let $(u_n)_{n\in\mathbb{N}}$ be the corresponding solutions of (A.4.6). It follows from Theorem A.4.11 that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(\Omega)$, and since $u_n \in C_0(\Omega)$ by Theorem A.4.28, u_n has a limit u in $C_0(\Omega)$. It is clear that u solves equation $-\triangle u + \lambda u = f$, in $\mathcal{D}'(\Omega)$.

Step 3. Conclusion. Property (i) follows from the equation. Property (ii) follows from Propositon A.4.10, since $u \in L^2_{\mathrm{loc}}(\Omega)$. Finally, given a solution u, it follows from uniqueness that u is the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ constructed in Step 2. Since $\lambda\|u_n\|_{L^\infty} \le \|f_n\|_{L^\infty}$ by Theorem A.4.11, one obtains (iii) by letting $n \to \infty$. This completes the proof. □

A.4.5. Eigenvalues of the Laplacian. Throughout this section, we assume that Ω is bounded. It follows from Poincaré's inequality that λ_1 defined by (A.4.1) is positive.

Let $f \in L^2(\Omega)$, and let $u \in H_0^1(\Omega)$ be the solution of the equation

$$-\triangle u = f, \text{ in } H^{-1}(\Omega).$$

Let us set $u = Kf$. By Theorem A.4.5, K is bounded $L^2(\Omega) \to H_0^1(\Omega)$. Therefore, by Theorem A.3.42, K is compact $L^2(\Omega) \to L^2(\Omega)$. We claim that K is self adjoint. Indeed, let $f, g \in L^2(\Omega)$ and let $u = Kf$, $v = Kg$. We have

$$(u,g)_{L^2} - (f,v)_{L^2} = -\langle \triangle v, u\rangle_{H^{-1},H_0^1} + \langle \triangle u, v\rangle_{H^{-1},H_0^1} = 0,$$

by (A.3.17). It is clear that $K^{-1}(0) = \{0\}$ and that $(Kf, f)_{L^2} \ge 0$, for every $f \in L^2(\Omega)$. Therefore (see Brezis [17], Theorem VI.11), $L^2(\Omega)$ possesses a Hilbert basis $(\varphi_n)_{n\ge1}$ of eigenvectors of K and the eigenvalues of K consist of a sequence $(\sigma_n)_{n\ge1} \subset (0,\infty)$ converging to 0, as $n \to \infty$. Let us set

$$\lambda_n = \frac{1}{\sigma_n}, \text{ for } n \ge 1.$$

We have $0 < \lambda_1 < \lambda_2 < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$. In addition, $\varphi_n \in H_0^1(\Omega)$ and

$$-\triangle \varphi_n = \lambda_n \varphi_n, \text{ in } H^{-1}(\Omega).$$

Below are some important properties concerning the spectral decomposition of $-\triangle$.

Proposition A.4.34. *Assume Ω is connected. The following properties hold:*

(i) $\varphi_n \in L^\infty(\Omega) \cap C^\infty(\Omega)$, *for every* $n \geq 1$*;*

(ii) λ_1 *is a simple eigenvalue;*

(iii) *one can chose* φ_1 *such that* $\varphi_1 > 0$ *on* Ω*;*

(iv) $\lambda_1 = \lambda_1(\Omega)$, *where* $\lambda_1(\Omega)$ *is defined by (A.4.1).*

Proof. We will prove these properties in a more general framework in Proposition A.4.35 below, except for property $\varphi_n \in C^\infty(\Omega)$. To see this property, write $-\triangle \varphi_n = -\lambda_n \varphi_n$. We have $\lambda_n \varphi_n \in H_0^1(\Omega)$, and it follows from Proposition A.4.10 that $\varphi_n \in H^3_{\text{loc}}(\Omega)$. Therefore, by applying again Proposition A.4.10, we obtain $\varphi_n \in H^5_{\text{loc}}(\Omega)$. An obvious iteration argument shows that $\varphi_n \in H^m_{\text{loc}}(\Omega)$, for every integer m. The result now follows from Sobolev's embedding theorem. □

We generalize the above observations. Consider $a \in L^\infty(\Omega)$ and $\alpha > -\lambda_1(-\triangle + a)$, where $\lambda_1(-\triangle + a)$ is defined by (A.4.13). Let $f \in L^2(\Omega)$, and let $u \in H_0^1(\Omega)$ be the solution of the equation

$$-\triangle u + au + \alpha u = f, \text{ in } H^{-1}(\Omega).$$

Let us set $u = K_a f$. By Theorem A.4.7, K_a is bounded $L^2(\Omega) \to H_0^1(\Omega)$. Therefore, by Theorem A.3.42, K_a is compact $L^2(\Omega) \to L^2(\Omega)$. We claim that K_a is self adjoint. Indeed, let $f, g \in L^2(\Omega)$ and let $u = K_a f$, $v = K_a g$. We have

$$(u, g)_{L^2} - (f, v)_{L^2} = \langle \triangle v, u \rangle_{H^{-1}, H_0^1} - \langle \triangle u, v \rangle_{H^{-1}, H_0^1} = 0,$$

by (A.3.17). It is clear that $K_a^{-1}(0) = \{0\}$ and that $(K_a f, f)_{L^2} \geq 0$, for every $f \in L^2(\Omega)$. Therefore (see Brezis [17], Theorem VI.11), $L^2(\Omega)$ possesses a Hilbert basis $(\varphi_n)_{n \geq 1}$ of eigenvectors of K_a and the eigenvalues of K_a consist of a sequence $(\sigma_n)_{n \geq 1} \subset (0, \infty)$ converging to 0, as $n \to \infty$. Let us set

$$\lambda_n = \frac{1}{\sigma_n} - \alpha, \text{ for } n \geq 1.$$

We have $\lambda_1 < \lambda_2 < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$. In addition, $\varphi_n \in H_0^1(\Omega)$ and

$$-\triangle \varphi_n + a \varphi_n = \lambda_n \varphi_n, \text{ in } H^{-1}(\Omega).$$

Below are some important properties concerning the spectral decomposition of $-\triangle + a$.

Proposition A.4.35. *Assume Ω is connected. If $a \in L^\infty(\Omega)$, then the following properties hold:*

(i) $\varphi_n \in L^\infty(\Omega) \cap C(\Omega)$, *for every* $n \geq 1$*;*

(ii) λ_1 *is a simple eigenvalue;*

(iii) *one can chose* φ_1 *such that* $\varphi_1 > 0$ *on* Ω*;*

(iv) $\lambda_1 = \lambda_1(-\triangle + a)$*, where* $\lambda_1(-\triangle + a)$ *is defined by (A.4.13).*

Proof. (i) Let $f_n = (1 + \lambda_n - a)\varphi_n$. We have

$$-\triangle\varphi_n + \varphi_n = f_n.$$

Note that $\varphi_n \in H_0^1(\Omega)$. If $N = 1$ or $N = 2$, we have in particular $f_n \in L^p(\Omega)$, for all $2 \le p < \infty$. Applying Theorem A.4.14 (i), it follows that $\varphi_n \in L^\infty(\Omega)$. If $N \ge 3$, let j be a nonnegative integer such that

$$\frac{2N}{N-2j} \le \frac{N}{2} < \frac{2N}{N-2(j+1)}.$$

Starting from the property $f_n \in L^2(\Omega)$ and applying iteratively Theorem A.4.14 (iii), it follows that $f_n \in L^{\frac{2N}{N-2(j+1)}}(\Omega)$. Applying now Theorem A.4.14 (i), we get $f_n \in L^\infty(\Omega)$. Continuity follows from the same estimates and by approximating f_n in $H_0^1(\Omega)$ by a sequence $(h_\ell)_{\ell\in\mathbb{N}} \subset C_0^\infty(\Omega)$. Hence (i).

Properties (ii) and (iii) are established in Gilbarg and Trudinger [54], Theorem 8.38.

(iv) It follows from formula (A.4.16) that

$$b(\varphi_1, \varphi_1) = \lambda_1 \int_\Omega |\varphi_1|^2\,dx;$$

and so $\lambda_1 \ge \lambda_1(-\triangle + a)$. Consider a minimizing sequence $(u_j)_{j\in\mathbb{N}}$ of (A.4.12). By coerciveness, $(u_j)_{j\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Therefore (Corollary A.3.10 and Theorem A.3.42) there exists $u \in H_0^1(\Omega)$ and a subsequence that we still denote by u_j such that $u_j \to u$ strongly in $L^2(\Omega)$, weakly in $H_0^1(\Omega)$ and almost everywhere. In particular, we have

$$\int_\Omega |u|^2\,dx = 1,$$

and

$$\liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2\,dx \le \int_\Omega |\nabla u|^2\,dx.$$

One shows easily that also

$$\liminf_{j\to\infty} \int_\Omega a|u_j|^2\,dx \le \int_\Omega a|u|^2\,dx$$

It follows that u achieves the minimum in (A.4.12); and so u solves the Euler equation

$$-\triangle u + au = \lambda_1(-\triangle + a)u.$$

Therefore, $\lambda_1(-\triangle + a)$ is an eigenvalue; and so $\lambda_1(-\triangle + a) \ge \lambda_1$. Hence (iv). This completes the proof. □

Remark A.4.36. Here are some comments concerning the above results.

(i) Connexity of Ω is required only for properties (ii) and (iii) of Proposition A.4.34 and Proposition A.4.35. Without connexity, these two properties may not hold, as shows the following example. Let $\Omega =$

$(0,\pi) \cap (\pi, 2\pi)$. Then $\lambda_1 = 1$, and the corresponding eigenspace is two-dimensional. More precisely, it is the spaces spanned by the two functions φ_1 and $\widetilde{\varphi}_1$ defined by

$$\varphi_1(x) = \begin{cases} \sin x \text{ if } 0 < x < \pi, \\ 0 \text{ if } \pi < x < 2\pi, \end{cases} \qquad \widetilde{\varphi}_1(x) = \begin{cases} 0 \text{ if } 0 < x < \pi, \\ -\sin x \text{ if } \pi < x < 2\pi. \end{cases}$$

In particular, both φ_1 and $\widetilde{\varphi}_1$ vanish on a connected component of Ω.

(ii) The conclusions of Theorem A.4.35 still hold in the case where $a \in L^\infty(\Omega) + L^p(\Omega)$, for some $p > 1$, $p > N/2$. The proof (i) has to be slightly modified. The rest of the proof is unchanged.

A.4.6. Complex-valued solutions. Throughout Section A.4, we considered real valued functions but a similar theory can be developped for complex valued functions, with obvious modifications. In particular, we have the following results.

Lemma A.4.37. *Let ν be defined by*

$$\nu = \inf \left\{ \int_\Omega |\nabla u|^2, u \in H^1_0(\Omega, \mathbb{C}), \int_\Omega |u|^2 = 1 \right\}.$$

Then $\nu = \lambda_1$, where λ_1 is defined by (A.4.1).

Proof. Since $H^1_0(\Omega, \mathbb{R}) \subset H^1_0(\Omega, \mathbb{C})$, it is clear that $\nu \le \lambda_1$. On the other hand, given $u \in H^1_0(\Omega, \mathbb{C})$ such that $\|u\|_{L^2} = 1$, let $v = |u|$. Then, $v \in H^1_0(\Omega, \mathbb{R})$, $\|u\|_{L^2} = 1$ and $|\nabla v| \le |\nabla u|$ almost everywhere (see Section A.3.7); and so, $\nu \ge \lambda_1$. □

Lemma A.4.38. *For every $f \in H^{-1}(\Omega, \mathbb{C})$, there exists a unique solution $u \in H^1_0(\Omega, \mathbb{C})$ of equation*

$$-\triangle u + u = f, \text{ in } H^{-1}(\Omega, \mathbb{C}).$$

Furthermore,

$$\|f\|_{H^{-1}} = \|u\|_{H^1}.$$

In addition,

$$\|u\|_{H^1} \le \|f\|_{L^2},$$

whenever $f \in L^2(\Omega)$.

Proof. The proof is the same as that of Lemma A.4.3. □

Remark A.4.39. The conclusions of Remark A.4.4 also hold in the complex case.

Theorem A.4.40. *Let λ_1 be defined by (A.4.1) and let $\lambda \in \mathbb{C}$. If $\mathrm{Re}\lambda > -\lambda_1$, then the following properties hold:*

(i) *For every $f \in H^{-1}(\Omega, \mathbb{C})$, there exists a unique element $u \in H^1_0(\Omega, \mathbb{C})$ such that*

$$-\triangle u + \lambda u = f, \text{ in } H^{-1}(\Omega, \mathbb{C});$$

(ii) $|||f||| = \|u\|_{H^1(\Omega)}$ *defines an equivalent norm on* $H^{-1}(\Omega, \mathbb{C})$*;*

(iii) $\mathrm{Re}\lambda \|u\|_{H^{-1}} \le \|f\|_{H^{-1}}$*;*

(iv) *if* $f \in L^2(\Omega, \mathbb{C})$*, then* $\triangle u \in L^2(\Omega, \mathbb{C})$*, the equation makes sense in* $L^2(\Omega, \mathbb{C})$ *and* $\mathrm{Re}\lambda \|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}$.

Proof. The proof is the same as that of Theorem A.4.5, by considering the bilinear functional

$$a(u, v) = \mathrm{Re} \left\{ \int_\Omega \{\nabla u \cdot \nabla \overline{v} + \lambda u \overline{v}\} \right\}.$$

Note that

$$a(u, u) = \int_\Omega |\nabla u|^2 + \mathrm{Re}(\lambda) \int_\Omega |u|^2,$$

for all $u \in H_0^1(\Omega, \mathbb{C})$. □

Consider now $a \in L^\sigma(\Omega, \mathbb{C}) + L^\infty(\Omega, \mathbb{C})$ for some σ verifying (A.4.8), and let

$$\nu(a) = \inf \left\{ \int_\Omega |\nabla u|^2 + \int_\Omega \mathrm{Re}(a)|u|^2;\ u \in H_0^1(\Omega, \mathbb{C}),\ \int_\Omega |u|^2 = 1 \right\}. \tag{A.4.45}$$

Note that $\nu(a) = \lambda_1(-\triangle + \mathrm{Re}(a))$, where $\lambda_1(\cdot)$ is defined by (A.4.13). In particular, $\nu(a)$ is finite. We have the following result.

Theorem A.4.41. *Let a be as above, let $\nu(a)$ be defined by (A.4.45) and let $\lambda \in \mathbb{C}$. If* $\mathrm{Re}\lambda > -\lambda_1(-\triangle + a)$*, then for every $f \in H^{-1}(\Omega, \mathbb{C})$, there exists a unique solution $u \in H_0^1(\Omega, \mathbb{C})$ of equation (A.4.15) in $H^{-1}(\Omega, \mathbb{C})$. In addition,*

$$\|u\|_{H^1} \le C \|f\|_{H^{-1}},$$

for some constant C independent of f.

Proof. The proof is the same as that of Theorem A.4.7, by considering the bilinear form

$$b(u, v) = \mathrm{Re} \left\{ \int_\Omega \{\nabla u \cdot \nabla \overline{v} + \lambda u \overline{v} + a u \overline{v}\} \right\}.$$

Note that

$$b(u, u) = \int_\Omega |\nabla u|^2 + \mathrm{Re}(\lambda) \int_\Omega |u|^2 + \int_\Omega \mathrm{Re}(a)|u|^2,$$

for all $u \in H_0^1(\Omega, \mathbb{C})$. □

Theorem A.4.42. *Let a be as in Theorem A.4.41 and let $\lambda \in \mathbb{C}$. Assume that* $\mathrm{Re}(a) \ge 0$ *almost everywhere and that* $\mathrm{Re}\lambda > 0$*. Given $f \in H^{-1}(\Omega, \mathbb{C})$, let $u \in H_0^1(\Omega, \mathbb{C})$ be the solution of (A.4.15) given by Theorem A.4.41. If $f \in L^p(\Omega, \mathbb{C})$ for some $1 \le p \le \infty$, then $u \in L^p(\Omega, \mathbb{C})$ and* $\mathrm{Re}(\lambda)\|u\|_{L^p} \le \|f\|_{L^p}$.

Proof. One proceeds as for Theorems A.4.11 and A.4.12, by using the following identity, which generalizes formula (A.4.24).

$$\mathrm{Re}\left(\nabla u \cdot \nabla (f(|u|)\overline{u})\right) = f(|u|)|\nabla u|^2 + \frac{f'(|u|)}{|u|} |\mathrm{Re}(u \overline{\nabla u})|^2 \text{ almost everywhere.} \tag{A.4.46}$$

Identity (A.4.46) holds for every $u \in H^1_0(\Omega, \mathbb{C})$ and for every smooth function $f : (0, \infty) \to [0, \infty)$ such that $f(s)$ and $sf'(s)$ are bounded on $(0, \infty)$. In particular, if $f, f' \geq 0$, then

$$\mathrm{Re}\left\{ \int_\Omega \nabla u \cdot \nabla (f(|u|)\overline{u}) \right\} \geq 0,$$

for all $u \in H^1_0(\Omega, \mathbb{C})$. If $1 \leq p \leq 2$, one takes $f(s) = (\varepsilon + s^2)^{\frac{p-2}{2}}$, and if $2 \leq p < \infty$, one takes $f(s) = \left(\dfrac{s^2}{1 + \varepsilon s^2} \right)^{\frac{p-2}{2}}$. One concludes as for Theorem A.4.11. □

Remark A.4.43. We summarize below more results of Section A.4 that still hold true for complex-valued solutions.

(i) The conclusions of Theorem A.4.8 and Proposition A.4.10 hold for complex-valued solutions, and for every $\lambda \in \mathbb{C}$. The proofs are essentially the same.

(ii) The conclusion of Theorem A.4.13 holds for complex-valued solutions, provided $\lambda \in \mathbb{R}$ and $a \geq 0$. This is easily seen by considering the real and imaginary parts of the solution.

(iii) The conclusions of Theorem A.4.14 hold for complex-valued solutions, when a and λ are as in Theorem A.4.42. The result is obtained by the same method, and by making use of formula (A.4.46) instead of formula (A.4.24).

(iv) The conclusion of Corollary A.4.17 holds for complex-valued solutions, when $\lambda \in \mathbb{C}$ and $\mathrm{Re}\lambda > 0$. The result is obtained by the same method.

(v) The conclusion of Theorem A.4.28 holds for complex-valued solutions, provided $\lambda \in \mathbb{R}$ and $a \geq 0$. This is obtained by considering the real and imaginary parts of the solution.

A.5. Inequalities. This section is devoted to various useful inequalities.

A.5.1. Jensen's inequality.

Theorem A.5.1. (Jensen's inequality) *Consider a set X endowed with a positive measure μ such that $\int_X d\mu = 1$, and let $F : \mathbb{R} \to \mathbb{R}$ be a convex function. Then for every $f \in L^1(X, d\mu)$ such that $F(f) \in L^1(X, d\mu)$, we have*

$$F\left(\int_X f(x)\, d\mu(x) \right) \leq \int_X F(f(x))\, d\mu(x).$$

Proof. Since F is convex, F has left and right derivatives $F^{\pm}(t)$ at every $t \in \mathbb{R}$; $F^{\pm}$ are nondecreasing functions and $F^-(t) \leq F^+(t)$, for every $t \in \mathbb{R}$. For $s < t$ we have

$$\frac{F(t) - F(s)}{t - s} \leq F^-(t) \leq F^+(t),$$

and for $s > t$ we have

$$\frac{F(s) - F(t)}{s - t} \geq F^+(t);$$

and so

$$F(t) - F(s) \leq F^+(t)(t - s), \text{ for every } s, t \in \mathbb{R}.$$

Take $t = \int_X f(x)\, d\mu(x)$ and $s = f(x)$, for $x \in X$. It follows that

$$F\left(\int_X f(x)\, d\mu(x)\right) \le F(f(x)) + F^+(t)\left(\int_X f(x)\, d\mu(x) - f(x)\right),$$

for almost all $x \in X$. Integrating the above inequality over X yields the desired estimate. □

Corollary A.5.2. *Let Ω be an open subset of $\mathbb{R}^N$, let φ be a nonnegative function of $L^1(\Omega)$ such that $\int_\Omega \varphi(x)\, dx = 1$ and let $F : \mathbb{R} \to \mathbb{R}$ be a convex function. Then,*

$$F\left(\int_\Omega f(x)\varphi(x)\, dx\right) \le \int_\Omega F(f(x))\varphi(x)\, dx,$$

for every $f \in L^1_{\mathrm{loc}}(\Omega)$ such that $f\varphi \in L^1(\Omega)$ and $F(f)\varphi \in L^1(\Omega)$.

Proof. Apply Theorem A.5.1 with $X = \Omega$ and $d\mu(x) = \varphi(x)dx$. □

A.5.2. A differential inequality.

Theorem A.5.3. *Let $0 < T \le \infty$ and let $\varphi \in C^1([0, T))$, $\varphi \ge 0$. If there exist $\alpha, A > 0$ such that*

$$\varphi'(t) + A\varphi(t)^{1+\alpha} \le 0,$$

for all $t \in [0, T)$, then

$$\varphi(t) \le \left(\frac{1}{\alpha A t}\right)^{\frac{1}{\alpha}},$$

for all $t \in (0, T)$.

Proof. Note that $\varphi' \le 0$ on $[0, T)$. Therefore, if $\varphi(t_0) = 0$ for some $t_0 \in [0, T)$, then $\varphi \equiv 0$ on $[t_0, T)$. Therefore, we may assume that there exists $t_0 \in (0, T)$ such that $\varphi > 0$ on $[0, t_0)$. It follows that

$$\left(At - \frac{1}{\alpha\varphi(t)^\alpha}\right)' \le 0,$$

on $[0, t_0)$. Integrating the above inequality, we obtain

$$At - \frac{1}{\alpha\varphi(t)^\alpha} \le -\frac{1}{\alpha\varphi(t_0)^\alpha} \le 0,$$

for all $t \in [0, t_0)$, from which the result follows. □

Remark. It is surprising (and very useful) that the estimate of φ does not depend on $\varphi(0)$.

A.5.3. Gronwall's lemma.

Theorem A.5.4. (Gronwall's lemma) *Let $T > 0$, $A \ge 0$ and let $f \in L^1(0, T)$ be a nonnegative function. Consider a nonnegative function $\varphi \in C([0, T])$ such that*

$$\varphi(t) \le A + \int_0^t f(s)\varphi(s)\, ds,$$

for every $t \in [0,T]$.Then,

$$\varphi(t) \le A \ \exp\left(\int_0^t f(s)\,ds\right),$$

for every $t \in [0,T]$.

Proof. Set $\psi(t) = A + \int_0^t f(s)\varphi(s)\,ds$ and $h(t) = \psi(t)\exp\left(-\int_0^t f(s)\,ds\right)$. $\psi, h \in W^{1,1}$ and (see Section A.2)

$$\begin{aligned} h'(t) &= (\psi'(t) - f(t)\psi(t))\exp\left(-\int_0^t f(s)\,ds\right) \\ &\le (f(t)\varphi(t) - f(t)\psi(t))\exp\left(-\int_0^t f(s)\,ds\right) \le 0. \end{aligned}$$

It follows that $h(t) \le h(0)$, from which the result follows. □

In fact, Theorem A.5.4 is a particular case of the following result.

Proposition A.5.5. *Let $T > 0$, $A \ge 0$ and let $f \in L^1(0,T)$ and $g \in C([0,T])$ be nonnegative functions. Consider a nonnegative function $\varphi \in C([0,T])$ such that*

$$\varphi(t) \le g(t) + \int_0^t f(s)\varphi(s)\,ds,$$

for every $t \in [0,T]$. Then,

$$\varphi(t) \le g(t) + \int_0^t f(s)g(s) \ \exp\left(\int_s^t f(\sigma)\,d\sigma\right) ds,$$

for every $t \in [0,T]$.

Proof. Let $\psi(t) = \int_0^t f(s)\varphi(s)\,ds$, for $t \in [0,T]$. We have $\psi \in W^{1,1}(0,T)$ and

$$\psi'(t) = f(t)\varphi(t) \le f(t)\psi(t) + f(t)g(t), \text{ for almost all } t \in [0,T].$$

Consider now

$$\theta(t) = \exp\left(-\int_0^t f(s)\,ds\right)\psi(t), \text{ for } t \in [0,T].$$

It follows that $\theta \in W^{1,1}(0,T)$ and that

$$\theta'(t) \le f(t)g(t) \ \exp\left(-\int_0^t f(s)\,ds\right) \text{ almost everywhere.}$$

The result follows by integrating the above inequality. □

Remark A.5.6. In particular, if φ verifies the hypotheses of Theorem A.5.4 with $A = 0$, then $\varphi \equiv 0$.

Theorem A.5.4 has many variants, in particular in the the case where the integrand is replaced by: An expression depending on also on t (and possibly with a singular behavior); expressions involving a nonlinear dependence in φ. We describe some of them below.

Proposition A.5.7. *Let $T > 0$, $A \geq 0$, $0 \leq \alpha, \beta \leq 1$ and let f be a nonnegative function with $f \in L^p(0,T)$ for some $p > 1$ such that $p' \max\{\alpha, \beta\} < 1$. Consider a nonnegative function $\varphi \in L^\infty(0,T)$ such that*

$$\varphi(t) \leq A\, t^{-\alpha} + \int_0^t (t-s)^{-\beta} f(s) \varphi(s)\, ds, \text{ for almost all } t \in [0,T].$$

Then there exists C, depending only on T, α, β, p and $\|f\|_{L^p}$ such that

$$\varphi(t) \leq A\, C\, t^{-\alpha},$$

for almost all $t \in [0,T]$.

Proof. Consider $t_0 \in [0,T]$ and $\delta \in (0,1)$ small enough so that

$$t_0^{\frac{1}{p'}-\beta} \|(1-\sigma)^{-\beta} \sigma^{-\alpha}\|_{L^{p'}(0,1)}\, \|f\|_{L^p(0,T)} \leq \frac{1}{2}, \tag{A.5.1}$$

$$T^{\frac{1}{p'}-\beta}\, \|f\|_{L^p(0,T)}\, (1-\delta)^{-\alpha}\, \|\sigma^{-\beta}\|_{L^{p'}(0,\delta T)} \leq \frac{1}{2}. \tag{A.5.2}$$

Let $\psi(t) = \text{ess sup}\{s^\alpha \varphi(s),\ s \in [0,t]\}$. We have

$$t^\alpha \varphi(t) \leq A + t^\alpha \int_0^t (t-s)^{-\beta} s^{-\alpha} f(s) \psi(s)\, ds. \tag{A.5.3}$$

For almost all $0 \leq t \leq t_0$, we have by (A.5.3) and (A.5.1)

$$\begin{aligned} t^\alpha \varphi(t) &\leq A + t^\alpha\, \|(t-s)^{-\beta} s^{-\alpha}\|_{L^{p'}(0,t)}\, \|f\|_{L^p(0,t)}\, \psi(t) \\ &\leq A + t^{\frac{1}{p'}-\beta}\, \|(1-\sigma)^{-\beta}\sigma^{-\alpha}\|_{L^{p'}(0,1)}\, \|f\|_{L^p(0,T)}\, \psi(t) \\ &\leq A + \frac{1}{2}\psi(t). \end{aligned} \tag{A.5.4}$$

For almost all $t_0 \leq t \leq T$, we have by (A.5.3)

$$\begin{aligned} t^\alpha \varphi(t) &\leq A + t^\alpha \int_0^{(1-\delta)t} (t-s)^{-\beta} s^{-\alpha} f(s) \psi(s)\, ds + t^\alpha\, \psi(t) \int_{(1-\delta)t}^t (t-s)^{-\beta} s^{-\alpha} f(s)\, ds \\ &\leq A + I_1 + I_2 \end{aligned} \tag{A.5.5}$$

On $(0,(1-\delta)t)$ we have $(t-s)^{-\beta} \leq (\delta t)^{-\beta} \leq (\delta t_0)^{-\beta}$; and so

$$I_1 \leq T^\alpha\, (\delta t_0)^{-\beta} \int_0^t s^{-\alpha} f(s) \psi(s)\, ds. \tag{A.5.6}$$

On $((1-\delta)t), t)$ we have $s^{-\alpha} \leq ((1-\delta)t)^{-\alpha}$; and so, by (A.5.2)

$$\begin{aligned} I_2 &\leq (1-\delta)^{-\alpha}\, \|(t-s)^{-\beta}\|_{L^{p'}((1-\delta)t,t)}\, \|f\|_{L^p(0,t)}\, \psi(t) \\ &\leq (1-\delta)^{-\alpha}\, t^{\frac{1}{p'}-\beta}\, \|\sigma^{-\beta}\|_{L^{p'}(0,\delta T)}\, \|f\|_{L^p(0,T)}\, \psi(t) \\ &\leq \frac{1}{2}\psi(t). \end{aligned} \tag{A.5.7}$$

It follows from (A.5.4), (A.5.5), (A.5.6) and (A.5.7) that

$$t^\alpha\, \varphi(t) \leq A + (\delta t_0)^{-\beta}\, T^\alpha \int_0^t s^{-\alpha} f(s) \psi(s)\, ds + \frac{1}{2}\psi(t);$$

and so,

$$\psi(t) \le A + (\delta t_0)^{-\beta}\, T^{\alpha} \int_0^t s^{-\alpha} f(s)\psi(s)\, ds + \frac{1}{2}\psi(t),$$

and we conclude with Theorem A.5.4, since $s^{-\alpha} f(s) \in L^1(0,T)$ by our assumption on f. □

Proposition A.5.8. *Let $F \in C([0,\infty),\mathbb{R})$ and assume that F is a nondecreasing function such that $F(0) = 0$ and $F(t) > 0$ for $t > 0$. Define the increasing function $H \in C(0,\infty)$ by*

$$H(t) = \int_1^t \frac{ds}{F(s)}.$$

Let $T > 0$, $A \ge 0$ and let $\varphi \in C([0,T])$ be a nonnegative function such that

$$\varphi(t) \le A + \int_0^t F(\varphi(s))\, ds, \text{ for all } t \in [0,T].$$

Then

(i) *if $H(t) \to \infty$ as $t \to \infty$, then $\varphi(t) \le H^{-1}(H(1+A)+t)$ for all $t \in [0,T]$;*

(ii) *if $A = 0$ and $H(t) \to -\infty$ as $t \to 0$, then $\varphi \equiv 0$.*

Proof. Let $\psi(t) = A + \int_0^t F(\varphi(s))\, ds$. We have $\psi \in W^{1,1}(0,T)$ and $\psi' \le F(\psi)$, $\psi(0) = A$.

Assume first that $A = 0$ and $H(t) \to -\infty$ as $t \to 0$. If $\psi \equiv 0$, then (ii) holds. Otherwise, there exists $t \in [0,T]$ such that $\psi(t) > 0$. Let t_0 be the infimum of such t's. Without loss of generality we may assume that $t_0 = 0$, and since ψ is nondecreasing, we have $\psi(t) > 0$ for $t > 0$. Since $(H(\psi(t))' \le 1$, we have for every $0 < s \le t < T$

$$H(\psi(t)) \le H(\psi(s)) + (t-s) \le H(\psi(s)) + t.$$

Letting $s \to 0$, we obtain $H(\psi(t)) = -\infty$, which is absurd. Hence (ii).

Assume now that $H(t) \to \infty$ as $t \to \infty$. Without loss of generality we may assume that $A > 0$. Therefore $\psi(t) > 0$ on $[0,T]$ and $(H(\psi(t))' \le 1$. It follows that

$$H(\psi(t)) \le H(A) + t.$$

Therefore, $\psi(t) \le H^{-1}(H(A)+t) \le H^{-1}(H(1+A)+t)$. This proves (i). □

Remark A.5.9. If $F(t) \le C(1 + |\mathrm{Log}(t)|)$ for all $t \ge 0$, then $H(t) \to +\infty$ as $t \to +\infty$ and $H(t) \to -\infty$ as $t \to 0$. Therefore we can apply both (i) and (ii).

Theorem A.5.10. *let $T > 0$ and $f, g \in C([0,T))$ with $f, g \ge 0$. Suppose there exist $0 \le \alpha < 1$, $1 < p < \dfrac{1}{\alpha}$ and $C \ge 0$ such that*

$$f(t+s) \le s^{-\alpha} g(t) + C + C\int_0^s (s-\sigma)^{-\alpha} f(t+\sigma)^p\, d\sigma,$$

for all $0 < t < t+s < T$. If

$$\limsup_{t \uparrow T} f(t) = +\infty,$$

then

$$\liminf_{t\uparrow T}(T-t)^{\gamma}g(t) > 0,$$

with $\gamma = \dfrac{1-\alpha p}{p-1}$.

Proof. We consider two cases.

Case 1. $\alpha > 0$. Without loss of generality, we may assume that $g(t) \geq 1$ for all $t \in [0,T)$ (otherwise, replace $g(t)$ by $g(t)+1$). Setting $\theta(s) = s^{\alpha} f(t+s)$, we deduce

$$\theta(s) \leq g(t) + Cs^{\alpha} + Cs^{\alpha}\int_0^s (s-\sigma)^{-\alpha}\sigma^{-p\alpha}\theta(\sigma)^p \, d\sigma.$$

Set now $\Theta(s) = \sup\limits_{0\leq\sigma\leq s} \theta(\sigma)$. We have

$$\theta(s) \leq g(t) + Cs^{\alpha} + Cs^{\alpha}\Theta(s)^p \int_0^s (s-\sigma)^{-\alpha}\sigma^{-p\alpha} \, d\sigma.$$

Since

$$s^{\alpha}\int_0^s (s-\sigma)^{-\alpha}\sigma^{-p\alpha} d\sigma = s^{1-p\alpha}\int_0^1 \frac{d\tau}{(1-\tau)^{\alpha}\tau^{p\alpha}} = as^{1-p\alpha},$$

for some constant a, we deduce that

$$\theta(s) \leq g(t) + Cs^{\alpha} + Cs^{1-p\alpha}\Theta(s)^p.$$

and so,

$$\Theta(s) \leq g(t) + Cs^{\alpha} + Cs^{1-p\alpha}\Theta(s)^p.$$

We have $\Theta(0) = 0$ and $\Theta(T-s) \underset{s\uparrow t}{\longrightarrow} \infty$. Therefore, there exists $\tau \in (0, T-t)$ such that $\Theta(\tau) = 2g(t)$. By applying the above inequality with $s = \tau$, we find

$$g(t) \leq C\tau^{\alpha} + C\tau^{1-p\alpha}g(t)^p \leq C(T-t)^{\alpha} + C(T-t)^{1-p\alpha}g(t)^p.$$

Note that $C(T-t)^{\alpha} \leq \dfrac{1}{2}g(t)$ for $T-t$ small enough; and so

$$\frac{1}{2}g(t) \leq C(T-t)^{1-p\alpha}g(t)^p,$$

which yields the desired estimate.

Case 2. $\alpha = 0$. In this case, we have

$$f(t+s) \leq g(t) + C + C\int_0^s f(t+\sigma)^p \, d\sigma.$$

Set

$$G(s) = g(t) + C + C\int_0^s f(t+\sigma)^p \, d\sigma,$$

so that

$$G'(s) = Cf(t+s)^p \leq CG(s)^p.$$

Integrating this inequality, we obtain

$$\frac{G^{1-p}(s) - G^{1-p}(0)}{1-p} \le Cs,$$

for every $0 < s < T - t$. Now we let $s \uparrow T - t$. Note that $G(s) \to +\infty$ as $s \uparrow T - t$. Indeed, G is nondecreasing and $\limsup_{s \uparrow T-t} G(s) \ge \limsup_{\tau \uparrow T} f(\tau) = +\infty$. Thus we obtain

$$G^{1-p}(0) \le C(p-1)(T-t).$$

On the other hand, $G(0) = g(t) + C$. This yields the desired conclusion. □

A.5.4. Interpolation inequalities. We begin with the well known Riesz-Thorin interpolation theorem.

Theorem A.5.11. *Let Ω be an open subset of $\mathbb{R}^N$, let $1 \le p_0, p_1, q_0, q_1 \le \infty$, and let $T : L^{p_0}(\Omega) \cap L^{p_1}(\Omega) \to L^{q_0}(\Omega) \cap L^{q_1}(\Omega)$ be a linear mapping. If there exist constants M_0, M_1 such that $\|Tu\|_{L^{q_j}} \le M_j \|u\|_{L^{p_j}}$ for $j = 0, 1$ and all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, then*

$$\|Tu\|_{L^{q_\theta}} \le M_0^{1-\theta} M_1^{\theta} \|u\|_{L^{p_\theta}},$$

for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ and all $0 < \theta < 1$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. See for example Bergh and Löfström [13], Theorem 5.1.1, p. 106. Note that the theorem is stated for L^p spaces of complex valued functions. However, if one considers real valued spaces, then one can define $\overline{T} : L^{p_i}(\Omega, \mathbb{C}) \to L^{q_i}(\Omega, \mathbb{C})$ by $\overline{T}u = Tf$ if $u = f + ig$. It is clear that $\|\overline{T}\|_{\mathcal{L}(L^{p_\theta}, L^{q_\theta})} \le C(p_\theta, q_\theta) \|T\|_{\mathcal{L}(L^{p_\theta}, L^{q_\theta})}$.
□

Concerning vector valued L^p spaces, we have the following results of Bergh and Löfström.

Theorem A.5.12. *Let Ω be an open subset of $\mathbb{R}^N$, let $1 \le p_0, p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$, and let I be an open interval of $\mathbb{R}$. let X be a Banach space and let $T : L^{p_0}(I, L^{q_0}(\Omega)) \cap L^{p_1}(I, L^{q_1}(\Omega)) \to X$ be a linear mapping. If there exist constants C_0, C_1 such that $\|Tf\|_X \le C_j \|f\|_{L^{p_j}(I, L^{q_j})}$ for all $f \in L^{p_0}(I, L^{q_0}(\Omega)) \cap L^{p_1}(I, L^{q_1}(\Omega))$ and $j = 1, 2$, then*

$$\|Tf\|_X \le C_0^{1-\theta} C_1^{\theta} \|f\|_{L^{p_\theta}(I, L^{q_\theta})},$$

for every $f \in L^{p_0}(I, L^{q_0}(\Omega)) \cap L^{p_1}(I, L^{q_1}(\Omega))$ and for every $0 < \theta < 1$, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. It follows from Bergh and Löfstrom [13], Theorem 4.4.1, p. 90 that T is continuous

$$(L^{p_0}(I, L^{q_0}(\Omega)), L^{p_1}(I, L^{q_1}(\Omega)))_{[\theta]} \to (X, X)_{[\theta]} = X,$$

with norm $C_0^{1-\theta} C_1^{\theta}$. Next, It follows from Bergh and Löfstrom [13], Theorems 5.1.1 and 5.1.2, pp. 106 and 107 that

$$(L^{p_0}(I, L^{q_0}(\Omega)), L^{p_1}(I, L^{q_1}(\Omega)))_{[\theta]} = L^{p_\theta}(I, (L^{q_0}(\Omega), L^{q_1}(\Omega))_{[\theta]}) = L^{p_\theta}(I, L^{q_\theta}(\Omega)).$$

Hence the result. □

Theorem A.5.13. *Let Ω be an open subset of $\mathbb{R}^N$, let $1 \le p_0^0, p_1^0, p_0^1, p_1^1 < \infty$ and $1 \le q_0^0, q_1^0, q_0^1, q_1^1 \le \infty$, and let I be an open interval of $\mathbb{R}$. let $T : L^{p_0^0}(I, L^{q_0^0}(\Omega)) \cap L^{p_1^0}(I, L^{q_1^0}(\Omega)) \to L^{p_0^1}(I, L^{q_0^1}(\Omega)) \cap L^{p_1^1}(I, L^{q_1^1}(\Omega))$ be a linear mapping. If there exist constants C_0, C_1 such that $\|Tf\|_{L^{p_j^1}(I,L^{q_j^1})} \le C_j \|f\|_{L^{p_j^0}(I,L^{q_j^0})}$ for all $f \in L^{p_0^0}(I, L^{q_0^0}(\Omega)) \cap L^{p_1^0}(I, L^{q_1^0}(\Omega))$ and $j = 1, 2$, then*

$$\|Tf\|_{L^{p_\theta^1}(I,L^{q_\theta^1})} \le C_0^{1-\theta} C_1^\theta \|f\|_{L^{p_\theta^0}(I,L^{q_\theta^0})},$$

for every $f \in L^{p_0^0}(I, L^{q_0^0}(\Omega)) \cap L^{p_1^0}(I, L^{q_1^0}(\Omega))$ and for every $0 < \theta < 1$, where $\dfrac{1}{p_\theta^j} = \dfrac{1-\theta}{p_0^j} + \dfrac{\theta}{p_1^j}$ and $\dfrac{1}{q_\theta^j} = \dfrac{1-\theta}{q_0^j} + \dfrac{\theta}{q_1^j}$.

Proof. The proof is the same as that of Theorem A.5.12. □

Remark A.5.14. Theorems A.5.12 and A.5.13 are valid with $p_j = \infty$ or $p_j^\ell = \infty$, provided one replaces the space $L^\infty(I, X)$ by the closure in $L^\infty(I, X)$ of the space spanned by the functions of the form 1_E where E is a measurable subset of I (see Bergh and Löfström [13]).

A.5.5. Convolution estimates. We begin with the well known Young's interpolation inequality.

Theorem A.5.15. *Let N be a positive integer and let $1 \le p, q, r \le \infty$ be such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then $f \star g \in L^r(\mathbb{R}^N)$ and

$$\|f \star g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q},$$

where

$$f \star g(x) = \int_{\mathbb{R}^N} f(y) g(x-y)\, dx,$$

for almost all $x \in \mathbb{R}^N$.

Note that one cannot apply Young's inequality to functions of the type $f(x) = |x|^{-\alpha}$, $\alpha > 0$ since this function does not belong to any space $L^p(\mathbb{R}^N)$. The following Riesz potentials inequality extends some Young's inequalities in this case.

Theorem A.5.16. *Let $0 < \alpha < N$. Given $u \in C_c(\mathbb{R}^N)$, define $I(u) \in C(\mathbb{R}^N)$ by*

$$I(u)(x) = \int_{\mathbb{R}^N} |x-y|^{-N+\alpha} u(y)\, dy = \left(|\cdot|^{-N+\alpha} \star u\right)(x).$$

Then $I(u) \in L^q(\mathbb{R}^N)$ for every $1 < q < \infty$. Moreover, for every $1 < p < q < \infty$ such that $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{N}$, there exists a constant $C(p, q)$ such that

$$\|I(u)\|_{L^q(\mathbb{R}^N)} \le C(p, q) \|u\|_{L^p(\mathbb{R}^N)},$$

for all $u \in C_c(\mathbb{R}^N)$.

The following corollary is useful for estimating solutions of nonhomogeneous evolution equations.

Corollary A.5.17. *Let* I *be an interval of* $\mathbb{R}$ *and let* X *be a Banach space. Let* $0 < \sigma < 1$ *and* $t_0 \in I$*, and given* $f \in C_c(I, X)$*, define* $\mathcal{I}_f \in C(I, X)$ *by*

$$\mathcal{I}_f(t) = \int_{t_0}^{t} |t-s|^{-\sigma} f(s)\, ds.$$

Then $\mathcal{I}_f \in L^q(I, X)$ *for every* $1 < q < \infty$*. Moreover, for every* $1 < p < q < \infty$ *such that* $\dfrac{1}{q} = \dfrac{1}{p} + \sigma - 1$*, there exists a constant* $C(p, q)$ *such that*

$$\|\mathcal{I}_f\|_{L^q(I,X)} \le C(p, q) \|f\|_{L^p(I,X)},$$

for all $f \in C_c(I, X)$.

Proof. For f as above, define $\widetilde{f} \in C_c(\mathbb{R}, X)$ by

$$\widetilde{f}(t) = \begin{cases} f(t), & \text{if } t \in I; \\ 0, & \text{if } t \notin I; \end{cases}$$

and $g \in C_c(\mathbb{R})$ by $g(t) = \|\widetilde{f}(t)\|$. We have

$$\|\mathcal{I}_f(t)\|_X \le \int_{-\infty}^{+\infty} |t-s|^{-\sigma} \|\widetilde{f}(s)\|\, ds = I(g)(t),$$

where $I(g)$ is defined in Theorem A.5.16. The result now follows by applying Theorem A.5.16 with $N = 1$ and $\alpha = 1 - \sigma$. □

Remark A.5.18. Note that the constant $C(p, q)$ in Corollary A.5.17 depends only on p, q. In particular, it is independent of I and $t_0 \in I$.

A.5.6. Kato's inequality. Even if $u : \mathbb{R}^N \to \mathbb{R}$ is smooth, $|u|$ may have singular second order derivatives. In particular, one cannot compare $\triangle u$ and $\triangle |u|$ as functions. However, one can prove an inequality in the sense of distributions. This is the object of the following result.

Theorem A.5.19. (Kato's inequality) *Let* Ω *be an open subset of* $\mathbb{R}^N$ *and define*

$$\operatorname{sign} x = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x = 0, \\ -1 \text{ if } x < 0. \end{cases}$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$ *is such that* $\triangle u \in L^1_{\mathrm{loc}}(\Omega)$*, then* $\triangle |u| \ge (\triangle u) \operatorname{sign} u$ *in* $\mathcal{D}'(\Omega)$.

Proof. Since the property is local, we may assume that $\Omega = \mathbb{R}^N$. The proof proceeds in three steps.

Step 1. If $j \in C^2(\mathbb{R}, \mathbb{R})$ is convex and if $u \in C^2(\mathbb{R}^N)$, then $\triangle j(u) \ge j'(u) \triangle u$. Indeed, an elementary calculation shows that $\triangle j(u) = j''(u) |\nabla u|^2 + j'(u) \triangle u \ge j'(u) \triangle u$.

Step 2. If $j \in C^2(\mathbb{R},\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ is convex and if $u \in L^1_{\rm loc}(\mathbb{R}^N)$ is such that $\triangle u \in L^1_{\rm loc}(\mathbb{R}^N)$, then $\triangle j(u) \ge j'(u)\triangle u$ in $\mathcal{D}'(\mathbb{R}^N)$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers and set $u_n = \rho_n \star u$. It follows that $u_n \in C^2(\mathbb{R}^N)$; and so, by Step 1, $\triangle j(u_n) \ge j'(u_n)\triangle u_n$. Consider now $0 < R < \infty$ and set $B = \{x \in \mathbb{R}^N;\, |x| < R\}$. It follows from Brezis [17], Théorème IV.22, p. 71 that $u_n \to u$ and $\triangle u_n = \rho_n \star \triangle u \to \triangle u$ in $L^1(B)$. We also may assume, by possibly substracting a subsequence that $u_n \to u$ and $\triangle u_n \to \triangle u$ almost everywhere and that there exists $f \in L^1(B)$ such that $|u_n| + |\triangle u_n| \le f$ almost everywhere. It follows that $j(u_n) \to j(u)$ in $L^1(B)$ and that $j'(u_n)\triangle u_n \to j'(u)\triangle u$ in $L^1(B)$. In particular, it follows that $\triangle j(u_n) - j'(u_n)\triangle u_n \to \triangle j(u) - j'(u)\triangle u$ in $\mathcal{D}'(B)$. Since R is arbitrary, it follows that $\triangle j(u_n) - j'(u_n)\triangle u_n \to \triangle j(u) - j'(u)\triangle u$ in $\mathcal{D}'(\mathbb{R}^N)$. Hence the result.

Step 3. Conclusion. Let u be as in the statement of the theorem and, given $\varepsilon > 0$, set $j_\varepsilon(x) = (\varepsilon^2 + t^2)^{1/2}$. It follows from Step 2 that $\triangle j_\varepsilon(u) - j'_\varepsilon(u)\triangle u \ge 0$ in $\mathcal{D}'(\mathbb{R}^N)$. The result follows, since $j_\varepsilon(x) \to |x|$ and $j'_\varepsilon(x) \to \operatorname{sign} x$ as $\varepsilon \downarrow 0$ (see the proof of Step 2). □

Kato's inequality has a parabolic version which we describe below.

Theorem A.5.20. *Let $T > 0$ and let Ω be an open subset of $\mathbb{R}^N$. If $u(t,x) \in L^1_{\rm loc}((0,T)\times\Omega)$ is such that $u_t \in L^1_{\rm loc}((0,T)\times\Omega)$ and $\triangle u \in L^1_{\rm loc}((0,T)\times\Omega)$, then $\frac{\partial}{\partial t}|u| - \triangle|u| \le \left(\frac{\partial u}{\partial t} - \triangle u\right)\operatorname{sign} u$ in $\mathcal{D}'((0,T)\times\Omega)$, where* sign *is as defined in Theorem A.5.19.*

Proof. The proof is the same as the proof of Theorem A.5.19. □

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