PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS AND HAMILTONIAN SYSTEMS

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Abstract. We consider the nonlinear vibrating string equation $u_{tt} - u_{xx} + h(u) = 0$ under Dirichlet boundary conditions on a finite interval. We assume that h is nondecreasing, $h(0) = 0$ and $\lim_{|u| \to \infty} [h(u)/u] = 0$. We prove that for T sufficiently large, there is a nontrivial T-periodic solution. A similar result holds for Hamiltonian systems.

0. Introduction. Consider the following nonlinear wave equation:

$$u_{tt} - u_{xx} + h(u) = 0 \qquad 0 < x < \pi, t \in \mathbf{R}. \tag{1}$$

under the boundary conditions:

$$u(0, t) = u(\pi, t) = 0, \tag{2}$$

where $h:\mathbf{R} \to \mathbf{R}$ is a continuous nondecreasing function such that $h(0) = 0$. We assume:

$$\lim_{|u| \to \infty} \frac{h(u)}{u} = 0 \tag{3}$$

There exists a constant R such that $h(u) \neq 0$ for $|u| \geq R$. (4)

We seek nontrivial solutions of (1), (2) which are T-periodic (in t). By "nontrivial" we mean that $h(u(x, t)) \neq 0$ on a set (x, t) of positive measure; in particular, $u(x, t) \neq 0$ on that set.

In Section 1 we prove the following

THEOREM 1. *There exists $T_0 > 0$ such that for every $T \geq T_0$, with T/π rational, Problem* (1), (2) *admits a nontrivial T-periodic* (*weak*) *solution $u \in L^\infty$.*

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By a result of [4], weak solutions are in fact smooth if h is smooth and strictly increasing.

The existence of nontrivial solutions for (1), (2) has been considered by several authors under assumptions which differ from ours (see [1, 4, 5, 7, 8]).

In Section 2 we discuss a comparable result for Hamiltonian systems.

Our investigation has been stimulated by the results of [3] (Section 4). Our technique relies on a duality device used in [6] for Hamiltonian systems and subsequently in [5] for the wave equation.

We thank P. Rabinowitz for helpful discussions.

Proof of Theorem 1. The proof is divided into five steps.

Step 1 Generalities about $Au = u_{tt} - u_{xx}$.
Step 2 Determination of T_0.
Step 3 Existence of a nontrivial solution for

$$Au + h(u) + \epsilon u = 0 \qquad (\epsilon > 0 \text{ small}).$$

Step 4 Estimates.
Step 5 Passage to the limit as $\epsilon \to 0$.

Step 1 *Generalities about* $Au = u_{tt} - u_{xx}$

Since $T/\pi \in \mathbf{Q}$ we may write $T = 2\pi b/a$ where a and b are coprime integers. Let $H = L^2(\Omega)$ with $\Omega = (0, \pi) \times (0, T)$. In H we consider the operator

$$Au = u_{tt} - u_{xx}$$

acting on functions satisfying (2) and which are T-periodic in t.

We summarize some of the main properties of A which we shall use (see e.g. [4] and the references in [4]):

i) $A^* = A$

ii) $N(A)$ consists of functions of the form

$$N(A) = \left\{p(t+x) - p(t-x), \text{ where } p \text{ has period } \frac{2\pi}{a} = \frac{T}{b} \text{ and } \int_0^{T/b} p = 0\right\}$$

iii) $R(A)$ is closed and $R(A) = N(A)^{\perp}$; whenever $u \in H$ we shall write $u = u_1 + u_2$ with $u_1 \in R(A)$, $u_2 \in N(A)$.

iv) The eigenvalues of A are $j^2 - [(2\pi/T)k]^2$, $j = 1, 2, 3, \ldots$ and $k = 0, 1, 2, \ldots$. The corresponding eigenfunctions are

$$\sin jx \sin\left(\frac{2\pi}{T}kt\right) \quad \text{and} \quad \sin jx \cos\left(\frac{2\pi}{T}kt\right).$$

We denote by $\lambda_{-1}(T)$ the first negative eigenvalue. Note that $\lambda_{-1}(T) \to 0$ as $T \to \infty$. Indeed, let $\mu = j^2 - [(2\pi/T)k]^2$ with $j = 1$ and $k = [T/2\pi] + 1$. We have $1 - [1 + (2\pi/T)]^2 \leq \mu < 0$ and so

$$|\lambda_{-1}(T)| \leq |\mu| \leq \frac{4\pi}{T}\left(1 + \frac{\pi}{T}\right).$$

v) Given $f \in R(A)$, there exists a unique $u \in R(A) \cap C(\overline{\Omega})$ such that $Au = f$.

We set

$$u = Kf = (A^{-1}f).$$

We have

$$||Kf||_{L^\infty} \leq C||f||_{L^1} \ \forall f \in R(A),$$

$$||Kf||_{H^1} \leq C||f||_{L^2} \ \forall f \in R(A).$$

K is a compact self-adjoint operator in $R(A)$.

Step 2 *Determination of* T_0

We set

$$H(u) = \int_0^u h(s)ds$$

$$H_\epsilon(u) = H(u) + \frac{\epsilon}{2}|u|^2 \qquad \epsilon > 0$$

so that H_ϵ is convex and we denote by H_ϵ^* its conjugate convex function (H_ϵ^* is C^1 and $(H_\epsilon^*)'$ is the inverse function of $h(u) + \epsilon u$). We shall use the same "duality" approach as in [5].

On $R(A)$ we define:

$$F_\epsilon(v) = \frac{1}{2}\int_\Omega Kv\cdot v + \int_\Omega H_\epsilon^*(v).$$

The following lemma plays a crucial role:

LEMMA 1. *There exists $T_0 > 0$ such that if $T > T_0$ and T/π is rational, then*

$$\operatorname*{Inf}_{R(A)} F_\epsilon \le -1 \qquad \forall \epsilon > 0.$$

Proof of Lemma 1. By (4) we may assume that

$$H(u) \ge \rho|u| - C \qquad \forall u$$

for some constants $\rho > 0$ and C. Hence

$$H_\epsilon(u) \ge \rho|u| - C \qquad \forall u$$

and

$$H_\epsilon^*(v) \le C \qquad \text{for } |v| \le \rho.$$

As a testing function for evaluating $\operatorname{Inf}_{R(A)} F_\epsilon$ we choose an eigenfunction of A corresponding to the eigenvalue $\lambda_{-1}(T)$. More precisely, let $v = \rho \sin jx \sin[(2\pi/T)kt]$ with $j^2 - [(2\pi/T)k]^2 = \lambda_{-1}(T)$. Thus,

$$F_\epsilon(v) \le -\frac{1}{2|\lambda_{-1}(T)|}\int_\Omega |v|^2 + C|\Omega|$$

$$= -\frac{\pi T\rho^2}{8|\lambda_{-1}(T)|} + C\pi T \le -1$$

provided $T \ge T_0$ for some large T_0.

In what follows we *fix* $T \ge T_0$.

Step 3 *Existence of a nontrivial solution for*

$$Au + h(u) + \epsilon u = 0 \qquad (\epsilon > 0 \text{ small})$$

We start with

LEMMA 2. *There exists constants $\alpha > 0$ and C (independent of ϵ) such that*

$$F_\epsilon(v) \geq \alpha \|v\|_{L^2}^2 - C \qquad \forall v \in R(A),\ \forall \epsilon \leq \tfrac{1}{4}|\lambda_{-1}|.$$

Proof of Lemma 2. Let $\delta = \frac{1}{4}|\lambda_{-1}|$. By (3) there is a constant C such that

$$H(u) \leq \frac{\delta}{2}|u|^2 + C \qquad \forall u.$$

Thus

$$H_\epsilon(u) \leq \frac{1}{4}|\lambda_{-1}||u|^2 + C \qquad \forall u$$

and

$$H_\epsilon^*(v) \geq \frac{1}{|\lambda_{-1}|}|v|^2 - C \qquad \forall v.$$

On the other hand,

$$\int_\Omega Kv\cdot v \geq -\frac{1}{|\lambda_{-1}|}\int_\Omega |v|^2 \qquad \forall v \in R(A)$$

and the conclusion follows.

It is now clear that for $\epsilon \leq \frac{1}{4}|\lambda_{-1}|$, $\mathrm{Min}_{R(A)} F_\epsilon$ is achieved at some v_ϵ. Indeed if v_n is a minimizing sequence, then by Lemma 2, v_n is bounded in L^2 and we may assume that v_n converges weakly to some v in L^2. Then $\lim \int Kv_n\cdot v_n = \int Kv\cdot v$ and $\underline{\lim} \int H_\epsilon^*(v_n) \geq \int H_\epsilon^*(v)$ (by the convexity of H_ϵ^*).

Clearly, we have

$$Kv_\epsilon + (H_\epsilon^*)'(v_\epsilon) = \chi \in N(A).$$

Set

$$u_\epsilon = (H_\epsilon^*)'(v_\epsilon)$$

so that $v_\epsilon = h(u_\epsilon) + \epsilon u_\epsilon$ and $Au_\epsilon + h(u_\epsilon) + \epsilon u_\epsilon = 0$. Note that $v_\epsilon \not\equiv 0$ since $F_\epsilon(v_\epsilon) \leq -1$.

Step 4. *Estimates*

In what follows we denote by C various constants independent of ϵ ($\epsilon \leq 1/4|\lambda_{-1}|$). By Lemma 2 we already know that $||v_\epsilon||_{L^2} \leq C$. Thus $||Au_\epsilon||_{L^2} \leq C$ and so $||u_{1\epsilon}||_{L^\infty} \leq C$.

We shall now prove

LEMMA 3. $||u_\epsilon||_{L^\infty} \leq C$.

Proof of Lemma 3. We follow the same technique as in [2]. We first prove that $||u_\epsilon||_{L^1} \leq C$. Indeed

$$h(u)\cdot u \geq H(u) \geq \rho|u| - C \qquad \forall u.$$

Therefore

$$\rho \int_\Omega |u_\epsilon| - C|\Omega| \leq \int h(u_\epsilon)\cdot u_\epsilon = \int (-Au_\epsilon - \epsilon u_\epsilon)u_\epsilon \leq C.$$

Next we show that $||u_{2\epsilon}||_{L^\infty} \leq C$. We write

$$u_{2\epsilon}(x, t) = p(t + x) - p(t - x)$$

where p has period T/b and $\int_0^{T/b} p = 0$. (p depends on ϵ, but we omit the subscript ϵ in order to simplify the notations.) Since $||u_{2\epsilon}||_{L^1} \leq C$ we have $||p||_{L^1} \leq C$. On the other hand, we recall that given $\psi \in L^2(\Omega)$, then $\psi \in N(A)^\perp$ if and only if

$$\sum_{k=0}^{b-1} \int_0^\pi \left[\psi\left(x, t + \frac{kT}{b} - x\right) - \psi\left(x, t + \frac{kT}{b} + x\right)\right] dx = 0 \text{ for a.e. } t$$

(indeed $\psi \in N(A)^\perp$ iff $\int_\Omega \psi(x, t)[q(t + x) - q(t - x)] = 0$ for every function q periodic with period T/b).

Since $g(u_\epsilon) + \epsilon u_{2\epsilon} \in N(A)^\perp$ it follows that

$$2\epsilon b\pi p(t) + \sum_{k=0}^{b-1} \int_0^\pi \left[g\left(u_\epsilon\left(x, t + \frac{kT}{b} - x\right)\right) - g\left(u_\epsilon\left(x, t + \frac{kT}{b} + x\right)\right)\right] dx = 0 \text{ for a.e. } t.$$

But

$$u_\epsilon\left(x, t + \frac{kT}{b} - x\right) \geq -C + p(t) - p(t - 2x)$$

and

$$u_\epsilon\left(x, t + \frac{kT}{b} + x\right) \leq C + p(t + 2x) - p(t)$$

(because $||u_{1\epsilon}||_{L^\infty} \leq C$ and p has period T/b). Therefore for a.e. t.

$$2\epsilon p(t) + \frac{1}{\pi}\int_0^\pi [g(-C + p(t) - p(t - 2x)) - g(C + p(t + 2x) - p(t))]dx \leq 0.$$

We conclude as in [2] that $||p||_{L^\infty} \leq C$.

Step 5. *Passage to the limit as* $\epsilon \to 0$.

Using the estimates of Step 4 we may extract a subsequence $\epsilon_n \to 0$ such that

$$\begin{aligned} u_{\epsilon_n} &\rightharpoonup u && \text{in } w^*L^\infty \\ h(u_{\epsilon_n}) &\rightharpoonup v && \text{in } w^*L^\infty, \\ Au_{\epsilon_n} &\rightharpoonup Au && \text{in } w^*L^\infty. \end{aligned}$$

Applying Minty's device as in [2] (or in [4]) we find that $h(u) = v$ and so $Au + h(u) = 0$.

Finally we prove that u is a nontrivial solution. Indeed we have

$$F_\epsilon(v_\epsilon) = \frac{1}{2}\int Kv_\epsilon \cdot v_\epsilon + \int H_\epsilon^*(v_\epsilon) \leq -1$$

and in particular

$$\frac{1}{2}\int Kv_\epsilon \cdot v_\epsilon \leq -1.$$

On the other hand, $v_{\epsilon_n} = h(u_{\epsilon_n}) + \epsilon u_{\epsilon_n} \to v$ and so $\frac{1}{2} \int Kv \cdot v \leq -1$. Therefore, $v \not\equiv 0$.

2. Nontrivial periodic solutions of Hamiltonian systems. Let $H: \mathbf{R}^{2n} \to \mathbf{R}$ be a C^1 convex function such that $H(0) = H_u(0) = 0$. Consider the Hamiltonian system

$$J\dot{u} + H_u(u) = 0 \qquad t \in \mathbf{R} \tag{5}$$

where

$$u = \begin{pmatrix} p \\ q \end{pmatrix} \qquad (p \text{ and } q \text{ are } n\text{-tuples}) \text{ and } J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We assume

$$\lim_{|u| \to \infty} \frac{H(u)}{|u|^2} = 0 \tag{6}$$

$$\lim_{|u| \to \infty} H(u) = \infty. \tag{7}$$

We seek nonconstant solutions of (5) which are T-periodic. Our main result is the following

THEOREM 2. *There exists $T_0 > 0$ such that for every $T > T_0$, Problem* (5) *possesses a solution with minimal period T.*

Remark. Theorem 2 is closely related to Theorem 4.7 in [3]. In [3] there is no convexity assumption; however, they assume (7) and

$$|H_u(u)| > 0, H(u) > 0 \qquad \forall u \in \mathbf{R}^{2n} \setminus \{0\}$$

$$|H_u(u)| \leq M \qquad \forall u \in \mathbf{R}^{2n}.$$

Theorem 2 is also related to the main result of [6] and our technique has been inspired by the duality device of [6]. Note, however, that we make no assumption about the behavior of H near 0; while the result of [6] requires the additional assumption

$$\lim_{|u|\to 0} \frac{H(u)}{|u|^2} > 0.$$

Proof of Theorem 2. The proof follows essentially the same pattern as the proof of Theorem 1 and we shall omit some details; it is somewhat simpler since dim $N(A) < \infty$.

In $H = L^2(0, T)^{2n}$ we consider the operator

$$Au = J\dot{u}$$

acting on functions which are T-periodic. We summarize some properties of A:

i) $A^* = A$.

ii) $N(A)$ consists of constants.

iii) $R(A)$ is closed and $R(A) = N(A)^\perp$; whenever $u \in H$ we shall write $u = u_1 + u_2$ with $u_1 \in R(A)$, $u_2 \in N(A)$.

iv) The eigenvalues of A are $(2\pi/T)k$, $k \in \mathbf{Z}$, and the corresponding eigenfunctions are

$$u(t) = a \sin\left(\frac{2\pi}{T} kt\right) + Ja \cos\left(\frac{2\pi}{T} kt\right)$$

where $a \in \mathbf{R}^{2n}$ is arbitrary ($a \neq 0$). Note that $\lambda_{-1} = -2\pi/T$.

v) Given $f \in R(A)$ there exists a unique $u \in R(A)$ such that $Au = f$.

We set

$$u = Kf = (A^{-1}f).$$

K is a compact self-adjoint operator in $R(A)$.

Given $\epsilon > 0$ we set

$$H_\epsilon(u) = H(u) + \frac{\epsilon}{2} |u|^2$$

and we denote by $H_\epsilon^*(v)$ its conjugate convex function. Note that H_ϵ^* is C^1 and that $(H_\epsilon^*)_v$ is the inverse mapping of $(H_u + \epsilon I)$.

On $R(A)$ we define

$$F_\epsilon(v) = \frac{1}{2}\int_0^T Kv\cdot v + \int_0^T H_\epsilon^*(v).$$

We first prove:

LEMMA 4. *There exists $T_0 > 0$ such that if $T > T_0$, then*

$$\operatorname*{Inf}_{R(A)} F_\epsilon \leq -1.$$

Proof. From (7) and the convexity of H we deduce that

$$H(u) \geq \rho|u| - C \qquad \forall u \in \mathbf{R}^{2n}$$

for some constants $\rho > 0$ and C.

Therefore

$$H_\epsilon^*(v) \leq C \qquad \text{for } |v| \leq \rho.$$

Let

$$v = \rho\left[-a \sin\left(\frac{2\pi}{T}t\right) + (Ja)\cos\left(\frac{2\pi}{T}t\right)\right]$$

where $a \in \mathbf{R}^{2n}$ is arbitrary with $|a| = 1$. So $Kv = -(T/2\pi)v$ and

$$F_\epsilon(v) \leq -\frac{T^2\rho^2}{4\pi} + TC \leq -1$$

provided $T \geq T_0$ for some large T_0.

In what follows we *fix* $T \geq T_0$. Next we observe (see Lemma 2) that

$$F_\epsilon(v) \geq \alpha\|v\|_{L^2}^2 - C \qquad \forall v \in R(A), \forall \epsilon \leq \frac{1}{4}|\lambda_{-1}|,$$

where $\alpha > 0$ and C are independent of ϵ. Therefore, $\operatorname{Min}_{R(A)} F_\epsilon$ is achieved at some v_ϵ ($\epsilon \leq 1/4\,|\lambda_{-1}|$) and we have

$$Kv_\epsilon + (H_\epsilon^*)_v(v_\epsilon) = \chi \in N(A).$$

Set

$$u_\epsilon = (H_\epsilon^*)_v(v_\epsilon)$$

so that

$$H_u(u_\epsilon) + \epsilon u_\epsilon = v_\epsilon$$

and

$$Au_\epsilon + H_u(u_\epsilon) + \epsilon u_\epsilon = 0.$$

Clearly, $||v_\epsilon||_{L^2} \leq C$; thus $||Au_\epsilon||_{L^2} \leq C$ and so $||u_{1\epsilon}||_{L^\infty} \leq C$.

Next we have, by the convexity of H and (7)

$$H_u(u)\cdot u \geq H(u) \geq \rho|u| - C \qquad \forall u \in \mathbf{R}^{2n}.$$

Therefore

$$\rho \int_0^T |u_\epsilon| - CT \leq \int_0^T H_u(u_\epsilon)\cdot u_\epsilon = -\int_0^T (Au_\epsilon + \epsilon u_\epsilon)u_\epsilon \leq C.$$

Consequently, $||u_\epsilon||_{L^\infty} \leq C$ [since $\dim N(A) < \infty$] and u_ϵ is even relatively compact in $C([0, T])^{2n}$. Therefore we may extract a subsequence $\epsilon_n \to 0$ such that $u_{\epsilon_n} \to u$ in $C([0, T])^{2n}$, and so

$$Au + H_u(u) = 0.$$

Moreover, we have

$$\frac{1}{2}\int Kv_\epsilon\cdot v_\epsilon + \int H_\epsilon^*(v_\epsilon) \leq \frac{1}{2}\int Kw\cdot w + \int H^*(w) \equiv F(w) \qquad \forall w \in R(A) \tag{8}$$

(since $H_\epsilon^* \leq H^*$). On the other hand, $v_{\epsilon_n} = H_u(u_{\epsilon_n}) + \epsilon_n u_{\epsilon_n} \to v = H_u(u)$ in $C([0, T])^{2n}$. It is easy to pass to the limit in (8) and we obtain

$$F(v) \leq F(w) \qquad \forall w \in R(A).$$

It follows as in [6] that v has minimal period T. Otherwise, suppose that v is periodic of period T/k for some integer $k > 1$.

Set $\tilde{v}(t) = v(t/k)$ for $0 < t < T$. Then we have $\tilde{v} \in R(A)$ and

$$F(\tilde{v}) = \frac{k}{2}\int_0^T Kv\cdot v + \int_0^T H^*(v) = F(v) + \frac{k-1}{2}\int_0^T Kv\cdot v < F(v)$$

a contradiction.

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