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Positive solutions of nonlinear elliptic equations in the case of critical Sobolev exponent

We report on a recent work with L. Nirenberg [5]. Consider the following problem. Assume $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded (smooth) domain. Find a (smooth) function u satisfying

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1)$$

where $p = \frac{N+2}{N-2}$ and f(u) is a "lower order perturbation" with $f(0) = 0$; a typical example is $f(u) = \lambda u$ $(\lambda \in \mathbb{R})$. The exponent $p = \frac{N+2}{N-2}$ is critical from the point of view of the variational formulation. Indeed, solutions of (1) correspond to critical points of the functional $\frac{1}{2}\int|\nabla u|^2 - \frac{1}{p+1}\int u^{p+1} - \int F(u)$ where F is a primitive of f and $p+1 = \frac{2N}{N-2}$ is the Sobolev exponent for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$.

Our lecture is divided as follows. First we recall some results concerning the *easy* case where $p < \frac{N+2}{N-2}$ and $f(u) = \lambda u$. Then, we consider the case where $p = \frac{N+2}{N-2}$ and $f(u) = \lambda u$. Finally we turn to the case where f is nonlinear.

Our interest in problem (1) comes from the fact that it presents some similarities with the Yamabe problem in geometry ; see e.g. Trudinger [11] and Th. Aubin [2].

1. THE CASE $p < \frac{N+2}{N-2}$.

Throughout Section 1 we assume that $p < \frac{N+2}{N-2}$. Clearly, there is a solution u of

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

Indeed, consider the following minimization problem

$$\operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^{p+1}}^2} \right\} \tag{3}$$

Since the injection $H_0^1 \subset L^{p+1}$ is compact, the infimum in (3) is achieved by some v_0. We may always assume that $v_0 \geq 0$ (otherwise replace v_0 by $|v_0|$) and that $\|v_0\|_{L^{p+1}} = 1$. Thus we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta v_0 = \mu {v_0}^p \tag{4}$$

and $\quad \mu = \int |\nabla v_0|^2 > 0.$

By stretching v_0 we find a function u satisfying

$$\begin{cases} u \geq 0 & \text{on } \Omega, \qquad u \not\equiv 0 \\ -\Delta u = u^p & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

[more precisely $u = kv_0$ satisfies (5) provided $k = \mu^{\frac{1}{p-1}}$]. It follows from the strong maximum principle that $u > 0$ in Ω.

(The question of uniqueness for problem (2) is open when Ω is starshaped. When Ω is an annulus and p is close to $\frac{N+2}{N-2}$ the solution of (2) need not be unique ; in fact there exist both spherical and non-spherical solutions, see [5]).

Let λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Consider now the following problem : find u such that

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

Then for each $\lambda \in (-\infty,\lambda_1)$ there is a solution of (6). Indeed

$$\operatorname*{Inf}_{v \in H^1_o} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{7}$$

is achieved by some v_o satisfying $v_o \geq 0$ on Ω and $\|v_o\|_{L^{p+1}} = 1$. Moreover there is a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta v_o - \lambda v_o = \mu v_o^p.$$

Thus $\mu = \int |\nabla v_o|^2 - \lambda \int v_o^2 > 0$ (since $\lambda < \lambda_1$). By stretching v_o as above we obtain a solution of (6).

The restriction $\lambda \in (-\infty,\lambda_1)$ is essential. Indeed suppose u is a solution of (6). Let $\phi_1 > 0$ in Ω be an eigenfunction corresponding to λ_1. We have

$$\lambda_1 \int u\phi_1 = \int u^p\phi_1 + \lambda \int u\phi_1 > \lambda \int u\phi_1$$

and thus $\lambda < \lambda_1$.

2. THE CASE $p = \frac{N+2}{N-2}$ AND $f(u) = \lambda u$

Throughout Sections 2 and 3 we assume that $p = \frac{N+2}{N-2}$. We consider now the following problem : find u such that

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + \lambda u & \text{in } \Omega \qquad\qquad (\lambda \in \mathbb{R}) \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8}$$

The argument we have used on Section 1 does not hold anymore since the injection $H_0^1 \subset L^{p+1}$ is *not* compact. In fact we know, by a result of Pohozaev [9], that if Ω is *starshaped* and $\lambda = 0$ there is *no* solution of (8). Using the same argument as in Pohozaev [9] one proves :

Theorem 0 : *Assume Ω is starshaped and $\lambda \leq 0$. Then there is no solution of* (8).

Remark 1 : On the other hand if Ω is an *annulus* then for every $\lambda \in (-\infty, \lambda_1)$ there is a spherical solution of (8). Indeed consider

$$\inf_{v \in H_r^1} \left\{ \frac{\|\nabla v\|_{L^2}^2 - \lambda \|v\|_{L^2}^2}{\|v\|_{L^{p+1}}^2} \right\} \tag{9}$$

where $H_r^1 = \{v \in H_0^1(\Omega)$ and v is spherically symmetric$\}$.

The infimum in (9) is achieved since the injection of H_r^1 into L^{p+1} is *compact.* Thus, after stretching, we obtain a spherical solution of (8).

Our main results are the following

Theorem 1 : *Assume* $\Omega \subset \mathbb{R}^N$, $N \geq 4$, *is any (smooth) bounded domain. Then for every* $\lambda \in (0,\lambda_1)$ *there exists a solution of* (8). *Moreover*

$$\operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{10}$$

is achieved.

Theorem 2 : *Assume* Ω *is a ball in* $\mathbb{R}^3$. *Then for every* $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ *there is a solution of* (8) ; *moreover the infimum in* (10) *is achieved. When* $\lambda \leq \frac{\lambda_1}{4}$ *there is no solution of* (8).

Remark 2 : The difference between dimension $N = 3$ and dimension $N \geq 4$ is quite suprizing. We have no simple explanation for it.

Remark 3 : When $N \geq 3$ and $\lambda \geq \lambda_1$ there is *no* solution of (8) (see Section 1). When $N \geq 3$ and Ω is *starshaped* there is no solution of (8) for $\lambda \leq 0$ (by Theorem 0).

Remark 4 : The generalization of Theorem 2 for starshaped domains is not known.

Before we sketch the proofs we present same facts about Sobolev spaces :

a) Define the best Sobolev constant S to be

$$S = \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \qquad (11)$$

In principle S depends on Ω ; but in fact S depends only on N. This is an easy consequence of the invariance under scaling of the ratio $\frac{\|\nabla v\|_{L^2}}{\|v\|_{L^{p+1}}}$ (that is, the ratio is unchanged if we replace u(x) by $u_k(x) = u(kx)$).

b) The infimum in (11) is never achieved, on any bounded domain. Indeed, suppose that the infimum in (11) is achieved by some function $v_o \geq 0$. Let $\tilde{\Omega}$ be a ball containing Ω and set

$$\tilde{v}_o = \begin{cases} v_o & \text{in } \Omega \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega \end{cases}$$

Thus, for $\tilde{\Omega}$ the infimum in (11) is achieved at v_o and we find

$$\begin{cases} -\Delta \tilde{v}_o = \mu v_o^p \text{ on } \Omega \\ v_o = 0 \text{ on } \partial\Omega \end{cases}$$

for some constant $\mu > 0$. This contradicts Pohozaev's Theorem.

c) When $\Omega = \mathbb{R}^N$ the infimum in (11) is achieved by the function $u(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}$ or - after scaling - by any of the functions

$$u(x) = \frac{1}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}} \quad (\varepsilon > 0).$$

see Aubin [3], Talenti [10], Lieb [8].

The following Lemma plays a crucial role in the proof of Theorem 1.

Lemma 1 : *Assume* $N \geq 4$. *Then, for every* $\lambda \in (0,\lambda_1)$ *we have*

$$S_\lambda \equiv \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} < S \tag{12}$$

The proof of Lemma 1 is rather technical ; for details see [5]. The main idea - borrowed from Aubin [1] - consists of estimating the ratio

$$Q(u) = \frac{\|\nabla u\|^2_{L^2} - \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}}$$

for $u(x) = u_\varepsilon(x) = \dfrac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$ where $\phi \in \mathcal{D}_+(\Omega)$ is a fixed function such that $\phi(x) \equiv 1$ near 0 (assuming $0 \in \Omega$).

A straightforward computation gives the following expansion as $\varepsilon \to 0$:

$$Q(u_\varepsilon) = S + O(\varepsilon^{\frac{N}{2}-1}) - \lambda C \varepsilon \qquad \text{when } N \geq 5$$

$$Q(u_\varepsilon) = S + O(\varepsilon) - \lambda C \varepsilon \, |\log \varepsilon| \quad \text{when } N = 4$$

where $C > 0$ is a constant. In both cases we see that $Q(u_\varepsilon) < S$ for $\varepsilon > 0$ sufficiently small. We shall also use the following measure theoretic lemma.

Lemma 2 : (Brezis-Lieb [4]) *Suppose* (v_j) *is a sequence in* $L^q(\Omega)$ *with* $1 \le q < \infty$ *such that* $\|v_j\|_{L^q}$ *remains bounded and* $v_j(x) \to v(x)$ *a.e. on* Ω. *Then*

$$\lim_{j\to\infty} \left\{ \int |v_j|^q - \int |v_j - v|^q \right\} = \int |v|^q \tag{13}$$

Proof of Theorem 1 : Choose a minimzing sequence (v_j) for (10) such that

$$v_j \ge 0 \text{ on } \Omega, \quad \|v_j\|_{L^{p+1}} = 1 \tag{14}$$

$$\int |\nabla v_j|^2 - \lambda \int v_j^2 = S_\lambda + o\ (1) \tag{15}$$

Since v_j is bounded in H_0^1 we may assume, for a subsequence, that

$$v_j \rightharpoonup v \quad \text{weakly in } H_0^1$$

$$v_j \to v \quad \text{a.e.} \quad \text{on } \Omega$$

$$v_j \to v \quad \text{strongly in } L^2$$

Set $w_j = v_j - v$ so that $w_j \rightharpoonup 0$ weakly in H_0^1.

By Lemma 2 (applied with $q = p+1$) we have

$$\lim \|w_j\|_{L^{p+1}}^{p+1} = 1 - \|v\|_{L^{p+1}}^{p+1}.$$

Thus

$$\begin{aligned} 1 &= (\|v\|_{L^{p+1}}^{p+1} + \|w_j\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}} + o\ (1) \le \\ &\le \|v^2\|_{L^{p+1}} + \|w_j\|_{L^{p+1}}^2 + o\ (1) \end{aligned} \tag{16}$$

On the other hand (since $w_j \rightharpoonup 0$ weakly in H_0^1) we have

$$\int |\nabla v_j|^2 = \int |\nabla v|^2 + \int |\nabla w_j|^2 + o\ (1) \tag{17}$$

Combining (15) (16) and (17) we obtain

$$\int |\nabla v|^2 - \lambda \int v^2 + \int |\nabla w_j|^2 \leq S_\lambda [\ \|v\|^2_{L^{p+1}} + \|w_j\|^2_{L^{p+1}}] + o\ (1) \tag{18}$$

By definition of S_λ we have $\int |\nabla v|^2 - \lambda \int v^2 \geq S_\lambda\ \|v\|^2_{L^{p+1}}$ and therefore

$$\begin{aligned} \int |\nabla w_j|^2 &\leq S_\lambda\ \|w_j\|^2_{L^{p+1}} + o\ (1) \\ &\leq \frac{S_\lambda}{S} \int |\nabla w_j|^2 + o\ (1). \end{aligned}$$

Since $S_\lambda < S$ (by Lemma 1), it follows that $\int |\nabla w_j|^2 \to 0$. Consequently $v_j \to v$ strongly in H_0^1 (and in L^{p+1}). Passing to the limit in (14) and (15) we conclude that the infimum in (11) is achieved by v. After stretching we obtain a solution of (8).

In the proof of Theorem 2 we use

Lemma 3 : *Assume* $\Omega = \{x \in \mathbb{R}^3\ ;\ |x| < 1\}$. *Then for each* $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ *we have*

$$S_\lambda \equiv \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^6}} \right\} < S \tag{19}$$

Proof : We estimate the ratio

$$Q(u) = \frac{\|\nabla u\|^2_{L^2} - \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^6}}$$

for $u(x) = u_\varepsilon(x) = \dfrac{\cos(\frac{\pi}{2}|x|)}{(\varepsilon+|x|^2)^{1/2}}$.

A technical computation (see [5]) gives the following expansion as $\varepsilon \to 0$:

$$Q(u_\varepsilon) = S + C\sqrt{\varepsilon}\ (\frac{\pi^2}{4} - \lambda) + 0(\sqrt{\varepsilon})$$

where $C > 0$ is a constant. Therefore if $\lambda > \frac{\lambda_1}{4}$ (note that here $\lambda_1 = \pi^2$) we see that $Q(u_\varepsilon) < S$ for $\varepsilon > 0$ sufficiently small.

Proof of Theorem 2 : The same argument as in the proof of Theorem 1 shows that for every $\lambda \in (\frac{\lambda_1}{4},\lambda_1)$ the infimum in (19) is achieved. Consequently we obtain a solution of (8) for each $\lambda \in (\frac{\lambda_1}{4},\lambda_1)$. Next we must show that no solution of (8) exists for $\lambda \geq \frac{\lambda_1}{4}$. By a result of Gidas-Ni-Nirenberg [7] we know that any solution u of (8) in a ball must be spherically symmetric. We write $u(x) = u(r)$ $(r = |x|)$ and so we have

$$- u'' - \frac{2}{r} u' = u^5 + \lambda u \quad \text{on } (0,1) \tag{20}$$

$$u'(0) = u(1) = 0. \tag{21}$$

Then we use an argument "à la Pohozaev" but with more complicated multipliers. Namely we multiply (20) through by

$$r^2[r \cos \pi r - b \sin \pi r]u'$$

and then by

$$r[-\frac{r}{2}(1+b\pi)\cos \pi r - \frac{\pi r^2}{2} \sin \pi r + b \sin \pi r]\, u$$

for some appropriate constant b. Integrating by parts and combining the two equalities leads to $\lambda > \frac{\pi^2}{4}$; for more details see [5].

3. THE GENERAL CASE, $-\Delta u = u^p + f(u)$ WITH $p = \frac{N+2}{N-2}$.

Here again we take $p = \frac{N+2}{N-2}$. Assume f is a C^1 function on $[0,+\infty)$ such that

$$f(0) = 0, \quad f(u) \geq 0 \quad \forall u \geq 0 \tag{22}$$

$$\lim_{u\to\infty} \frac{f(u)}{u^p} = 0 \tag{23}$$

$$f'(0) < \lambda_1 \tag{24}$$

The problem is to find a function u satisfying

$$\begin{cases} u > 0 & \text{on } \Omega \\ -\Delta u = u^p + f(u) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{25}$$

Our main results are the following

Theorem 3 : *Assume* $N \geq 5$, (22), (23), (24) *and*

$$f \not\equiv 0 \tag{26}$$

Then there is a solution of (25).

Theorem 4 : *Assume* $N = 4$, (22), (23), (24) *and either*

$$f'(0) > 0 \tag{27}$$

or

$$\liminf_{u \to +\infty} \frac{f(u)}{u} > 0 \tag{28}$$

Then there is a solution of (25).

Theorem 5 : *Assume* $N = 3$, (22), (23), (24)

$$\lim_{u \to +\infty} \frac{f(u)}{u^3} = +\infty$$

Then there is a solution of (25).

Remark 5 : Theorem 3, 4 and 5 admit appropriate extensions to the case where f depends also on x with $f(x,0) \equiv 0$. They may be used in order to prove the following :

<u>Theorem 6</u> : *Assume* $N \geq 3$. *Then there is a constant* $\lambda^* > 0$ *such that the problem*

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = \lambda(1+u)^p & \text{in } \Omega \qquad p = \frac{N+2}{N-2} \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{29}$$

has at least two solutions for each $\lambda \in (0,\lambda^*)$ *and no solution for* $\lambda > \lambda^*$.

(A similar result for $p < \frac{N+2}{N-2}$ had been obtained earlier by Crandall-Rabinowitz [6]).

The idea of the proof is the following. Firstly one obtains (easily) a minimal solution $\underline{u}$ of (29) for every $\lambda \in (0,\lambda^*)$ (see e.g. [6]). Then one looks for a second solution of (29) of the form $u = \underline{u} + v$ with $v > 0$ on Ω. Thus v satisfies $-\Delta v = \lambda(1+\underline{u}+v)^p - \lambda(1+\underline{u})^p \equiv \lambda v^p + f(x,v)$ and we are reduced to a problem of the type (25).

The proofs of Theorems 3, 4 and 5 involve two ingredients. Firstly a geometrical result which is a variant of the Ambrosetti-Rabinowitz [1] mountain pass Lemma <u>without the (PS) condition</u> (see Lemma 4). Secondly a technical Lemma which has the same flavour as Lemma 1 or Lemma 3 (see Lemma 5).

<u>Lemma 4</u> : *Assume* Φ *is a* C^1 *function on a Banach space* E *such that*

$$\Phi(0) = 0 \tag{30}$$

$$\begin{cases} \textit{there exist constants } \rho > 0 \textit{ and } r > 0 \textit{ such that } \Phi(u) \geq \rho \\ \textit{for every } u \in E \textit{ with } \|u\| = r \end{cases} \tag{31}$$

$\Phi(v) \leq 0$ *for some* $v \in E$ *with* $\|v\| > r$. (32)

Set

$$c = \operatorname*{Inf}_{P \in \mathcal{P}} \operatorname*{Sup}_{p \in P} \Phi(p) \tag{33}$$

where $\mathcal{P}$ *denotes the class of all paths joining* 0 *to* v. *Then there exists a sequence* (u_j) *in* E *such that* $\Phi(u_j) \to c$ *and* $\Phi'(u_j) \to 0$ *in* E^*.

The proof of Lemma 4 is essentially the same as the proof given in [1]. In order to prove Theorems 3,4 and 5 we apply Lemma 4 in $E = H_0^1$ to the functional

$$\Phi(u) = \frac{1}{2}\int|\nabla u|^2 - \frac{1}{p+1}\int(u^+)^{p+1} - \int F(u^+) \tag{34}$$

where $F(u) = \int_0^u f(t)dt$. Property (31) is an easy consequence of assumption (24). For every $u \in H_0^1$, $u \geq 0$ in Ω, $u \not\equiv 0$ we have $\lim_{t\to+\infty} \Phi(tu) = -\infty$. Hence, there are many v's satisfying (32). However it is essential to make a *special choice* of v in order to be able to use properly the sequence (u_j) given by Lemma 4. More precisely we have

Lemma 5 : *Under the assumptions of Theorems 3,4 and 5 there is some* $u_0 \in H_0^1$, $u_0 \geq 0$ *in* Ω, $u_0 \not\equiv 0$ *and*

$$\operatorname*{Sup}_{t\geq 0} \Phi(t\, u_0) < \frac{1}{N} S^{N/2}. \tag{35}$$

The proof is rather technical. Let $u_\varepsilon(x) = \dfrac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$ with $\phi \in \mathcal{D}_+(\Omega)$ and $\phi \equiv 1$ near $x = 0$ (assuming $0 \in \Omega$). We show by an expansion method (as in Lemma 1) that u_ε satisfies (35) provided $\varepsilon > 0$ is sufficiently small. As was already observed, the expansion technique is sensitive to the dimension N ; the cases N = 3, N = 4 and N ≥ 5 must be considered separately. (See the details in [5]).

<u>Proofs of Theorem 3, 4 and 5</u> : By Lemma 5 there is some $v \in H_0^1$ such that $\|v\| > r$, $\Phi(v) \leq 0$ and

$$\sup_{t\geq 0} \Phi(tv) < \frac{1}{N} S^{N/2} \tag{36}$$

We apply Lemma 4 with such a v. From (33) it follows that

$$c < \frac{1}{N} \; S^{N/2} \tag{37}$$

Let (u_j) be the sequence given by Lemma 4. We have

$$\frac{1}{2}\int|\nabla u_j|^2 - \frac{1}{p+1}\int(u_j^+)^{p+1} - \int F(u_j^+) = c + o\ (1) \tag{38}$$

$$-\Delta u_j = (u_j^+)^p + f(u_j^+) + \zeta_j \tag{39}$$

with $\zeta_j \to 0$ in H^{-1}.

Combining (38) and (39) it is easy to show that $\|u_j\|_{H^1}$ remains bounded.

Thus we may assume that

$$u_j \rightharpoonup u \quad \text{weakly in } H_o^1$$

$$u_j \to u \quad \text{a.e.}$$

From (39) we deduce that

$$-\Delta u = (u^+)^p + f(u^+) \qquad \text{in } H^{-1} \tag{40}$$

By the maximum principle we have $u \geq 0$ and so

$$-\Delta u = u^{p+1} + f(u).$$

It remains to prove that $u \not\equiv 0$. Suppose by contradiction that $u \equiv 0$. Using (23) we obtain

$$\int F(u_j^+) \to 0, \quad \int f(u_j^+)u_j \to 0 \tag{41}$$

We may always assume that

$$\int |\nabla u_j|^2 \to \ell \tag{42}$$

and by (39)

$$\int (u_j^+)^{p+1} \to \ell \tag{43}$$

From (38) we deduce that

$$\frac{1}{N} \ell = c > 0. \tag{44}$$

On the other hand we have (by Sobolev inequality)

$$\int |\nabla u_j|^2 \geq S\|u_j\|^2_{L^{p+1}} \geq S\|u_j^+\|^2_{L^{p+1}}$$

and therefore at the limit

$$\ell \geq S\ell^{\frac{2}{p+1}}$$

Thus

$$\ell \geq S^{N/2}$$

and (by (44))

$$c \geq \frac{1}{N} S^{N/2}$$

a contradiction with(37). Therefore $u \not\equiv 0$.

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