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Symmetry in Nonlinear PDE's

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The question of symmetry in nonlinear partial differential equations has been the subject of intensive investigations over the past 25 years. The general theme is the following. Suppose the domain Ω, as well as the boundary condition on $\partial\Omega$, has some symmetry, for example radial symmetry, axial symmetry or symmetry with respect to some hyperplane. Do solutions of nonlinear partial differential equations in Ω inherit these symmetries? In a related direction, one may consider overdetermined problems on a general domain Ω, for example the solution of a second order PDE satisfying both a constant Dirichlet and a constant Neumann condition on $\partial\Omega$. Does this imply that the domain Ω is a ball or the complement of a ball?

Remarkable progress has been achieved through the work of Louis Nirenberg and his collaborators, especially on the first question. I will review some of their basic results. They are concerned with *positive* solutions of a *single* PDE. Related questions may be asked for *systems*. Some suggestive partial results have been obtained but the general situation is still far from satisfactory. I will describe some outstanding open problems.

1. Symmetry via moving planes.

The main result in the celebrated paper by B. Gidas, W. M. Ni and L. Nirenberg [GNN1] from 1979 is the following:

Theorem 1. *Let $\Omega = B$ be the open unit ball in $\mathbb{R}^n$. Assume $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where f is C^1. Then u is radially symmetric and the radial derivative $u'(r)$ is negative for $0 < r < 1$.

As we will see the method of proof relies on the maximum principle used in conjunction with the method of moving planes due to A. D. Alexandroff [Al]. This

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type of argument had been initiated by J. Serrin [S] in 1972 in proving the radial symmetry of the *domain* for overdetermined problems. The paper of B. Gidas, W. M. Ni and L. Nirenberg [GNN1] has become enormously popular for several reasons:

a) It established radial symmetry of the solution for a large class of problems. It became an incentive for investigating symmetry in numerous other situations.
b) The method is very flexible.

In fact, it has been adapted with success to a large variety of questions arising in concrete problems. For example, C. J. Amick and L. E. Fraenkel [AF] have used it in connection with vortex rings. W. Craig and P. Sternberg [CS] have used it to settle an open problem on water waves. H. Berestycki and L. Nirenberg [BN2] have used it in connection with problems arising in combustion.

A beautiful generalization of Theorem 1 is the following

THEOREM 2 [BN1]. *Let Ω be a general bounded convex set in $\mathbb{R}^n$ which is symmetric about some hyperplane, say $x_1 = 0$. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (1) with f locally Lipschitz. Then u is symmetric with respect to x_1 and $\dfrac{\partial u}{\partial x_1} < 0$ for $0 < x_1$ in Ω.*

Theorem 2 was originally proved in [GNN1] under additional assumptions, for example $\partial\Omega$ had to be of class C^2; in particular, the simple case of a cube could not be handled. These restrictions were lifted by H. Berestycki and L. Nirenberg [BN1] who also gave a very elegant proof. I cannot resist the pleasure of describing their argument which deserves to become part of the classical literature.

The proof uses Stampacchia's version of the maximum principle. The standard form of the maximum principle asserts that if a function w satisfies

$$-\Delta w + c(x)w \le 0 \quad \text{in } \omega \tag{2}$$

$$w \le 0 \quad \text{on } \partial\omega \tag{3}$$

with

$$c(x) \ge 0 \quad \text{in } \omega, \tag{4}$$

then

$$w \le 0 \quad \text{in } \omega.$$

In Stampacchia's form assumption (3) is weakened. One merely assumes that $c^- = \max(-c, 0)$ is small in some appropriate L^p norm. Suppose, for simplicity, that $n \ge 3$ and let S_n be the best Sobolev constant in $\mathbb{R}^n$, i.e.,

$$S_n = \operatorname*{Inf}_{\varphi \in H_0^1(\omega)} \|\nabla\varphi\|_2^2 / \|\varphi\|_{2n/(n-2)}^2.$$

This number, which is independent of ω (and depends only on n), can be computed explicitly (see e.g. [Au]).

Lemma 1. *Assume w satisfies (2)-(3) with*

$$\|c^-\|_{n/2} < S_n. \tag{5}$$

Then $w \le 0$ in ω.

In particular (5) always holds in small domains provided the sup-norm of c^- is bounded. For example the maximum principle is valid whenever

$$\|c^-\|_\infty |\omega|^{2/n} < S_n. \tag{6}$$

Another formulation suggested by S.R.S. Varadhan may be found in [BN1].

Proof of Lemma 1. Multiplying (2) by $w^+ = \max(w, 0)$ and integrating by parts yields

$$\int |\nabla w^+|^2 + \int c^+(w^+)^2 - \int c^-(w^+)^2 \le 0.$$

Thus

$$S_n \|w^+\|^2_{2n/(n-2)} \le \int c^-(w^+)^2 \le \|c^-\|_{n/2} \|w^+\|^2_{2n/(n-2)}.$$

Applying (5) we find that $w^+ = 0$, i.e., $w \le 0$ in ω.

Proof of Theorem 2. Write $x = (x_1, y)$ with $y = (x_2, x_3, \dots, x_n)$ and set

$$a = \max\{x_1; (x_1, y) \in \Omega\}.$$

We will prove that

$$u(x_1, y) < u(x_1', y) \quad \forall x = (x_1, y) \in \Omega \text{ with } x_1 > 0 \text{ and } \forall x_1' \text{ with } |x_1'| < x_1. \tag{7}$$

Inequality (7) yields

$$u(x_1, y) \le u(-x_2, y), \tag{8}$$

and applying (8) to $\tilde{u}(x_1, y) = u(-x_1, y)$, which is also a solution of (1) one finds that $u(-x_1, y) = u(x_1, y)$, i.e., u is symmetric with respect to x_1. The fact that $\frac{\partial u}{\partial x_1} < 0$ for $0 < x_1$ in Ω is an easy consequence of (7).

For $0 < \lambda < a$, set

$$\Sigma(\lambda) = \{x = (x_1, y) \in \Omega;\ x_1 > \lambda\}$$

and

$$w^\lambda(x) = u(2\lambda - x_1, y) - u(x_1, y) \quad \text{for } x \in \Sigma(\lambda).$$

Note that w^λ is well defined on $\Sigma(\lambda)$ since Ω is convex and symmetric about the hyperplane $x_1 = 0$.

Inequality (7), to be proved, is equivalent to

$$w^\lambda(x) > 0 \quad \forall x \in \Sigma(\lambda), \quad \forall \lambda \in (0, a). \tag{9}$$

The function w^λ satisfies

$$-\Delta w^\lambda + c^\lambda(x) w^\lambda = 0 \quad \text{in } \Sigma(\lambda)$$

where

$$c^\lambda(x) = \begin{cases} \dfrac{f(u(x_1,y)) - f(u(2\lambda - x_1, y))}{w^\lambda(x)} & \text{if } w^\lambda(x) \neq 0 \\ 0 & \text{if } w^\lambda(x) = 0. \end{cases}$$

Clearly $\|c^\lambda\|_\infty \leq L$ where L is the Lipschitz constant of f on the interval $[-\|u\|_\infty, +\|u\|_\infty]$. Moreover

$$\begin{aligned} w^\lambda &\geq 0 \quad \text{on } \partial\Sigma(\lambda), \\ w^\lambda &\not\equiv 0 \quad \text{on } \partial\Sigma(\lambda). \end{aligned}$$

For λ near a, $\Sigma(\lambda)$ has small measure and we deduce from Lemma 1 that $w^\lambda \geq 0$ in $\Sigma(\lambda)$.

Let

$$\Lambda = \{\lambda \in (0,a);\ w^\lambda \geq 0 \quad \text{in } \Sigma(\lambda)\},$$

so that Λ is not empty. Clearly Λ is closed in $(0,a)$. We claim that Λ is open.

Fix some $\lambda \in \Lambda$. By the strong maximum principle applied to w^λ in $\Sigma(\lambda)$ we see that

$$w^\lambda > 0 \quad \text{in } \Sigma(\Lambda).$$

Let K be any (smooth) compact set in $\Sigma(\lambda)$ such that $|\Sigma(\mu)\backslash K|$ is sufficiently small for all μ near λ. Sufficiently small refers to Lemma 1 applied in $\Sigma(\mu)\backslash K$ with $\|c\|_\infty \leq L$ (the Lipschitz constant of f).

Since

$$w^\lambda(x) \geq \delta > 0 \quad \text{in } K$$

we have, by continuity,

$$w^\mu(x) \geq 0 \quad \text{in } K$$

for all μ near λ. In particular

$$w^\mu(x) \geq 0 \quad \text{on the boundary of } \Sigma(\mu)\backslash K.$$

Applying Lemma 1 to w^μ in $\Sigma(\mu)\backslash K$ we see that $w^\mu \geq 0$ in $\Sigma(\mu)\backslash K$ and thus $w^\mu \geq 0$ in $\Sigma(\mu)$. Hence $\mu \in \Lambda$ for all μ near λ, i.e., Λ is open.

REMARK 1. The assumption that Ω is convex is essential. For example if Ω is an annulus, radial symmetry may fail. (In fact, nonradial solutions can have lower energy than the radial ones.) We have constructed in [BrN1] (see also [D]) *nonradial* positive solutions of

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega = \text{annulus} \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Radial symmetry of solutions of (1) when Ω is all of $\mathbb{R}^n$ (with $u(x) \to 0$ as $|x| \to \infty$) has been originally studied in [GNN2]; important extensions may be found in [CGS], [Li], [LN], [CL] and [Z]. In particular, the radial symmetry is useful in order to give a complete description of all positive solutions of $-\Delta u = u^{(n+2)/(n-2)}$ in $\mathbb{R}^n$. These functions are the extremals for the Sobolev inequality $\displaystyle\int |\nabla\varphi|^2 \geq S\|\varphi\|^2_{2n/(n-2)}$. This classification plays an important role—after blow-up—in the study of the Yamabe problem, in particular in the work of R. Schoen [Sc].

In the case where Ω is a half-space $\Omega = \{x \in \mathbb{R}^n; x_n > 0\}$, with zero Dirichlet condition, H. Berestycki, L. A. Caffarelli and L. Nirenberg [BCN] have established symmetry (i.e., $u = u(x_n)$) and monotonicity provided u is bounded and $f(\sup u) \le 0$. The case of a half-space Ω with a nonlinear Neumann condition has been investigated in [CFS], [E], and [LZ] and [T]. Such results have applications to prescribed curvature problems on manifolds with boundary.

Symmetry and monotonicity in infinite cylindrical domains has been studied by H. Berestycki and L. Nirenberg [BN2] in connection with travelling front solutions arising in combustion. The case where Ω is the exterior of a ball has been considered in [AB]. The interested reader will find further variations on this theme in the expository paper [Be].

The moving plane method has also been applied to establish symmetry of solutions for some classes of *systems* of PDE's; see [Ba], [DF], and [Tr]. In particular, for the Liouville system,

$$-\Delta u_i = \exp(\Sigma_{j=1}^n a_{ij} u_j) \quad \text{in } \mathbb{R}^2, \ \ 1 \le i \le n,$$

with $u_i > 0$. M. Chipot, I. Shafrir and G. Wolansky [CSW] have proved under mild assumptions, that each u_i is radially symmetric and decreasing about some point x_i in $\mathbb{R}^2$. (An interesting earlier approach by S. Chanillo and M. Kiessling [CK], based solely on an isoperimetric inequality and the Pohozaev identity, led to similar conclusions under stronger assumptions.) However, the application of the moving plane method to systems has been, so far, very limited. I would like to describe next, two types of systems where other techniques have been successful.

2. Questions of symmetry for the Ginzburg-Landau system.

The Ginzburg-Landau system consists of a coupled system of 2 equations in $\mathbb{R}^2$,

$$-\Delta u = u(1 - |u|^2) \quad \text{in } \mathbb{R}^2. \tag{10}$$

Here u takes its values in $\mathbb{R}^2$ and it is also convenient to view u as a complex number. Despite its simple appearance, problem (10) has a rich structure, which is not yet fully understood. It is an interesting laboratory for testing new methods.

One is concerned with solutions of (10) satisfying

$$|u(x)| \to 1 \quad \text{as } |x| \to \infty. \tag{11}$$

It is easy to construct solutions of (10)–(11) in polar coordinates, using separation of variables. Given any integer $q \in \mathbb{Z}$ the function

$$u = u(r, \theta) = e^{iq\theta} f(r) \tag{12}$$

is a solution of (10)–(11) provided the real valued function f satisfies the ordinary differential equation

$$\begin{cases} -f'' - \dfrac{1}{r} f' + \dfrac{q^2}{r^2} f = f(1 - f^2) \text{ on } \quad (0, \infty) \\ f(0) = 0 \text{ and } \quad f(\infty) = 1. \end{cases} \tag{13}$$

It is not difficult to see that for every integer q problem (13) has a unique solution f_q (see e.g. Appendices II, III in [BBH] and also [HH]). Hence, we obtain a family of special solutions $u_q = e^{iq\theta} f_q(r)$ for $q \in \mathbb{Z}$. An outstanding open problem is whether these are the only solutions of (10)–(11):

Open Problem 1. Let u be any solution of (10)–(11). Is $u = u_q$ for some $q \in \mathbb{Z}$, modulo translation and rotation?

An unusual quantization phenomenon takes place for solutions of (10) having the property that $|u(x)| \to 1$ as $|x| \to \infty$, fast enough so that

$$\int_{\mathbb{R}^2} (|u|^2 - 1)^2 < \infty. \tag{14}$$

Remark 2. It is easy to show that if u is any solution of (10) satisfying (14) then $|u(x)| \to 1$ as $|x| \to \infty$. The converse is not known:

Open Problem 2. Suppose u is a solution of (10) such that $|u(x)| \to 1$ as $|x| \to \infty$. Does (14) hold?

Theorem 3 ([BMR]). *Let u be a solution of (10) satisfying (14). Let $q = \deg(u, \infty)$ be the degree of u at infinity, i.e., the winding number of the map*

$$x \in S^1 \mapsto \frac{u(Rx)}{|u(Rx)|} \in S^1 \quad \textit{for large } R.$$

Then

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} (|u|^2 - 1)^2 = q^2. \tag{15}$$

The "radial" solution $u_q = e^{iq\theta} f_q(r)$ described above satisfies (15). It is not known, for general q, whether the only solution of (10) satisfying (15) is u_q, modulo translation, rotation and complex conjugation. The answer is positive for $q = 0$ and $q = 1$. The case $q = 0$ is an easy consequence of Liouvile theorem. The case $q = 1$ is a remarkable result of P. Mironescu ([M2]) described in Theorem 4 below.

Sketch of the proof of Theorem 3. The main ingredient is the Pohozaev identity applied to (10). It asserts that

$$\int_{B_R} (|u|^2 - 1)^2 = \frac{R}{2} \int_{S_R} (|u|^2 - 1)^2 + R \int_{S_R} (|u_t|^2 - |u_n|^2). \tag{16}$$

Here $B_R = \{x \in \mathbb{R}^2; |x| < R\}$, $S_R = \{x \in \mathbb{R}^2; |x| = R\}$, u_t and u_n denote respectively the tangential and normal derivatives of u along S_R. Identity (16) is obtained, as usual, through the multiplication of (10) by $xu_x + yu_y = ru_r$ and integration on B_R. Using (14) one shows (see e.g. [Sh1] and [Br]) that, as $R \to \infty$,

$$R \int_{S_R} (|u|^2 - 1)^2 \to 0 \quad \text{and } R \int_{S_R} |u_n|^2 \to 0.$$

The important term in (16)—the one which "carries" the degree—is u_t. More precisely one shows that, as $R \to \infty$,

$$R \int_{S_R} |u_t|^2 \to 2\pi q^2.$$

Now, to the result of Mironescu:

THEOREM 4. *Let u be a solution of (10) satisfying (15) with $q = 1$. Then u has radial symmetry, i.e., $u = e^{i\theta} f_1(r)$ modulo rotation, translation and complex conjugation.*

Sketch of the proof of Theorem 4. Let $f(r) = f_1(r)$ be the unique solution of (13) corresponding to $q = 1$. Since $\deg(u, \infty) \neq 0$ the function u must have at least one zero. After translation we may assume that $u(0) = 0$. Set $v = u/f$. Using (10) and (13) it is easy to derive a PDE satisfied by v:

$$-\Delta v - \frac{2f'}{f} v_r - \frac{v}{r^2} = f^2 v(1 - |v|^2) \tag{17}$$

where v_r is the radial derivative of v, i.e., $v_r = \frac{1}{|x|}(x \cdot \nabla v)$.

Applying the Pohozaev identity to (17) yields

$$\int_{B_R} \left[\frac{2rf'}{f} |v_r|^2 + \frac{1}{2}(f^2 + rff')(|v|^2 - 1)^2 \right] = \int_{S_R} [\cdots] \tag{18}$$

where $[\cdots]$ is a lengthy expression involving v, v_r, f and f'. The solution f of (13) is known to be monotone increasing (see e.g. [HH]), so that the integrand on the left-hand side of (18) is nonnegative. A careful asymptotic analysis of $u(x)$ as $|x| \to \infty$ (see [Sh1] and [Br]) combined with Theorem 3 shows that the right-hand side in (18) tends to 0 as $r \to \infty$. Thus $v_r \equiv 0$ and $|v| \equiv 1$. Going back to (17) we obtain $v_{\theta\theta} + v = 0$, i.e., $v = e^{i(\theta+\theta_0)}$ or $v = e^{-i(\theta+\theta_0)}$. Returning to u we find $u = e^{i\theta} f(r)$ or $u = e^{-i\theta} f(r)$, modulo a rotation.

REMARK 3. The Pohozaev identity seems to play a distinguished role in proving symmetry for 2-dimensional problems. P. L. Lions [Lio] has given a proof of Theorem 1 in 2-d which does not make use of the moving plane method. It relies on a clever combination of the Pohozaev identity with an isoperimetric inequality. Related ideas may be found in [Ba], [CK1] and [CK2]. Unfortunately the method seems to be restricted to 2-dimensional problems. One may consider the analogue of (10) in higher dimension and there no symmetry result is known:

Open Problem 3. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a solution of

$$-\Delta u = u(1 - |u|^2) \quad \text{on } \mathbb{R}^n, \ \ n \geq 3$$

with $|u(x)| \to 1$ as $|x| \to \infty$ (possibly with a "good" rate of convergence). Assume $\deg(u, \infty) = \pm 1$. Does u have the form

$$u(x) = \frac{x}{|x|} f(r)$$

(modulo translation and isometry), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function, such that $f(0) = 0$ and $f(\infty) = 1$?

REMARK 4. The proof of Theorem 4 provides some information for general values of q. Let u be a solution of (10) satisfying (15) with $q \geq 2$. Assume that u has only *one* zero (of degree q). Then $u = e^{iq\theta} f_q(r)$. This result raises an interesting variant of Problem 1:

Open Problem 4. Let u be a solution of (10) with $|u(x)| \to 1$ as $|x| \to \infty$. Can u have more than one zero?

Theorem 4 has important implications, for example

Theorem 5 ([M2]). *Let u be a solution of (10) which is a local minimizer of the energy*

$$E(v,\Omega) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4}\int_\Omega (|v|^2-1)^2$$

in the sense that for every bounded domain $\Omega \subset \mathbb{R}^2$,

$$E(u,\Omega) \le E(v,\Omega), \quad \forall v \quad \text{such that } v = u \text{ on } \partial\Omega.$$

Then either u is a constant or $u = e^{i\theta} f_1(r)$ (modulo translation, rotation and complex conjugation).

The proof of Theorem 5 uses Theorem 4 in conjunction with a result of E. Sandier [Sa2] and I. Shafrir [Sh1] (u a local minimizer $\Rightarrow \frac{1}{2\pi}\int_{\mathbb{R}^2}(|u|^2-1)^2 = 1$).

Theorem 5 is very useful in analyzing the structure of the Ginzburg-Landau vortices near the vortex core. Let Ω be a bounded domain in $\mathbb{R}^2$ and let $g : \partial\Omega \to S^1$ be a smooth boundary condition of degree $d > 0$. Let u_ε be a minimizer of the Ginzburg-Landau energy

$$E_\varepsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (|v|^2-1)^2$$

with boundary condition $v = g$ on $\partial\Omega$. One of the main results in [BBH] asserts that for ε small, u_ε has exactly d zeroes $a_\varepsilon^1, a_\varepsilon^2, \dots, a_\varepsilon^d$ and that (along a subsequence)

$$u_\varepsilon(z) \to u_\star(z) = e^{i\psi(z)} \prod_{i=1}^d \frac{z-a^i}{|z-a^i|} \tag{19}$$

where $a^i = \lim_{\varepsilon\to 0} a_\varepsilon^i$ and ψ is a real-valued harmonic function in Ω. The convergence in (19) holds in $C^k_{\text{loc}}(\Omega \backslash \{a_1, a_2, \dots, a_d\})$, for every k. However, there was no information in [BBH] about the mode of convergence of u_ε to $u_\star$ near its singularities. As a consequence of Theorem 5 we now have

Theorem 6 ([Sh2], [M2]). *Let $U(z) = \frac{z}{|z|} f_1(z)$ where f_1 is the solution of (13) with $q = 1$. Then*

$$\lim_{\varepsilon\to 0} \Big\| u_\varepsilon(z) - e^{i\psi(z)} \prod_{i=1}^d U\Big(\frac{z-a_\varepsilon^i}{\varepsilon}\Big)\Big\|_{L^\infty(\Omega)} = 0 \tag{20}$$

The product $\prod_{i=1}^d$ in (20) denotes the product of complex numbers. Theorem 6 is derived from Theorem 5 via a blow-up argument. One shows that, as $\varepsilon \to 0$, $u_\varepsilon(\varepsilon z + a_\varepsilon^i)$ converges to a solution of (10) which is a local minimizer of the energy in the sense of Theorem 5 and we may now identify this blow-up limit as $U(z)$ (modulo a rotation).

One may ask similar questions on the disc but the situation is widely open. Consider, for example, the equation

$$\begin{cases} -\Delta u = au(1-|u|^2) & \text{in } B = \text{the unit disc in } \mathbb{R}^2 \\ u(x) = x & \text{on } \partial B \end{cases} \tag{21}$$

where $a > 0$ is a constant and $u : B \to \mathbb{C} = \mathbb{R}^2$. It is easy to construct a "radial" solution

$$u = u(r,\theta) = e^{i\theta} f(r)$$

where f satisfies the ordinary differential equation

$$\begin{cases} -f'' - \frac{1}{r}f' + \frac{1}{r^2}f = af(1-f^2) \quad \text{in } (0,1) \\ f(0) = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{and } f(1) = 1. \end{cases}$$

This f is uniquely determined (see [BBH] and [HH]).

Open Problem 5. Is the radial solution $e^{i\theta}f(r)$ the only solution of (21)?

If $a \le \lambda_1$ (the first eigenvalue of $-\Delta$ on B with zero Dirichlet condition) the answer is positive since the energy functional

$$E(v) = \frac{1}{2}\int_B |\nabla v|^2 + \frac{a}{4}\int_B (|v|^2 - 1)^2$$

is strictly convex and thus (21) has a solution.

Remark 5. The argument described in the proof of Theorem 4 is still valid provided u/f makes sense at 0, i.e., $u(0) = 0$. Thus, another formulation of Open Problem 5 is

Open Problem 5′. Does any solution u of (21) vanish at 0?

A weaker form of Open Problem 5, which I find quite intriguing is

Open Problem 6. Is the radial solution $u = e^{i\theta}f(r)$ a minimizer of the energy E? More generally, if B is the unit ball in $\mathbb{R}^n, n \ge 2$ and $u : B \to \mathbb{R}^n$, is there a minimizer of E of the form $\frac{x}{|x|}f(|x|)$?

P. Mironescu [M1] (see also [LL]) has given a partial answer. He proves that the radial solution u is a local minimizer in the sense that $E(u) \le E(v)$ for all $v \in H^1$ such that $v(x) = x$ on ∂B and $\|v - u\|_{H^1}$ is small. In the scalar case rearrangement techniques (see e.g. [Ba]) are often used to prove that minimizers have radial symmetry. But in the vector-valued case no such method is available (see, however, the discussion after Theorem 9 below). Therefore it would be very interesting to settle Open Problem 6.

3. Questions of symmetry for minimizing harmonic maps.

Another simple nonlinear PDE system which has received much attention in recent years is the system of harmonic maps. Since we are interested in questions of symmetry we will assume that the target space is a sphere, say $S^{k-1}, k \ge 2$. The unknown u is a map from a domain $\Omega \subset \mathbb{R}^n$ with values into $\mathbb{R}^k$ satisfying

$$(22) \qquad \begin{cases} -\Delta u = u|\nabla u|^2 & \text{in } \Omega \\ |u| = 1 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $g : \partial\Omega \to S^{k-1}$ is a given (smooth) boundary condition. The solutions of (22) arise as critical points of the Dirichlet integral

$$E(v) = \int_\Omega |\nabla v|^2$$

subject to the constraint

$$v \in H^1_g(\Omega; S^{k-1}) = \{v : \Omega \to \mathbb{R}^k; \int_\Omega |\nabla v|^2 < \infty, |v| = 1 \quad \text{in } \Omega \text{ and } v = g \text{ on } \partial\Omega\}.$$

Of particular interest are minimizing harmonic maps, i.e., minimizers of E in $H^1_g(\Omega; S^{k-1})$. *Minimizing* harmonic maps seem to inherit some symmetry properties of the data. However, general harmonic maps, i.e., arbitrary (weak) solutions of (22) usually break symmetry. Here are some results.

THEOREM 7. *Let $\Omega = B$ be the unit ball in $\mathbb{R}^n$, $n \geq 3$. Assume $k = n$ and $g(x) = x$. Then $u(x) = x/|x|$ is a minimizing harmonic map; in fact it is the unique minimizer of E in $H^1_g(\Omega; S^{n-1})$.*

This result was originally proved by W. Jäger and H. Kaul [JK] when $n \geq 7$ (they even show that $x/|x|$ is a minimizer in $H^1_g(\Omega; S^n)$, where S^{n-1} is identified with an equator of S^n). Theorem 7 is due to H. Brezis, J. M. Coron and E. Lieb [BCL] when $n = 3$ and to F. H. Lin [Lin] for general $n \geq 3$. The proof of F. H. Lin is especially ingenious and elegant. The restriction $n \geq 3$ is needed. When $n = 2$ the class of testing functions $H^1_g(\Omega; S^1)$ is empty; this is a consequence of the fact that there is a degree theory for maps in $H^{1/2}(S^1; S^1)$ (see [BBH] and [BrN2]).

REMARK 6. There is no hope to prove that general (i.e., nonminimizing) harmonic maps inherit the radial symmetry of the boundary condition. In fact, T. Rivière [R] has constructed an abundance of weird solutions of (22) when Ω is the unit ball in $\mathbb{R}^3$, $k = 3$ and g is any nonconstant boundary condition (in particular $g(x) = x$).

REMARK 7. F. Almgren and E. Lieb [AL] have pointed out that natural notions of symmetry may be broken, even for minimizing harmonic maps. Consider, for example in 3-d the notion of mirror symmetry through the xy plane, i.e.,

$$\begin{aligned} u_1(x, y, -z) &= u_1(x, y, z) \\ u_2(x, y, -z) &= u_2(x, y, z) \\ u_3(x, y, -z) &= -u_3(x, y, z). \end{aligned}$$

They have constructed an example where Ω is the unit ball in $\mathbb{R}^3$, $k = 3$, the boundary condition g has mirror symmetry, but no minimizer has mirror symmetry.

When Ω is a 2-dimensional domain and $k \geq 3$ there seems to be a better chance for symmetry. Here are some situations where symmetry holds.

Let Ω be the unit disc in $\mathbb{R}^2$ and consider maps $u : \Omega \to S^2$. We say that u has radial symmetry if it can be written in the form

$$u(x, y) = (a(r)x, a(r)y, b(r))$$

where $r = (x^2 + y^2)^{1/2}$, $a(r)$ and $b(r)$ are real valued functions such that $a^2(r) + b^2(r) = 1$.

THEOREM 8 ([BC]). *Let Ω be the unit disc in $\mathbb{R}^2$ and let*

$$g(x,y) = (Rx, Ry, \sqrt{1-R^2}) \quad \text{for } (x,y) \in \partial\Omega \text{ with } 0 < R \le 1.$$

Then any minimizer of E in $H^1_g(\Omega; B^2)$ has radial symmetry. (In fact, there are precisely two minimizers.)

Open Problem 7. Is the same conclusion true for general (nonminimizing) solutions of (22)?

THEOREM 9. *Let $0 < \rho < 1$ and consider the annulus*

$$\Omega_\rho = \{(x,y) \in \mathbb{R}^2; \rho^2 < x^2 + y^2 < 1\}.$$

Consider the boundary condition

$$g(x,y) = (x,y,0) \quad \text{on } \partial\Omega_\rho.$$

Then any minimizer of E in $H^1_g(\Omega_\rho; S^2)$ has radial symmetry.

Theorem 9 was originally proved by E. Sandier [Sa1] in connection with results of F. Bethuel, H. Brezis, B. Coleman and F. Hélein [BBCH]. A new proof was given by S. Kaniel and I. Shafrir [KS]. It relies on a very interesting symmetrization device, which could possibly be useful for other vector-valued problems. It is also quite unexpected to have radial symmetry in the annulus since there are examples of broken symmetry for the annulus even in the scalar case (see Remark 1).

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