B. Ruf

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS.
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H BRÉZIS

Laser beams and limiting cases of Sobolev inequalities

INTRODUCTION

I will first describe a joint work with T. Gallouet [2] concerning the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} - \Delta u + k|u|^2u = 0 \quad \text{on } \Omega \times (0,+\infty) \quad \text{where} \quad \Omega \subset \mathbb{R}2$$

In order to solve this problem we rely on a limiting case of Sobolev's inequality in dimension two.

In the second part of my talk, I will describe a joint work with S. Wainger[3] dealing with more general limiting cases of Sobolev type inequalities in dimension N.

1. A NON LINEAR SCHRÖDINGER EQUATION

Let $R \subset R^2$ be a smooth domain. In practice Ω will be either a bounded domain, or an exterior domain, or a half-plane etc...

We look for a function $u(x,t) : \Omega \times (0,+\infty) \to C$ such that

$$i\frac{\partial u}{\partial t} - \Delta u + k|u|^2u = 0 \qquad \text{on} \quad \Omega \times (0,+\infty) \qquad (1)$$

$$u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,+\infty) \qquad (2)$$

$$u(x,0) = u_0(x) \qquad \text{on} \quad \Omega ,$$

where $k \in \mathbb{R}$ is a given constatit and $u_0(x)$ is prescribed.

This kind of problem occurs in nonlinear optics and has been extensively studied when $\Omega = \mathbb{R}^2$ (see [1], [4], [5], [8], [12]), but nothing seems to have been known for $\Omega \neq \mathbb{R}^2$.

On main result is the following

Theorem 1. *Assume* $u_0 \in H^2 \cap H_0^1$. *In addition assume that either* $k \geq 0$ *or* $k < 0$ *with* $|k| \int |u_0|^2 < 4$. *Then there exists a unique solution* u *of* (1), (2), (3) *such that*

$$u \in C([0,\infty); H^2 \cap H_0^1) \cap C^1([0,\infty); L^2).$$

Remarks. 1) If $k < 0$, it may happen for some initial conditions that the solution of (1), (2), (3), which exists for a small time interval, blows up in finite time (the example of Glassey [6]) can be extended to some exterior domains; it is not known whether blow-up may occur in finite time for bounded domains).

2) When $R = \mathbb{R}^2$ and $k \geq 0$, it ir known (see [4]) that $\|u(.,t)\|_{L^\infty} = O(\frac{\log t}{t})$ ar $t \to \infty$. Such a result presumably holds when Ω ir an exterior domain. However *if* Ω is *bounded*, $\|u(.,t)\|_{L^\infty}$ does not tend *to* 0 as $t \to \infty$ (except when $u_0 = 0$), since $\|u(.,t)\|_{L^2} = \|u_0\|_{L^2}$ (see the proof of Theorem 1). However *it* is plausible that $\|u(.,t)\|_{L^\infty}$ remains bounded as $t \to \infty$. (The proof of Theorem 1 shows only that $\|u(.,t)\|_{L^\infty} \leq C e^{ot}$).

Our next Lemma is an essential tool for the proof of Theorem 1.

Lemma 1. *There exists a constant* C *- depending only on* Ω *- such that*

$$\|u\|_{L^\infty} \leq C\,[1 + \log(1 + \|u\|_{H^2})]^{1/2}$$

for every $u \in H^2$ *with* $\|u\|_{H^1} \leq 1$.

Sketch of the proof of Lemma 1

Using standard extension methods we can reduce the proof of Lema 1 to the case where $\Omega = \mathbb{R}^2$. Let $\hat{u}$ denote the Fourier transform of u . We have

$$\|u\|_{L^\infty} \leq C\|\hat{u}\|_{L^1}$$

Let $R > 0$ (to be chosen later); we write

$$\|\hat{u}\|_{L^1} = \int_{|\xi|<R}(1+|\xi|)|\hat{u}(\xi)|\frac{1}{(1+|\xi|)}\,d\xi + \int_{|\xi|>R}(1+|\xi|^2)|\hat{u}(\xi)|\frac{1}{(1+|\xi|^2)}\,d\xi$$

$$\leq \|(1+|\xi|)\hat{u}(\xi)\|_{L^2}\left[\int_{|\xi|<R}\frac{1}{(1+|\xi^2|)^2}\,d\xi\right]^{1/2}$$

$$+\|(1+|\xi|^2)\hat{u}(\xi)\|_{L^2}\left[\int_{|\xi|>R}\frac{1}{(1+|\xi|^2)^2}\,d\xi\right]^{1/2}$$

From which it follows easily that

$$\|\hat{u}\|_{L^1} \leq C[\log(1+R)]^{1/2} + C\|u\|_{H^2}(1+R)^{-1}$$

Since $\|u\|_{H^1} = \|(1+|\xi|)\hat{u}(\xi)\|_{L^2} \leq 1$.

We conclude by choosing $R = \|u\|_{H^2}$.

In order to prove the existence of a local solution we shall rely on the following convenient variant of a classical result of I. Segal [10] :

Lemma 2. *Let* X *be a Banach space and let* $-A : D(A) \subset X \to X$ *be a linear unbounded operator which generates a continuous semi-group of contractions* $S(t)$.

Assume $F : D(A) \to D(A)$ *is a nonlinear mapping which is lipschitzian on bounded sets of* $D(A)$ *(for the graph norm).*

Then, for every $u_o \in D(A)$, *there exists a unique local solution of the equation*

$$\frac{du}{dt} + Au = Fu \quad \text{on} \quad [0,T)$$

$$u(0) = u_o$$

with $u \in C([0,T);D(A)) \cap C^1([0,T);X)$.

In addition the solution u *can be extended to a maximal interval* $[0,T_{max})$ *with the alternative : either* $T_{max} = \infty$, *or* $T_{max} < \infty$ *and* $\lim_{t \uparrow T_{max}} (\|u(t)\| + \|Au(t)\|) = \infty$.

Sketch of the proof of Theorem 1. We apply Lemma 2 in $X = L^2(\Omega;\mathbb{C})$ with $Au = i\Delta u$, $D(A) = H^2 \cap H_o^1$ and $Fu = ik|u|^2 u$. We obtain a solution defined on $[0,T_{max})$ with the alternative : *either* $T_{max} = \infty$ *or* $T_{max} < \infty$ and $\lim_{t \uparrow T_{max}} \|u(.,t)\|_{H^2} = \infty$

We shall now prove that the second part of the alternative is excluded by showing that $\|u(.,t)\|_{H^2}$ is bounded on every finite time interval.

First note that

$$\|u(.,t)\|_{L^2} = \|u_o\|_{L^2}, \quad \forall t \tag{4}$$

(It suffices to multiply (1) by $\bar{u}$, inttgrate by parts and use the imaginary part).

Next we have :

$$\frac{1}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx + \frac{k}{4} \int_\Omega |u(x,t)|^4 dx \equiv E_0, \quad \forall t . \qquad (5)$$

(It suffices to multiply (1) by $\frac{\partial \bar{u}}{\partial t}$, integrate over Ω and use the real part).

It follows thrt $\|u(.,t)\|_{H^1}$ remains bounded. Indeed this is clear when $k \geq 0$. Otherwise when $k < 0$ we have, by *(5)* and by an inequality of Gagliardo-Nirenberg 19]

$$\frac{1}{2} \int_\Omega |\nabla u(x,t)|^2 dx \leq C + \frac{|k|}{4} \int_\Omega |u(x,t)|^4 dx$$

$$\leq C + \frac{|k|}{8} \int_\Omega |u(x,t)|^2 \, dx \int_\Omega |\nabla u(x,t)|^2 dx$$

$$= C + \frac{|k|}{8} \|u_0\|^2_{L^2} \int_\Omega |\nabla u(x,t)|^2 dx$$

and the conclusion follows since $|k| \, \|u_0\|^2_{L^2} < 4$.

Finally we use the relation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Fu(s)ds.$$

and thus

$$Au(t) = S(t) \, Au_0 + \int_0^t S(t-s)AFu(s)ds$$

Taking the L^2 norm on both sides we find

$$\|u(.,t)\|_{H^2} \leq \|u_0\|_{H^2} + \int_0^t \|Fu(s)\|_{H^2} \, ds \qquad (6)$$

it is easy to check. using again an inequality of Gagliardo-Nirenberg that

$$\|Fu\|_{H^2} \leq C\|u\|^2_{L^\infty} \|u\|_{H^2} \qquad \forall\, u \in H^2 \tag{7}$$

It follows from (6) and (7) that

$$\|u(.t)\|_{H^2} \leq C + C \int_0^t \|u(.,s)\|^2_{L^\infty} \|u(.s)\|_{H^2}\, ds. \tag{8}$$

On the other hand we deduce from Lema 1,which holds sincc $\|u(.,t)\|_{H^1} \leq C$, that

$$\|u(.,s)\|^2_{L^\infty} \leq C\,[1+\log(1+\|u(.,s)\|_{H^2})] \tag{9}$$

Combining (8) and *(9)* leads to

$$\phi(t) \leq C + C \int_0^t [1+\log(1+\phi(s))]\ \phi(s)ds$$

where $\phi(s) = \|u(.,s)\|_{H^2}$. We conclude (as in Gronwall's inequality) that $\phi(t) \leq e^{\alpha e^{\beta t}}$ for some constants α and β.

2. REMARKS ON SOBOLEV TYPE INEQUALITIES IN THE LIMITING CASE

We start with an extension of Lema 1 to dimension $N \geq 2$. Let $\Omega \subset \mathbb{R}^N$

Theorem 2. (see [3]) *Let* $1 \leq p < \infty$, $1 \leq q \leq \infty$, j *and* k *be such that* $\frac{1}{p} - \frac{j}{N} = 0$, $\frac{1}{q} - \frac{k}{N} < 0$. *Then there exists a constant* C - *depending only on* R, p, q, j *and* k *such that*

$$\|u\|_{L^\infty} \leq C\,[1+\log(1+\|u\|_{W^{k,q}})]^{\frac{N-1}{N}},$$

$$\forall\ u \in W^{k,q} \text{ such that } \|u\|_{W^{j,p}} \leq 1 . \tag{10}$$

. Sketch of the proof We use the same method as **in Lemma** 1 with a **"continuous"** decomposition.

Let $\phi \in \mathcal{D}(\mathbb{R}^N)$ with' $\phi(0) = 1$ and $\operatorname{Supp} \phi \subset B(0,1)$. Let $\psi = 1-\phi$.

Write

$$u(x) = \int e^{ix\xi}\hat{u}(\xi)d\xi = \int e^{ix\xi}\hat{u}(\xi) \ [\phi(\tfrac{\xi}{R}) + \psi(\tfrac{\xi}{R})]d\xi = I_1 + I_2 .$$

Next we establish the following estimates

$$|I_1| \leq C\,[\log(1+R)]^{\frac{N-1}{N}}$$

and

$$|I_2| \leq C\,\|u\|_{W^{k,q}}(1+R)^{-\delta}$$

for some $\delta > 0$

Indeed we write

$$I_1 = \int e^{ix\xi}(1+|\xi|^2)^{j/2}\,\hat{u}(\xi)\ \frac{1}{(1+|\xi|^2)^{j/2}}\ \phi(\tfrac{\xi}{R})d\xi$$

$$= \mathcal{F}((1+|\xi|^2)^{j/2}\,\hat{u}(\xi)) * \mathcal{F}(\frac{1}{(1+|\xi|^2)^{j/2}}\ \phi(\tfrac{\xi}{R}))$$

and by Hölder

$$|I_1| \leq \|u\|_{W^{1,p}}\ \|\mathcal{F}(\frac{1}{(1+|\xi|^2)^{j/2}}\ \phi(\tfrac{\xi}{R}))\|_{L^{p'}}$$

Next one proves (see [3]) that

$$\| \mathcal{F}(\frac{1}{(1+|\xi|^2)^{j/2}} \phi(\frac{\xi}{R})) \|_{L^{p'}} \le C[\log(1+R)]^{\frac{N-1}{N}}$$

Similarly

$$|I_2| \le C \|u\|_{W^{k,q}} \| \mathcal{F}(\frac{1}{(1+|\xi|^2)^{k/2}} \psi(\frac{\xi}{R})) \|_{L^{q'}}$$

Next one proves (see [3]) that

$$\| \mathcal{F}(\frac{1}{(1+|\xi|^2)^{k/2}} \psi(\frac{\xi}{R})) \|_{L^{q'}} \le C(1+R)^{-\delta}$$

Finally we choose $R = \|u\|_{W^{k,q}}^{\frac{1}{\delta}}$.

~~Remark~~. It would be interesting to find a proof of Theorem 2 which does not aake use of Fourier transform.

We have tried-without success-to connect Theorem 2 with the well-known inequalities of Trudinger and Strichartz (see below). We have made nevertheless some new observations. First we recall these inequalities. For simplicity we assume that Ω is bounded.

Theorem 3. (Trudinger [14]) . *Let* $u \in W^{1,N}$; *then* $e^{|u|^{\frac{N}{N-1}}} \in L1$. *In addition this "injection" is "optimal".*

We deduce directly

Corollary 4. *Let* $u \in W^{j,p}$ *with* $j \ge 1$, $\frac{1}{p} - \frac{j}{N} = 0$;
Then $e^{|u|^{\frac{N}{N-1}}} \in L^1$.

Indeed, by Sobolev $W^{j,p} \subset W^{1,N}$.

A few years later, Strichartz [13] has improved Corollary 4 and shown

Theorem 5. *Let* $u \in W^{j,p}$ *with* $j \geq 1$ *and* $\frac{1}{p} - \frac{j}{N} = 0$; *then* $e^{|u|^{\frac{p}{p-1}}} \in L^1$.
In addition this "injection" is "optimal".

We have tried to understand *why* Corollary 4 did not provide the optimal result. While Sobolev's and Trudinger's inequalities are in some sense sharp. The reason is that Sobolev's inequality can be "slightly" improved by working in Lorentz spaces instead of Lebesgue space. True, the improvement is microscopic; however in some cases this improvement is "magnified". For example in our situation we use *first* Sobolev and *then* Trudinger. Here Trudinger's inequality plays the role of a "magnifying glass".
More precisely let us recall first the definition of Lorentz spaces :
Let $f(x)$ be a measurable function on Ω. Set

$$\alpha(t) = \text{meas}\,\{x \in \Omega \;\; ; \; |f(x)| > t\} \quad , \qquad t > 0$$

We denote by $f^*(t)$ the reciprocal function of α : $f^* = \alpha^{-1}$.
Given $1 < p < \infty$ and $1 \leq q \leq \infty$ we define the Lorentz space

$$L(p,q) = \{f;\; t^{1/p} f^*(t) \in L^q(0,\infty;\, \tfrac{dt}{t})\}$$

The properties of Lorentz spaces are discussed in [7] and [11]. Recall that

$$L(p,p) = L^p$$

$$L(p,\infty) = M^p$$

(M^p = Marcinkiewicz space = weak L^p space) ,

$$L(p,q_1) \subset L(p,q_2) \quad \text{if} \quad q_1 < q_2 \; .$$

The following result improves Sobolev's Theorem

<u>Theorem 6</u>. *Let* $u \in W^{j,p}$ *with* $p < N/j$, *then* $u \in L(p^*,p)$ *with* $\frac{1}{p^*} = \frac{1}{p} - \frac{j}{N}$

Note that the usual conclusion asserts that $u \in L^{p^*} = L(p^*,p^*)$ - a larger space than $L(p^*,p)$.

Theorem 6 is an easy consequence of the results of Stein and O'Neil concerning convolutions in Lorentz spaces.

Next we need the following version of Trudinger's inequality in Lorentr spaces :

<u>Theorem 7</u>. *Assume* $Du \in L(N,\alpha)$ *for some* $1 \le \alpha \le \infty$. *Then* $e^{|u|^{\frac{\alpha}{\alpha-1}}} \in L^1$ ($u \in L^\infty$ *when* $\alpha = 1$) .

The proof of Theorem 7 is based on a new convolution inequality in Lorentr spaces in the limiting case (which was not examined by Stein and O'Neil) : Assume $f \in L(p,q_1)$, $g \in L(p',q_2)$, then $u = f * g$ satisfies $e^{|u|^r} \in L^1$ with $\frac{1}{r} = 1 - \frac{1}{q_1} - \frac{1}{q_2}$ provided $\frac{1}{q_1} + \frac{1}{q_2} < 1$ (for the proof, see [3]) .
Combining Theorems 6 and 7 we deduce directly Theorem 5.

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