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Équations aux dérivées partielles/*Partial Differential Equations*

H^1 versus C^1 local minimizers

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Abstract – We consider functionals of the form $\Phi(u)=(1/2)\int_\Omega |\nabla u|^2-\int_\Omega F(x,u)$. Under suitable assumptions we prove that a local minimizer of Φ in the C^1 topology must be a local minimizer in the H^1 topology. This result is especially useful when the corresponding equation admits a sub and super solution.

Minima locaux relatifs à C^1 et H^1

Résumé – *On considère des fonctionnelles de la forme* $\Phi(u)=(1/2)\int_\Omega |\nabla u|^2-\int_\Omega F(x,u)$. *Sous des hypothèses convenables on prouve qu'un minimum local de* Φ *au sens de la norme* C^1 *est nécessairement un minimum local au sens de la norme* H^1. *Ce résultat est particulièrement utile dans le cas où l'équation correspondante admet une sous-solution et une sur-solution.*

Version française abrégée – On considère la fonctionnelle

$$\Phi(u)=\int_\Omega \frac{1}{2}|\nabla u|^2-\int_\Omega F(x,u)$$

définie sur $H_0^1(\Omega)$ où Ω est un ouvert borné régulier de $\mathbb{R}^n$ et $F(x,u)=\int_0^u f(x,s)\,ds$. On suppose que f est mesurable en x, continue en u et vérifie la condition naturelle de croissance

$$(1)\qquad |f(x,u)|\leqq C(1+|u|^p)\qquad \text{avec}\quad p\leqq (n+2)/(n-2).$$

Notre résultat principal est le suivant :

Théorème 1. – *On suppose que* $u_0\in H_0^1(\Omega)$ *est un minimum local de* Φ *pour la topologie* C^1, *c'est-à-dire qu'il existe* $r>0$ *tel que*

$$(2)\qquad \Phi(u_0)\leqq\Phi(u_0+v),\qquad \forall v\in C_0^1(\bar\Omega)\quad \textit{avec}\quad \|v\|_{C^1}\leqq r.$$

Alors u_0 *est aussi un minimum local de* Φ *pour la topologie* H^1, *c'est-à-dire qu'il existe* $\varepsilon_0>0$ *tel que*

$$(3)\qquad \Phi(u_0)\leqq\Phi(u_0+v),\qquad \forall v\in H_0^1(\Omega)\quad \textit{avec}\quad \|v\|_{H^1}\leqq \varepsilon_0.$$

Le théorème 1 est particulièrement utile lorsque l'équation d'Euler associée à Φ,

$$(4)\qquad \left|\begin{aligned} -\Delta u&=f(x,u)\quad \text{sur }\Omega\\ u&=0\quad \text{sur }\partial\Omega\end{aligned}\right.$$

admet une sous-solution et une sur-solution. Plus précisément, on suppose qu'il existe deux fonctions $\underline{u}$, $\bar u\in C(\bar\Omega)$ vérifiant, au sens des distributions,

$$-\Delta\underline{u}-f(x,\underline{u})\leqq 0\leqq -\Delta\bar u-f(x,\bar u)\quad \text{sur }\Omega$$

ainsi que $\underline{u}\leqq 0\leqq\bar u$ sur $\partial\Omega$. On suppose que $\underline{u}$ et $\bar u$ ne sont pas solutions de (4). On suppose enfin qu'il existe une constante k telle que

$$(5)\qquad f(x,u)+ku \text{ soit croissante en } u,\quad \text{p.p. en } x.$$

Note présentée par Haïm Brezis.

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THÉORÈME 2. — *Sous les hypothèses précédentes il existe une solution* u_0 *de* (4) *avec* $\underline{u}<u_0<\bar{u}$, *telle que, de plus,* u_0 *soit un minimum local de* Φ *dans* $H_0^1(\Omega)$.

For functions u in $H_0^1(\Omega)$ in a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, we consider the functional

$$\Phi(u)=\int_\Omega \frac{1}{2}|\nabla u|^2-F(x, u).$$

Here $F(x, u)=\int_0^u f(x, s)\,ds$ and we assume the natural growth condition

$$|f(x, u)|\leqq C(1+|u|^p) \quad \text{with} \quad p\leqq(n+2)/(n-2), \tag{1}$$

as well as the usual assumptions that f is measurable in x and continuous in u.

Our main result is the following:

THEOREM 1. — *Assume* $u_0\in H_0^1(\Omega)$ *is a local minimizer of* Φ *in the* C^1 *topology; this means that there is some* $r>0$ *such that*

$$\Phi(u_0)\leqq\Phi(u_0+v), \qquad \forall v\in C_0^1(\bar{\Omega}) \quad \textit{with} \quad \|v\|_{C^1}\leqq r. \tag{2}$$

Then u_0 *is a local minimizer of* Φ *in the* H_0^1 *topology*, i.e. *there exists* $\varepsilon_0>0$ *such that*

$$\Phi(u_0)\leqq\Phi(u_0+v), \qquad \forall v\in H_0^1(\Omega) \quad \textit{with} \quad \|v\|_{H^1}\leqq\varepsilon_0. \tag{3}$$

The theorem is somewhat surprising since an $H_0^1(\Omega)$ neighbourhood is much bigger than a C_0^1 neighbourhood. The proof involves the special structure of Φ – the claim would be false for a general functional Φ.

The proof is divided in 3 steps.

Step 1. — We claim that $u_0\in C^{1,\alpha}(\bar{\Omega})$, $\forall\alpha<1$. — Recall that u_0 satisfies in the weak sense

$$\left|\begin{array}{l} -\Delta u_0=f(x, u_0) \quad \text{in } \Omega, \\ u_0=0 \quad \text{on } \partial\Omega. \end{array}\right. \tag{4}$$

In case $p<(n+2)/(n-2)$ it is easy to prove the regularity of u_0 by a bootstrap argument (*see* e.g. [5]). For $p=(n+2)/(n-2)$ we present the argument – the standard bootstrap procedure does not work. We write $f(x, u_0)$ in the form

$$f(x, u_0)=a(x)u_0+b(x)$$

with

$$a(x)=\left|\begin{array}{ll} f(x, u_0(x))/u_0(x) & \text{where } |u_0(x)|>1 \\ 0 & \text{where } |u_0(x)|\leqq 1 \end{array}\right.$$

and

$$b(x)=\left|\begin{array}{ll} 0 & \text{where } |u_0(x)|>1 \\ f(x, u_0(x)) & \text{where } |u_0(x)|\leqq 1. \end{array}\right.$$

From (1) we have $|a(x)|\leqq C|u_0(x)|^{p-1}$. By Sobolev, $u_0\in L^{2n/(n-2)}$ and thus $a\in L^{n/2}$. On the other hand $b\in L^\infty$. Applying Theorem 2.3 in [6] we infer that $u_0\in L^q$, $\forall q<\infty$. Hence $f(x, u_0)\in L^q$, $\forall q<\infty$. From (4) we deduce that $u_0\in W^{2,q}$, $\forall q<\infty$. The claim is proved.

Without loss of generality we may now assume that $u_0=0$.

Step 2. — *Proof of Theorem* 1 *in the subcritical case* $p<(n+2)/(n-2)$. — Suppose the conclusion (3) does not hold. Then

$$\forall\varepsilon>0, \quad \exists v_\varepsilon\in B_\varepsilon \quad \text{such that } \Phi(v_\varepsilon)<\Phi(0) \tag{5}$$

where $B_\varepsilon = \{u \in H_0^1;\ \|u\|_{H^1} \leqq \varepsilon\}$. By a standard lower semicontinuity argument $\min_{B_\varepsilon} \Phi$ is achieved at some point which we may still denote by v_ε. We shall prove that $v_\varepsilon \to 0$ in C^1, but then (2) and (5) are contradictory (a similar argument is used in [11]). The corresponding Euler equation for v_ε involves a Lagrange multiplier $\mu_\varepsilon \leqq 0$, namely, v_ε satisfies

$$\langle \Phi'(v_\varepsilon), \zeta \rangle_{H^{-1}, H_0^1} = \mu_\varepsilon (v_\varepsilon, \zeta)_{H_0^1}, \qquad \forall \zeta \in H_0^1,$$

i. e.

$$\int_\Omega \nabla v_\varepsilon \cdot \nabla \zeta - f(x, v_\varepsilon)\zeta = \mu_\varepsilon \int_\Omega \nabla v_\varepsilon \cdot \nabla \zeta, \qquad \forall \zeta \in H_0^1.$$

This means

(6) $$-(1-\mu_\varepsilon)\,\Delta v_\varepsilon = f(x, v_\varepsilon).$$

Using (6) together with the assumption (1) with $p < (n+2)/(n-2)$ and the essential fact that $\mu_\varepsilon \leqq 0$, one may bootstrap the bound $\|v_\varepsilon\|_{H^1} \leqq C$ to $\|v_\varepsilon\|_{C^{1,\alpha}} \leqq C$ (independent of ε). Since $v_\varepsilon \to 0$ in H^1, $v_\varepsilon \to 0$ in C^1 (by Ascoli). This concludes the proof in the subcritical case.

Step 3. – *Proof of Theorem* 1 *in the critical case* $p = (n+2)/(n-2)$. – It is much more delicate, because the standard bootstrap argument mentioned above does not work. We rely once more on Theorem 2.3 of [6] in conjunction with the additional fact that $\|v_\varepsilon\|_{H^1} \to 0$ as $\varepsilon \to 0$. Suppose the conclusion (3) fails. Then (5) holds. For every j consider the truncation map

$$T_j(r) = \begin{cases} -j & \text{if } r \leqq -j, \\ r & \text{if } -j < r < j, \\ j & \text{if } r \geqq j. \end{cases}$$

Set

$$f_j(x, s) = f(x, T_j(s)), \qquad F_j(x, u) = \int_0^u f_j(x, s)\, ds$$

and

$$\Phi_j(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - F_j(x, u).$$

Note that, for each $u \in H_0^1$, $\Phi_j(u) \to \Phi(u)$ as $j \to \infty$. Hence, for each $\varepsilon > 0$ there is some $j = j(\varepsilon) \geqq 1$ such that $\Phi_j(v_\varepsilon) < \Phi(0)$. Clearly, $\min_{B_\varepsilon} \Phi_{j(\varepsilon)}$ is achieved at some point, say w_ε.

We have

(7) $$\Phi_{j(\varepsilon)}(w_\varepsilon) \leqq \Phi_{j(\varepsilon)}(v_\varepsilon) < \Phi(0).$$

CLAIM. – *One has* $w_\varepsilon \in C_0^1$ *and* $w_\varepsilon \to 0$ *in* C^1.

Assuming the Claim we see that, for ε small enough,

$$\Phi(w_\varepsilon) = \Phi_{j(\varepsilon)}(w_\varepsilon) < \Phi(0)$$

and this contradicts (2).

Proof of the Claim. – The Euler equation for w_ε is

(8) $$-(1-\mu_\varepsilon)\,\Delta w_\varepsilon = f_j(x, w_\varepsilon).$$

Note that

(9) $$|f_j(x, u)| \leqq C(1 + |u|^p)$$

with $p=(n+2)/(n-2)$ and C independent of j. Since $w_\varepsilon \to 0$ in H_0^1, it also converges in $L^{2n/(n-2)}$ and thus there is some fixed function $h \in L^{2n/(n-2)}$ such that, for a subsequence, still denoted (w_ε),

$$|w_\varepsilon| \leqq h$$

(*see* e. g. [4], Théorème IV.9). Therefore, by (9),

$$|f_j(x, w_\varepsilon)| \leqq C(1+a|w_\varepsilon|)$$

where $a=h^{4/(n-2)} \in L^{n/2}$. This implies, as before, that (w_ε) is bounded in any L^q space. Going back to (8), and using (9), we see that (w_ε) is bounded in $C^{1,\alpha}$. Consequently, $w_\varepsilon \to 0$ in C^1 since $w_\varepsilon \to 0$ in H_0^1. Theorem 1 is proved.

Next, we present a simple, useful, application of Theorem 1.

Consider Φ as in Theorem 1 with f such that for some constant k,

$$f(x, u)+ku \text{ is nondecreasing in } u \quad \text{for a.e. } x.$$

Assume we have $C(\bar{\Omega})$ sub and supersolutions $\underline{u}$ and $\bar{u}$, *i.e.* in the distribution sense

$$-\Delta\underline{u}-f(x, \underline{u}) \leqq 0 \leqq -\Delta\bar{u}-f(x, \bar{u}) \quad \text{in } \Omega,$$
$$\underline{u} \leqq 0 \leqq \bar{u} \quad \text{on } \partial\Omega.$$

Moreover, assume that neither $\underline{u}$ nor $\bar{u}$ is a solution of (4).

THEOREM 2. — *Under the assumptions above there is a solution u_0 of (4), $\underline{u}<u_0<\bar{u}$, such that, in addition, u_0 is a local minimum of Φ in H_0^1.*

The proof relies on Theorem 1 as well as on the following

THEOREM 3. — *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $u \in L^1_{loc}(\Omega)$ and assume that, for some $k \geqq 0$, u satisfies*

$$-\Delta u+ku \geqq 0 \quad \textit{in } \Omega,$$

$$u \geqq 0 \quad \textit{on } \Omega.$$

Then either $u \equiv 0$, or there exists $\varepsilon>0$ such that

(10) $$u(x) \geqq \varepsilon \operatorname{dist}(x, \partial\Omega) \quad \textit{in } \Omega.$$

Proof of Theorem 3. — The measure $\mu=-\Delta u+ku$ is nonnegative in Ω. We may assume $u \not\equiv 0$.

Case 1: $\mu \equiv 0$. In this case $u \in C^\infty(\Omega)$,

$$-\Delta u+ku=0, \qquad u \geqq 0 \quad \text{in } \Omega.$$

Since $u \not\equiv 0$, $u \geqq \delta>0$ in some closed ball B in Ω. Let Ω_j be an expanding sequence of subdomains of Ω with smooth boundaries and $\bigcup_j \Omega_j=\Omega$; suppose $B \subset \Omega_j$, $\forall j$. Let h_j be the solution in $\Omega_j \setminus B$ of

$$(-\Delta+k)h_j=0 \quad \text{in } \Omega_j \setminus B$$
$$h_j=\delta \quad \text{on } \partial B,$$
$$h_j=0 \quad \text{on } \partial\Omega_j.$$

Then $u \geqq h_j$ in $\Omega_j \setminus B$. As $j \to \infty$, we find

$$u \geqq h \quad \text{in } \Omega \setminus B,$$

where h solves

$$(-\Delta+k)h=0 \quad \text{in } \Omega \setminus B$$
$$h=\delta \quad \text{on } \partial B,$$
$$h=0 \quad \text{on } \partial\Omega.$$

Using the Hopf lemma one finds

$$h(x) \geqq \varepsilon \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega \setminus B$$

for some $\varepsilon > 0$. The conclusion of Theorem 3 then follows directly.

Case 2: $\mu \not\equiv 0$. Let $\zeta \in C_0^\infty(\Omega)$ be a cutoff function, $0 \leqq \zeta \leqq 1$, such that $\zeta\mu \not\equiv 0$. Let v be the solution of

$$\begin{aligned} (-\Delta + k)\, v &= \zeta\mu \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since v is smooth outside a compact set $K \subset \Omega$, it follows, as above, by the Hopf lemma that

$$v(x) \geqq \varepsilon \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega$$

for some $\varepsilon > 0$. The conclusion of Theorem 3 is a direct consequence of the following.

CLAIM. – *One has $u \geqq v$ in Ω.*

Proof of the Claim. – Given any $\alpha > 0$ we will prove that

$$\bar{u} = u + \alpha \geqq v \quad \text{in } \Omega.$$

The Claim then follows.

Note that

$$w = \bar{u} - v$$

satisfies

$$(11) \qquad (-\Delta + k)\, w = (1 - \zeta)\, \mu + k\alpha \geqq 0 \quad \text{in } \Omega$$

$$(12) \qquad w \geqq 0 \quad \text{in } N_\eta = \{ x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < \eta \}$$

provided η is sufficiently small (depending on α). The last property (12) follows from the fact that v is smooth near $\partial\Omega$ and $v = 0$ on $\partial\Omega$. Let (ρ_j) be a sequence of mollifiers with $\operatorname{supp} \rho_j \subset B(0, 1/j)$. Set $w_j(x) = \int_\Omega \rho_j(x - y)\, w(y)$.

Clearly w_j is smooth, and by (11) we have

$$(-\Delta + k)\, w_j \geqq 0 \quad \text{in } \Omega \setminus \bar{N}_{1/j}.$$

On the other hand, we deduce from (12) that

$$w_j \geqq 0 \quad \text{in } N_{(\eta - 1/j)}$$

and in particular

$$w_j \geqq 0 \quad \text{on } \partial(\Omega \setminus \bar{N}_{1/j})$$

provided $\eta > 2/j$. The maximum principle implies that

$$w_j \geqq 0 \quad \text{in } \Omega \setminus \bar{N}_{1/j}$$

when $\eta > 2/j$. Passing to the limit as $j \to \infty$ we see that

$$w \geqq 0 \quad \text{in } \Omega$$

which is the desired conclusion.

Proof of Theorem 2. – We introduce an auxiliary functional. Set

$$\tilde{f}(x, s) = \begin{cases} f(x, \underline{u}(x)) & \text{if } s < \underline{u}(x), \\ f(x, s) & \text{if } \underline{u}(x) \leqq s \leqq \bar{u}(x) \\ f(x, \bar{u}(x)) & \text{if } s > \bar{u}(x); \end{cases}$$

it is continuous in s. Then set

$$\tilde{F}(x, u)=\int_0^u \tilde{f}(x, s)\, ds$$

and

$$\tilde{\Phi}(u)=\int_\Omega \frac{1}{2}|\nabla u|^2-\tilde{F}(x, u).$$

Let u_0 be a minimizer of $\tilde{\Phi}$ on $H_0^1(\Omega)$; it is easily seen that the minimum is achieved and satisfies

$$-\Delta u_0=\tilde{f}(x, u_0) \quad \text{in } \Omega.$$

Thus $u_0 \in W^{2,p}(\Omega)$, $\forall p<\infty$. We claim that $\underline{u} \leqq u_0 \leqq \bar{u}$; we will just prove the first inequality. Indeed we have

$$(13) \qquad -\Delta(\underline{u}-u_0) \leqq f(x, \underline{u})-\tilde{f}(x, u_0)$$

and in particular

$$-\Delta(\underline{u}-u_0) \leqq 0 \quad \text{in } A=\{x \in \Omega;\ u_0(x)<\underline{u}(x)\}.$$

Since $\underline{u}-u_0 \leqq 0$ on ∂A, it follows from the maximum principle that $\underline{u}-u_0 \leqq 0$ in A. Therefore $A=\varnothing$ and the claim is proved.

Returning to (13) we have

$$-\Delta(\underline{u}-u_0)+k(\underline{u}-u_0) \leqq (f(x, \underline{u})+k\underline{u})-(f(x, u_0)+ku_0) \leqq 0.$$

Since $\underline{u}$ is not a solution, it follows from Theorem 3 that there is some $\varepsilon>0$ such that

$$\underline{u}(x)-u_0(x) \leqq -\varepsilon \operatorname{dist}(x, \partial\Omega), \qquad \forall x \in \Omega.$$

Similarly for $\bar{u}$; thus

$$\underline{u}(x)+\varepsilon \operatorname{dist}(x, \partial\Omega) \leqq u_0(x) \leqq \bar{u}(x)-\varepsilon \operatorname{dist}(x, \partial\Omega). \qquad \forall x \in \Omega.$$

It follows that if $u \in C_0^1(\bar{\Omega})$ and $\|u-u_0\|_{C^1} \leqq \varepsilon$ then

$$\underline{u} \leqq u \leqq \bar{u} \quad \text{in } \Omega.$$

Next, we use the fact that $\tilde{F}(x, u)-F(x, u)$ is a function of x alone for $u \in [\underline{u}(x), \bar{u}(x)]$. In particular, $\Phi(u)-\tilde{\Phi}(u)$ is constant for $\|u-u_0\|_{C^1} \leqq \varepsilon$. Hence, u_0 is a local minimum of Φ in the C^1 topology (since it is a global minimum for $\tilde{\Phi}$). Now, we invoke Theorem 1 to claim that u_0 is also a local minimum of Φ in the H_0^1 topology. This completes the proof of Theorem 2.

Remark 1. – The proof of existence of a solution between a sub and a supersolution by minimizing the modified functional $\tilde{\Phi}$ – or by minimizing Φ on the convex set $\{\underline{u} \leqq u \leqq \bar{u}\}$ – is standard (*see* e. g. [14], [12], [8], [9], [16]). This yields a local minimizer of Φ in the C^1 topology. The point of Theorem 2 is that it is a local minimizer of Φ in the H^1 topology.

Remark 2. – Another standard approach is via a monotone iteration (*see* e. g. [2]). In this way one obtains a minimal solution u_1 and a maximal solution u_2 between $\underline{u}$ and $\bar{u}$. They both satisfy (*see* e. g. [7])

$$\lambda_1(-\Delta-f_u(x, u_i)) \geqq 0, \qquad i=1, 2$$

where $\lambda_1(\ )$ denotes the first eigenvalue of the corresponding linearized problem. However, in principle, u_1 and u_2 need not be local minima of Φ.

Remark 3. — Theorem 2 holds if $\underline{u}$ and $\bar{u}$ belong to $H^1(\Omega)$ instead of $C(\bar{\Omega})$. The proof involves a slight modification of the above argument using Stampacchia's form of the maximum principle.

In many instances, one proves the existence of multiple solutions for problems of the form (4). A first solution is obtained via sub and super solutions and a second solution is then obtained with the aid of the mountain pass lemma (*see* e.g. [10], [11], [15]). Here are some other examples:

Example 1 ([7], [13]). — Consider the problem

$$\left| \begin{array}{l} -\Delta u = \lambda f(u), \quad u>0, \quad \text{in } \Omega, \\ \quad u=0 \quad \text{on } \partial\Omega, \end{array} \right. \tag{14}$$

with f in C^1, $f(0)=f'(0)=0$, $f>0$ on $(0, a)$ and $f<0$ on $(a, +\infty)$.
Then, there is some $0<\lambda^*<\infty$ such that

a) for every $\lambda>\lambda^*$, (14) has at least two solutions $u_1<u_2$,

b) for $\lambda<\lambda^*$, (14) has no solution.

Example 2 [3]. — Consider the problem

$$\left| \begin{array}{l} -\Delta u = \lambda u^q + u^p, \quad u>0, \quad \text{in } \Omega, \\ \quad u=0 \quad \text{on } \partial\Omega \end{array} \right. \tag{15}$$

with $0<q<1<p\leqq(n+2)/(n-2)$. Then, there is some $0<\lambda^*<\infty$ such that

a) for every $0<\lambda<\lambda^*$, (15) has at least two solutions $u_1<u_2$,

b) for $\lambda>\lambda^*$, (15) has no solution.

Example 3 [1]. — Consider the problem

$$\left| \begin{array}{l} -\Delta u - \lambda u = W(x) u^p, \quad u>0, \quad \text{in } \Omega, \\ \qquad u=0 \quad \text{on } \partial\Omega \end{array} \right. \tag{16}$$

with $1<p<(n+2)/(n-2)$, W changing sign and $\int_\Omega W e_1^{p+1}<0$ where e_1 is the principle positive eigenfunction of $-\Delta$. Then, there is some $\lambda^*>\lambda_1$ such that

a) for $\lambda\in(\lambda_1, \lambda^*)$ (16) has at least two solutions,

b) for $\lambda=\lambda_1$, (16) has at least one solution,

c) for $\lambda>\lambda^*$, (16) has no solution.

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