MULTIPLE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION ON $\mathbb{R}^N$ WITH NONLINEAR DEPENDENCE ON THE GRADIENT

By

Chao–Nien Chen

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MULTIPLE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION ON $\mathbb{R}^N$ WITH NONLINEAR DEPENDENCE ON THE GRADIENT

CHAO-NIEN CHEN*

§0. Introduction. The existence of solutions of the semilinear equation

$$\begin{aligned} -\Delta u = g(x,u) \quad , \quad x \in \mathbb{R}^N \, , \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty \end{aligned} \tag{0.1}$$

has received a great deal of attention in the last decade. Equations of this type arise in various applications, for example in the description of waves for nonlinear Schrödinger or Klein–Gordon equations [2,4,19,38], in the study of various stellar dynamics models [1,32] and in determining metrics which realize given scalar or Gaussian curvatures [5,23,26,30,31]. Problem (0.1) has also been studied for the function g having the form $g(x,u) = \lambda u + h(x,u)u$ with $h(x,0) = 0$; in this situation $u \equiv 0$ is a trivial solution for every $\lambda \in \mathbb{R}$ and the linearization about $u \equiv 0$ has a continuous spectrum. Related questions about nontrivial solutions bifurcate from the line of trivial solutions have been investigated [6–9, 17–18,20,24,39,40].

Existence of infinitely many solutions to the boundary value problems like (0.1) has been obtained by several authors [2,4,6,8,18,37,38] using variational methods. For $N \geq 2$, the variational approach in general does not provide any information about the shape of the solutions except for the existence of at least one positive solution. However, for a class of ordinary differential equations, Heinz [16] obtained an interesting relation between the Ljusternik–Schnirelman critical levels associated with a variational functional and nodal properties of the solutions. More recently, considering both the one-dimensional case and the radial case in higher dimensions, the author [6,8] used a Nehari's idea [29] to obtain the existence of solutions with any prescribed number of nodes. An additional advantage of this approach is that one can also handle equations with nonlinearities not necessary to be odd.

Besides the variational argument, the existence of radial solutions to (0.1), together with information about their nodal properties, can be obtained by shooting method [3,7,11, 17,22, 27,30,32]. In some situations, the shooting method also provides information for showing uniqueness results for the solutions [7,11,25,28]. However, in applying this method, one needs to count the number of zeros when it jumps. There is an example [8] which shows the number of zeros can jump in steps larger than one. Thus it could be a formidable task in counting the nodes of the solutions.

*Department of Mathematics, Indiana University, Bloomington, IN 47405. Part of results were obtained at the Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455

In this paper we are concerned with the existence of multiple solutions of

$$(I)\qquad \begin{aligned} &-u'' - \tfrac{N-1}{\rho}u' = \lambda u - F(\rho,u,u')u \ , \ 0<\rho<\infty \ , && (0.2)\\ &u'(0)=0 \quad , \quad \int_0^\infty u^2(\rho)\rho^{N-1}d\rho < \infty \ ; && (0.3) \end{aligned}$$

that is, we study the radially symmetric solutions to a class of nonlinear elliptic eigenvalue problems of the following type:

$$\begin{aligned} -\Delta u = \lambda u - \widetilde{F}(x,u,\nabla u)u \quad &, \quad x\in \mathbb{R}^N \ , \\ u \in L^2(\mathbb{R}^N) \ . \end{aligned}$$

Variational approach is doomed because of the dependence of F on u'. It seems to be difficult to apply shooting argument to equations having complicated nonlinear terms. We thus intend to use an approximation scheme to obtain the existence of nontrivial solutions of problem (I).

It is assume that the function F satisfies the following conditions.

(F1) $\lim\limits_{|y|+|z|\to 0} F(\rho,y,z)=0$ uniformly on compact subsets of $[0,\infty)$.

(F2) F is continuous on $[0,\infty)\times \mathbb{R}^2$ and is locally Lipschitz continuous in y and z.

(F3) For fixed $y\in\mathbb{R}$, $F(\rho,y,0)$ is a nondecreasing function of ρ.

(F4) For fixed $\rho\in[0,\infty)$, $F(\rho,y,0)$ is an increasing function of y if $y\geq 0$ and a decreasing function of y if $y\leq 0$.

Furthermore, we assume that F can be decomposed into the sum of two nonnegative functions $H(\rho,y)$ and $G(\rho,y,z)$ with H being continuous and locally Lipschitz in y and satisfying the following conditions:

(H1) There are positive numbers σ_i and continuous functions $w_i:[0,\infty)\to(0,\infty)$ which satisfy $\int_0^\infty w_i^{-2/\sigma_i}\rho^{N-1}d\rho<\infty$, $i=1,2$, such that $H(\rho,y)\geq w_1(\rho)|y|^{\sigma_1}$ for $\rho\in[0,\infty)$, $y\geq 0$ and $H(\rho,y)\geq w_2(\rho)|y|^{\sigma_2}$ for $\rho\in[0,\infty)$, $y\leq 0$.

(H2) For fixed $\rho\in[0,\infty)$, $H(\rho,y)$ is an increasing function of y if $y\geq 0$ and a decreasing function of y if $y\leq 0$.

Under the above assumptions, the existence of positive and negative solutions of problem (I) will be established.

Our next aim is concerned with the existence of solutions with multiple nodes. If $G(\rho,y,z)\equiv 0$ this question has been fairly understood [6,8,9,17,18,20]. However, it is left open if $G(\rho, y, z) \not\equiv 0$; even in the case G does not depend on z. It seems to be a reasonable starting point to treat G as a perturbation. A task here is to assure that there are no nodes degenerating at infinity as the approximate solutions passing to the limit. Indeed, the phenomenon of "losing nodes at infinity" can happen in some situations as an example [8,9] illustrated. Motivated by a recent paper [9], we achieve this task by making use of comparison argument to show that the nodes of the approximate solutions are uniformly bounded away from the infinity. In doing so, we look at the functions $y = \mathcal{L}_+(\lambda, \rho)$ and $\mathcal{L}_-(\lambda, x)$ defined by the positive and negative solutions of $\mathcal{H}(\rho, y) = \lambda$, where the function $\mathcal{H}$ is defined by

$$\mathcal{H}(\rho, y) = H(\rho, \rho^{(1-N)/2} y) \tag{0.4}$$

for $(\rho, y) \in (0, \infty) \times \mathbb{R}$. For every $\lambda > 0$, the functions $\mathcal{L}_+(\lambda, \cdot)$ and $\mathcal{L}_-(\lambda, \cdot)$ are well-defined by (H2). Furthermore, by the implicit function theorem $\mathcal{L}_+(\lambda, \cdot)$ and $\mathcal{L}_-(\lambda, \cdot)$ are continuously differentiable functions if $\mathcal{H}$ is continuously differentiable and $y \cdot \frac{\partial}{\partial y} \mathcal{H}(\rho, y) > 0$ for $\rho > 0$ and $y \neq 0$. We will show the existence of solutions with any prescribed number of nodes if the following additional assumptions are satisfied.

($\mathcal{L}$1) For any $\lambda > 0$, there is a $\delta \in (0, \lambda)$ and an $X > 0$ such that if $\rho \geq X$ then $\mathcal{L}'_-(\lambda + \delta, \rho) > 0$, $\mathcal{L}'_+(\lambda + \delta, \rho) < 0$, $\mathcal{L}'_+(\lambda - \delta, \rho) \leq (\lambda + \delta)^{1/2} \mathcal{L}_-(\lambda + \delta, \rho)$ and $\mathcal{L}'_-(\lambda - \delta, \rho) \geq (\lambda + \delta)^{1/2} \mathcal{L}_+(\lambda + \delta, \rho)$.

(G1) For any $\delta, \mu, M > 0$, there is an $X_1 > 0$ such that if $\rho > X_1$, $|y| \leq \rho^{(1-N)/2} \mathcal{L}(\mu, \rho)$ and $|z| \leq M \rho^{(1-N)/2} \mathcal{L}(\mu, \rho)$ then $G(\rho, y, z) \leq \frac{\delta}{2}$, where $\mathcal{L}(\mu, \rho) = \mathrm{Max}(\mathcal{L}_+(\mu, \rho), -\mathcal{L}_-(\mu, \rho))$.

Note that the decomposition of F into H and G may not be unique. For example, if $F(\rho, y, z) = w(\rho)|y| + |y|^2 + |z|^2$, we can take $H(\rho, y)$ to be $w(\rho)|y|$ or $w(\rho)|y| + |y|^2$. Let us assume $w(\rho) = \rho^{(N-1)/2} e^{\rho^2}$ for $\rho \geq 1$ and choose $H(\rho, y) = w(\rho)|y|$. Then direct computation yields $\mathcal{L}_+(\lambda, \rho) = -\mathcal{L}_-(\lambda, \rho) = \lambda e^{-\rho^2}$ and $\mathcal{L}'_-(\lambda, \rho) = -\mathcal{L}'_+(\lambda, \rho) = 2\lambda \rho e^{-\rho^2}$ for $\rho \geq 1$. It is easy to check that condition ($\mathcal{L}$1) is satisfied. Condition (G1) evidently indicates that the existence result persists under "small" perturbation such as $|u|^2 + |u'|^2$. Indeed, for this H a much "stronger" perturbation like $r_1(\rho)|u|^{p_1} + r_2(\rho)|u'|^{p_2} + r(\rho)|u|^p|u'|^q$ still satisfies (G1), provided that

$$\lim_{\rho \to \infty} [(r_1(\rho))^{1/p_1} + (r_2(\rho))^{1/p_2} + (r(\rho))^{1/(p+q)}] \rho^{(1-N)/2} e^{-\rho^2} = 0 \ .$$

Some examples for functions H satisfying condition ($\mathcal{L}$1) can be found in [9].

There is no need to assume (F3) and (F4) if the function F does not depend on z. This generalize an existence result of [9] in the direction that F needs not to be monotone in y.

§1 **Main Results.** Let $L^2_\rho[0,\infty)$ be the weighted Hilbert space of u such that $\int_0^\infty u^2(\rho)\rho^{N-1}d\rho < \infty$. Define $H^1_\rho[0,\infty)$ by $u \in H^1_\rho[0,\infty)$ if and only if $u \in L^2_\rho[0,\infty)$ and $u' \in L^2_\rho[0,\infty)$. By a solution of (I) we mean $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ which satisfies (0.2), (0.3) and

$$\lim_{\rho\to\infty} \rho^{(N-1)/2}u(\rho) = \lim_{\rho\to\infty} \rho^{(N-1)/2}u'(\rho) = 0 \ . \tag{1.1}$$

THEOREM 1. *Suppose that (F1)–(F4) and (H1) are satisfied. For every $\lambda > 0$, there is a positive solution of (I).*

REMARK 1. (a) A negative solution for problem (I) can be obtained by the same argument.
(b) If $G(\rho, y, z) \equiv 0$, it has been proved that the positive solution as well as the negative solution for problem (I) is unique. Such uniqueness result also holds for bounded interval case under certain boundary conditions (see e.g. [6, Theorem 4.17]).
(c) It is easy to verify that $u \equiv 0$ is the only solution of (I) if $\lambda \le 0$.

For $\lambda > 0$ and $n \in \mathbb{N}$, let $S_n^+(\lambda)$ denote the set of $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ such that u is a solution of (I), $u > 0$ in a deleted neighborhood of $\rho = 0$ and u has exactly $n-1$ simple zeros in $(0,\infty)$. Similarly $S_n^-(\lambda)$ denotes the set $u < 0$ in a deleted neighborhood of $\rho = 0$.

THEOREM 2. *Assume that (F1)–(F4), (H1), (H2), (G1) and ($\mathcal{L}$1) are satisfied. Then for every $\lambda > 0$ and $n \in \mathbb{N}$, $S_n^+(\lambda)$ and $S_n^-(\lambda)$ are nonempty.*

REMARK 2. If $N = 1$ or the function F does not depend on z, there is no need to assume (F3) and (F4) in Theorems 1 and 2. This will be clear in the proof and a further discussion will be given in section 3.

To prove Theorems 1 and 2, we need some preliminary. Our strategy for obtaining solutions of (I) is taking as approximate solutions those of

$$(I)_{b,\varepsilon} \quad \begin{aligned} &-((\rho+\varepsilon)^{N-1}u')' = \lambda(\rho+\varepsilon)^{N-1}u - (\rho+\varepsilon)^{N-1}F(\rho,u,u')u \ , \ 0<\rho<b \ , && (1.2)\\ &u'(0) = u(b) = 0 \ , && (1.3)\end{aligned}$$

where $\varepsilon > 0$ and $b \in (0,\infty)$. We note that equation (1.2) is equivalent to

$$-u'' - \frac{N-1}{\rho+\varepsilon}\, u' = \lambda u - F(\rho,u,u')u \ . \tag{1.4}$$

If $(I)_{b,\varepsilon}$ is linearized about the trivial solution $u \equiv 0$, we get

$$\begin{aligned} -((\rho+\varepsilon)^{N-1}v')' &= \lambda(\rho+\varepsilon)^{N-1}v \ , \\ v'(0) &= v(b) = 0 \ . \end{aligned} \tag{1.5}$$

It is known [12] that (1.5) possesses a sequence of eigenvalues

$$0 < \mu_1(b,\varepsilon) < \mu_2(b,\varepsilon) < \cdots < \mu_n(b,\varepsilon) < \cdots \ . \tag{1.6}$$

The functions μ_n are continuous in b and ε, and are decreasing functions of b if ε is fixed, and satisfy

$$\lim_{\substack{\varepsilon\to 0^+ \\ b\to 0^+}} \mu_n(b,\varepsilon) = \infty \tag{1.7}$$

and

$$\lim_{\substack{\varepsilon\to 0^+ \\ b\to \infty}} \mu_n(b,\varepsilon) = 0 \ . \tag{1.8}$$

For $\lambda > 0$, let $S^+_{b,\varepsilon,n}(\lambda)$ denote the set of $u \in C^1[0,b]$ such that u satisfies $(I)_{b,\varepsilon}$, $u > 0$ in a deleted neighborhood of $\rho = 0$ and u has exactly $n-1$ simple zeros in $(0,b)$. Similarly $S^-_{b,\varepsilon,n}(\lambda)$ denotes the set $u < 0$ in a deleted neighborhood of $\rho = 0$. If (F1) and (F2) are satisfied, a direct application of the global bifurcation theorem of Rabinowitz [36, Chap. 4] shows that $(I)_{b,\varepsilon}$ possesses unbounded components $C^+_n(b,\varepsilon)$ and $C^-_n(b,\varepsilon)$ of solutions in $\mathbb{R} \times C^1[0,b]$, containing $(\mu_n(b,\varepsilon),\ 0)$ and having the property that if $(\lambda,u) \in C^\pm_n(b,\varepsilon)$ and $u \not\equiv 0$ then $u \in S^\pm_{b,\varepsilon,n}(\lambda)$. Moreover, under the assumption that $F(\rho,y,z) \geq 0$, we have some information about the solutions of $(I)_{b,\varepsilon}$ as follows.

PROPOSITION 1. *(i) If $\lambda \leq \mu_1(b,\varepsilon)$ and u is a solution of $(I)_{b,\varepsilon}$ then $u \equiv 0$.*
(ii) If $\lambda \leq \mu_n(b,\varepsilon)$ and u is a solution of $(I)_{b,\varepsilon}$ then $u \notin S^\pm_{b,\varepsilon,n}(\lambda)$.

The proof may proceed like that of [13, Corollary 2.11] by making use of standard comparison theorem. We omit it.

We now state some results concerning upper bounds for u and u'.

LEMMA 1. *Assume that (F2) and (H1) are satisfied. Let $\varepsilon_0 > 0$ and $\lambda > 0$ be fixed. For all $\varepsilon \in [0,\varepsilon_0]$ and $b \geq 1$ there are constant K_1 and K_2, not depending on b and ε, such that if u is a solution of $(I)_{b,\varepsilon}$ then*

$$\int_0^b (\rho+\varepsilon)^{N-1}(u^2 + u'^2)\,d\rho \leq K_1 \tag{1.9}$$

and

$$\|u\|_{L^\infty[0,b]} \leq K_2 \ . \tag{1.10}$$

We omit the proof, which only requires slight modifications in those of [6, Lemma 4.8 and Lemma 4.12]. The next two lemmas will be proved in section 2.

LEMMA 2. *Assume that (F2)–(F4) and (H1) are satisfied. Let $\lambda_0 > 0$ be fixed. There are ε_0 and K_3 such that for any $\lambda \le \lambda_0$ and $b \ge 1$, if $\varepsilon \in [0, \varepsilon_0]$ and u is a solution of $(I)_{b,\varepsilon}$ then*

$$\|u'\|_{L^\infty[0,b]} \le K_3 \ . \tag{1.11}$$

LEMMA 3. *Assume that (F2) – (F4) and (H1) are satisfied. For any fixed $\gamma \ge 1$, there are ε_0 and K_4 such that if $\varepsilon \in [0, \varepsilon_0]$, $b \ge \gamma$ and u is a solution of $(I)_{b,\varepsilon}$ then*

$$|u'(\rho)| \ \le K_4 \rho \tag{1.12}$$

for $\rho \in [0, \gamma]$. The number ε_0 can be chosen independent of γ.

The estimates established in the above lemmas together with arguments used in [36, Proposition 4.32] yields the existence of solutions of $(I)_{b,\varepsilon}$ stated as follows.

PROPOSITION 2. *Assume that (F1) – (F4) and (H1) are satisfied. For fixed $\lambda > 0$, there are positive numbers ε_0 and β_n such that $S^{\pm}_{b,\varepsilon,n}(\lambda) \ne \phi$ if $\varepsilon \in [0, \varepsilon_0]$ and $b \ge \beta_n$.*

The proof is straight, followed by the idea of [36, Proposition 4.32], and we omit it. The next proposition provides somewhat barrier effect to prevent the limit of the approximate solutions from degenerating to the trivial solution.

PROPOSITION 3. *Assume that (F1) and (F2) are satisfied. Let $\mu_n(b, \varepsilon)$ be defined as in (1.6) and b_n be the number such that $\mu_n(b_n, 0) = \lambda$. Let $U(\xi, \varepsilon, \cdot)$ be the solution of the initial value problem*

$$-u'' - \frac{N-1}{\rho+\varepsilon}\, u' = \lambda u - F(\rho, u, u')u \ ,$$
$$u(0) = \xi \ , \ u'(0) = 0 \ ,$$

which is understood to be extended to its maximal interval of definition. For every $n \in \mathbb{N}$, there are positive numbers ε_n and δ_n such that if $b > b_n$, $\varepsilon \in [0, \varepsilon_n]$ and $0 < |\xi| \ < \delta_n$, then $U(\xi, \varepsilon, \cdot)$ has at least n zeros in $(0, b)$.

The proof is similar to that of [7, Proposition 1.4]. We omit it.

Proof of Theorem 1. Let $\{b_k\}$ be an increasing sequence and $\{\varepsilon_k\}$ be a decreasing sequence such that $b_k \to \infty$ and $\varepsilon_k \to 0$ as $k \to \infty$. For each k, by Proposition 2 there is a function $u_k \in S_{b_k,\varepsilon_k,1}(\lambda)$; that is, a positive solution of $(I)_{b_k,\varepsilon_k}$. For each fixed l, we know from Lemma 1 and Lemma 3 that

$$\|u_k\|_{C^1[0,b_l]} \le C_1$$

and

$$\operatorname*{Max}_{\rho\varepsilon[0,b_l]} \left| \frac{u_k'(\rho)}{\rho + \varepsilon_k} \right| \ \le C_2$$

for all $k \geq l$, where C_1, C_2 are constant. By the same lines of reasoning as in the proof of [6, Theorem 4.18], there exist a subsequence $\{u_{k_j}\}$ and a function $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$ such that

$$u_{k_j} \xrightarrow{C^2} u \qquad \text{uniformly on compact subsets of } [0,\infty). \tag{1.13}$$

This together with (1.12) shows $u'(0) = 0$. Also the same sort of arguments used in the proof of [6, Theorem 4.18] shows that (1.1) holds.

We now show that $u > 0$ on $[0,\infty)$. Let $\xi_j = u_{k_j}(0)$. It follows from Proposition 3 that $\xi_j \geq \delta_1$ for all large j. Since $u(0) = \lim_{j\to\infty} \xi_j \geq \delta_1 > 0$, we know $u \not\equiv 0$. Next we note that $u \geq 0$ on the whole interval $[0,\infty)$ by virtue of (1.13) and the positivity of u_{k_j}. If $u(\rho) = 0$ for some $\rho \in (0,\infty)$, we would have $u'(\rho) = 0$. Then a modified version of [36, Lemma 4.12] would imply $u \equiv 0$. This is impossible and hence completes the proof.

Proof of Theorem 2. Arguing like the proof of Theorem 1, with only $u_k \in S^+_{b_k,\varepsilon_k,1}(\lambda)$ changed to $u_k \in S^\pm_{b_k,\varepsilon_k,n}(\lambda)$, we obtain a subsequence, still denoted by $\{u_k\}$, and a function $u \in C^2[0,\infty) \cap H^1_\rho[0,\infty)$, satisfying (1.1) and $u'(0) = 0$, such that

$$u_k \xrightarrow{C^2} u \qquad \text{uniformly on compact subsets of } [0,\infty). \tag{1.14}$$

Furthermore, we see that $u \not\equiv 0$ and all zeros of u must be simple zeros. Thus it remains to show that u has exactly $n-1$ zeros in $(0,\infty)$.

Indeed, using (1.14) and the fact that u_k has exactly $n-1$ zeros in $(0, b_k)$, we conclude that u has at most $n-1$ zeros in $(0,\infty)$. The above argument also shows $S^\pm_{b,0,n}(\lambda) \neq \phi$ for all large b if we choose $b_k = b$ for all k and let $\varepsilon_k \to 0$ as $k \to \infty$. Thus we may let $\varepsilon_k = 0$ for all k in the beginning of the proof. To show u has exactly $n-1$ zeros in $(0,\infty)$, we need

LEMMA 4. *Assume that (F1) – (F2), (H1), (H2), (G1) and ($\mathcal{L}$1) are satisfied. For any fixed subinterval $[a_1, a_2]$ of $(0,\infty)$, there is a positive number β, depending on a_2 only, such that for $b \geq \beta$ and $a \in [a_1, a_2]$, if U_- is a negative solution of*

$$(II)_{b,c} \qquad \begin{gathered} -(\rho^{N-1}u')' = \lambda\rho^{N-1}u - \rho^{N-1}F(\rho,u,u')u\ ,\ b < \rho < c\ , \\ u(b) = u(c) = 0 \end{gathered}$$

and U_+ is a positive solution of $(II)_{a,b}$ then

$$U'_+(b) < U'_-(b)\ . \tag{1.15}$$

The inequality (1.15) also holds if U_+ is a positive solution of

$$\begin{gathered} -(\rho^{N-1}u')' = \lambda\rho^{N-1}u - \rho^{N-1}F(\rho,u,u')u\ ,\ 0 < \rho < b\ , \\ u'(0) = u(b) = 0\ . \end{gathered}$$

We postpone the proof of Lemma 4 to the next section and continue the proof of Theorem 2 now. Let $n \geq 2$. Suppose that u has exactly $j-1$ zeros in $(0,\infty)$ for some $j < n$. Let $z_1, z_2, \ldots, z_{j-1}$ be the zeros of u in $(0,\infty)$ and $z_1(k)$, $z_2(k), \ldots, z_{n-1}(k)$ be the zeros of u_k in $(0, b_k)$, $k = 1,2,3\ldots$. It follows from (1.14) that

$$\lim_{k\to\infty} z_i(k) = z_i \ , \ 1 \leq i \leq j-1$$

and

$$\lim_{k\to\infty} z_i(k) = \infty \ , \ j \leq i \leq n-1 \ .$$

Let $z_{j-1}(k) = 0$ if $j = 1$ and $z_{j+1}(k) = b_k$ if $j = n-1$. We may assume without loss of generality that $u_k(\rho) > 0$ for $\rho \in (z_{j-1}(k),\ z_j(k))$ and $u_k(\rho) < 0$ for $\rho \in (z_j(k)\ ,\ z_{j+1}(k))$, since the other case can be treated similarly. For each k, letting $U_+(\rho) = u_k(\rho)$ for $\rho \in [z_{j-1}(k),\ z_j(k)]$ and $U_-(\rho) = u_k(\rho)$ for $\rho \in [z_j(k), z_{j+1}(k)]$, we see that

$$U'_+(z_j(k)) = u'_k(z_j(k)) = U'_-(z_j(k)) \ . \tag{1.16}$$

However, if k is sufficiently large, it follows from Lemma 4 that $U'_+(z_j(k)) < U'_-(z_j(k))$, which is incompatible with (1.16). Therefore u must have exactly $n-1$ zeros in $(0,\infty)$ and the proof is complete.

§2. **Proofs of Preliminary Lemmas.** We now prove several lemmas stated in section 1. Since the proof of Lemma 2 is more involved, we prove Lemma 3 first.

Proof of Lemma 3. Integrating (1.2) over $[0,\rho]$ together with (1.3) yields

$$(\rho+\varepsilon)^{N-1}u'(\rho) = \int_0^\rho [F(t,u,u')u - \lambda u](t+\varepsilon)^{N-1}dt \ .$$

Invoking the mean value theorem for integrals, we obtain

$$(\rho+\varepsilon)^{N-1}u'(\rho) = A(s)(s+\varepsilon)^{N-1}\rho \ ,$$

where $A(s) = [F(s,u(s),u'(s)) - \lambda]u(s)$ for some $s \in [0,\rho]$. Thus (1.12) easily follows by letting

$$K_4 = \operatorname*{Max}_{\substack{0\leq t\leq\gamma \\ 0\leq y\leq K_2 \\ 0\leq z\leq K_3}} [F(t,y,z)+\lambda]|y| \ ,$$

where K_2, K_3 are constant appeared in (1.10) and (1.11).

Proof of Lemma 2. If $u \equiv 0$, (1.11) is clearly satisfied. Let u be a nontrivial solution of $(I)_{b,\varepsilon}$ and $R(\rho) = \lambda u^2(\rho) + u'^2(\rho)$. It follows, with the aid of Eq. (1.4), that

$$\frac{1}{2}\ R'(\rho) = F(\rho,u(\rho),u'(\rho))u(\rho)u'(\rho) - \frac{N-1}{\rho+\varepsilon}\ u'^2(\rho) \ . \tag{2.1}$$

Let s_1 be the first zero of u in $(0, b]$. We may assume without loss of generality that $u(\rho) > 0$ for $\rho \in (0, s_1)$, since the case $u(\rho) < 0$ for $\rho \in (0, s_1)$ can be treated similarly. We claim

$$u'(\rho) \leq 0 \quad \text{for} \quad \rho \in (0, s_1] \ . \tag{2.2}$$

Assuming (2.2) for now and using the hypothesis $F \geq 0$, we conclude that $R'(\rho) \leq 0$ and hence

$$u'^2(\rho) \leq R(\rho) \leq R(0) = \lambda u^2(0) \leq \lambda \|u\|^2_{L^\infty[0,b]}$$

for $\rho \in [0, s_1]$. Invoking Lemma 1 yields

$$\|u'\|_{L^\infty[0,s_1]} \leq \sqrt{\lambda}\ K_2 \ . \tag{2.3}$$

To show (2.2), we prove indirectly. Suppose $u'(\rho_1) > 0$ for some $\rho_1 \in (0, s_1]$. By the continuity of u', we know that $u' > 0$ on an interval containing the point ρ_1. Let $[\rho_2, \rho_3]$ be the largest subinterval of $[0, s_1]$ such that $\rho_1 \in [\rho_2, \rho_3]$ and $u'(\rho) \geq 0$ if $\rho \in [\rho_2, \rho_3]$. Note that $u'(s_1) < 0$, since $u'(s_1) = 0$ would imply $u \equiv 0$. Thus $\rho_3 < s_1$ and u has a local maximum at ρ_3. Hence we have $u'(\rho_3) = 0$ and $u''(\rho_3) \leq 0$. It follows from Eq. (1.4) that

$$F(\rho_3, u(\rho_3), 0) \leq \lambda \ . \tag{2.4}$$

Since $u'(\rho_1) > 0$ and $u'(\rho) \geq 0$ for $\rho \in [\rho_2, \rho_3]$, we know that $u(\rho_2) < u(\rho_3)$. This together with (F4), (F3) and (2.4) yields

$$F(\rho_2, u(\rho_2), 0) < F(\rho_2, u(\rho_3), 0) \leq F(\rho_3, u(\rho_3), 0) \leq \lambda \ . \tag{2.5}$$

Since $u'(0) = 0$, we know from the definition of $[\rho_2, \rho_3]$ that $u'(\rho_2) = 0$ and u has a local minimum at ρ_2. On the other hand, it follows from (1.4) and (2.5) that $u''(\rho_2) < 0$. Clearly, a contradiction occurs and thus we conclude that (2.2) must be valid.

To prove the lemma for the case $\rho > s_1$, we make the transformation $v(\rho) = (\rho + \varepsilon)^{(N-1)/2} u(\rho)$. Then (1.4) takes the form

$$-v'' = [\lambda - \frac{(N-1)(N-3)}{4(\rho+\varepsilon)^2} - \mathcal{F}(\rho, v, v')]v \ , \tag{2.6}$$

where the function $\mathcal{F}$ is defined by

$$\mathcal{F}(\rho, y, z) = F(\rho, (\rho+\varepsilon)^{(1-N)/2} y, (\rho+\varepsilon)^{(1-N)/2} z - \frac{(N-1)}{2}(\rho+\varepsilon)^{-(N+1)/2} y) \ . \tag{2.7}$$

We have several estimates for v and v' stated as follows.

LEMMA 5. *Assume that (F2) and (H1) are satisfied. Let $b > a > 0$. If v satisfies (2.6) and $v(a) = v(b) = 0$, then*

$$\|v\|_{L^2[a,b]} \le K_5 \ , \tag{2.8}$$

$$\|v'\|_{L^2[a,b]} \le K_6 \ , \tag{2.9}$$

$$\|v\|_{L^\infty[a,b]} \le K_7 \tag{2.10}$$

and

$$\|v'\|_{L^\infty[a,b]} \le K_8 \ , \tag{2.11}$$

where

$$K_5 = K_5(\lambda, a, \varepsilon) = \sum_{i=1}^{2} \left(\lambda + \frac{1}{4a^2}\right)^{1/\sigma_i} \left(\int_a^\infty w_i^{-2/\sigma_i}(\rho+\varepsilon)^{N-1} d\rho\right)^{1/2} , \tag{2.12}$$

$$K_6 = \left(\lambda + \frac{1}{4a^2}\right)^{1/2} K_5 \ , \tag{2.13}$$

$$K_7 = (2K_5K_6)^{1/2} \tag{2.14}$$

and

$$K_8 = \left(\lambda + \frac{1}{4a^2}\right)^{1/2} K_7 \ . \tag{2.15}$$

Assuming Lemma 5 for now, we continue the proof of Lemma 2. Let $\lambda \le \lambda_0$. We claim the following:

> There are positive numbers ε_0 and s_0 such that if $\varepsilon \in [0, \varepsilon_0]$ and u_1 is a positive or a negative solution of $(I)_{b,\varepsilon}$ then $b \ge s_0$. (2.16)

To see this, we note that the function $\mu_1(b, \varepsilon)$, defined in (1.6), is continuous and satisfies (1.7). Thus there are positive numbers ε_0 and s_0 such that $\mu_1(b, \varepsilon) > \lambda_0$ if $\varepsilon \in [0, \varepsilon_0]$ and $b \in [0, s_0]$. By Proposition 1 (i), $(I)_{b,\varepsilon}$ cannot possess a nontrivial solution if $\varepsilon \in [0, \varepsilon_0]$ and $b \in [0, s_0]$. Therefore (1.29) must hold.

By direct computation, $u'(\rho) = \frac{(1-N)}{2}(\rho+\varepsilon)^{-(N+1)/2}v(\rho) + (\rho+\varepsilon)^{(1-N)/2}v'(\rho)$. Hence we have

$$\|u'\|_{L^\infty[s_1,b]} \le \frac{(N-1)}{2} \ s_1^{-(N+1)/2}\|v\|_{L^\infty[s_1,b]} + s_1^{(1-N)/2}\|v'\|_{L^\infty[s_1,b]} \ . \tag{2.17}$$

For $\varepsilon \in [0, \varepsilon_0]$, we know from (2.16) that if u is a nontrivial solution of $(I)_{b,\varepsilon}$ and s_1 is the first zero of u then $s_1 \ge s_0$. It follows from (2.17) and Lemma 5 that

$$\|u'\|_{L^\infty[s_1,b]} \le \frac{(N-1)}{2} \ s_0^{-(N+1)/2} K_7(\lambda, s_0, \varepsilon_0) + s_0^{(1-N)/2} K_8(\lambda, s_0, \varepsilon_0) \ .$$

This together with (2.3) completes the proof of Lemma 2.

Proof of Lemma 5. Suppose first $v > 0$ in (a, b). Since $v(a) = v(b) = 0$, multiplying (2.6) by v and integrating it by parts lead to

$$\int_a^b v'^2 d\rho + \int_a^b \mathcal{F}(\rho, v, v') v^2 d\rho = \int_a^b \left(\lambda - \frac{(N-1)(N-3)}{4(\rho+\varepsilon)^2} \right) v^2 d\rho \,. \tag{2.18}$$

From (2.7) and (H1) we know that

$$\mathcal{F}(\rho, v, v') = F(\rho, u, u') \geq H(\rho, u) \geq w_1(\rho)|u|^{\sigma_1} = w_1(\rho)|v|^{\sigma_1}(\rho+\varepsilon)^{(1-N)\sigma_1/2} \,.$$

This together with (2.18) yields

$$\int_a^b v'^2 d\rho + \int_a^b w_1(\rho)|v|^{\sigma_1+2}(\rho+\varepsilon)^{(1-N)\sigma_1/2} d\rho \leq \left(\lambda + \frac{1}{4a^2} \right) \int_a^b v^2 d\rho \,.$$

Applying Hölder inequality and arguing like the proof of [17, Lemma 3.6], we get

$$\int_a^b v^2 d\rho \leq \left(\lambda + \frac{1}{4a^2} \right)^{2/\sigma_1} \int_a^b w_1^{-2/\sigma_1} (\rho+\varepsilon)^{N-1} d\rho$$

and

$$\int_a^b v'^2 d\rho \leq \left(\lambda + \frac{1}{4a^2} \right)^{(2+\sigma_1)/\sigma_1} \int_a^b w_1^{-2/\sigma_1} (\rho+\varepsilon)^{N-1} d\rho \,.$$

Likewise, if $v < 0$ in (a, b) the same proof shows that

$$\int_a^b v^2 d\rho \leq \left(\lambda + \frac{1}{4a^2} \right)^{2/\sigma_2} \int_a^b w_2^{-2/\sigma_2} (\rho+\varepsilon)^{N-1} d\rho$$

and

$$\int_a^b v'^2 d\rho \leq \left(\lambda + \frac{1}{4a^2} \right)^{(2+\sigma_2)/\sigma_2} \int_a^b w_2^{-2/\sigma_2} (\rho+\varepsilon)^{N-1} d\rho \,.$$

Then the argument used in the proof [6, Lemma 1.45] shows that (2.8) and (2.9) are valid even if v changes sign in (a, b).

To obtain L^∞ bounds for v and v', we note that a slightly modified version of the proof of [6, Lemma 1.48] shows

$$\|v'\|_{L^\infty[a,b]} \leq \left(\lambda + \frac{1}{4a^2} \right)^{1/2} \|v\|_{L^\infty[a,b]} \,. \tag{2.19}$$

Then (2.10) and (2.11) follow from the same lines of reasoning as in the proof of [6, Lemma 1.45].

The proof of Lemma 4 is based on comparison argument. This requires some preliminary. Let $\eta_+(\rho) = \rho^{(N-1)/2}U_+(\rho)$ and $\eta_-(\rho) = \rho^{(N-1)/2}U_-(\rho)$. Since $\eta'_+(b) = b^{(N-1)/2}U'_+(b)$ and $\eta'_-(b) = b^{(N-1)/2}U'_-(b)$, we will show

$$\eta'_+(b) < \eta'_-(b) \tag{2.20}$$

to establish (1.15). The following lemma provides a lower bound for $\eta'_-(b)$ in terms of the function $\mathcal{L}_-$.

LEMMA 6. *Assume that (F1), (F2), (H1) and (H2) are satisfied. Let $\delta > 0$ be given. If b is sufficiently large, there is a $\hat{\rho} \in (b, c)$ such that*

$$\eta'_-(b) > (\lambda + \delta)^{1/2}\mathcal{L}_-(\lambda + \delta, \hat{\rho}) \ . \tag{2.21}$$

Proof. Since $\eta_-(b) = \eta_-(c) = 0$ and $\eta_-(\rho) < 0$ for $\rho \in (b, c)$, η_- must attain its minimum at a point $\hat{\rho} \in (b, c)$. Note that η_- satisfies the equation

$$-\eta'' = \left[\lambda - \frac{(N-1)(N-3)}{4\rho^2} - \widetilde{\mathcal{F}}(\rho, \eta, \eta')\right] \eta \ , \tag{2.22}$$

where the function $\widetilde{\mathcal{F}}$ is defined by

$$\widetilde{\mathcal{F}}(\rho, y, z) = F(\rho, \rho^{(1-N)/2}y \ , \ \rho^{(1-N)/2}z - \frac{(N-1)}{2} \ \rho^{-(N+1)/2}y) \ . \tag{2.23}$$

For simplicity in notation, we drop the subscript from η_- in the remaining of the proof. Given $\delta > 0$, we choose b large enough so that $\left|\frac{(N-1)(N-3)}{4b^2}\right| < \delta$. Then we see from (2.22) that

$$-\eta'' > [(\lambda + \delta) - \widetilde{\mathcal{F}}(\rho, \eta, \eta')]\eta$$

if $\rho \in (b, c)$. It follows from $\eta(\hat{\rho}) < 0$, $\eta'(\hat{\rho}) = 0$ and $\eta''(\hat{\rho}) \geq 0$ that

$$(\lambda + \delta) - \widetilde{\mathcal{F}}(\hat{\rho}, \eta(\hat{\rho}), 0) > -\frac{\eta''(\hat{\rho})}{\eta(\hat{\rho})} \geq 0 \ .$$

Since $\widetilde{\mathcal{F}}(\rho, y, 0) = F(\rho, \rho^{(1-N)/2}y \ , -\frac{(N-1)}{2} \ \rho^{-(N+1)/2}y) \geq H(\rho, \rho^{(1-N)/2}y) = \mathcal{H}(\rho, y)$, we get $(\lambda + \delta) > \mathcal{H}(\hat{\rho}, \eta(\hat{\rho}))$. It follows from the definition of $\mathcal{L}_-$ and (H2) that

$$\eta(\hat{\rho}) > \mathcal{L}_-(\lambda + \delta, \hat{\rho}) \ . \tag{2.24}$$

Note that a modified version of [6, Lemma 1.48] shows

$$\|\eta'\|_{L^\infty[b,c]} \le \left(\lambda + \frac{1}{4b^2}\right)^{1/2} \|\eta\|_{L^\infty[b,c]} \ . \tag{2.25}$$

This together with $\delta > \frac{1}{4b^2}$, $-\eta'(b) \le \|\eta'\|_{L^\infty[b,c]}$ and $-\eta(\hat{\rho}) = \|\eta\|_{L^\infty[b,c]}$ leads to

$$\eta'(b) > (\lambda+\delta)^{\frac{1}{2}}\eta(\hat{\rho}) \ . \tag{2.26}$$

Combining (2.24) with (2.26), we have (2.21).

Our strategy in obtaining estimates for $\eta'_+(b)$ is to make use of a positive solution of

$$(III)_{\alpha,b} \qquad \begin{aligned} -v'' &= (\lambda-\delta)v - \mathcal{H}(\rho,v)v \quad , \quad \alpha<\rho<b \ , && (2.27)\\ v(\alpha) &= v(b) = 0 && (2.28)\end{aligned}$$

as a comparison function. It has been shown (e.g. [6]) that problem $(III)_{\alpha,b}$ has a unique positive solution, which will be denoted by $V_+(\lambda-\delta,\alpha,b,\cdot)$, provided that $\delta<\lambda$ and $b-\alpha$ is sufficiently large. We will choose α to be sufficiently large. Moreover, as a direct consequence of [9, Lemma 5 and Lemma 6], we have the following estimates for $V'_+(\lambda-\delta,\alpha,b,b)$.

LEMMA 7. *Assume that (F1), (F2), (H1) and (H2) are satisfied. Let $\delta \in (0,\lambda)$ be fixed. For any fixed $\alpha>0$, there is a number $\beta>\alpha$ such that if $b\ge\beta$ then*

$$V'_+(\lambda-\delta,\alpha,b,b) < \mathcal{L}'_+(\lambda-\delta,\rho_0) \tag{2.29}$$

for some $\rho_0 \in (\alpha,b)$.

We are now in the position for the

Proof of Lemma 4. As we mentioned earlier, the proof can be done by showing

$$\eta'_+(b) < \eta'_-(b) \qquad \text{for all sufficiently large } b. \tag{2.30}$$

We first claim the following:

For every fixed $\alpha > X$ there is a $\beta=\beta(\alpha)$ such that
$\eta'_-(b) > V'_+(\lambda-\delta,\alpha,b,b)$ if $b\ge\beta$. (2.31)

Indeed, it follows from $(\mathcal{L}1)$ that

$$\mathcal{L}'_+(\lambda-\delta,\rho_0) \le (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta,\rho_0) \tag{2.32}$$

and

$$\mathcal{L}'_-(\lambda+\delta,\rho)>0 \tag{2.33}$$

if $\rho \geq \alpha$. Let $\hat{\rho}$ and ρ_0 be the numbers appeared in Lemma 6 and Lemma 7. Since $\alpha < \rho_0 < b < \hat{\rho} < c$, it follows from (2.33) that

$$\mathcal{L}_-(\lambda+\delta,\rho_0) < \mathcal{L}_-(\lambda+\delta,\hat{\rho}) \ . \tag{2.34}$$

Putting (2.21), (2.34), (2.32) and (2.29) together yields

$$\begin{aligned}\eta'_-(b) &> (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta,\hat{\rho}) > (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta,\rho_0)\\ &\geq \mathcal{L}'_+(\lambda-\delta,\rho_0) > V'_+(\lambda-\delta,\alpha,b,b),\end{aligned}$$

provided that $b \geq \beta$, where β is a positive number such that (2.21) and (2.29) are valid if $b \geq \beta$.

We now prove (2.30) by arguing indirectly. Suppose (2.30) is false. Then for any fixed $\alpha > X$, by virtue of (2.31), we can find a number b, as large as we pleased, such that

$$\eta'_+(b) > V'_+(\lambda-\delta,\alpha,b,b) \ . \tag{2.35}$$

Note that α can be chosen arbitrarily large in advance. Thus we are free to let $\alpha > \mathrm{Max}(a_2, X_1)$. Throughout the proof, we let $v = V_+(\lambda-\delta,\alpha,b,\cdot)$ and $\eta = \eta_+$ for simplifying notation. Since $\eta(b) = v(b) = 0$, it follows from (2.35) that $\eta(\rho) < v(\rho)$ for $\rho \in (b-\varepsilon, b)$ and some $\varepsilon > 0$. Since $v(\alpha) = 0$, $v(\rho) > 0$ for $\rho \in (\alpha, b)$ and $\eta(\rho) > 0$ for $\rho \in [\alpha, b)$, there is a $t \in (\alpha, b)$ such that

$$\eta(t) = v(t) \tag{2.36}$$

and $\eta(\rho) < v(\rho)$ for $\rho \in (t, b)$. It is clear that

$$\eta'(t) \leq v'(t) \ . \tag{2.37}$$

Since $\eta(b) = 0 < \mathcal{L}_+(\lambda-\delta, b)$, by the continuity of the functions η and $\mathcal{L}_+(\lambda-\delta,\cdot)$ we know that $\eta(\rho) < \mathcal{L}_+(\lambda-\delta,\rho)$ for $\rho \in (b-\varepsilon, b)$ and some $\varepsilon > 0$. Define

$$s = \mathrm{Inf}\{\tau | \eta(\rho) < \mathcal{L}_+(\lambda-\delta,\rho) \quad \text{if} \quad \rho \in (\tau, b)\} \ . \tag{2.38}$$

We first consider the case $s \leq t$, in which we have

$$\lambda - \delta \geq \mathcal{H}(\rho, \eta(\rho)) \qquad \text{if} \qquad \rho \in [t, b] \ . \tag{2.39}$$

We claim that if α is sufficiently large and $s \leq t$ for such an α then

$$-\eta''(\rho) > [(\lambda-\delta) - \mathcal{H}(\rho,\eta(\rho))]\eta(\rho) \qquad \text{for} \quad \rho \in [t, b) \ . \tag{2.40}$$

Let us assume (2.40) for now and recall that v satisfies Eq. (2.27). Multiplying (2.40) by v and (2.27) by $-\eta$ and adding together, we obtain

$$v''(\rho)\eta(\rho) - \eta''(\rho)v(\rho) > [\mathcal{H}(\rho, v(\rho)) - \mathcal{H}(\rho, \eta(\rho))]\eta(\rho)v(\rho)$$

by making use of the fact that $0 < \eta(\rho) < v(\rho)$ if $\rho \in (t, b)$. This together with (H2) and the identity $v''\eta - \eta''v = (v'\eta - \eta'v)'$ yields

$$(v'\eta - \eta'v)' > 0 \quad \text{on} \quad (t, b) \ . \tag{2.41}$$

Since $v'(b)\eta(b) - \eta'(b)v(b) = 0$, it follows from (2.41) that $v'(t)\eta(t) - \eta'(t)v(t) < 0$. On the other hand, by virtue of (2.36) and (2.37), we see that $v'(t)\eta(t) - \eta'(t)v(t) = [v'(t) - \eta'(t)]\eta(t) \geq 0$. Thus we conclude from the above contradition that $s \leq t$ is not possible for sufficiently large α, provided that (2.40) is valid.

We now prove (2.40). By (2.38) we know that

$$\eta(\rho) \leq \mathcal{L}_+(\lambda - \delta, \rho) \tag{2.42}$$

if $\rho \in [s, b]$. Let us assume for now that

$$|\eta'(\rho)| \leq (\lambda + \delta)^{1/2} J(\rho) \tag{2.43}$$

if $\rho \in [t, b]$, where

$$J(\rho) = \mathrm{Max}(\mathcal{L}_+(\lambda - \delta, \rho), -\mathcal{L}_-(\lambda + \delta, \rho)) \ .$$

We recall that $U_+(\rho) = \rho^{(1-N)/2}\eta(\rho)$ and $U_+'(\rho) = \frac{1-N}{2\rho}\, U_+(\rho) + \rho^{(1-N)/2}\eta'(\rho)$. From (H2) we see that $J(\rho) \leq \mathcal{L}(\lambda + \delta, \rho)$. It follows from (2.42), (2.43) and (G1) that

$$\begin{aligned} \widetilde{\mathcal{F}}(\rho, \eta(\rho), \eta'(\rho)) - \mathcal{H}(\rho, \eta(\rho)) &= F(\rho, U_+(\rho), U_+'(\rho)) - H(\rho, U_+(\rho)) \\ &= G(\rho, U_+(\rho), U_+'(\rho)) \leq \frac{\delta}{2} \end{aligned} \tag{2.46}$$

if $\rho \in [t, b)$. Moreover, we know that

$$\left| \frac{(N-1)(N-3)}{4\rho^2} \right| \leq \left| \frac{(N-1)(N-3)}{4\alpha^2} \right| < \frac{\delta}{2} \tag{2.47}$$

if $\rho \geq \alpha$ and α is large. Since η satisfies Eq. (2.22), we obtain (2.40) by using (2.46) and (2.47).

It remains to show (2.43) to complete the proof of (2.40). We first note that for any $\rho \in [t, b)$, if $|\eta'(\rho)| \leq (\lambda + \delta)^{1/2} J(\rho)$ the preceding proof shows

$$-\eta''(\rho) > [(\lambda - \delta) - \mathcal{H}(\rho, \eta(\rho))]\eta(\rho) \ . \tag{2.48}$$

This together with (2.39) implies that

$$\eta''(\rho) < 0 \text{ if } \rho \in [t, b) \text{ and } |\eta'(\rho)| \ \le (\lambda+\delta)^{1/2} J(\rho) \ . \tag{2.49}$$

In particular, we have

$$\eta''(\rho) < 0 \text{ if } \rho \in [t, b) \text{ and } \eta'(\rho) = 0 \ . \tag{2.50}$$

By Lemma 6 we know that $\eta'_-(b) > (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta, \hat{\rho})$ for some $\hat{\rho} > b$. Since we assume (2.30) is false, it follows that

$$\eta'(b) \ge \eta'_-(b) > (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta, \hat{\rho}) \ .$$

Using the fact $\mathcal{L}'_-(\lambda+\delta, \rho) > 0$ if $\rho \ge \alpha$, we conclude that

$$\eta'(b) > (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta, \rho) \tag{2.51}$$

if $\rho \in [\alpha, b]$. By the continuity of η', there is an $\varepsilon > 0$ such that

$$0 \ge \eta'(\rho) > (\lambda+\delta)^{1/2}\mathcal{L}_-(\lambda+\delta, \rho) \tag{2.52}$$

if $\rho \in (b-\varepsilon, b)$. Then $\eta''(\rho) < 0$ for such ρ as noted in (2.49). Suppose that $\eta'(\rho) \le 0$ for all $\rho \in [t, b]$. Defining

$$t_1 = \operatorname{Inf}\{\tau | \eta''(\rho) \le 0 \quad \text{if} \quad \rho \in [\tau, b) \subset [t, b)\} \ , \tag{2.53}$$

we have $0 \ge \eta'(\rho) \ge \eta'(b)$ for $\rho \in [t_1, b]$. This together with (2.51) implies that (2.52) and hence (2.43) holds for $\rho \in [t_1, b]$. Then it follows that $\eta''(t_1) < 0$ as noted in (2.49). But this contradicts (2.53) unless $t_1 = t$. Therefore we have (2.43) for $\rho \in [t, b]$.

Next we consider the case that $\eta'(\rho) > 0$ for some $\rho \in [t, b)$. From (2.50) and $\eta'(b) < 0$, we know that η has exactly one critical point $\tilde{t} \in (t, b)$ and $\eta'(\rho) > 0$ for $\rho \in [t, \tilde{t})$, $\eta'(\rho) < 0$ for $\rho \in (\tilde{t}, b]$. Defining $t_2 = \operatorname{Inf}\{\tau|\ \eta''(\rho) \le 0 \text{ if } \rho \in [\tau, b) \subset [\tilde{t}, b)\}$ and arguing like the proof of $t_1 = t$, we obtain (2.43) for $\rho \in [\tilde{t}, b]$. Moreover, an argument used in the proof of [6, Lemma 1.48] shows

$$|\eta'(\rho)| \le (\lambda+\delta)^{1/2}\eta(\tilde{t})$$

if $\rho \in [t, \tilde{t}]$. This together with (2.42) and the fact that $\mathcal{L}'_+(\lambda-\delta, \rho) < 0$ yields

$$|\eta'(\rho)| \le (\lambda+\delta)^{1/2}\mathcal{L}_+(\lambda-\delta, \tilde{t}) \le (\lambda+\delta)^{1/2}\mathcal{L}_+(\lambda-\delta, \rho)$$

if $\rho \in [t, \tilde{t}]$. So we have completed the proof of (2.43).

To complete the proof of the lemma, it remains to show that $s > t$ is not possible either, where we recall that s was defined in (2.38) and

$$t = \operatorname{Inf}\{\tau | \eta(\rho) < v(\rho) \quad \text{if} \quad \rho \in (\tau, b)\} \ .$$

In this case we have

$$\eta''(\rho) < 0 \text{ if } \rho \in [s,b) \text{ and } \eta'(\rho) = 0 \quad , \tag{2.54}$$

by the same reasoning as showing (2.50). In view of (2.38), we know that

$$\eta'(s) \leq \mathcal{L}'_{+}(\lambda - \delta, s) \ . \tag{2.55}$$

Since it follows from ($\mathcal{L}1$) that $\mathcal{L}'_{+}(\lambda - \delta, s) < 0$, we have $\eta'(s) < 0$. It is easy to see from (2.54) that if $\rho \in (s,b)$ and $\eta'(\rho) = 0$ then $\eta(\rho)$ must be the unique absolute maximum of η in (s,b). Since $\eta'(s) < 0$ and $\eta'(b) < 0$, we conclude that $\eta'(\rho) < 0$ for all $\rho \in [s,b]$. Letting $t_3 = \text{Inf}\{\tau | \eta''(\rho) \leq 0 \text{ if } \rho \in [\tau, b) \subset [s,b)\}$ and arguing like the proof of $t_1 = t$, we get $t_3 = s$ and hence (2.52) for $\rho \in [s,b]$. In particular, we have

$$\eta'(s) > (\lambda + \delta)^{1/2} \mathcal{L}_{-}(\lambda + \delta, s) \ . \tag{2.56}$$

From ($\mathcal{L}1$) we know that

$$\mathcal{L}'_{+}(\lambda - \delta, s) \leq (\lambda + \delta)^{1/2} \mathcal{L}_{-}(\lambda + \delta, s) \ . \tag{2.57}$$

Putting (2.55) – (2.57) together yields $\eta'(s) > \eta'(s)$, which is absurd. Thus the proof is complete.

§3 Final Remarks. The analysis for $N = 1$ is simpler. In this case, L^∞-estimates of u' easily follows from (1.10) and a modified version of [6, Lemma 1.48]; that is, a simpler version of Lemma 2 can be obtained under the assumptions (F2) and (H1) only. Furthermore Lemma 3 is redundant and thus we see that there is no need to assume (F3) and (F4) in Theorems 1 and 2 if $N = 1$.

For $N > 1$, the assumptions (F3) and (F4) can also be dropped if $\operatorname{Sup}_{z \in \mathbb{R}} F(\rho, y, z)$ is uniformly bounded on compact subsets of $[0, \infty) \times \mathbb{R}$; the reason is Lemma 3 can be established without invoking Lemma 2. This is so in particular if F does not depend on z.

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