SUBLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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1. Introduction

We are concerned with the question of existence (or nonexistence) and uniqueness of solutions of the problem

$$(1) \qquad -\Delta u = \rho(x)u^{\alpha} \quad \text{in} \quad \mathbb{R}^n, \ n \geq 3$$

with $0 < \alpha < 1$ and $\rho(x) \geq 0$, ρ not identically zero. We shall assume throughout the paper that $\rho \in L^{\infty}_{loc}$. We look for a solution $u \geq 0$, u not identically zero, so that, by the strong maximum principle, if such a solution exists then $u > 0$ in $\mathbb{R}^n$.

We shall often use the following:

<u>Definition</u>: We say that a function $\rho \in L^{\infty}_{loc}(\mathbb{R}^n)$, $\rho \geq 0$, has the property (H) if the linear problem

$$(2) \qquad -\Delta U = \rho \quad \text{in} \quad \mathbb{R}^n$$

has a bounded solution.

Our main result is

<u>Theorem 1</u>. Problem (1) has a bounded solution iff ρ satisfies (H). Moreover there is a minimal positive solution of (1).

This minimal positive solution of (1) tends to zero at infinity in a sense to be precised later. Moreover it is the unique positive solution of (1) which tends to zero at infinity (see Theorem 2 below).

In Section 2 we prove Theorem 1 and in Section 3 we present uniqueness results for (1). In Appendix I we summarize some properties of the linear Poisson equation (2). In Appendix II we review the uniqueness question for equation (1) in bounded domains.

Problem (1) for <u>bounded domains</u> with zero Dirichlet condition has been extensively studied (even for more general sublinear functions). We refer in particular to Krasnoselskii [9] (Theorem 7.14 and 7.15) and [1] (see also the references therein). Problem (1) in all of space has been considered in [3], [4], and [10] under more restrictive conditions on ρ (ρ is equivalent to a radial function for large $|x|$).

The study of (1) is also related to the asymptotic behavior (as $t \longrightarrow \infty$) of the solution of

$$\rho(x)\,\frac{\partial u}{\partial t} = \Delta u^m \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \tag{3}$$

with $m = 1/\alpha > 1$ which has recently been studied by Eidus [5] (see also [6]) for a class of functions ρ tending to zero at infinity. In fact, separating variables, we have a solution $u(x,t)$ of (3) of the form $u(x,t) = C\, v^{1/m}(x)(t + \tau)^{-1/(m-1)}$ provided $v(x)$ is a solution of (1).

<u>Proof of Theorem 1</u>

A. <u>Sufficient condition</u>:

Let

$$B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$$

and let u_R be the solution of

$$(4) \qquad \begin{cases} -\Delta u = \rho\, u^\alpha & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

It is well–known that u_R exists and is unique (see e.g. [9], [1] or Appendix II). The sequence u_R is increasing with R. Indeed, let $R' > R$. Then $u_{R'}$ is a supersolution for the R–problem. We now construct a subsolution $\underline{u}$ for the R–problem with $\underline{u} \le u_{R'}$. This will imply that there is a solution u for the R–problem between $\underline{u}$ and $u_{R'}$. Since the unique solution is u_R it follows that $u_R \le u_{R'}$ in B_R. For $\underline{u}$ we may take $\varepsilon\varphi_1$ where φ_1 satisfies

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\, \rho\, \varphi_1 & \text{in } B_R\;, \\ \varphi_1 = 0 & \text{on } \partial B_R\;. \end{cases}$$

We now prove that the sequence u_R remains bounded as $R \longrightarrow \infty$. In fact

$$u_R \le C\, U$$

for some appropriate constant C. Indeed, C U is a supersolution for the R–problem since

$$-\Delta\,(CU) = C\rho \ge \rho\,(CU)^\alpha$$

provided

$$C^{1-\alpha} \ge \|U\|_\infty^\alpha\,.$$

Therefore $u = \lim_{R\to\infty} u_R$ exists and u is a solution of (1) satisfying

$$(5) \qquad u \le C\,U\,.$$

Clearly u is the minimal solution; indeed if $\bar{u}$ is another solution of (1) then $u_R \le \bar{u}$ on B_R by the above argument and thus $u \le \bar{u}$.

B. Necessary condition

Suppose u is bounded positive solution of (1) and set

$$v = \frac{1}{1-\alpha} u^{1-\alpha} .$$

Then

$$-\Delta v = \alpha u^{-\alpha-1} |\nabla u|^2 + \rho \geq \rho .$$

The solution w_R of the problem

$$\text{(6)} \qquad \begin{cases} -\Delta w_R = \rho & \text{in } B_R , \\ w_R = 0 & \text{on } \partial B_R \end{cases}$$

satisfies $w_R \leq v$. Thus w_R increases as $R \to \infty$ to a bounded solution of (2).

The meaning of Theorem 1 is that if $\rho(x)$ decays fast enough at infinity then Problem (1) has a solution. It need not exist if $\rho(x)$ has a slow decay at infinity. As we see in the next example, if $\rho(x)$ decays like a power, the critical exponent is two.

Example 1: Assume

$$\rho(x) = \frac{1}{1 + |x|^p} \qquad \text{with } p > 2$$

or

$$\rho(x) = \frac{1}{(1 + |x|^2)|\log(2+|x|)|^p} \qquad \text{with } p > 2$$

then Problem (1) has a bounded solution. Indeed the Poisson integral $\frac{c}{|x|^{n-2}} * \rho$ provides a bounded positive solution of (2) where $c/|x|^{n-2}$ is the fundamental solution of $-\Delta$.

Example 2: Assume

$$\rho(x) = \frac{1}{1 + |x|^p} \qquad \text{with } p \leq 2$$

then Problem (1) has no solution. In fact a stronger nonexistence result holds. Assume

$$(7) \qquad \int_{|x|\geq 1} \frac{\rho(x)}{|x|^{n-2}}\, dx = \infty \, ,$$

then there is no function $u \in L^1_{loc}(\mathbb{R}^n)$ satisfying

$$(8) \qquad \begin{cases} -\Delta u = \rho u^\alpha & \text{in } \mathcal{D}'(\mathbb{R}^n) \\ u \geq 0 \end{cases}$$

except $u \equiv 0$. Indeed, assume we have a solution of (8). By local regularity, $u \in W^{2,q}_{loc}$ for all $q < \infty$ and if u is not identically zero then $u > 0$ in $\mathbb{R}^n$. As above, set

$$v = \frac{1}{1-\alpha} u^{1-\alpha}$$

so that $-\Delta v \geq \rho$. It follows that

$$(9) \qquad w_R \leq v$$

where w_R is defined by (6). As $R \uparrow \infty$, $w_R \uparrow \infty$ because of (7) (see Appendix I). This is impossible by (9).

<u>Remark 1</u>. The minimal solution u obtained in Theorem 1 satisfies

$$(10) \qquad u(x) = c \int_{\mathbb{R}^n} \frac{\rho(y)u^\alpha(y)}{|x-y|^{n-2}}\, dy$$

and also

$$(11) \qquad \lim_{R\to\infty} \fint_{S_R} u = 0$$

where $\fint_{S_R} u$ denotes the average of u on the sphere of radius R (centered at 0). Indeed, u satisfies (5) for any positive solution U of (2); in particular we can take $U = \frac{c}{|x|^{n-2}} * \rho$. We now apply Lemma A.4 in Appendix I to conclude that (11) holds. As a consequence of (11) we have

$$\liminf_{|x| \to \infty} u(x) = 0 .$$

Next, let $f = \rho u^{\alpha}$. The linear equation $-\Delta v = f$ in $\mathbb{R}^n$ has a unique solution satisfying

$$\lim_{R\to\infty} \fint_{S_R} v = 0 ,$$

namely $v = \frac{c}{|x|^{n-2}} * f$. Since u satisfies the same equation and also (11) we obtain (10).

Remark 2. The minimal solution u of (1) depends monotonically on ρ. Indeed let $\rho_1 \leq \rho_2$ and let u_1, u_2 be the corresponding minimal solutions of (1). Then u_2 is a supersolution for the equation

$$\begin{aligned} -\Delta u &= \rho_1 u^{\alpha} && \text{in } B_R \\ u &= 0 && \text{on } \partial B_R . \end{aligned}$$

Thus $u_{1,R} \leq u_2$ in B_R. Passing to the limit as $R \longrightarrow \infty$ we find that $u_1 \leq u_2$.

Remark 3. The minimal solution u obtained in Theorem satisfies $C_1 U^{1-\alpha} \leq u \leq C_2 U$. In general these bounds are sharp. For example if ρ has compact support then both u and U behave at infinity like the fundamental solution. However if $\rho(x) \sim |x|^{-p}$ at infinity with $2+(n-2)(1-\alpha) < p < n$ then a simple computation shows that $U(x) \sim |x|^{-(p-2)}$ and $u(x) \sim |x|^{-(p-2)(1-\alpha)}$.

3. Uniqueness

As we have noted the minimal solution u constructed above satisfies

(12) $$\liminf_{|x| \to \infty} u(x) = 0 .$$

Our main uniqueness result is

Theorem 2. Assuming ρ has property (H), then there is exactly one bounded positive solution of (1) satisfying (12).

Remark 4. There exist other bounded positive solutions of (1) which do not satisfy (12). In fact, given any positive constant a, there exists a solution of (1) satisfying

$$\liminf_{|x| \to \infty} u(x) = a.$$

Indeed, consider the problem

(13) $$\begin{cases} - \Delta u = \rho\, u^{\alpha} & \text{in } B_R , \\ u = a & \text{on } \partial B_R . \end{cases}$$

As subsolution for (13) we may take a and as supersolution we may take $(CU + a)$ where $U = \frac{c}{|x|^{n-2}} * \rho$ with C is large enough. We then let $R \longrightarrow \infty$.

The proof of Theorem 2 is divided into 3 steps:

Step 1. Assume $\rho_1 \le \rho_2$ and that they satisfy property (H). Given any bounded positive solution u_1 of

(14) $$\begin{cases} - \Delta u_1 = \rho_1\, u_1^{\alpha} & \text{in } \mathbb{R}^n \\ \lim_{R \to \infty} ⨍_{S_R} u_1 = 0 \end{cases}$$

then there exists a bounded positive solution u_2 of

$$(15)\qquad \begin{cases} -\Delta u_2 = \rho_2\, u_2^{\alpha} & \text{in } \mathbb{R}^n \\ \lim_{R\to\infty} \fint_{S_R} u_2 = 0 \end{cases}$$

such that $u_1 \leq u_2$.

Proof. Clearly u_1 is a subsolution for (15) in the sense that

$$-\Delta u_1 \leq \rho_2\, u_1^{\alpha}.$$

Since u_1 is bounded we have

$$-\Delta u_1 \leq C\, \rho_2$$

and by Lemma A.6 we find that

$$u_1 \leq C\, \left(\frac{1}{|x|^{n-2}} * \rho_2\right).$$

The right-hand side is a supersolution for (15) provided C is large enough. Using the standard monotone iteration technique (directly in $\mathbb{R}^n$) we obtain a solution u_2 of (15) such that

$$u_1 \leq u_2 \leq C\left(\frac{1}{|x|^{n-2}} * \rho_2\right).$$

The only difference with the usual case of bounded domains is that the Dirichlet condition is replaced by the condition at infinity $\lim_{R\to\infty} \fint_{S_R} u = 0$. The standard maximum principle is replaced at each stage by Lemma A.6.

We shall now show that it suffices to prove Theorem 2 in the case $\rho > 0$.

Step 2. Assume we have proved uniqueness for any $\rho > 0$, then we also have uniqueness for a general $\rho \geq 0$.

Proof. Let $\rho_\varepsilon = \rho + \varepsilon h$ where $h \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $h > 0$.

Let u_ε be the unique solution of

(16) $$\begin{cases} -\Delta u_\varepsilon = \rho_\varepsilon u_\varepsilon^\alpha & \text{in } \mathbb{R}^n \\ \lim\limits_{R\to\infty} \fint_{S_R} u_\varepsilon = 0 \end{cases}$$

Let u be any solution of

(17) $$\begin{cases} -\Delta u = \rho u^\alpha & \text{in } \mathbb{R}^n \\ \lim\limits_{R\to\infty} \fint_{S_R} u = 0 . \end{cases}$$

By Step 1 (and by the uniqueness of u_ε) we know that

(18) $$u \le u_\varepsilon$$

We prove that, as $\varepsilon \downarrow 0$, $u_\varepsilon \downarrow \underline{u}$ where $\underline{u}$ is the minimal solution constructed in Theorem 1. Indeed let $u_{\varepsilon,R}$ and u_R be the positive solutions of

(19) $$-\Delta u_{\varepsilon,R} = \rho_\varepsilon u_{\varepsilon,R}^\alpha \quad \text{in } B_R$$

with $$u_{\varepsilon,R} = 0 \quad \text{on } \partial B_R$$

and

(20) $$-\Delta u_R = \rho u_R^\alpha \quad \text{in } B_R$$

with $$u_R = 0 \quad \text{on } \partial B_R .$$

We now use the same device as in Appendix II (method II), namely, we multiply (19) by u_R and (20) by $u_{\varepsilon,R}$. Integrating by parts we find

$$\int_{B_R} \rho\, u_{\varepsilon,R}^\alpha u_R^\alpha (u_{\varepsilon,R}^{1-\alpha} - u_R^{1-\alpha}) = \int_{B_R} (\rho_\varepsilon - \rho) u_R u_{\varepsilon,R}^\alpha$$

and thus

$$\int_{B_R} \rho\, u_{\varepsilon,R}^\alpha u_R^\alpha (u_{\varepsilon,R}^{1-\alpha} - u_R^{1-\alpha}) \le C\varepsilon$$

where C is independent of R. Passing to the limit as $R \to \infty$ (and using Fatou) we obtain

$$\int_{\mathbb{R}^n} \rho\, u_\varepsilon^{\alpha}\, \underline{u}^{\alpha}\, (u_\varepsilon^{1-\alpha} - \underline{u}^{1-\alpha}) \leq C\varepsilon .$$

Using (18) we have

$$\int \rho\, u^{\alpha}\, \underline{u}^{\alpha}\, (u^{1-\alpha} - \underline{u}^{1-\alpha}) = 0$$

and thus $\rho u^{\alpha} = \rho \underline{u}^{\alpha}$. Hence $\Delta(u - \underline{u}) = 0$ and therefore $u = \underline{u}$ (by the condition at infinity).

The last step involves the use of parabolic equations as in [8]. As we already mentioned in the Introduction if $u(x)$ is a solution of (1) then

$$v(x, t) = \frac{C u^{1/m}(x)}{(t + \tau)^{1/(m-1)}}$$

satisfies

(21) $$\rho \frac{\partial v}{\partial t} = \Delta v^m$$

where $m = 1/\alpha$ and $C = (m-1)^{-1/(m-1)}$. Our proof of uniqueness for problem (1), (12) relies heavily on existence, uniqueness and comparison properties of solution of (21).

Step 3. We recall first a well-known fact about bounded domains (see e.g. [2]).

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\rho \in L^\infty(\Omega)$, $\rho \geq \delta > 0$ on Ω. Then given any $v_0 \geq 0$ on Ω, $v_0 \in L^\infty(\Omega)$, there exists a unique solution $v(x,t)$ of the problem

$$(22)\qquad \begin{cases} \rho \dfrac{\partial v}{\partial t} - \Delta v^m = 0 & \text{in } \Omega \times (0,\infty) \\ v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } \Omega \end{cases}$$

Moreover if there is another solution $\tilde{v}(x,t)$ of (22) with $\tilde{v}(x,t) \geq 0$ on $\partial\Omega\times(0,\infty)$ and $\tilde{v}(x,0) \geq v_0(x)$ then $\tilde{v}(x,t) \geq v(x,t)$.

Let $\underline{u}$ be the minimal positive solution of (1) in the sense of Theorem 1. Let u be any bounded positive solution of (1) satisfying (12). By Appendix I we know that

$$\lim_{R\to\infty} \fint_{S_R} u = 0 .$$

Let v_R be the solution of

$$\begin{cases} \rho \dfrac{\partial v_R}{\partial t} - \Delta v_R^{\,m} = 0 & \text{in } B_R \times (0,\infty) \\ v_R(x,t) = 0 & \text{on } \partial B_R \times (0,\infty) \\ v_R(x,0) = u(x) & \text{in } B_R \end{cases} .$$

By comparison in bounded domains we see that

$$(23)\qquad v_R(x,t) \leq \frac{u^{1/m}(x)}{(t+1)^{1/(m-1)}}$$

and also

$$(24)\qquad v_R(x,t) \leq \frac{\underline{u}^{1/m}(x)}{t^{1/(m-1)}} .$$

As $R \uparrow \infty$ the sequence v_R increases to some limit $v_\infty(x,t)$ which satisfies

$$(25)\qquad \rho \frac{\partial v_\infty}{\partial t} - \Delta v_\infty^{\,m} = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

and

$$(26)\qquad v_\infty(x,0) = u(x) .$$

Moreover we have

(27) $$v_\infty(x,t) \le \frac{u^{1/m}(x)}{(t+1)^{1/(m-1)}} .$$

We already have a solution of (25), (26) namely $\dfrac{u^{1/m}(x)}{(t+1)^{1/(m-1)}}$. We claim that

(28) $$v_\infty(x,t) = \frac{u^{1/m}(x)}{(t+1)^{1/(m-1)}} \equiv \hat{v}(x,t) .$$

For this purpose we multiply

$$\rho \frac{\partial}{\partial t}(\hat{v} - v_\infty) - \Delta(\hat{v}^m - v_\infty^m) = 0$$

by the function $K(x) = c \left[\dfrac{1}{|x|^{n-2}} - \dfrac{1}{R^{n-2}}\right]$ and integrate over $B_R \times (0,T)$.

We find

$$\int_{B_R} \rho(x)\, (\hat{v} - v_\infty)K(x)dx_{|t=T} + \int_0^T (\hat{v}^m - v_\infty{}^m)dt_{|x=0}$$

$$= - \int_0^T \int_{\partial B_R} (\hat{v}^m - v_\infty{}^m)\, \frac{\partial K}{\partial \nu}\, dS\, dt .$$

The integral on the right hand side is bounded by

$$C\, T ⨍_{S_R} u$$

which tends to zero as $R \to \infty$. Thus $\hat{v} = v_\infty$ (since $\rho > 0$). Passing to the limit in (24) we find

$$\frac{u^{1/m}(x)}{(t+1)^{1/(m-1)}} \le \frac{\underline{u}^{1/m}(x)}{t^{1/(m-1)}} .$$

Letting $t \longrightarrow \infty$ we conclude that $u \le \underline{u}$.

Remark 5. Assume ρ has property (H). As we know from Appendix I

$$\lim_{R\to\infty} ⨍_{S_R} U = 0$$

where $U = \frac{c}{|x|^{n-2}} * \rho$, and thus $\liminf_{|x|\to\infty} U = 0$. It may happen that $U(x)$ does not tend to zero as $|x| \longrightarrow \infty$. Here is a simple example for $n \geq 4$. Let $\psi(x')$ be the solution of

$$\begin{cases} -\Delta_{x'} \psi = \rho(x') & \text{in } \mathbb{R}^{n-1} \\ \lim_{|x'|\to\infty} \psi = 0 \end{cases}$$

where $\rho \in C_0^\infty(\mathbb{R}^{n-1})$, $\rho \geq 0$ and ρ not identically zero. Then

$$U(x) = \psi(x') \qquad x = (x_1, x')$$

provides such an example since $U(x_1, 0) = \psi(0)$ does not tend to zero as $|x_1| \longrightarrow \infty$. In such a situation there is no solution u of (1) which tends to zero at infinity because of the estimate from below $\underline{u}^{1-\alpha} \geq (1-\alpha)\, U$ (see the proof of necessary condition in Theorem 1).

The uniqueness question becomes easier under a stronger assumption

Theorem 2′. Assume there is a solution U of (2) such that

$$\lim_{|x|\to\infty} U(x) = 0 . \tag{29}$$

Then there exists a unique positive solution u of (1) such that

$$\lim_{|x|\to\infty} u(x) = 0 .$$

Proof. The existence part is clear since we already know that there is a solution u of (1) such that $u \leq CU$. For the uniqueness we could invoke Theorem 2 but we present instead a simple argument due to Louis Nirenberg.

First we change the unknown. As in the proof of Theorem 1 we set

$$v = \frac{1}{1-\alpha} \quad u^{1-\alpha}$$

so that we find

$$(30) \qquad -\Delta v - \frac{C}{v}|\nabla v|^2 = \rho$$

for some positive constant C (depending on α). Uniqueness holds for (30) since the function $1/v$ is decreasing in v. More precisely, suppose we have two solutions v_1, v_2 of (30) with $\lim_{|x|\to\infty} v_1 = \lim_{|x|\to\infty} v_2 = 0$. Then $w = v_1 - v_2$ satisfies

$$-\Delta w - \frac{C}{v_1}\nabla(v_1 + v_2) \cdot \nabla w + \frac{C}{v_1 v_2}|\nabla v_2|^2 w = 0 \ .$$

Since the coefficient of w is nonnegative we may use the maximum principle to conclude that $w = 0$.

Remark 6. Clearly if ρ is a radial function satisfying (H) then (29) holds. It also holds if ρ is bounded by a radial function satisfying (H).

4. Some generalization

Our methods extend to more general problems of the form

$$-\Delta u = \rho(x)\ f(u) \qquad \text{in } \mathbb{R}^n$$

under suitable assumptions of f and in particular $f(u)$ behaves like u^α near $u = 0$. For simplicity we restrict our attention to the model problem $f(u) = u^\alpha(1-u)$, i.e.

$$(31) \qquad -\Delta u = \rho(x)\ u^\alpha(1-u) \qquad \text{in } \mathbb{R}^n \ .$$

Theorem 3. Assume ρ satisfies (H). Then there is a unique solution u, $0 < u < 1$ of (31) such that

(32) $$\liminf_{|x| \to \infty} u(x) = 0 .$$

Proof. For the existence part we proceed as in the proof of Theorem 1 (sufficient condition). We obtain a minimal solution $\underline{u}$ with $\underline{u} \leq 1$ and $\underline{u} \leq CU$. For the uniqueness we proceed in two steps.

Step 1. Let u be any solution of (31), (32). Then there exists some $\varepsilon > 0$ such that

(33) $$\varepsilon u \leq \underline{u} .$$

It is useful to introduce the unique positive solution v of the problem

(34) $$\begin{cases} -\Delta v = \rho v^\alpha & \text{in } \mathbb{R}^n \\ ⨍_{S_R} v \longrightarrow 0 & \text{as } R \longrightarrow \infty \end{cases} .$$

Note that u is a subsolution for (32) since

$$u^\alpha(1-u) \leq u^\alpha$$

and therefore, by monotone iteration and uniqueness of v, we obtain

$$u \leq v .$$

Next, we note that for $\varepsilon > 0$ small enough εv is a subsolution for (31) since

$$-\Delta(\varepsilon v) = \varepsilon \rho v^\alpha \leq \rho(\varepsilon v)^\alpha(1-\varepsilon v).$$

It follows that $\varepsilon v \leq \underline{u}$, the minimal solution of (31) (to justify this we use comparison in B_R and then let $R \longrightarrow \infty$). Thus (33) holds.

Step 2. We now follow the same technique as in Method III of Appendix II. Let u be any solution of (31), (32) and let

$$\Lambda = \{t \in [0, 1]; \ \ tu \leq \underline{u}\} .$$

We claim that $1 \in \Lambda$. Suppose not, that

$$t_0 = \sup \Lambda < 1.$$

By Step 1 we know that $t_0 > 0$. Fix K large enough so that the function $f(t) + Kt$ is increasing on $[0, 1]$. We have

$$-\Delta(\underline{u} - t_0 u) + K\rho(\underline{u} - t_0 u) \geq \rho[f(t_0 u) - t_0 f(u)].$$

Note that for $\varepsilon > 0$ small enough

$$f(t_0 u) - t_0 f(u) \geq \varepsilon f(u)$$

since

$$t_0^\alpha \geq t_0 + \varepsilon .$$

Thus we obtain

$$-\Delta(\underline{u} - t_0 u - \varepsilon u) \geq 0$$

and by Appendix I we conclude that $\underline{u} - t_0 u - \varepsilon u \geq 0$. Hence $t_0 + \varepsilon \in \Lambda$, which contradicts the maximality of t_0.

Appendix I

Throughout the paper we have often used the property (H), namely that the equation

(A.1) $$-\Delta U = f \qquad \text{in } \mathbb{R}^n$$

has a bounded solution. We discuss here some equivalent forms and some consequences. In what follows we always assume that $f \in L^\infty_{loc}(\mathbb{R}^n)$, $f \geq 0$ a.e. and that f is not identically zero. Let u_R be the solution of

$$\text{(A.2)} \qquad \begin{cases} -\Delta u_R = f & \text{in } B_R \\ u_R = 0 & \text{on } \partial B_R \end{cases}.$$

Note that u_R is a nondecreasing sequence of positive functions (in B_R) for R large enough. Moreover u_R is given by

$$\text{(A.3)} \qquad u_R(x) = \int_{B_R} G_R(x,y)\, f(y)\, dy$$

where G_R is the Green's function relative to B_R and zero boundary condition. Let

$$u_\infty(x) = \lim_{R\uparrow\infty} u_R(x) \qquad \text{(possibly } +\infty).$$

Note that, by monotone convergence of G_R,

$$u_\infty(x) = c\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)dy = \frac{c}{|x|^{n-2}} * f$$

(possibly $+\infty$), where $c/|x|^{n-2}$ is the fundamental solution. Remark that there are only two possibilities, either $u_\infty(x) = +\infty \ \forall x$ or $u_\infty(x) < +\infty \ \forall x$. Indeed suppose for example that $u_\infty(0) < +\infty$.

Write

$$u_\infty(x) = c\int_{|y| \le 2|x|} \frac{f(y)}{|x-y|^{n-2}} + c\int_{|y| > 2|x|} \frac{f(y)}{|x-y|^{n-2}}.$$

The first integral is finite (for each fixed x) while the second integral is bounded by $2^{n-2} c \int \frac{f(y)}{|y|^{n-2}} dy$. Hence $u_\infty(x) < \infty$.

If we make the assumption that

$$u_\infty(0) = c\int \frac{f(y)}{|y|^{n-2}} dy < \infty$$

then $u_\infty(x)$ is finite for each fixed x but it need <u>not</u> be uniformly bounded on $\mathbb{R}^n$.

Lemma A.1. f satisfies property (H) iff

$$\frac{c}{|x|^{n-2}} * f \in L^\infty(\mathbb{R}^n) . \tag{A.4}$$

Proof. Suppose first that (H) holds. By adding a constant we may always assume that $U \geq 0$ in $\mathbb{R}^n$. By the maximum principle

$$u_R \leq U \qquad \text{on} \quad B_R$$

and therefore

$$u_\infty = \frac{c}{|x|^{n-2}} * f \leq U . \tag{A.5}$$

Conversely, the function $\frac{c}{|x|^{n-2}} * f$ provides a bounded solution of (A.1).

Since U could be any nonnegative solution of (A.1) we have

Corollary A.2. u_∞ is the minimal positive solution of (A.1).

As a consequence of minimality we have

Corollary A.3.

$$\liminf_{|x| \to \infty} u_\infty(x) = 0 .$$

In fact, any bounded solution U of (A.1) such that

$$\liminf_{|x| \to \infty} U(x) = 0$$

coincides with u_∞. This follows from the fact that the difference of any two bounded solutions of (A.1) is a bounded harmonic function and thus it is a constant.

A stronger way of expressing that u_∞ tends to zero at infinity is the following

Lemma A.4.

$$\lim_{R\to\infty} ⨍_{S_R} u_\infty = 0$$

where $⨍_{S_R}$ denotes the average on the sphere of radius R.

Proof. By Fubini we have

$$\frac{1}{R^{n-1}} \int_{S_R} u_\infty(y)\, dS_y = c \int_{\mathbb{R}^n} f(x) \frac{1}{R^{n-1}} \left[\int_{|y|=R} \frac{dS_y}{|x-y|^{n-2}} \right] dx \,.$$

Note that

$$I(x) = \int_{|x|=R} \frac{dS_y}{|x-y|^{n-2}} = \begin{cases} CR(\frac{R}{|x|})^{n-2} & \text{if } |x| > R \\ I(0) & \text{if } |x| < R \end{cases}$$

with

$$I(0) = \int_{|y|=R} \frac{dS_y}{|y|^{n-2}} = CR$$

(this is a consequence of the fact that $I(x)$ is harmonic in $|x| < R$ and in $|x| > R$; moreover $I(x) = I(|x|)$ and in addition $I(\infty) = 0$).

Hence we have

$$⨍_{S_R} u_\infty = \frac{C}{R^{n-2}} \int_{|x|<R} f(x)\, dx + c \int_{|x|>R} \frac{f(x)}{|x|^{n-2}}\, dx \,.$$

Clearly the second integral tends to zero as $R \to \infty$. We estimate the first one by

$$\frac{C}{R^{n-2}} \int_{|x|<R_0} f(x)\, dx + C \int_{R_0<|x|<R} \frac{f(x)}{|x|^{n-2}}\, dx \,.$$

We first choose R_0 so that

$$C \int_{R_0<|x|} \frac{f(x)}{|x|^{n-2}}\, dx < \varepsilon$$

and then R large enough so that

$$\frac{C}{R^{n-2}} \int_{|x|<R_0} f(x)\, dx < \varepsilon .$$

Lemma A5. Any bounded solution U of (A.1) such that

$$⨍_{S_R} U \longrightarrow 0 \qquad \text{as} \quad R \longrightarrow \infty$$

coincides with u_∞.

This is clear since the difference of two bounded solutions of (A.1) is a constant.

Lemma A.6. Assume $U \in L^\infty$ with $\Delta U \in L^\infty_{loc}$ satisfies

$$-\Delta U \leq \rho \qquad \text{in } \mathbb{R}^n$$

and

$$⨍_{S_R} U \longrightarrow 0 \qquad \text{as} \quad R \longrightarrow \infty .$$

Then $U \leq u_\infty$.

Proof. Set

$$g = -\Delta(u_\infty - U) \geq 0 .$$

Since

$$⨍_{S_R} (u_\infty - U) \longrightarrow 0 \qquad \text{as} \quad R \longrightarrow \infty$$

we may apply Lemma A.5 to conclude that

$$u_\infty - U = \frac{c}{|x|^{n-2}} * g \geq 0.$$

Appendix II

Here we briefly review several proofs of uniqueness for the problem

$$\text{(A.6)}\qquad \begin{cases} -\Delta u = \rho(x)\, f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \psi \geq 0 & \text{on } \partial\Omega \end{cases}$$

under the assumptions that $\frac{f(t)}{t}$ is decreasing, Ω is a smooth bounded domain and $\rho \geq 0$.

Method I. This is the method introduced in [1]. Let u_1 and u_2 be two solutions of (A.6). We have

$$\text{(A.7)}\qquad -\frac{\Delta u_1}{u_1} + \frac{\Delta u_2}{u_2} = \rho\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)$$

Multiplying (A.7) by $(u_1^2 - u_2^2)$ we obtain

$$\int |\nabla u_1 - \frac{u_1}{u_2} \nabla u_2|^2 + |\nabla u_2 - \frac{u_2}{u_1} \nabla u_1|^2 = \int \rho\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2) .$$

It follows that $u_1 = u_2$ on the set $[\rho > 0]$. In particular $\rho f(u_1) = \rho f(u_2)$ on Ω. Going back to (A.6) we see that $u_1 = u_2$.

<u>Method II</u>. Let u_1 and u_2 be two solution of (A.6). We have

$$-(\Delta u_1)\, u_2 + (\Delta u_2)\, u_1 = \rho u_1 u_2 \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right) . \tag{A.8}$$

Integrating (A.8) on the set $[u_1 > u_2] = E$ we obtain formally

$$-\int_{\partial E} \frac{\partial u_1}{\partial \nu} u_2 + \int_{\partial E} \frac{\partial u_2}{\partial \nu} u_1 = \int_E \rho\, u_1 u_2 \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)$$

Note that $u_1 = u_2$ and $\frac{\partial}{\partial \nu}(u_1 - u_2) \le 0$ on ∂E. Thus the lefthand side is nonnegative while the integrand on the righthand side is nonpositive. Similarly, using $F = [u_1 < u_2]$, we are led to

$$\int_\Omega \rho\, u_1 u_2 \left|\frac{f(u_1)}{u_1} \frac{f(u_2)}{u_2}\right| = 0 .$$

We conclude as above.

To make this argument rigorous we proceed as follows. Let θ be a smooth nondecreasing function such that $\theta(0) = 0$ and $\theta(t) = 1$ for $t \ge 1$, $\theta(t) = -1$ for $t \le -1$. Set

$$\theta_\varepsilon(t) = \theta(t/\varepsilon)$$

Multiplying (A.8) by $\theta_\varepsilon(u_1 - u_2)$ and integrating we obtain

$$\text{(A.9)} \qquad \begin{cases} \int [(\nabla u_1)\cdot u_2 - (\nabla u_2)\cdot u_1]\ \theta'_\varepsilon(u_1 - u_2)\cdot\nabla(u_1 - u_2) \\ = \int \rho\, u_1 u_2 \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right) \theta_\varepsilon(u_1 - u_2) \end{cases} .$$

Clearly $$\text{LHS} \geq \int (\nabla u_2)(u_2 - u_1)\ \theta_\varepsilon'(u_1 - u_2)\cdot\nabla(u_1 - u_2)\ .$$

Note that

$$\int \nabla u_2(u_2 - u_1)\theta_\varepsilon'(u_1 - u_2)\nabla(u_1 - u_2) = - \int \nabla u_2 \nabla \gamma_\varepsilon(u_1 - u_2)$$

where $$\gamma_\varepsilon(t) = \int_0^t s\ \theta_\varepsilon'(s)\ ds\ .$$

Since $|\gamma_\varepsilon(t)| \leq C\ \varepsilon$ and $\Delta u_2 \in L^\infty$ we see that

$$\text{LHS} \geq - C\ \varepsilon\ .$$

Going back to (A.9) we obtain, as $\varepsilon \to 0$,

$$\int \rho\ u_1 u_2\ |\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}| = 0\ .$$

<u>Method III</u>. This is a variant of Krasnoselkii's method [9]. Let u_1 and u_2 be two solutions. Let

$$\Lambda = \{t \in [0,1];\ \ tu_1 \leq u_2 \ \text{ on } \ \Omega\}\ .$$

Clearly Λ contains a neighbourhood of 0. We claim that $1 \in \Lambda$. Suppose not, that

$$t_0 = \sup \Lambda < 1\ .$$

Then

$$- \Delta(u_2 - t_0 u_1) = \rho\ f(u_2) - t_0\ \rho\ f(u_1).$$

Fix a positive constant K large enough so that $f(t) + Kt$ is increasing on $[0,\text{Max } u_2]$. Then

$$- \Delta(u_2 - t_0 u_1) + K\rho(u_2 - t_0 u_1) = \rho[f(u_2) + Ku_2 - t_0(f(u_1) + Ku_1)]$$

$$\geq \rho[f(t_0 u_1) + Kt_0 u_1 - t_0(f(u_1) + Ku_1)] = \rho[f(t_0 u_1) - t_0 f(u_1)] \geq 0$$

(the last inequality follows from the fact that $f(u)/u$ is decreasing). On $\partial\Omega$ we have $u_2 - t_0 u_1 = (1 - t_0)\ \varphi \geq 0$.

We distinguish two cases:

Case 1: $\varphi \equiv 0$. Using the strong maximum principle we see that either $u_2 - t_0 u_1 > 0$ on Ω with $\frac{\partial}{\partial \nu}(u_2 - t_0 u_1) < 0$ on $\partial\Omega$. Then, clearly there is some $\varepsilon > 0$ such that $u_2 - t_0 u_1 \geq \varepsilon\ u_1$. Thus $t_0 + \varepsilon \in \Lambda$. Impossible. Or $u_2 - t_0 u_1 \equiv 0$. This case is also impossible since we would have, by the equation $\rho f(u_2) = t_0\ \rho f(u_1)$, but $f(t_0 u_1) > t_0 f(u_1)$.

Case 2: φ is not identically zero. We claim that there is some $\varepsilon > 0$ such that

$$w \equiv u_2 - t_0 u_1 \geq \varepsilon u_1 .$$

Suppose not, that for every $\varepsilon > 0$ there is some point $x_\varepsilon \in \bar{\Omega}$ such that

$$w(x_\varepsilon) < \varepsilon u_1(x_\varepsilon) .$$

Clearly $x_\varepsilon \notin \partial\Omega$ (for ε small). Choosing a point of minimum for the function $(w - \varepsilon u_1)$ we may also assume that

$$\nabla w(x_\varepsilon) = \varepsilon\ \nabla u_1(x_\varepsilon) .$$

As $\varepsilon \longrightarrow 0$ (through an appropriate sequence) $x_\varepsilon \longrightarrow x_0 \in \bar{\Omega}$ such that

$$w(x_0) \leq 0 \quad \text{and} \quad \nabla w(x_0) = 0 .$$

It follows that $w(x_0) = 0$ and thus $x_0 \in \partial\Omega$. This contradicts the strong maximum principle since we have

$$\begin{cases} -\Delta w + K\rho w \geq 0 & \text{in} \quad \Omega\ , \\ w \geq 0 & \text{on} \quad \partial\Omega\ , \\ w \quad \text{not identically zero}\ . \end{cases}$$

Method IV. This is a variant of Nirenberg's method already presented in the proof of Theorem 2′. It requires further restrictions on f, namely, f is positive, concave and $\int_0^\delta \frac{dt}{f(t)} < \infty$.

We use the new unknown

$$v = \int_0^u \frac{dt}{f(t)}\ ,$$

or in other words $u = h(v)$ where h satisfies

$$h'(s) = f(h(s))\ .$$

The equation for v becomes

$$-\Delta v - f'(h(v))|\nabla v|^2 = \rho\ .$$

Uniqueness holds provided the function $f'(h(v))$ is nonincreasing in v (see the proof of Theorem 2′). This follows from the assumptions on f.

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