Further results on Liouville type theorems for some conformally invariant fully nonlinear equations

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For $n \geq 3$, let $\mathcal{S}^{n\times n}$ be the set of $n \times n$ real symmetric matrices, $\mathcal{S}^{n\times n}_+ \subset \mathcal{S}^{n\times n}$ be the set of positive definite matrices, $O(n)$ be the set of $n\times n$ real orthogonal matrices.

For a positive C^2 function u, let

$$A^u := -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

where I is the $n \times n$ identity matrix.

Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying

$$O^{-1}UO = U, \qquad \forall\ O \in O(n), \tag{1}$$

and

$$U \cap \{M + tN \mid 0 < t < \infty\} \text{ is convex} \qquad \forall\ M \in \mathcal{S}^{n\times n}, N \in \mathcal{S}^{n\times n}_+. \tag{2}$$

Let $F \in C^\infty(U)$ satisfy

$$F(O^{-1}MO) = F(M), \qquad \forall\ M \in U, \tag{3}$$

$$(F_{ij}(M)) > 0, \qquad \forall\ M \in U, \tag{4}$$

where $F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M)$.

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For $n \geq 3$, $-\infty < p \leq \frac{n+2}{n-2}$, we consider

$$F(A^u) = u^{p-\frac{n+2}{n-2}}, \quad A^u \in U, \quad u > 0 \quad \text{on } \mathbb{R}^n. \tag{5}$$

Our main theorem is

Theorem 1 *For $n \geq 3$, let $U \subset \mathcal{S}^{n\times n}$ satisfy (1), (2), and let $F \in C^2(U)$ satisfy (3), (4). Assume that $u \in C^2(\mathbb{R}^n)$ is a superharmonic solution of (5) for some $-\infty < p \leq \frac{n+2}{n-2}$. Then either $u \equiv$ constant or $p = \frac{n+2}{n-2}$ and for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b satisfying $2b^2a^{-2}I \in U$ and $F(2b^2a^{-2}I) = 1$,*

$$u(x) \equiv \left(\frac{a}{1 + b^2|x-\bar{x}|^2}\right)^{\frac{n-2}{2}}, \quad \forall\, x \in \mathbb{R}^n. \tag{6}$$

Remark 1 *About half a year ago, we established a slightly weaker version of Theorem 1 for $p < \frac{n+2}{n-2}$, and the proof was different than the one in the present paper. The weaker result requires some additional (though minor, e.g., F being homogeneous of degree 1 would be enough) assumptions on (F, U).*

Theorem 1 for $p = \frac{n+2}{n-2}$ was established in [9], which extends earlier Liouville type theorems for conformally invariant equations by Obata ([11]), Gidas, Ni and Nirenberg ([4]), Caffarelli, Gadis and Spruck ([1]), Viaclovsky ([12] and [13]), Chang, Gursky and Yang ([2] and [3]), and Li and Li ([6], [7], [8] and [9]).

The proof of Theorem 1 for $p = \frac{n+2}{n-2}$ in the present paper is simplier than that in our earlier paper [9], though the most crucial ideas are the same. Theorem 1 for $-\infty < p < \frac{n+2}{n-2}$ extends the corresponding result of Gidas and Spruck in [5]. The proof of Theorem 1 for $-\infty < p < \frac{n+2}{n-2}$ is essentially the same as our simplified proof of Theorem 1 for $p = \frac{n+2}{n-2}$ in this paper. Our proof of Theorem 1 makes use of the following lemma used in our first proof of Theorem 1 for $p = \frac{n+2}{n-2}$ (see theorem 1 in [9]).

Lemma 1 *([9]) For $n \geq 1$, $R > 0$, let $u \in C^2(B_R \setminus \{0\})$ satisfying $\Delta u \leq 0$ in $B_R \setminus \{0\}$. Assume that there exist $w,\ v \in C^1(B_R)$ satisfying*

$$w(0) = v(0), \quad \nabla w(0) \neq \nabla v(0),$$

and

$$u \geq w, \quad u \geq v, \quad \text{in } B_R \setminus \{0\}.$$

Then

$$\liminf_{x\to 0} u(x) > w(0).$$

In fact, the above lemma was stated as lemma 2 in [9] under additional hypotheses ($w, v \in C^2(B_R)$ and $\Delta w \leq 0$, $\Delta v \leq 0$ in B_R). However the proof of lemma 2 in [9] did not use these extra hypotheses. Indeed, lemma 1 in [9] was first established and hypothesis (11) there was not used in the proof. So the proof of lemma 2 in [9] actually establishes Lemma 1 above.

Proof of Theorem 1 for $p = \frac{n+2}{n-2}$. Since u is a positive superharmonic function, we have, by the maximum principle, that

$$u(x) \geq \frac{\min_{\partial B_1} u}{|x|^{n-2}}, \quad \forall\ |x| \geq 1.$$

In particular

$$\liminf_{|x|\to\infty}(|x|^{n-2}u(x)) > 0. \tag{7}$$

Lemma 2 *For any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq u(y), \quad \forall\ |y-x| \geq \lambda,\ 0 < \lambda < \lambda_0(x).$$

Proof of Lemma 2. This follows from the proof of lemma 2.1 in [10].

□

For any $x \in \mathbb{R}^n$, set

$$\bar{\lambda}(x) := \sup\{\mu \mid u_{x,\lambda}(y) \leq u(y),\ \forall\ |y-x| \geq \lambda,\ 0 < \lambda < \mu\}.$$

Let

$$\alpha := \liminf_{|x|\to\infty}(|x|^{n-2}u(x)). \tag{8}$$

Because of (7),

$$0 < \alpha \leq \infty. \tag{9}$$

If $\alpha = \infty$, then the moving sphere procedure can never stop and therefore $\bar{\lambda}(x) = \infty$ for any $x \in \mathbb{R}^n$. This follows from arguments in [10], [7] and [8]. By the definition of $\bar{\lambda}(x)$ and the fact $\bar{\lambda}(x) = \infty$, we have,

$$u_{x,\lambda}(y) \leq u(y), \quad \forall\ |y-x| \geq \lambda > 0.$$

By a calculus lemma (see e.g., lemma 11.2 in [10]), $u \equiv$ *constant*, and Theorem 1 for $p = \frac{n+2}{n-2}$ is proved in this case (i.e. $\alpha = \infty$). So, from now on, we assume

$$0 < \alpha < \infty. \tag{10}$$

By the definition of $\bar{\lambda}(x)$,

$$u_{x,\lambda}(y) \le u(y), \quad \forall\ |y-x| \ge \lambda,\ 0 < \lambda < \bar{\lambda}(x).$$

Multiplying the above by $|y|^{n-2}$ and sending $|y| \to \infty$, we have,

$$\alpha \ge \lambda^{n-2} u(x), \qquad \forall\ 0 < \lambda < \bar{\lambda}(x).$$

Sending $\lambda \to \bar{\lambda}(x)$, we have (using (10)),

$$\infty > \alpha \ge \bar{\lambda}(x)^{n-2} u(x), \quad \forall\ x \in \mathbb{R}^n. \tag{11}$$

Since the moving sphere procedure stops at $\bar{\lambda}(x)$, we must have, by using the arguments in [10], [7] and [8],

$$\liminf_{|y|\to\infty} (u(y) - u_{x,\bar{\lambda}(x)}(y))|y|^{n-2} = 0, \tag{12}$$

i.e.,

$$\alpha = \bar{\lambda}(x)^{n-2} u(x), \quad \forall\ x \in \mathbb{R}^n. \tag{13}$$

Let us switch to some more convenient notations. For a Mobius transformation ϕ, we use notation

$$u_\phi := |J_\phi|^{\frac{n-2}{2n}} (u \circ \phi),$$

where J_ϕ denotes the Jacobian of ϕ.

For $x \in \mathbb{R}^n$, let

$$\phi^{(x)}(y) := x + \frac{\bar{\lambda}(x)^2 (y-x)}{|y-x|^2},$$

we know that $u_{\phi^{(x)}} = u_{x,\bar{\lambda}(x)}$.

Let $\psi(y) := \frac{y}{|y|^2}$, and let

$$w^{(x)} := (u_{\phi^{(x)}})_\psi = u_{\phi^{(x)} \circ \psi}.$$

For $x \in \mathbb{R}^n$, the only possible singularity for $w^{(x)}$ (on $\mathbb{R}^n \cup \{\infty\}$) is $\frac{x}{|x|^2}$. In particular, $y = 0$ is a regular point of $w^{(x)}$. A direct calculation yields

$$w^{(x)}(0) = \bar{\lambda}(x)^{n-2} u(x),$$

and therefore, by (13),

$$w^{(x)}(0) = \alpha, \quad \forall\, x \in \mathbb{R}^n. \tag{14}$$

Clearly, $u_\psi \in C^2(\mathbb{R}^n \setminus \{0\})$, $\Delta u_\psi \le 0$ in $\mathbb{R}^n \setminus \{0\}$. We also know that

$$w^{(x)}(0) = \alpha \quad \forall\, x \in \mathbb{R}^n, \qquad \liminf_{y\to 0} u_\psi(y) = \alpha,$$

and, for some $\delta(x) > 0$,

$$w^{(x)} \in C^2(B_{\delta(x)}), \quad \forall\, x \in \mathbb{R}^n,$$

$$u_\psi \ge w^{(x)} \quad \text{in } B_{\delta(x)} \setminus \{0\}, \quad \forall\, x \in \mathbb{R}^n,$$

$$\Delta w^{(x)} \le 0 \quad \text{in } B_{\delta(x)}, \quad \forall\, x \in \mathbb{R}^n.$$

Lemma 3 *$\nabla w^{(x)}(0) = \nabla w^{(0)}(0)$, i.e., $\nabla w^{(x)}(0)$ is independent of $x \in \mathbb{R}^n$.*

Proof of Lemma 3. This follows from Lemma 1. Indeed, for any $x,\ \tilde{x} \in \mathbb{R}^n$, let

$$v := w^{(x)}, \quad w := w^{(\tilde{x})}, \quad u := u_\psi.$$

We know that $w(0) = v(0)$, $u_\psi \ge w$ and $u_\psi \ge v$ near the origin, and we also know that $\liminf_{y\to 0} u_\psi(y) = w(0)$, so, by Lemma 1, we must have $\nabla v(0) = \nabla w(0)$, i.e., $\nabla w^{(x)}(0) = \nabla w^{(\tilde{x})}(0)$. Lemma 3 is established.

□

For $x \in \mathbb{R}^n$,

$$\begin{aligned}
w^{(x)}(y) &= \frac{1}{|y|^{n-2}}\Big\{\Big(\frac{\bar\lambda(x)}{|\frac{y}{|y|^2} - x|}\Big)^{n-2} u\Big(x + \frac{\bar\lambda(x)^2(\frac{y}{|y|^2} - x)}{|\frac{y}{|y|^2} - x|^2}\Big)\Big\} \\
&= \Big(\frac{\bar\lambda(x)}{|\frac{y}{|y|} - |y|x|}\Big)^{n-2} u\Big(x + \frac{\bar\lambda(x)^2(y - |y|^2 x)}{|\frac{y}{|y|} - |y|x|^2}\Big) \\
&= \Big(\frac{\bar\lambda(x)^2}{1 - 2x\cdot y + |y|^2 x}\Big)^{\frac{n-2}{2}} u\Big(x + \frac{\bar\lambda(x)^2(y - |y|^2 x)}{1 - 2x\cdot y + |y|^2|x|^2}\Big).
\end{aligned}$$

So, for $|y|$ small,

$$w^{(x)}(y) = \bar\lambda(x)^{n-2}(1 + (n-2)x\cdot y)u(x + \bar\lambda(x)^2 y) + O(|y|^2),$$

and, using (13),

$$\nabla w^{(x)}(0) = (n-2)\bar\lambda(x)^{n-2}u(x)x + \bar\lambda(x)^n \nabla u(x) = (n-2)\alpha x + \alpha^{\frac{n}{n-2}} u(x)^{\frac{n}{2-n}} \nabla u(x).$$

By Lemma 3, $\vec{V} := \nabla w^{(x)}(0)$ is a constant vector in $\mathbb{R}^n$, so we have,

$$\nabla_x\Big(\frac{n-2}{2}\alpha^{\frac{n}{n-2}}u(x)^{-\frac{2}{n-2}} - \frac{(n-2)\alpha}{2}|x|^2 + \vec{V}\cdot x\Big) \equiv 0.$$

Consequently, for some $\bar{x} \in \mathbb{R}^n$ and $d \in R$,

$$u(x)^{-\frac{2}{n-2}} \equiv \alpha^{-\frac{2}{n-2}}|x-\bar{x}|^2 + d\alpha^{-\frac{2}{n-2}}.$$

Since $u > 0$, we must have $d > 0$. Thus

$$u(x) \equiv \Big(\frac{\alpha^{\frac{2}{n-2}}}{d + |x-\bar{x}|^2}\Big)^{\frac{n-2}{2}}.$$

Let $a = \alpha^{\frac{2}{n-2}}d^{-1}$ and $b = d^{-\frac{1}{2}}$. Then u is of the form (6). Clearly $A^u(0) = 2b^2a^{-2}I$, so $2b^2a^{-2}I \in U$ and $F(2b^2a^{-2}I) = 1$. Theorem 1 in the case $p = \frac{n+2}{n-2}$ is established.

$\square$

Proof of Theorem 1 for $-\infty < p < \frac{n+2}{n-2}$. In this case, the equation satisfied by u is no longer conformally invariant, but it transforms to our advantage when making reflections with respect to spheres, i.e., the inequalities have the right direction so that the strong maximum principle and the Hopf lemma can still be applied. First, we still have (7) since this only requires the superharmonicity and the positivity of u. Lemma 2 still holds since it only uses (7) and the C^1 regularity of u in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, we still define $\bar{\lambda}(x)$ in the same way. We also define α as in (8) and we still have (9).
For $x \in \mathbb{R}^n$, $\lambda > 0$, the equation of $u_{x,\lambda}$ now takes the form

$$F(A^{u_{x,\lambda}}(y)) = \Big(\frac{\lambda}{|y-x|}\Big)^{(n-2)(\frac{n+2}{n-2}-p)}u_{x,\lambda}(y)^{p-\frac{n+2}{n-2}}, \quad A^{u_{x,\lambda}}(y) \in U, \quad \forall\ y \neq x. \tag{15}$$

Lemma 4 *If $\alpha = \infty$, then $\bar{\lambda}(x) = \infty$ for any $x \in \mathbb{R}^n$.*

Proof of Lemma 4. Suppose the contrary, $\bar{\lambda}(\bar{x}) < \infty$ for some $\bar{x} \in \mathbb{R}^n$. Without loss of generality, we may assume $\bar{x} = 0$, and we use notations

$$\bar{\lambda} := \bar{\lambda}(0), \quad u_\lambda := u_{0,\lambda}, \quad B_\lambda := B_\lambda(0).$$

By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}} \le u \quad \text{on} \ \ \mathbb{R}^n \setminus B_{\bar{\lambda}}. \tag{16}$$

By (15),

$$F(A^{u_{\bar\lambda}}) \le u_{\bar\lambda}^{p-\frac{n+2}{n-2}}, \quad A^{u_{\bar\lambda}} \in U, \quad \text{on } \mathbb{R}^n \setminus B_{\bar\lambda}. \tag{17}$$

Recall that u satisfies

$$F(A^u) = u^{p-\frac{n+2}{n-2}}, \quad A^u \in U, \quad \text{on } \mathbb{R}^n \setminus B_{\bar\lambda}. \tag{18}$$

By (17) and (18),

$$F(A^{u_{\bar\lambda}}) - F(A^u) - (u_{\bar\lambda}^{p-\frac{n+2}{n-2}} - u^{p-\frac{n+2}{n-2}}) \le 0, \quad A^{u_{\bar\lambda}} \in U,\ A^u \in U, \quad \text{on } \mathbb{R}^n \setminus B_{\bar\lambda}. \tag{19}$$

Since $\alpha = \infty$, we have

$$\liminf_{|y|\to\infty} |y|^{n-2}(u - u_{\bar\lambda})(y) > 0. \tag{20}$$

The inequality in (19) goes the right direction. Thus, with (20), the arguments for $p = \frac{n+2}{n-2}$ work essentially in the same way here and we obtain a contradiction by continuing the moving sphere procedure a little bit further. This deserves some explanations. Because of (20), and using arguments in [7] and [8], we only need to show that

$$u_{\bar\lambda}(y) < u(y), \quad \forall\ |y| > \bar\lambda, \tag{21}$$

and

$$\frac{d}{dr}(u - u_{\bar\lambda})|_{\partial B_{\bar\lambda}} > 0, \tag{22}$$

where $\frac{d}{dr}$ denotes the differentiation in the outer normal direction with repect to $\partial B_{\bar\lambda}$.

If $u_{\bar\lambda}(\bar y) = u(\bar y)$ for some $|\bar y| > \bar\lambda$, then, using (19) as in the proof of lemma 2.1 in [7], we know that $u_{\bar\lambda} - u$ satisfies that

$$L(u_{\bar\lambda} - u) \le 0,$$

where $L = -a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$ with $(a_{ij}) > 0$ continuous and b_i, c continuous. Since $u_{\bar\lambda} - u \le 0$ near $\bar y$, we have, by the strong maximum principle, $u_{\bar\lambda} \equiv u$ near $\bar y$. For the same reason, $u_{\bar\lambda}(y) \equiv u(y)$ for any $|y| \ge \bar\lambda$, violating (20). (21) has been checked. Estimate (22) can be established in a similar way by using the Hopf lemma (see the proof of lemma 2.1 in [7]). Thus Lemma 4 is established.

□

By Lemma 4 and the usual arguments, we know that if $\alpha = \infty$, u must be a constant, and Theorem 1 for $-\infty < p < \frac{n+2}{n-2}$ is also proved in this case.

¿From now on, we always assume (10). As before, we obtain (11). Since the inequality in (17) goes the right direction, the arguments for $p = \frac{n+2}{n-2}$ (see also the arguments in the proof of Lemma 4) essentially apply and we still have (12) and (13). The rest of the arguments for $p = \frac{n+2}{n-2}$ apply and we have that u is of the form (6) with some positive constants a and b. However, we know that, for u of the form (6), $A^u \equiv 2b^2 a^{-2} I$ and $F(A^u) \equiv constant$. This violates (5) since $u^{p-\frac{n+2}{n-2}}$ is not a constant (recall that $p < \frac{n+2}{n-2}$). Theorem 1 for $-\infty < p < \frac{n+2}{n-2}$ is established.

$\square$

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