A general Liouville type theorem for some conformally invariant fully nonlinear equations

Aobing Li & YanYan Li*
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd.
Piscataway, NJ 08854

Various Liouville type theorems for conformally invariant equations have been obtained by Obata ([9]), Gidas, Ni and Nirenberg ([4]), Caffarelli, Gidas and Spruck ([1]), Viaclovsky ([10] and [11]), Chang, Gursky and Yang ([2] and [3]), and Li and Li ([5], [6] and [7]). See e. g. theorem 1.3 and remark 1.6 in [6] where these results (except for the one in [7]) are stated more precisely.

In this paper we give a general Liouville type theorem for conformally invariant fully nonlinear equations. This extends the above mentioned Liouville type theorems.

For $n \geq 3$, let $\mathcal{S}^{n\times n}$ be the set of $n\times n$ real symmetric matrices, $\mathcal{S}_+^{n\times n} \subset \mathcal{S}^{n\times n}$ be the set of positive definite matrices, and let $O(n)$ be the set of $n\times n$ real orthogonal matrices.

For a positive C^2 function u, let

$$A^u := -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

where I is the $n\times n$ identity matrix.

Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying

$$O^{-1}UO = U, \quad \forall\ O \in O(n) \tag{1}$$

and

$$U \cap \{M + tN \mid 0 < t < \infty\} \ \text{ is convex} \quad \text{for } \forall\ M \in \mathcal{S}^{n\times n}, N \in \mathcal{S}_+^{n\times n}. \tag{2}$$

*Partially supported by NSF Grant DMS-0100819.

Let $F \in C^1(U)$ satisfy

$$F(O^{-1}MO) = F(M), \quad \forall M \in U, O \in O(n) \tag{3}$$

and

$$(F_{ij}(M)) > 0, \quad \forall M \in U, \tag{4}$$

where $F_{ij}(M) = \frac{\partial F}{\partial M_{ij}}(M)$.

Theorem 1 *For $n \geq 3$, let $U \subset \mathcal{S}^{n\times n}$ be open and satisfy (1) and (2), and let $F \in C^1(U)$ satisfy (3) and (4). Assume that $u \in C^2(R^n)$ is a positive function satisfying*

$$F(A^u) = 1, \quad A^u \in U, \quad on \ \ R^n, \tag{5}$$

and

$$\Delta u \leq 0, \quad on \ \ R^n. \tag{6}$$

Then for some $\bar{x} \in R^n$ and some constants $a > 0$ and $b \geq 0$ satisfying $2b^2a^{-2}I \in U$ and $F(2b^2a^{-2}I) = 1$

$$u(x) \equiv \Big(\frac{a}{1 + b^2|x - \bar{x}|^2}\Big)^{\frac{n-2}{2}}, \quad \forall x \in R^n. \tag{7}$$

Remark 1 *If U has the property that*

$$Trace(M) := \sum_{i=1}^{n} M_{ii} \geq 0, \quad \forall \ M \in U, \tag{8}$$

then any positive solution u of (5) automatically satisfies (6).

Remark 2 *When $b = 0$ in (7), then $u \equiv$Constant, $A^u \equiv 0$, $0 \in U$, and $F(0) = 1$.*

Let $B_R(x) \subset R^n$ denote the ball of radius R centered at x, and let $B_R = B_R(0)$.

Lemma 1 *For $n \geq 1$, $R > 0$, let $\xi \in C^2(B_R \setminus \{0\})$ satisfy*

$$\Delta\xi \geq 0 \quad in \ B_R \setminus \{0\}, \tag{9}$$

and

$$\inf_{B_R\setminus\{0\}} \xi > -\infty. \tag{10}$$

Assume that there exist $\eta,\ \zeta \in C^1(B_R)$ *satisfying*

$$\Delta\eta \geq 0, \quad \Delta\zeta \geq 0, \quad \text{in } B_R \ \text{ in the distribution sense,} \tag{11}$$

$$\eta(0) = \zeta(0), \tag{12}$$

$$\nabla\eta(0) \neq \nabla\zeta(0), \tag{13}$$

and

$$\xi \leq \eta, \quad \xi \leq \zeta, \quad \text{in } B_R \setminus \{0\}. \tag{14}$$

Then

$$\limsup_{x\to 0} \xi(x) < \eta(0). \tag{15}$$

Remark 3 *If we further assume that* $\eta, \zeta \in C^2(B_R)$*, then hypothesis (10) is not needed in Lemma 1. This can be deduced easily from Lemma 2 by letting* $\xi = -u$*,* $\eta = -w$ *and* $\zeta = -v$*.*

Proof of Lemma 1. Replacing ξ, η and ζ by

$$\tilde{\xi}(x) = \xi(x) - \nabla\eta(0)\cdot x + |\nabla\eta(0)|R + 1 - \inf_{B_R\setminus\{0\}} \xi,$$

$$\tilde{\eta}(x) = \eta(x) - \nabla\eta(0)\cdot x + |\nabla\eta(0)|R + 1 - \inf_{B_R\setminus\{0\}} \xi,$$

and

$$\tilde{\zeta}(x) = \zeta(x) - \nabla\eta(0)\cdot x + |\nabla\eta(0)|R + 1 - \inf_{B_R\setminus\{0\}} \xi$$

respectively, we may further assume that

$$\nabla\eta(0) = 0, \tag{16}$$

$$\xi \geq 1 \quad \text{in } B_R \setminus \{0\}. \tag{17}$$

Without loss of generality, we may assume that $R = 1$. By (13) and (16), $\nabla\zeta(0) \neq 0$. After making a rotation, we may assume that

$$\frac{\partial\zeta}{\partial x_1}(0) = -|\nabla\zeta(0)| < 0. \tag{18}$$

Since $\xi \in L^\infty_{loc}(B_1)$ and $\Delta\xi \geq 0$ in $B_1 \setminus \{0\}$, we know that $\Delta\xi \geq 0$ in B_1 in the distribution sense. Consequently,

$$\xi(y) \leq \frac{1}{|B_r(y)|}\int_{B_r(y)} \xi, \quad \forall\ 0 < |y| < 1,\ \forall\ 0 < r < 1 - |y|. \tag{19}$$

Since $\zeta \in C^1(B_1)$ satisfies (18), there exists $0 < \delta < \frac{1}{2}$ such that for $\forall\ e \in R^n$ with $|e| = 1$ and $e \cdot e_1 \geq 1 - \delta$, we have

$$\nabla \zeta(x) \cdot e < -\delta, \quad \forall\ |x| < \delta. \tag{20}$$

where $e_1 = (1, 0, \cdots, 0)$.
Let $S_\delta := \{x \in R^n \setminus \{0\} |\ \frac{x}{|x|} \cdot e_1 > 1 - \delta\}$. Now we fix the value of δ. In the following, we will choose small positive numbers r and t satisfying $0 < t < \frac{r}{10} < \frac{\delta}{40}$, and we will show that for some positive constant c, depending only on δ, n and r, we have

$$\xi(y) \leq \eta(0) - c, \quad \forall\ 0 < |y| < t. \tag{21}$$

For $0 < |y| < t$, we have, by using (14), (16), (19), that

$$\begin{aligned} \xi(y) \ &\leq \ \frac{1}{|B_r(y)|} \int_{B_r(y)} \xi \leq \frac{1}{|B_r(y)|} \{ \int_{B_r(y) \setminus S_\delta} \eta + \int_{B_r(y) \cap S_\delta} \zeta \} \\ &= \frac{1}{|B_r(y)|} \{ \int_{B_r(y) \setminus S_\delta} (\eta(0) + o(r)) + \int_{B_r(y) \cap S_\delta} \zeta \}, \end{aligned}$$

where $o(r)$ satisfying $\lim_{r \to 0} \frac{|o(r)|}{r} = 0$.
First recall that $|y| < t < \frac{r}{10}$,

$$\int_{B_r(y) \setminus S_\delta} (\eta(0) + o(r)) = \eta(0) |B_r(y) \setminus S_\delta| + o(r^{n+1}).$$

Next recall that ζ satisfies (20), $1 \leq \xi \leq \zeta$ and $|y| < t < \frac{r}{10}$,

$$\begin{aligned} \int_{B_r(y) \cap S_\delta} \zeta \ &\leq \ \int_{B_{r+|y|} \cap S_\delta} \zeta = \int_0^{r+|y|} (\int_{\partial B_s \cap S_\delta} \zeta)\ ds \\ &\leq \ \int_0^{r+|y|} \Big(\int_{\partial B_s \cap S_\delta} (\zeta(0) - \delta s) \Big)\ ds \\ &= \ \zeta(0) |B_{r+|y|} \cap S_\delta| - \frac{\delta}{n+1} |\partial B_1 \cap S_\delta| (r + |y|)^{n+1} \\ &\leq \ \zeta(0) |B_{r+2|y|}(y) \cap S_\delta| - \frac{\delta}{n+1} |\partial B_1 \cap S_\delta| r^{n+1} \end{aligned}$$

Since $\zeta(0) = \eta(0)$, we deduce from the above that

$$\xi(y) \ \leq \ \frac{1}{|B_r(y)|} \{ \eta(0) |B_r(y) \setminus S_\delta| + o(r^{n+1}) + \eta(0) |B_{r+2|y|}(y) \cap S_\delta|$$

$$
\begin{aligned}
& -\frac{\delta}{n+1}|\partial B_1 \cap S_\delta| r^{n+1}\} \\
= \quad & \eta(0) + \frac{1}{|B_r(y)|}\{\eta(0)|(B_{r+2|y|}(y) \setminus B_r(y)) \cap S_\delta| \\
& +o(r^{n+1}) - \frac{\delta}{n+1}|\partial B_1 \cap S_\delta| r^{n+1}\}
\end{aligned}
$$

Now fix some small r satisfying $0 < r < \frac{\delta}{4}$ and $o(r^{n+1}) - \frac{\delta}{2(n+1)}|\partial B_1 \cap S_\delta| r^{n+1} \le 0$. Since

$$|B_{r+2|y|}(y) \setminus B_r(y)| \le C(n) r^{n-1}|y|,$$

we can fix a smaller t satisfying $0 < t < \frac{r}{10}$ such that

$$C(n)\eta(0) r^{n-1} t - \frac{\delta}{4(n+1)}|\partial B_1 \cap S_\delta| r^{n+1} \le 0.$$

With these choices of r and t, we have

$$\xi(y) \le \eta(0) - \frac{\delta}{4(n+1)}|\partial B_1 \cap S_\delta| r^{n+1}.$$

Estimate (15) follows from the above. Lemma 1 is established.

□

Lemma 2 *For $n \ge 1$, $R > 0$, let $u \in C^2(B_R \setminus \{0\})$ satisfy $\Delta u \le 0$ in $B_R \setminus \{0\}$. Assume that there exist $w,\ v \in C^2(B_R)$ satisfying*

$$\Delta w \le 0, \quad \Delta v \le 0 \quad in\ B_R,$$

$$w(0) = v(0), \quad \nabla w(0) \ne \nabla v(0),$$

and

$$u \ge w, \quad u \ge v, \quad in\ B_R \setminus \{0\}.$$

Then

$$\liminf_{x \to 0} u(x) > w(0).$$

Proof of Lemma 2. By adding a large constnat to u, w and v, we may assume that

$$v \ge 1, \quad w \ge 1, \quad \text{in } B_{\frac{R}{2}} \setminus \{0\}.$$

Let $\xi = \frac{1}{u}$, $\eta = \frac{1}{w}$ and $\zeta = \frac{1}{v}$. Since $\Delta u \leq 0$ in $B_R \setminus \{0\}$, a straight forward calculation yields

$$\Delta \xi = -u^{-2}\Delta u + 2u^{-3}|\nabla u|^2 \geq 0, \quad \text{in } B_{\frac{R}{2}} \setminus \{0\}.$$

Similarly, we have

$$\Delta \eta \geq 0, \quad \Delta \zeta \geq 0, \quad \text{in } B_{\frac{R}{2}}.$$

Clearly, $\eta(0) = \zeta(0)$, $\nabla\eta(0) \neq \nabla\zeta(0)$ and $\xi > 0$ in $B_{\frac{R}{2}} \setminus \{0\}$. It follows from Lemma 1 that $\limsup_{x\to 0} \xi(x) < \eta(0)$, i.e., $\liminf_{x\to 0} u(x) > w(0)$. Lemma 2 is established.

□

Proposition 1 *For $n \geq 3$, let $U \subset \mathcal{S}^{n\times n}$ be open and satisfy (1) and (2), and let $F \in C^1(U)$ satisfy (3) and (4). Assume that $u \in C^2(R^n \setminus \{0\})$ is a positive function satisfying*

$$F(A^u) = 1, \quad A^u \in U, \quad \text{in } R^n \setminus \{0\}, \tag{22}$$

$$\Delta u \leq 0, \quad \text{in } R^n \setminus \{0\}, \tag{23}$$

and

$$u_{0,1} \quad \text{can be extended to a } C^2 \text{ function near the origin,} \tag{24}$$

where $u_{0,1}(x) := \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2})$.
We further assume that there exist some constant $\delta > 0$ and $v \in C^2(B_\delta)$ such that

$$\Delta v \leq 0 \quad \text{in } B_\delta. \tag{25}$$

$$\nabla v(0) = 0, \tag{26}$$

$$u - v \geq 0 \quad \text{in } B_\delta \setminus \{0\}, \tag{27}$$

$$\liminf_{x\to 0} (u - v)(x) = 0. \tag{28}$$

Then u is radially symmetric, i.e,

$$u(x) = u(y), \quad \forall \ |x| = |y| > 0. \tag{29}$$

Moreover $u'(r) < 0$ for $\forall \ r > 0$, where we have used $u(r)$ to denoted the radially symmetric function u.

Lemma 3 *Let $u \in C^0(B_2 \setminus \{0\})$ satisfy*

$$\Delta u \le 0 \quad \text{in } B_2 \setminus \{0\} \text{ in the distribution sense,}$$

and $\inf_{B_2 \setminus \{0\}} u > -\infty$. *Then*

$$u \ge \min_{\partial B_1} u \quad \text{on } B_1 \setminus \{0\}.$$

Proof of Lemma 3. For $\epsilon > 0$, consider $v_\epsilon(x) := \epsilon(1 - \frac{1}{|x|^{n-2}}) + \min_{\partial B_1} u$. Then

$$\Delta(v_\epsilon - u) \ge 0 \quad \text{ in } B_1 \setminus \{0\}, \quad (v_\epsilon - u) \le 0 \quad \text{ on } \partial B_1.$$

Since $\limsup_{x \to 0}(v_\epsilon(x) - u(x)) = -\infty$, we deduce from the maximum principle that

$$v_\epsilon - u \le 0 \quad \text{on } B_1 \setminus \{0\}.$$

Fix any x in $B_1 \setminus \{0\}$, and send $\epsilon \to 0$, we have $u(x) \ge \min_{\partial B_1} u$. Lemma 3 is established.

□

Proof of Proposition 1. By the positivity of u and by (23), we have $u_{0,1} > 0$ and $\Delta u_{0,1} \le 0$ on $R^n \setminus \{0\}$. By Lemma 3,

$$\inf_{B_1 \setminus \{0\}} u > 0, \quad \min_{B_1} u_{0,1} > 0. \tag{30}$$

If u can be extended to a C^1 function near the origin, then, by theorem 1.2 in [6], u is of the form (7) for some $\bar{x} \in R^n$ and some positive constants a and b. By (27), (28) and (26), $\nabla u(0) = 0$, and therefore $\bar{x} = 0$. Proposition 1 is proved in this case. *In the rest of the proof of Proposition 1, we always assume that u can not be extended to a C^1 function near the origin.*

By (30) and the repeatedly used arguments in [8], [6] and [7], we can prove that $\forall\, x \in R^n \setminus \{0\}$, there exists $\lambda_0(x) > 0$ such that

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \le u(y), \quad \forall\, 0 < \lambda < \lambda_0(x),\ |y-x| \ge \lambda,\ y \ne 0.$$

Set

$$\bar{\lambda}(x) = \sup\{0 < \mu < |x| \mid u_{x,\lambda}(y) \le u(y),\ \forall\, |y-x| \ge \lambda,\ y \ne 0,\ 0 < \lambda \le \mu\}.$$

We distinguish into two cases.
Case 1. $\exists\ \bar{x} \in R^n \setminus \{0\}$ such that $\bar\lambda(\bar{x}) < |\bar{x}|$.
Case 2. $\bar\lambda(x) = |x|$ for $\forall\ x \in R^n \setminus \{0\}$.

In Case 1, we have

$$u_{\bar{x},\lambda}(y) \le u(y), \quad \forall\ 0 < \lambda < \bar\lambda(\bar{x}),\ |y-\bar{x}| \ge \lambda,\ y \ne 0. \tag{31}$$

After a rotation, we may assume that $\bar{x} = \bar{x}_1 e_1$ with $\bar{x}_1 > 0$.

Lemma 4 $\nabla u_{\bar{x},\bar\lambda(\bar{x})}(0) \ne 0$.

Proof of Lemma 4. Suppose the contrary,

$$\nabla u_{\bar{x},\bar\lambda(\bar{x})}(0) = 0. \tag{32}$$

A direct calculation yields that

$$\partial_{y_1} u_{\bar{x},\bar\lambda(\bar{x})}(0) = (n-2)\bar\lambda(\bar{x})^{n-2}|\bar{x}|^{1-n}u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}) - \bar\lambda(\bar{x})^n|\bar{x}|^{-n}\partial_1 u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}).$$

By (32),

$$(n-2)|\bar{x}|u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}) = \bar\lambda(\bar{x})^2\partial_1 u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}). \tag{33}$$

Consider $w(s) := u(\bar{x} - s\frac{\bar{x}}{|\bar{x}|})$ for $s > 0$. By (31) with $y = \bar{x} - s\frac{\bar{x}}{|\bar{x}|}$,

$$(\frac{\lambda}{s})^{n-2} w(\frac{\lambda^2}{s}) \le w(s), \quad \forall\ \lambda \le s < |\bar{x}|,\ \forall\ 0 < \lambda \le \bar\lambda(\bar{x}).$$

It follows (with $t = \frac{\lambda^2}{s}$) that $t^{\frac{n-2}{2}}w(t) \le s^{\frac{n-2}{2}}w(s)\ \forall\ 0 < t \le s \le \bar\lambda(\bar{x})$, and therefore (note that $\frac{\bar\lambda(\bar{x})^2}{|\bar{x}|} < \bar\lambda(\bar{x})$)

$$\frac{d}{ds}(s^{\frac{n-2}{2}}w(s))\big|_{s=\frac{\bar\lambda(\bar{x})^2}{|\bar{x}|}} \ge 0,$$

i.e.,

$$\frac{n-2}{2}u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}) \ge \frac{\bar\lambda(\bar{x})^2}{|\bar{x}|}\partial_1 u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}). \tag{34}$$

By (33) and (34), $\frac{n-2}{2}u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x}) \ge (n-2)u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x})$. This is a contradiction, since $u((1-(\frac{\bar\lambda(\bar{x})}{|\bar{x}|})^2)\bar{x})$ and $n-2>0$. Lemma 4 is established.

□

Since $\bar{\lambda}(\bar{x}) < |\bar{x}|$, we have, by (31), that $u_{\bar{x},\bar{\lambda}(\bar{x})} \le u$ in an open neighborhood of the origin. Since u is a C^2 superharmonic function in $R^n \setminus \{0\}$, $u_{\bar{x},\bar{\lambda}(\bar{x})}(\bar{x})$ is a superharmonic function in an open neighborhood of the origin. We first show that

$$\liminf_{|y|\to\infty} |y|^{n-2}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(y) > 0. \tag{35}$$

Indeed, let $\xi(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2})$ and $\eta(x) = \frac{1}{|x|^{n-2}} u_{\bar{x},\bar{\lambda}(\bar{x})}(\frac{x}{|x|^2})$. By the hypothesis on u, both ξ and η can be extended as a C^2 positive function near the origin. Since the equation satisfied by u is conformally invariant, we have

$$F(A^\xi) = F(A^\eta) = 1, \quad A^\xi,\ A^\eta \in U, \quad \text{in an open neighborhood of the origin.}$$

We also know that $\xi \ge \eta$ in an open neighborhood of the origin. If (35) does not hold, then $\xi(0) = \eta(0)$. By the arguments in the proof of lemma 2.1 in [6] which are based on the strong maximum principle while using only the fairly weak ellipticity hypotheses (2) and (4), we have $\xi \equiv \eta$ near the origin, i.e., $u(y) \equiv u_{\bar{x},\bar{\lambda}(\bar{x})}(y)$ for large $|y|$. Again, by the same arguments, $u \equiv u_{\bar{x},\bar{\lambda}(\bar{x})}$, and in particular u can be extended as a C^2 function near the origin, violating our assumption that u does not have such an extension. We have proved (35).
Similarly, also using arguments in the proof of lemma 2.1 in [6] (based on the Hopf lemma and the strong maximum principle), we have

$$\frac{d}{dr}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})|_{\partial B_{\bar{\lambda}(\bar{x})}(\bar{x})} > 0, \tag{36}$$

where $\frac{d}{dr}$ denotes the outer normal differentiation with respect to $B_{\bar{\lambda}(\bar{x})}(\bar{x})$.
Again, by using the strong maximum principle as in the proof of lemma 2.1 in [6] (recall that we always assume that u can not be extended as a C^1 function near the origin), we have

$$(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(y) > 0, \quad \forall\ |y - \bar{x}| > \bar{\lambda}(\bar{x}),\ y \ne 0. \tag{37}$$

Because of (35), (36), and the definition of $\bar{\lambda}(\bar{x})$, we must have, as usual,

$$\liminf_{y\to 0}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(y) = 0. \tag{38}$$

On the other hand, applying Lemma 2 to u with $w = u_{\bar{x},\bar{\lambda}(\bar{x})}$ (note that $\nabla u_{\bar{x},\bar{\lambda}(\bar{x})}(0) \ne \nabla v(0)$ due to (26) and Lemma 4), we have $\liminf_{x\to 0}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(x) > 0$, violating (38). Case 1 is settled.

In Case 2, we have, $\forall\ x \in R^n \setminus \{0\}$,

$$u_{x,\lambda}(y) \le u(y), \quad \forall\ |y-x| \ge \lambda,\ y \ne 0,\ 0 < \lambda < |x|. \tag{39}$$

For $e \in R^n$ with $\|e\| = 1$ and $\mu > 0$, let

$$\Sigma_\mu(e) := \{y \in R^n |\ y \cdot e < \mu\}, \quad u^{e,\mu}(y) := u(y^{e,\mu}),$$

where $y^{e,\mu}$ denotes the mirror symmetry point of y with respect to the plane $\partial\Sigma_\mu(e)$.

Lemma 5 *$\forall\ e \in R^n$ with $\|e\| = 1$ and $\forall\ \mu > 0$, we have*

$$u^{e,\mu}(y) \le u(y), \quad \forall\ y \in \Sigma_\mu(e) \setminus \{0\}.$$

Proof of Lemma 5. Without loss of generality, we may assume $e = e_1$. For any fixed $\mu > 0$, let $x = x(R) = Re_1$ for $R > \mu$, and let $\lambda = \lambda(R) = R - \mu$. By (39),

$$u_{x,\lambda}(y) \le u(y), \quad \forall\ y \in \Sigma_\mu(e_1) \setminus \{0\}.$$

Fix $y \in \Sigma_\mu(e_1)$, we deduce from the above that

$$u(y) \ge \lim_{R\to\infty} u_{x,\lambda}(y) = \lim_{R\to\infty} (\frac{\lambda}{|y-x|})^{n-2} u(x + \frac{\lambda^2(y-x)}{|y-x|^2}) = u(y^{e_1,\mu}).$$

Here we have used the fact that $\lim\limits_{R\to\infty}(x + \frac{\lambda^2(y-x)}{|y-x|^2}) = y^{e_1,\mu}$. Lemma 5 is established.

□

It follows from Lemma 5 that w is radially symmetric, and as usual, by the Hopf Lemma (as in the proof of lemma 2.1 in [6], using only the fairly weak ellipticity hypotheses (2) and (4)), we have $u'(r) < 0$ for $\forall\ r > 0$. Proposition 1 is established.

□

Proposition 2 *For $n \ge 3$, let $U \subset \mathcal{S}^{n\times n}$ be open and satisfy (1) and (2) and let $F \in C^1(U)$ satisfy (3) and (4). Assume that $u \in C^2(R^n \setminus \{0\})$ is a positive radially symmetric function satisfying (22), (24) and*

$$u'(r) \le 0, \quad \forall\ 0 < r < \infty. \tag{40}$$

Then either $u(r) \equiv \frac{constant}{|r|^{n-2}}$ or u is of the form (7) with $\bar{x} = 0$ and some positive constants a and b satisfying $2b^2a^{-2}I \in U$ and $F(2b^2a^{-2}I) = 1$.

Proof of Proposition 2. If we know $\lim_{r\to 0^+}(r|u'(r)|) = 0$, then, by theorem 1.2 in [6], u is of the form (7). By the radial symmetry of u, $\bar{x} = 0$. Since ∞ is regular point of u, b must be positive. Proposition 2 is proved in this case. In the following, we assume that

$$\limsup_{r\to 0}(-ru'(r)) = \limsup_{r\to 0}(r|u'(r)|) > \delta > 0, \tag{41}$$

and we will show that $u(r) \equiv \frac{constant}{|r|^{n-2}}$.
By (41), we can find $r_i \to 0^+$ such that

$$-r_i u'(r_i) \geq \delta, \quad \forall\ i. \tag{42}$$

Since u is positive in $R^n \setminus \{0\}$ and $u'(r) \leq 0$ for $\forall\ r > 0$, we have $\inf_{0<r<1} u(r) \geq u(1) > 0$. By (24), ∞ is a regular point of u. As usual we have, for large $\lambda > 0$, that

$$u_\lambda(x) := (\frac{\lambda}{|x|})^{n-2} u(\frac{\lambda^2 x}{|x|^2}) \leq u(x), \quad \forall\ 0 < |x| \leq \lambda.$$

Here and below we have abused notation slightly by writing $u(x) = u(|x|)$.

For any fixed i, set

$$\bar{\lambda}_i := \{\mu > r_i \mid u_\lambda(x) \leq u(x),\ forall\ r_i \leq |x| \leq \lambda,\ \forall\ \lambda \geq \mu\}.$$

Lemma 6 $\lim_{i\to\infty} \bar{\lambda}_i = 0.$

Proof of Lemma 6. Suppose not, then for some positive constant $\delta_1 > 0$ and along a subsequence, we have $\bar{\lambda}_i > \delta > r_i$. By the usual arguments based on the strong maximum principle, the Hopf lemma and our ellipticity hypothesis, a touching must occur at $r = r_i$, i.e., $u_{\bar{\lambda}_i}(r_i) = u(r_i)$. Recall that $u_{\bar{\lambda}_i}(r) \leq u(r)$ for $\forall\ r_i \leq r < \bar{\lambda}_i$. Thus

$$u'(r_i) \geq u'_{\bar{\lambda}_i}(r_i). \tag{43}$$

Since u is regular at ∞ $\Big((24)\Big)$ and $\bar{\lambda}_i \geq \delta_1 > 0$, we have

$$|u'_{\bar{\lambda}_i}(r_i)| \leq C \tag{44}$$

for some constant $C > 0$ independent of i. On the other hand, we have, by (42),

$$\lim_{i\to\infty} u'(r_i) = -\infty. \tag{45}$$

We reach a contradiction from (43), (44) and (45). Lemma 6 is established.

□

Lemma 7 $\lim_{r\to 0^+} u(r) = \infty$.

Proof of Lemma 7. For any fixed $\lambda > 0$, we have, by Lemma 6, $\bar\lambda_i < \lambda$ for large i. By the definition of $\bar\lambda_i$, we have, for large i,

$$u_\lambda(x) \le u(x), \quad \forall\ r_i \le |x| \le \lambda.$$

For any fixed $x \in \bar B_\lambda \setminus \{0\}$, send $i \to \infty$, we have $u_\lambda(x) \le u(x)$. It follows that for any fixed $\lambda > 0$, we have

$$\begin{aligned} \liminf_{|x|\to 0} u(x) &\ge \lim_{|x|\to 0} u_\lambda(x) = \lim_{|x|\to 0} (\frac{\lambda}{|x|})^{n-2} u(\frac{\lambda^2 x}{|x|^2}) \\ &= \lim_{|x|\to 0} \lambda^{2-n} u_{0,1}(\frac{x}{\lambda^2}) = \lambda^{2-n} u_{0,1}(0). \end{aligned}$$

Here we have used (24).
Sending $\lambda \to 0$, we have established Lemma 7.

□

By Lemma 7,

$$\liminf_{|x|\to\infty} (|x|^{n-2} u_{0,1}(x)) = \infty. \tag{46}$$

We also know $u_{0,1} \in C^2(R^n)$ is a positive solution of

$$F(A^{u_{0,1}}) = 1, \quad A^{u_{0,1}} \in U, \quad \text{on } R^n.$$

Let $w = u_{0,1}$. Starting from any point $x \in R^n$, the moving phere procedure can get started and can never stop due to (46). This follows from our usual arguments (see [8], [6], [7]). Thus we have

$$w_{x,\lambda}(y) \le w(y), \quad \forall\ x \in R^n,\ 0 < \lambda < \infty,\ |y - x| \ge \lambda.$$

By a calculus lemma (see, e.g., lemma 11.2 in [8]), $w \equiv$constant, i.e., $u(r) \equiv \frac{constant}{r^{n-2}}$. Proposition 2 is established.

□

Proof of Theorem 1. Using the positivity and the superharmonicity of u on R^n, we have, by the maximum principle, $\liminf_{|x|\to\infty}(|x|^{n-2} u(x)) \ge \min_{\partial B_1} u > 0$. With this, we have, as usual, that for any $x \in R^n$, there exists some $\lambda_0(x) > 0$ such that

$$u_{x,\lambda}(y) \le u(y), \quad \forall\ |y - x| \ge \lambda,\ 0 < \lambda \le \lambda_0(x).$$

Set, for $x \in R^n$,

$$\bar{\lambda}(x) := \{\mu > 0 \mid u_{x,\lambda}(y) \le u(y), \ \forall\, |y-x| \ge \lambda, \ 0 < \lambda < \mu\}.$$

If $\bar{\lambda}(x) = \infty$ for any $x \in R^n$, then, as usual, $u \equiv$constant. We're done ($b = 0$ in (7)). So, we only need to deal with the situation that $0 < \bar{\lambda}(x) < \infty$ for some $\bar{x} \in R^n$. The moving sphere procedure stops at $\lambda = \bar{\lambda}(\bar{x})$, therefore, as usual, we have that

$$\liminf_{|y|\to\infty} |y|^{n-2}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(y) = 0, \tag{47}$$

$$(u - u_{\bar{x},\bar{\lambda}(\bar{x})})(y) \ge 0, \quad \forall\, |y-\bar{x}| \ge \bar{\lambda}(\bar{x}). \tag{48}$$

Let $\phi_1(x) := \bar{x} + \frac{\bar{\lambda}(\bar{x})^2(x-\bar{x})}{|x-\bar{x}|^2}$, we know $u_{\phi_1} = u_{\bar{x},\bar{\lambda}(\bar{x})}$, where $u_{\phi_1} := |J_{\phi_1}|^{\frac{n-2}{2n}}(u \circ \phi_1)$, J_{ϕ_1} denotes the Jacobian of ϕ_1. Pick any $\tilde{x} \ne \bar{x}$ and let

$$\phi_2(x) := \tilde{x} + \frac{x-\tilde{x}}{|x-\tilde{x}|^2}, \quad \tilde{u} := u_{\phi_2}, \quad \tilde{v} := (u_{\phi_1})_{\phi_2} = u_{\phi_1\circ\phi_2}.$$

Then $\tilde{u} \in C^2(R^n \setminus \{\tilde{x}\})$, ∞ is a regular point of $\tilde{u}$ (i.e., $\frac{1}{|x|^{n-2}}\tilde{u}(\frac{x}{|x|^2})$ can be extended to a positive C^2 function near the origin), $\Delta\tilde{u} \le 0$ in $R^n\setminus\{\tilde{x}\}$, $\tilde{v} \in C^2(R^n\setminus\{\phi_2^{-1}(\bar{x})\})$, ∞ is a regular point of $\tilde{v}$ (since $\bar{x} \ne \tilde{x}$), $\Delta\tilde{v} \le 0$ in $R^n \setminus \{\phi_2^{-1}(\bar{x})\}$, $\tilde{u} \ge \tilde{v}$ in an open neighborhood of $\tilde{x}$ (because of (48)), and $\liminf_{x\to\tilde{x}}(\tilde{u}-\tilde{v})(x) = 0$ (because of (47)). By (1) and the conformal invariance of the equation satisfied by u, we have

$$F(A^{\tilde{u}}) = 1, \quad A^{\tilde{u}} \in U, \quad \text{in } R^n \setminus \{\tilde{x}\}.$$

Since $\tilde{x} \ne \bar{x}$, we have $\phi_2^{-1}(\bar{x}) \ne \tilde{x}$, therefore $\tilde{v}$ is a positive C^2 function near $\tilde{x}$. If $\nabla\tilde{v}(\tilde{x}) = 0$, then, by applying Proposition 1 to $\hat{u}(x) := \tilde{u}(\tilde{x}+x)$, $\hat{u}$ is radially symmetric and

$$\hat{u}'(r) < 0, \quad \forall\, 0 < r < \infty.$$

Next, by applying Proposition 2 to $\hat{u}$, we have either

$$\hat{u}(x) \equiv \frac{constant}{|x|^{n-2}}, \tag{49}$$

or, for some positive constants a and b,

$$\hat{u}(r) \equiv \left(\frac{a}{1+b^2r^2}\right)^{\frac{n-2}{2}}. \tag{50}$$

If (49) occurs, then $u \equiv$constant, i.e., u is of the form (7) with $b = 0$ and some $a > 0$. If (50) occurs, then

$$u(y) = \frac{1}{|y - \tilde{x}|^{n-2}} \hat{u}(\frac{1}{|y - \tilde{x}|}) = (\frac{a}{b^2 + |y - \tilde{x}|^2})^{\frac{n-2}{2}},$$

and therefore u is of the form (7). Thus we have proved Theorem 1 provided that $\nabla \tilde{v}(\tilde{x}) = 0$. If $\nabla \tilde{v}(\tilde{x}) \neq 0$, we will make a suitable Möbius transformation to reduce it to the situation with $\nabla \tilde{v}(\tilde{x}) = 0$. For this, we need the following fact (used in the proof of theoerm 1.1 in [6]).

Lemma 8 *Let $s > 0$, $y, p \in R^n \setminus \{0\}$ with $n \geq 3$ and $y = \frac{(2-n)s}{|p|^2} p$. Assume that ξ is a C^1 function near y satisfying $\xi(y) = s$ and $\nabla \xi(y) = p$. Then*

$$(\nabla \xi_\psi)(\psi^{-1}(y)) = 0,$$

where $\psi(x) := \frac{\lambda^2 x}{|x|^2}$ for any fixed $\lambda > 0$.

Lemma 8 follows from a direct computation.

Back to the proof of Theorem 1, when $\nabla \tilde{v}(\tilde{x}) \neq 0$, let $s = \tilde{v}(\tilde{x}) > 0$, $p = \nabla \tilde{v}(\tilde{x}) \neq 0$, and $y = \frac{(2-n)s}{|p|^2} p$. Define $\xi(x) := \tilde{v}(x - y + \tilde{x})$, $\psi(x) := \frac{|y|^2 x}{|x|^2}$. By Lemma 8, $(\nabla \xi_\psi)(\psi^{-1}(y)) = 0$. Now let

$$\eta(x) = \tilde{u}(x - y + \tilde{x}), \quad \hat{u} = \eta_\psi, \quad \hat{v} = \xi_\psi.$$

Then $\hat{u} \in C^2(R^n \setminus \{\psi^{-1}(y)\})$, ∞ is a regular point of $\hat{u}$, $\Delta \hat{u} \leq 0$ in $R^n \setminus \{\psi^{-1}(y)\}$, $\hat{v}$ is a positive C^2 superharmonic function in an open neighborhood of $\psi^{-1}(y)$, $\hat{u} \geq \hat{v}$ in an open neighborhood of $\psi^{-1}(y)$,

$$\liminf_{x \to \psi^{-1}(y)} (\hat{u} - \hat{v})(x) = 0,$$

and

$$F(A^{\hat{u}}) = 1, \quad A^{\hat{u}} \in U, \quad \text{in } R^n \setminus \{\psi^{-1}(y)\}.$$

Now we also know that $\nabla \hat{v}(\psi^{-1}(y)) = 0$. So we have, by applying Proposition 1 to $u^*(x) := \hat{u}(x + \psi^{-1}(y))$, that u^* is radially symmetric and

$$(u^*)'(r) < 0, \quad \forall\ 0 < r < \infty.$$

Applying Proposition 2 to u^*, we have either

$$u^*(x) \equiv \frac{constant}{|x|^{n-2}}, \tag{51}$$

or, for some positive constants a and b,

$$u^*(r) \equiv (\frac{a}{1+b^2r^2})^{\frac{n-2}{2}}. \tag{52}$$

If (51) occurs, we have $u \equiv$constant. If (52) occurs, u is of the form (7) and u is not a constant. Theorem 1 is established.

□

References

[1] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), 271-297.

[2] S.Y. A. Chang, M. Gursky and P. Yang,pear . An a priori estimate for a fully nonlinear equation on four-manifolds, preprint.

[3] S.Y. A. Chang, M. Gursky and P. Yang, Entire solutions of a fully nonlinear equation, preprint.

[4] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209-243.

[5] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations, C. R. Acad. Sci. Paris, Ser. I 334 (2002), 1-6.

[6] A. Li and Y.Y. Li, On some conformally invariant fully nonlinear equations, preprint.

[7] A. Li and Y.Y. Li, A Liouville type theorem for some conformally invariant fully nonlinear equations, arXiv:math.AP/0212376 v1 30 Dec 2002.

[8] Y.Y. Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, Journal d'Analyse Mathematique, to appear.

[9] M. Obata, The conjecture on conformal transformations of Riemannian manifolds, J. Diff. Geom. 6 (1971), 247-258.

[10] J. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, Duke Math. J. 101 (2000), 283-316.

[11] J. Viaclovsky, Conformally invariant Monge-Ampere equations: global solutions, Trans. Amer. Math. Soc. 352 (2000), 4371-4379.