Some recent work on elliptic systems from composite material

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Dedicated to Louis Nirenberg with admiration and friendship

In this talk I describe some recent joint work with Louis. Please see [8] for details.

Let D be a bounded domain in $\mathbb{R}^n$. For positive constants $0 < \lambda \leq \Lambda < \infty$, let $\mathcal{A}(\lambda, \Lambda)$ denote the class of measurable coefficients $\{A_{ij}^{\alpha\beta}(x)\}, 1 \leq \alpha, \beta \leq n, 1 \leq i, j \leq N$, which satisfy

$$|A_{ij}^{\alpha\beta}(x)| \leq \Lambda \qquad \forall\ \alpha, \beta, i, j, \text{and } x \in D, \tag{1}$$

and

$$\int_D A_{ij}^{\alpha\beta} \partial_\alpha \varphi^i \partial_\beta \varphi^j \geq \lambda \int_D |\nabla \varphi|^2, \qquad \forall\ \varphi \in H_0^1(D, \mathbb{R}^N). \tag{2}$$

We are interested in solutions $u \in H^1(D, \mathbb{R}^N)$ of

$$\partial_\alpha \left(A_{ij}^{\alpha\beta}(x) \partial_\beta u^j \right) = 0, \qquad 1 \leq i \leq N, \qquad \text{in } D. \tag{3}$$

System (3) is called strongly elliptic if the coefficients satisfy

$$A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \geq \lambda |\xi|^2, \qquad \forall\ x \in D \text{ and } \forall\ n \times N \text{ matrix } \xi.$$

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It is called a linear system of elasticity if the coefficients satisfy

$$\begin{cases} n = N, \qquad A_{ij}^{\alpha\beta}(x) = A_{ji}^{\beta\alpha}(x) = A_{\alpha j}^{i\beta} \quad \forall\ \alpha, \beta, i, j, \text{and } x \in D, \\ A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \ge \lambda|\xi|^2, \qquad \forall\ \textit{symmetric}\ n \times n \text{ matrix } \xi \text{ and } \forall\ x \in D. \end{cases}$$

It is known that strongly elliptic systems and linear systems of elasticity satisfy (2). A necessary condition for (2) is

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha \xi_\beta \eta^i \eta^j \ge \lambda|\xi|^2|\eta|^2, \qquad \forall\ \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^N, x \in D.$$

If coefficients $\{A_{ij}^{\alpha\beta}\}$ satisfy (1) and (2), and are smooth in D, then H^1 solutions of (3) are smooth. If the coefficients satisfy (1) and (2) only, then the classical result of De Giorgi and Nash says that for $N = 1$ (scalar equation) H^1 solutions are Hölder continuous (in general not Lipschitz). The situation is very different for systems ($N \ge 2$). A well known example of De Giorgi shows that H^1 solutions to strongly elliptic systems (with L^∞ coefficients) are not necessarily bounded. An example of this nature was later given by Necas and Stipl for linear systems of elasticity.

Let D be a bounded domain in $\mathbb{R}^n$ containing L disjoint subdomains $D_1, \cdots, D_L$, with $D = (\cup \overline{D}_m) \setminus \partial D$. If a point in D lies on some ∂D_m then we assume for that m, ∂D_m is smooth. This implies that any point $x \in D$ belongs to the boundaries of at most two of the D_m. Thus if the boundaries of two D_m touch, then they touch on a whole component of such boundary. We assume that $\{\partial D_m\}$, $1 \le m \le L$, are C^2 and the principal curvatures are bounded by some constant K. We assume that $A \in \mathcal{A}(\lambda, \Lambda)$ and A is smooth in every $\overline{D}_m$. The above assumption arise naturally from composite material. In $\overline{D}$ we consider a composite media whose physical characteristics are smooth in the closure of each region D_m but possibly discontinuous across their boundaries. The physical properties are described in terms of a linear system of elasticity. Thus the coefficients of the system are smooth in each $\overline{D}_m$ but not across their boundaries. In engineering, one is interested in obtaining bounds on the stresses represented by ∇u.

Theorem 1 *([8]) Assume the above. For any $\epsilon > 0$, there exists some constant C, depending only on $n, N, \lambda, \Lambda, L, \epsilon, K$, and $\|A\|_{C^1(\overline{D}_m)}$, such that if $u \in H^1(D, \mathbb{R}^N)$ is a solution of (3), then*

$$\|u\|_{C^{1,\frac{1}{4}}(\overline{D}_m \cap D_\epsilon)} \le C\|u\|_{L^2(D)}, \qquad \forall\ m, \tag{4}$$

where $D_\epsilon = \{x \in D \mid dist(x, \partial D) > \epsilon\}$. Consequently,

$$\|\nabla u\|_{L^\infty(D_\epsilon)} \le C\|u\|_{L^2(D)}. \tag{5}$$

The constant C in the above theorem does not depend on the distances between $\{\partial D_m\}$. As a result, by moving the D_m slightly, more general domains are allowed. For $N = 1$ (scalar equation) the above theorem was established by Li and Vogelius in [9] for a smaller Hölder exponent in (4) ($C^{1,\alpha'}$ for $\alpha' < \frac{1}{2n}$ instead of $C^{1,\frac{1}{4}}$). The proof made use of the estimates of De Girogi and Nash which are not available to systems. Babuska et al [2] were interested in elliptic systems arising in elasticity. They observed numerically that, for certain homogeneous isotropic linear systems of elasticity, $|\nabla u|$ is bounded independent of how close the regions were to each other.

The question of higher regularity of solutions (higher than $C^{1,\frac{1}{4}}$ in (4)) remains largely open. A very special case in dimension $n = 2$ was examined by Bonnetier and Vogelius in [3] and then by Li and Vogelius in [9]: Let D_1, D_2 be unit disks in $\mathbb{R}^2$ centered at $(0, -1)$ and $(0, 1)$–so their closures touch at the origin, and let $a(x) \equiv 1$ in $B_R \setminus (D_1 \cup D_2)$, $a(x) = a_0 \neq 1$ in D_1 and D_2, here a_0 is a positive constant. Consider scalar equation

$$\partial_i \left(a(x)\partial_i u\right) = 0 \quad \text{in } B_R,\ u \in H^1(B_R).$$

It is shown in [9] that if R is sufficiently large, derivatives of u (of any order) are bounded in D_1, D_2, and in $D_0 \cap B_{R-1}$. The proof made use of conformal mapping. It is worth pointing out that the same problem in higher dimensions is open.

Let $\{D_m\}$ be disjoint sub-domains of a flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, as described above. Based on Theorem 1 and the method in [1], we have the following extension of a result of Avellaneda and Lin in [1].

Theorem 2 *([8]) Let $\{D_m\}$ be as above and let $A \in \mathcal{A}(\lambda, \Lambda)$ be "piecewise Hölder continuous" as described earlier. Assume that A is 1-periodic in each variable, and $u \in H^1(B_1, \mathbb{R}^N)$ is a solution of*

$$\partial_\alpha \left(A_{ij}^{\alpha\beta}(\frac{x}{\epsilon})\partial_\beta u^j \right) = 0, \quad B_1.$$

Then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^2(B_1)},$$

where B_1 is the unit ball of $\mathbb{R}^n$ and C is independent of ϵ and the distances between the $\{\partial D_m\}$.

$W^{1,\infty}$ estimate is given in the above theorem. $W^{1,p}$ estimate for $p < \infty$ is due to Caffarelli and Peral ([6]). Under a stronger hypothesis that A is Hölder on T^n, the $W^{1,\infty}$ estimate is due to Avellaneda and Lin ([1]).

In the remaining of the note, we outline our proof ([8]) of the $C^{1,\frac{1}{4}}$ estimates. The proof makes use of ideas of L. Caffarelli of [4] and [5]. To estimate $|\nabla u(x)|$ at a point x in D_ϵ we need only consider the case that x is close to some ∂D_m, otherwise standard interior estimates yield the result. In that case we approximate the problem by a laminar one which has been studied by Chipot et al [7]. For a laminar system, we consider D to be the cube

$$\Omega = \{x \ ; \ |x_i| < 1\}, \quad \text{with } x = (x', x_n),$$

divided into Ω_m's; however the Ω_m are different, they are "strips":

$$\Omega_m = \{x \in \Omega \ ; \ c_{m-1} < x_n < c_m\},$$

where the c_m are increasing constants lying between -1 and 1. We consider system (3), for a vector-valued function v,

$$\partial_\alpha \left(\overline{A}_{ij}^{\alpha\beta} \partial_\beta v^j \right) = \overline{H}_i + \partial_\alpha (\overline{G}_\alpha^i), \quad i = 1, \cdots, N.$$

The coefficients $\overline{A}$ are constant in each Ω_m and satisfy (1) and (2). The $\overline{H}$ and $\overline{G}$ are also assumed to be constant in each Ω_m. The following estimate can be deduced from [7]: For any $\epsilon > 0$, any $k \geq 0$, and any m,

$$\|v\|_{C^k(\overline{\Omega}_m \cap (1-\epsilon)\Omega)} \leq C \left(\|v\|_{L^2(\Omega)} + \|\overline{H}\|_{L^\infty(\Omega)} + \|\overline{G}\|_{L^\infty(\Omega)} \right).$$

where $C = C(\epsilon, k, n, N, \lambda, \Lambda)$.

We establish a general perturbation result which asserts, roughly, the following: Suppose u is a solution of system

$$\partial(A \partial u) = \partial g$$

in a cube Ω. Suppose that B are the coefficients of a similar system also satisfying (1) and (2) with the L^1 norm of $(A - B) \leq \epsilon$ small. Then in $\frac{3}{4}\Omega$, there is an H^1 solution of the "B system"

$$\partial(B \partial v) = 0 \qquad \text{in } \frac{3}{4}\Omega$$

with

$$\|u - v\|_{H^1(\frac{1}{2}\Omega)} \leq C \left(\|g\|_{L^2(\Omega)} + \epsilon^\gamma \|u\|_{L^2(\Omega)} \right)$$

for some universal constant $\gamma > 0$ and some C.

To estimate $|\nabla u(x)|$, we only need to use the perturbation lemma in the case that $B = \overline{A}$ is a laminar one. For simplicity, we take x as the origin. By suitable rotation and scaling, we may suppose that a number of the ∂D_m lie in the usual cube Ω and that these take the form

$$x_n = f_j(x') \quad \forall\, x' \in [-1,1]^{n-1}, j = 1, \cdots, l,$$

with

$$-1 < f_1(x') < f_2(x') < \cdots < f_l(x') < 1,$$

and with the f_j in $C^2([-1,1]^{n-1})$. We set $f_0(x') = -1$, $f_{l+1} = 1$, and have $l+1$ regions:

$$D_m = \{x \in \Omega \mid f_{m-1}(x') < x_n < f_m(x')\}, \qquad 1 \le m \le l+1.$$

We may suppose that $f_{m_0+1}(0') < 0 < f_{m_0}(0')$, and the closest point on D_{m_0} to the origin is $(0', f_{m_0+1}(0'))$. Thus $\nabla' f_{m_0+1}(0') = 0$.

Our system (3) still takes the same form, with (1) and (2) still holding. The coefficients A are smooth in every $\overline{D}_m \cap \Omega$. Our desired estimate for $\nabla u(0)$ is given by

$$|\nabla u(0)| \le C\|u\|_{L^2(\Omega)}. \tag{6}$$

We define the coefficients $\overline{A}$ as

$$\overline{A}(x) = \begin{cases} \lim_{y\in D_m, y\to(0,f_{m-1}(0'))} A(y), & x \in \Omega_m, m > m_0, \\ A(0), & x \in \Omega_{m_0}, \\ \lim_{y\in D_m, y\to(0,f_m(0'))} A(y), & x \in \Omega_m, m < m_0. \end{cases}$$

By the regularity of the f_m, there exists some constant C such that

$$\left(\fint_{r\Omega} |A - \overline{A}|^2\right)^{\frac{1}{2}} \le C r^{\frac{1}{4}}, \qquad \forall 0 < r < 1/2.$$

In fact, by a harmless scaling, C can be assumed to be some small ϵ_0. Applying the previously mentioned perturbation lemma, with $B = \overline{A}$, we obtain a solution w_0 of the $\overline{A}$ system

$$\partial\left(\overline{A}\partial w_0\right) = 0$$

with

$$\|u - w_0\|_{L^2(\frac{1}{2}\Omega)} \le (\frac{1}{4})^{\frac{n}{2}+\frac{5}{4}}.$$

In addition, using the previously mentioned result on laminar systems we show that

$$\|\nabla w_0\|_{L^\infty(\frac{1}{4}\Omega)} \le C.$$

By repeated use of the perturbation result, applied first to $u - w_0$, in smaller and smaller cubes, and by scaling, we obtain a sequence of functions $w_1, w_2, \cdots$ satisfying, with C a fixed constant,

$$\|\nabla w_k\|_{L^\infty(4^{-(k+1)}\Omega)} \leq C4^{-k/4}, |w_k(0)| \leq C4^{-5k/4}, \tag{7}$$

and

$$\|u - \sum_{j=0}^{k} w_j\|_{L^2(4^{-k}\Omega)} \leq C4^{-(k+1)(\frac{n}{2}+\frac{5}{4})}. \tag{8}$$

Using (7) and (8) we finally obtain

$$\|u - \sum_{j=0}^{\infty} w_j(0)\|_{L^2(4^{-(k+1)}\Omega)} \leq C4^{-(k+1)\frac{n+2}{2}}$$

which yields (6).

Next we describe the proof of the Hölder continuity of ∇u. Take two points in some D_{m_0}, one of them we take as the origin while the other we call it x. Pick a point on $\cup_m \partial D_m$ such that the distance of the origin to this point is the shortest distance of the origin to $\cup_m \partial D_m$. Let the line going through this point and the origin be the x_n-axis. With $|x|$ small, we establish

$$|\nabla u(0) - \nabla u(x)| \leq C|x|^{\frac{1}{4}}. \tag{9}$$

To do this we compare ∇u at 0 and x with ∇u at two other points $\bar{x}, \bar{z}$, as in [9].

Since the number of regions D_m is finite we may find $\bar{x}$ on the x_n-axis such that $|\bar{x}| \sim |x|$ and $\bar{x} + 8|x|\Omega$ lies entirely in some D_m. We prove that

$$|\nabla u(\bar{x}) - T\nabla u(0)| \leq C|x|^{\frac{1}{4}},$$

where T is some invertible linear transformation with $\|T\|$ and $\|T^{-1}\|$ bounded from above by some universal constant. Similarly, we can find $\bar{z}$ with $|\bar{z} - \bar{x}| \leq 2|x|$ and

$$|\nabla u(\bar{z}) - T\nabla u(x)| \leq C|x|^{\frac{1}{4}}.$$

Finally we show that

$$|\nabla u(\bar{x}) - \nabla u(y)| \leq C|x|^{\frac{1}{4}} \qquad \forall\ y \in \bar{x} + 6|x|\Omega.$$

In particular,

$$|\nabla u(\bar{x}) - \nabla u(\bar{z})| \leq C|x|^{\frac{1}{4}}.$$

The desired estimate (9) follows from the above.

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