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Multiplicity of positive solutions for a class of quasilinear nonhomogeneous Neumann problems

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Abstract

In this paper we study the existence, nonexistence and multiplicity of positive solutions for nonhomogeneous Neumann boundary value problem of the type

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $1 < p < n$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p - 1 < q \leqslant p^* - 1$, $p^* = np/(n-p)$, $\varphi \in C^{\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, $\varphi \not\equiv 0$, $\varphi(x) \geqslant 0$ and λ is a real parameter. The proofs of our main results rely on different methods: lower and upper solutions and variational approach.

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Keywords: Nonlinear elliptic problems; Neumann boundary value problems; Positive solutions; Lower and upper solutions; Variational methods; p-Laplacian; Critical exponent

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1. Introduction

In this paper we deal with quasilinear elliptic problems of the form

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1_λ}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\varphi \in C^\alpha(\overline{\Omega})$, $0<\alpha<1$, $\varphi \not\equiv 0$, $\varphi(x) \geqslant 0$, $1<p<n$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p-1<q\leqslant p^*-1$, $p^* = np/(n-p)$ is the critical exponent for the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and λ is a real parameter.

When $p=2$, (1_λ) becomes the second-order semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $1<q\leqslant 2^*-1 = (n-2)/(n+2)$.

The study of semilinear elliptic problems involving critical growth and Neumann boundary conditions has received considerable attention in recent years. First we would like to mention the progress for problems involving homogeneous boundary conditions, which correspond to $\varphi \equiv 0$ in (1.1). They have been studied for instance in [1,2,13,16], among others. Problem (1.1) with nonhomogeneous Neumann boundary conditions, which correspond to $\varphi \not\equiv 0$, has been investigated by Deng–Peng [9]. In the present paper we will improve the main results in [9]. We prove that there exists $\lambda^*>0$ such that problem (1_λ) has at least two positive solutions if $\lambda>\lambda^*$, has at least one positive solution if $\lambda=\lambda^*$ and has no positive solution if $\lambda<\lambda^*$. The proofs of our main results rely on different methods: lower and upper solutions method and variational approach.

The special features of this class of problems, considered in this paper, are they involve critical growth and a nonlinear operator. The arguments used in [9] to prove the existence of the second solutions cannot be carried out for a quasilinear problem as (1_λ). Moreover, because we are dealing with p-Laplacian equations, it is technically much involved than in [9], in our case some estimates involving the minimax level become more subtle to be established.

Next we describe in a more precise way our main results.

Theorem 1.1. *For each $q \in (p-1, p^*-1]$, there exists $\lambda^*>0$ such that*:

(i) *problem (1_λ) possesses a minimal positive solution u_λ if $\lambda \in [\lambda^*, \infty)$ and there is no positive solution if $\lambda<\lambda^*$.*
(ii) *u_λ is decreasing with respect to λ* if *$\lambda \in [\lambda^*, \infty)$.*
(iii) *u_λ is bounded uniformly in $W^{1,p}(\Omega)$ and $u_\lambda \to 0$ as $\lambda \to \infty$.*

Theorem 1.2. *For each $\lambda \in (\lambda^*, +\infty)$ and $q \in (p-1, p^*-1]$, problem (1_λ) possesses at least two positive solutions v_λ and w_λ.*

The rest of this paper is organized as follows. The existence of minimal solution u_λ for (1_λ) is obtained in Section 2. The main tool is a general method of lower- and upper-solutions described in Section 2, similar to that given in [7]. Section 3 is devoted to proving Theorem 1.2.

The underlying idea for proving Theorem 1.2 is first to show with the help of the minimal solution u_λ that there exists a solution v_λ, which is a local minimum of the associated functional J_λ to problem (1_λ) in $W^{1,p}(\Omega)$. For proving the existence of the second solution, we consider the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$. We prove that this functional has the mountain pass geometry and using the Ekeland variational principle we obtain a Palais–Smale sequence at this mountain pass level $c(v_\lambda)$ of I_λ. Finally, doing an argument similar in spirit to that used in the classical result due to Brezis–Nirenberg [6], we obtain a nontrivial critical point u of I_λ. Thus, $w_\lambda = u + v_\lambda$ is a second solution of problem (1_λ).

Notation: In this paper we make use of the following notations:

If $p \in (1, \infty)$, p' denotes the number $p/(p-1)$ so that $p' \in (1, \infty)$ and $1/p + 1/p' = 1$;

$L^p(\Omega)$ denotes Lebesgue spaces with the norm $\|.\|_{L^p(\Omega)}$;

$W^{1,p}(\Omega)$ denotes Sobolev spaces with the norm $\|.\|_{1,p}$;

$C^{k,\alpha}(\Omega)$, with k a nonnegative integer and $0 \leqslant \alpha < 1$ denotes Hölder spaces;

$C, C_0, C_1, C_2, \ldots$ denote (possibly different) positive constants;

$|A|$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^n$;

ω_{n-1} is the $(n-1)$-dimensional measure of the $n-1$ unit sphere in $\mathbb{R}^n$;

We denote by $\mathbb{R}^n_+$ the half-space, that is, $\mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$;

$D^{1,p}_o(\Omega)$ is the completeness of $C^\infty_o(\Omega)$ with respect to the norm $\|u\| := \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)$.

We denote by S the best constant to the Sobolev embedding $D^{1,p}_o(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that is,

$$S = \inf_{D^{1,p}_o(\Omega)} \left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x;\ \int_\Omega |u|^{p^*} \,\mathrm{d}x = 1 \right\}.$$

We remark also that S is independent of Ω and depends only on n. Moreover, when $\Omega = \mathbb{R}^n$ this infimum S is achieved by the functions u_ε given by

$$u_\varepsilon(x) = C_n \varepsilon^{(n-p)/p^2} (\varepsilon + |x|^{p/(p-1)})^{(p-n)/p},$$

where the constant C_n is chosen of the form that

$$-\Delta_p u_\varepsilon = u_\varepsilon^{p^*-1} \quad \text{in } \mathbb{R}^n.$$

Thus,

$$S = \frac{K_1}{K_2^{(n-p)/n}}$$

with

$$K_1 := \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^p \,\mathrm{d}x \quad \text{and} \quad K_2 := \int_{\mathbb{R}^n} |u_\varepsilon|^{p^*} \,\mathrm{d}x. \tag{1.2}$$

2. Proof of Theorem 1.1

Our argument to prove the existence of the first solution to problem (1_λ) relies on the lower and upper solution methods. Our first solution is a minimal solution u_λ of problem (1_λ), in the sense that $u_\lambda \leqslant w$, for all w solutions of (1_λ). The main focus of our next subsection is to prove the existence of such a minimal solution.

2.1. The existence of minimal solution

Let us first recall some definitions. We say that $u \in W^{1,p}(\Omega)$ is a *weak solution* of problem (1_λ) if for all $v \in W^{1,p}(\Omega)$ we have

$$\int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v + \lambda |u|^{p-2} u v]\,\mathrm{d}x = \int_\Omega |u|^{q-1} u v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y. \tag{2.3}$$

Hence, the weak solutions of (1_λ) correspond to nontrivial critical points of the energy functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega [|\nabla u|^p + \lambda |u|^p]\,\mathrm{d}x - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x - \int_{\partial\Omega} \varphi u\,\mathrm{d}\sigma_y, \quad u \in W^{1,p}(\Omega).$$

A function $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a *lower solution* of (1_λ) if

$$\int_\Omega [|\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla v + \lambda |\underline{u}|^{p-2} \underline{u} v]\,\mathrm{d}x \leqslant \int_\Omega |\underline{u}|^{q-1} \underline{u} v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y$$

for all $v \in W^{1,p}(\Omega)$, $v \geqslant 0$. In the same way, a function $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a *upper solution* of (1_λ) if

$$\int_\Omega [|\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla v + \lambda |\overline{u}|^{p-2} \overline{u} v]\,\mathrm{d}x \geqslant \int_\Omega |\overline{u}|^{q-1} \overline{u} v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y$$

for all $v \in W^{1,p}(\Omega)$, $v \geqslant 0$.

Lemma 2.1 (*Maximum Principle*). *Let $\lambda > 0$ and $u_1, u_2 \in W^{1,p}(\Omega)$ be nonnegative functions such that for all $v \in W^{1,p}(\Omega)$, $v \geqslant 0$ we have*

$$\int_\Omega [|\nabla u_1|^{p-2}\nabla u_1 \nabla v + \lambda u_1^{p-1} v]\,\mathrm{d}x \leqslant \int_\Omega [|\nabla u_2|^{p-2}\nabla u_2 \nabla v + \lambda u_2^{p-1} v]\,\mathrm{d}x. \tag{2.4}$$

Then $u_1 \leqslant u_2$ almost everywhere in Ω.

For a proof of Lemma 2.1, see (Tolksdorf, 1983 [18] Lemma 3.4) for example.

Our next result concerns the existence of solutions for problem (2_λ) and some properties of the associated solution operator.

Lemma 2.2. *If $\varphi \in C^\alpha(\overline{\Omega})$, $0<\alpha<1$, $\varphi \not\equiv 0$ and $\varphi \geqslant 0$, then for each nonnegative function $f \in L^{p'}(\Omega)$, problem (2_λ) possesses a unique weak positive solution $w_\lambda \in C^{1,\alpha}(\bar{\Omega})$ for all $\lambda>0$. Moreover, the associated operator $T_\lambda : L^{p'}(\Omega) \to W^{1,p}(\Omega)$, $f \mapsto w_\lambda$ is continuous and nondecreasing.*

Proof. First we use variational argument to prove the existence of the solution. More precisely, we use minimization argument to the associated energy functional of the problem (2_λ),

$$I_\lambda(w) = \frac{1}{p}\int_\Omega [|\nabla w|^p + \lambda |w|^p]\,\mathrm{d}x - \int_\Omega f w\,\mathrm{d}x - \int_{\partial\Omega} \varphi w\,\mathrm{d}\sigma_y,$$

defined on the reflexive Banach space $W^{1,p}(\Omega)$. Note that I_λ is coercive. Indeed,

$$\begin{aligned} I_\lambda(w) &\geqslant C_1\|w\|_{1,p}^p - \|f\|_{L^{p'}(\Omega)}\|w\|_{L^p(\Omega)} - \|\varphi\|_{L^{p'}(\partial\Omega)}\|w\|_{L^p(\partial\Omega)} \\ &\geqslant C_2\|w\|_{1,p}^p - C_3, \end{aligned}$$

where above we have used Holder inequality, Sobolev embedding and trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

Now, we proceed to prove that I_λ is sequentially weakly lower semicontinuous. To this end it is sufficient to show that for $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ we have

$$\int_\Omega f u_n\,\mathrm{d}x \to \int_\Omega f u\,\mathrm{d}x \tag{2.5}$$

and

$$\int_{\partial\Omega} \varphi u_n\,\mathrm{d}\sigma_y \to \int_{\partial\Omega} \varphi u\,\mathrm{d}\sigma_y. \tag{2.6}$$

Since $f \in L^{p'}(\Omega)$, (2.5) follows from the definition of weak convergence. Finally, (2.6) follows from the trace embedding.

Let u_i be a weak solution of (2_λ) associated to $f_i \in L^{p'}(\Omega)$, that is

$$\int_\Omega [|\nabla u_i|^{p-2}\nabla u_i \nabla v + \lambda |u_i|^{p-1} u_i v]\,\mathrm{d}x = \int_\Omega f_i v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y$$

for all $v \in W^{1,p}(\Omega)$ and $i = 1, 2$.

If $f_1 \leqslant f_2$, using Lemma 2.1, we obtain that $u_1 \leqslant u_2$. From this we get the uniqueness and that T_λ is nondecreasing.

Using the regularity result due to Lieberman [12] we may prove that $u \in C^{1,\alpha}(\overline{\Omega})$. Finally, by the maximum principle or Hanark's inequality it is standard to prove that $u > 0$ (see [14,15]). This completes the proof of Lemma 2.2. $\square$

Proposition 2.3. *Let $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\overline{\Omega})$ be, respectively, a lower solution and an upper solution of problem (1_λ), with $0 \leqslant \underline{u}(x) \leqslant \overline{u}(x)$ almost everywhere in Ω. Then, there exists a minimal (and, respectively, a maximal) weak solution u_* (resp. u^*) for problem (1_λ).*

Proof. Consider the interval $[\underline{u}, \overline{u}]$ with the topology of $W^{1,p}(\Omega)$ and the operator $S\colon [\underline{u}, \overline{u}] \to L^{p'}(\Omega)$ defined by $Sv := v^q$. Since $\overline{u} \in L^\infty(\Omega)$, we see that S is well defined. Moreover, for $u_n, u \in [\underline{u}, \overline{u}]$ with $u_n \to u$ in $W^{1,p}(\Omega)$, we have that $\|Su_n - S_u\|_{L^{p'}(\Omega)} \to 0$, and hence S is continuous.

Considering the operators, $[\underline{u}, \overline{u}] \xrightarrow{S} L^{p'}(\Omega) \xrightarrow{T_\lambda} W^{1,p}(\Omega)$, we can define $F : [\underline{u}, \overline{u}] \mapsto W^{1,p}(\Omega)$ given by $F = T_\lambda \circ S$, where $F(v) = w$ is the unique weak positive solution of the boundary value problem

$$\begin{cases} -\Delta_p w + \lambda w^{p-1} = v^q & \text{in } \Omega, \\ |\nabla w|^{p-2} \dfrac{\partial w}{\partial \eta} = \varphi & \text{on } \partial\Omega. \end{cases}$$

It is clear that F is continuous and nondecreasing.

Writing $u_1 = F(\underline{u})$ and $u^1 = F(\overline{u})$, for all $v \in W^{1,p}(\Omega)$ with $v \geqslant 0$, we have

$$\begin{aligned} \int_\Omega [|\nabla u_1|^{p-2} \nabla u_1 \nabla v + \lambda u_1^{p-1} v]\,\mathrm{d}x &= \int_\Omega \underline{u}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y \\ &\geqslant \int_\Omega [|\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla v + \lambda \underline{u}^{p-1} v]\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int_\Omega [|\nabla u^1|^{p-2} \nabla u^1 \nabla v + \lambda (u^1)^{p-1} v]\,\mathrm{d}x &= \int_\Omega \overline{u}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y \\ &\leqslant \int_\Omega [|\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla v + \lambda \overline{u}^{p-1} v]\,\mathrm{d}x. \end{aligned}$$

Thus, applying Lemma 2.1 and taking into account that F is nondecreasing, we get

$$\underline{u} \leqslant F(\underline{u}) \leqslant F(u) \leqslant F(\overline{u}) \leqslant \overline{u}, \text{ a.e.} \quad \text{in } \Omega.$$

Repeating the same reasoning, we can obtain the existence of sequences (u^n) and (u_n) in $W^{1,p}(\Omega)$ satisfying

$$\begin{aligned} &u^0 = \overline{u}, \quad u^{n+1} = F(u^n), \\ &u_0 = \underline{u}, \quad u_{n+1} = F(u_n) \end{aligned}$$

and for every weak solution $u \in [\underline{u}, \overline{u}]$ of problem (1_λ), we have

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant u \leqslant u^n \leqslant \cdots \leqslant u^1 \leqslant u^0 \text{ a.e.} \quad \text{in } \Omega.$$

Since

$$\begin{aligned}\int_\Omega [|\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\nabla v + \lambda u_{n+1}^{p-1} v]\,\mathrm{d}x &= \int_\Omega u_n^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y \\ &\leqslant \int_\Omega \overline{u}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y\end{aligned}$$

and

$$\begin{aligned}\int_\Omega [|\nabla u^{n+1}|^{p-2}\nabla u^{n+1}\nabla v + \lambda (u^{n+1})^{p-1} v]\,\mathrm{d}x &= \int_\Omega (u^n)^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y \\ &\leqslant \int_\Omega \overline{u}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y,\end{aligned}$$

we obtain that (u^n) and (u_n) are bounded in $W^{1,p}(\Omega)$. Therefore, up to subsequences, we have $u_n \rightharpoonup u_*$, $u^n \rightharpoonup u^*$ weakly in $W^{1,p}(\Omega)$, $u_n \to u_*$, $u_n \to u^*$ in $L^r(\Omega)$ for $1 \leqslant r < p^*$ and $u_n \to u_*$, $u^n \to u^*$ almost everywhere in Ω. Moreover, by construction we have $u_*,\ u^* \in [\underline{u}, \overline{u}]$ and $u_* \leqslant u^*$ almost everywhere in Ω. Now, using $S(u_n) \to S(u_*)$, $S(u^n) \to S(u^*)$ and the continuity of T_λ we conclude that $u_{n+1} = F(u_n) \to F(u_*)$ and $u^{n+1} = F(u^n) \to F(u^*)$ in $W^{1,p}(\Omega)$. Thus, $u_*, u^* \in W^{1,p}(\Omega)$ with $u_* = F(u_*)$, $u^* = F(u^*)$. This completes the proof of Proposition 2.3. □

Lemma 2.4. *There exists $\lambda^* \geqslant 0$, such that problem (1_λ) possesses a minimal positive solution for each $\lambda \in (\lambda^*, +\infty)$ and (1_λ) has no positive solution for $\lambda \in (-\infty, \lambda^*)$.*

Proof. Notice that $\underline{u} \equiv 0$ is a lower solution of (1_λ) for all $\lambda \geqslant 0$. Now, we take w_1 the positive solution of problem (2_λ) with $f \equiv 0$ and $\lambda = 1$. Thus, $\overline{u} = w_1$ is an upper solution of (1_{λ_0}) with $\lambda_0 = 1 + \max_{x\in\overline{\Omega}} w_1^{q-p+1}$. Using Proposition 2.3 we get a minimal solution u_{λ_0} of (1_{λ_0}). Finally, by Harnack's inequality (see [15, Theorem 1.2]) we have $\underline{u} \equiv 0 < u_{\lambda_0} < \overline{u}$. Thus,

$$\Lambda = \{\lambda \in \mathbb{R} : (1)_\lambda \text{ possesses at least one positive solution}\} \tag{2.7}$$

is a nonempty set. Notice that u_{λ_0} is an upper solution of (1_λ) for all $\lambda \geqslant \lambda_o$. Thus, using the same argument above we conclude that $[\lambda_o, \infty) \subset \Lambda$. Moreover, $u_{\lambda_1} \leqslant u_{\lambda_2}$ if $\lambda_2 \leqslant \lambda_1$ and $\Lambda \subset [0, +\infty)$, because for u_λ solution of (1_λ) then u_λ satisfies (2.3) and taking $v = 1$ as test function we get

$$\lambda \int_\Omega u_\lambda^{p-1}\,\mathrm{d}x = \int_\Omega u_\lambda^q\,\mathrm{d}x + \int_{\partial\Omega} \varphi\,\mathrm{d}\sigma_y > 0,$$

which implies that $\lambda > 0$. Consequently, setting

$$\lambda^* = \inf \Lambda,$$

we have $\lambda^* \in [0, +\infty)$. Moreover, for all $\lambda \in (\lambda^*, \infty)$, (1_λ) possesses one minimal solution and has no solution if $\lambda \in (-\infty, \lambda^*)$. □

Lemma 2.5. *λ^* is positive real number and the problem (1_{λ^*}) possesses a minimal positive solution.*

Proof. Our goal is to prove that λ^* is attained. To this end, let us take (λ_j) a decreasing sequence in (λ^*, ∞), satisfying $\lim_{j\to\infty} \lambda_j = \lambda^*$ and (u_j) in $W^{1,p}(\Omega)$ the correspondent sequence of minimal positive solutions of problem (1_{λ_j}) given in Lemma 2.4. We claim that (u_j) is bounded in $W^{1,p}(\Omega)$. Indeed, suppose by contradiction (up to subsequences) that $\|u_j\|_{1,p} \to +\infty$, as $j \to +\infty$. From this we will prove that

$$\int_\Omega u_j^{p-1}\,\mathrm{d}x \to \infty \quad \text{as} \quad j \longrightarrow +\infty. \tag{2.8}$$

Setting $w_j = u_j/\|u_j\|_{1,p}$, we have $\|w_j\|_{1,p} = 1$ and $w_j > 0$ in Ω. Thus, (up to subsequences) there exists $w \in W^{1,p}(\Omega)$ such that $w_j \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, $w_j \to w$ in $L^r(\Omega)$ for $1 \leqslant r < p^*$ and $w_j \to w$ almost everywhere in Ω. Taking $v = w/\|u_j\|_{1,p}^{p-1}$ as a test function in (2.3), we obtain

$$\int_\Omega |\nabla w_j|^{p-2}\nabla w_j \nabla w\,\mathrm{d}x + \int_\Omega \frac{(\lambda_j u_j^{p-1} - u_j^q)}{\|u_j\|_{1,p}^{p-1}}\, w\,\mathrm{d}x = \frac{1}{\|u_j\|_{1,p}^{p-1}} \int_{\partial\Omega} \varphi w\,\mathrm{d}\sigma_y. \tag{2.9}$$

Passing to the limit in (2.9) and using a convergence result due to Lucio–Bocardo (see [5, Theorem 2.1]) we concluded that

$$\int_\Omega \frac{(\lambda_j u_j^{p-1} - u_j^q)}{\|u_j\|_{1,p}^{p-1}} w\,\mathrm{d}x \to \int_\Omega |\nabla w|^p\,\mathrm{d}x. \tag{2.10}$$

Similarly, taking $v = w_j/\|u_j\|_{1,p}^{p-1}$ in (2.3) and passing to the limit we obtain

$$\int_\Omega |\nabla w_j|^p\,\mathrm{d}x - \int_\Omega \frac{(u_j^q - \lambda_j u_j^{p-1})}{\|u_j\|_{1,p}^{p-1}}\, w_j\,\mathrm{d}x \to 0. \tag{2.11}$$

From (2.10)–(2.11) we conclude that

$$\|\nabla w_j\|_{L^p} \to \|\nabla w\|_{L^p}. \tag{2.12}$$

Now, observe that w_j satisfies

$$\begin{cases} -\Delta_p w_j + \lambda w_j^{p-1} = f_j & \text{in } \Omega, \\ |\nabla w_j|^{p-2}\dfrac{\partial w_j}{\partial \eta} = \varphi_j & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

where $f_j = u_j^q/\|u_j\|_{1,p}^{p-1}$ and $\varphi_j = \varphi/\|u_j\|_{1,p}^{p-1}$. It is not difficult to see that $f_j \rightharpoonup f$ weakly in $L^p(\Omega)$, and $\varphi_j \to 0$ almost everywhere in $\partial\Omega$. By a convergence result due to Lucio–Bocardo (see [5, Theorem 2.1]) and Brézis–Lieb's Lemma (see [17]), we conclude that $\nabla w_j \to \nabla w$ strongly in $(L^p(\Omega))^n$. This fact implies that $w_j \to w$ strongly in $L^{p^*}(\Omega)$. Since Ω is a bounded domain, we conclude that $w_j \to w$ strongly in $W^{1,p}(\Omega)$. Observe that $w \geqslant 0$ and $w \not\equiv 0$. Therefore, there exists a subset $\mathscr{V} \subset \Omega$ of positive Lebesgue measure such that $w > 0$ almost everywhere in $\mathscr{V}$. Thus, there exists j_o such that for all $j \geqslant j_o$ we have $u_j \to +\infty$ almost everywhere in $\mathscr{V}$. Therefore, given $M > 0$ there exists j_o such that $u_j(x) \geqslant M$ for all $j \geqslant j_o$ and almost everywhere in $\mathscr{V}$. So, for each $1 \leqslant r \leqslant p^*$, we have

$$M^r |\mathscr{V}| \leqslant \int_{\mathscr{V}} u_j^r \, \mathrm{d}x \leqslant \int_\Omega u_j^r \, \mathrm{d}x.$$

Thus, making $M \to +\infty$, we obtain (2.8).

On the other hand, choosing $v = 1$ in (2.3) and using the Holder's inequality we have

$$C(\Omega, q, p) \left(\int_\Omega u_j^{p-1} \, \mathrm{d}x \right)^{\frac{q}{p-1}} \leqslant \int_\Omega u_j^q \, \mathrm{d}x = \lambda_j \int_\Omega u_j^{p-1} \, \mathrm{d}x - \int_{\partial\Omega} \varphi(y) \, \mathrm{d}\sigma_y, \quad (2.14)$$

where $C = C(\Omega, q, p) > 0$, which is a contradiction of (2.8). Since (u_j) is bounded in $W^{1,p}(\Omega)$, taking subsequence if necessary, we can assume that there exists a function $u \in W^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$ weakly in the spaces $W^{1,p}(\Omega)$, $L^{p+1}(\Omega))^*$, $L^p(\partial\Omega)$ and $L^q(\Omega)$ for each $q \in (1, p^*)$. Since u_j satisfies (1_{λ_j}), we have

$$\int_\Omega [|\nabla u_j|^{p-2} \nabla u_j \nabla v + \lambda_j |u_j|^{p-2} u_j v] \, \mathrm{d}x = \int_\Omega u_j^q v \, \mathrm{d}x + \int_{\partial\Omega} \varphi v \, \mathrm{d}\sigma_y. \quad (2.15)$$

Hence, using a convergence result due to Lucio–Bocardo (see [5, Theorem 2.1]) we have $\nabla w_n \to \nabla w$ strongly. Moreover, by Brezis–Lieb's Lemma, we have after taking the limit

$$\int_\Omega [|\nabla u|^{p-2} \nabla u \nabla v + \lambda^* |u|^{p-2} u v] \, \mathrm{d}x = \int_\Omega u^q v \, \mathrm{d}x + \int_{\partial\Omega} \varphi v \, \mathrm{d}\sigma_y. \quad (2.16)$$

Therefore, u is a weak solution of $(1)_{\lambda^*}$. Finally, applying Proposition 2.3 and using the fact that $\underline{u} \equiv 0$ is a lower solution of $(1)_{\lambda^*}$, we conclude that there exists a minimal solution u_{λ^*} of $(1)_{\lambda^*}$. $\square$

We notice that until this moment we have proved the items (i) and (ii) of Theorem 1.1.

2.2. *Asymptotic behavior of the minimal solution*

Next we are going to prove the last item of Theorem 1.1. For this end firstly we observe that taking $v = u_\lambda$ as a test function in (2.3) we obtain

$$\|\nabla u_\lambda\|_{L^p(\Omega)}^p = \int_{\partial\Omega} \varphi(y) u_\lambda \, \mathrm{d}\sigma_y + \int_\Omega (u_\lambda^{q+1} - \lambda u_\lambda^p) \, \mathrm{d}x. \quad (2.17)$$

Let λ_1 be a fixed element in Λ. From (ii) in Theorem 1.1 follows that for each $\lambda \geqslant \lambda_1$, the respective minimal solution u_λ satisfies $u_\lambda \leqslant u_{\lambda_1}$ in Ω. Thus, using this fact and the Hölder's inequality with $1/p' + 1/p = 1$, we obtain

$$\|\nabla u_\lambda\|^p_{L^p(\Omega)} \leqslant \|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{L^p(\partial\Omega)} + \int_{\{u_\lambda \leqslant 1\}} 1 \,\mathrm{d}x + \int_{\{u_\lambda \geqslant 1\}} u_{\lambda_1}^{q+1} \,\mathrm{d}x - \lambda \int_\Omega u_\lambda^p \,\mathrm{d}x. \tag{2.18}$$

Now, applying the trace embedding theorem and Young's inequality, we have

$$\begin{aligned}\|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{L^p(\partial\Omega)} &\leqslant \|\varphi\|_{L^{p'}(\partial\Omega)} \|u_\lambda\|_{1,p} \\ &\leqslant C_\varepsilon \|\varphi\|^{p'}_{L^{p'}(\partial\Omega)} + \varepsilon(\|\nabla u_\lambda\|^p_{L^p(\Omega)} + \|u_\lambda\|^p_{L^p(\Omega)}),\end{aligned} \tag{2.19}$$

which together with (2.18) and (2.19) implies that

$$\begin{aligned}(1-\varepsilon)\|\nabla u_\lambda\|^p_{L^p(\Omega)} \leqslant C_\varepsilon \|\varphi\|^{p'}_{L^{p'}(\partial\Omega)} &+ \varepsilon\left(\int_{\{u_\lambda \leqslant 1\}} \mathrm{d}x + \int_{\{u_\lambda \geqslant 1\}} u_{\lambda_1}^p \,\mathrm{d}x\right) \\ &+ \int_{\{u_\lambda \leqslant 1\}} \mathrm{d}x + \int_{\{u_\lambda \geqslant 1\}} u_{\lambda_1}^{q+1} \,\mathrm{d}x - \lambda \int_\Omega u_\lambda^p \,\mathrm{d}x.\end{aligned} \tag{2.20}$$

Therefore, taking $\varepsilon \in (0, 1)$ and using (2.20), we conclude that $u_\lambda \to 0$ as $\lambda \to \infty$ in $L^p(\Omega)$. Since $u_\lambda \in C^{1,\alpha}$, we deduce that $u_\lambda \to 0$ as $\lambda \to \infty$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In order to prove Theorem 1.2 we first show with the help of the minimal solution u_λ that there exists a solution v_λ, which is a local minimum of the associated functional J_λ to problem (1_λ) in $W^{1,p}(\Omega)$. This is necessary because the minimal solution u_λ is not a variational solution. So it is not clear how to get an estimate to its the energy level. For proving the existence of the second solution we consider the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$ and we prove that this functional has the mountain pass geometry. Using the Ekeland variational principle, we obtain a Palais–Smale sequence at this mountain pass level $c(v_\lambda)$ of I_λ. Finally, doing an argument similar in spirit to that used in the classical result due to Brezis–Nirenberg [6], we obtain a nontrivial critical point u of I_λ. Thus, $w_\lambda = u + v_\lambda$ is a second solution of problem (1_λ).

3.1. Existence of a local minimum

Here we are going to prove the existence of a local minimum of the energy functional J_λ for all $\lambda > \lambda^*$. To do that, the existence of the minimal solution obtained in the last section is crucial to our argument.

Proposition 3.1. *For each $\lambda \in (\lambda^*, +\infty)$, the functional J_λ has a local minimum v_λ* in $W^{1,p}(\Omega)$.

Proof. Fixed $\lambda \in (\lambda^*, +\infty)$, we can take real numbers $\lambda_1, \lambda_2 \geqslant \lambda^*$ such that $\lambda_2 < \lambda < \lambda_1$. Let u_{λ_i} be the positive minimal solution associated to the problem (1_{λ_i}), for $i \in \{1,2\}$ given by Theorem 1.1. Thus,

$$0 < u_{\lambda_1} \leqslant u_{\lambda_2}. \tag{3.21}$$

Since $\lambda_2 < \lambda < \lambda_1$, for all $v \geqslant 0$ we have

$$\begin{aligned}
&\int_\Omega [|\nabla u_{\lambda_1}|^{p-2}\nabla u_{\lambda_1}\nabla v + \lambda u_{\lambda_1}^{p-1} v]\,\mathrm{d}x \\
&\quad < \int_\Omega [|\nabla u_{\lambda_1}|^{p-2}\nabla u_{\lambda_1}\nabla v + \lambda_1 u_{\lambda_1}^{p-1} v]\,\mathrm{d}x \\
&\quad = \int_\Omega u_{\lambda_1}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\int_\Omega u_{\lambda_2}^q v\,\mathrm{d}x + \int_{\partial\Omega} \varphi v\,\mathrm{d}\sigma_y &= \int_\Omega [|\nabla u_{\lambda_2}|^{p-2}\nabla u_{\lambda_2}\nabla v + \lambda_2 u_{\lambda_2}^{p-1} v]\,\mathrm{d}x \\
&< \int_\Omega [|\nabla u_{\lambda_2}|^{p-2}\nabla u_{\lambda_2}\nabla v + \lambda u_{\lambda_2}^{p-1} v]\,\mathrm{d}x.
\end{aligned} \tag{3.23}$$

Thus, using (3.21)–(3.23), for all $v \geqslant 0$, we get

$$\begin{aligned}
&\int_\Omega [|\nabla u_{\lambda_1}|^{p-2}\nabla u_{\lambda_1}\nabla v + \lambda u_{\lambda_1}^{p-1} v]\,\mathrm{d}x \\
&\quad < \int_\Omega [|\nabla u_{\lambda_2}|^{p-2}\nabla u_{\lambda_2}\nabla v + \lambda u_{\lambda_2}^{p-1} v]\,\mathrm{d}x.
\end{aligned} \tag{3.24}$$

Next, we apply the minimization methods to the Euler Lagrange functional

$$\tilde{J}_\lambda(u) = \frac{1}{p}\int_\Omega [|\nabla u|^p + \lambda |u|^p]\,\mathrm{d}x - \int_\Omega \tilde{F}(u_+)\,\mathrm{d}x - \int_{\partial\Omega} \varphi u_+ \mathrm{d}\sigma_y,$$

associated to the problem

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = \tilde{f}(u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{F}(t) = \int_0^t \tilde{f}(s)\,\mathrm{d}s$ is the primitive of function

$$\tilde{f}(u(x)) = \begin{cases} u_{\lambda_1}^q(x) & \text{if } u(x) \leqslant u_{\lambda_1}(x), \\ u^q(x) & \text{if } u_{\lambda_1}(x) \leqslant u(x) \leqslant u_{\lambda_2}(x), \\ u_{\lambda_2}^q(x) & \text{if } u_{\lambda_2}(x) \leqslant u(x). \end{cases}$$

It is not difficult to prove that the functional $\tilde{J}_\lambda$ is coercive and bounded below on $W^{1,p}(\Omega)$. Indeed, it is enough to observe that

$$\int_\Omega \tilde{F}(u_{\lambda_1}(x))\,\mathrm{d}x \leqslant \int_\Omega \tilde{F}(u(x))\,\mathrm{d}x \leqslant \int_\Omega \tilde{F}(u_{\lambda_2}(x))\,\mathrm{d}x.$$

Therefore, we get a minimizer v_λ to $\tilde{J}_\lambda$ in $W^{1,p}(\Omega)$, from which without loss of generality we can assume that v_λ is positive. By regularity theory $v_\lambda \in C^{1,\alpha}$. Moreover,

$$-\Delta_p u_{\lambda_1} + \lambda u_{\lambda_1}^{p-1} \leqslant \tilde{f}(u_{\lambda_1}) \leqslant \tilde{f}(v_\lambda) \leqslant f(u_{\lambda_2}) \leqslant -\Delta_p u_{\lambda_2} + \lambda u_{\lambda_2}^{p-1}.$$

Thus, by weak comparison principle (see Lemma 2.1), we have

$$u_{\lambda_1} \leqslant v_\lambda \leqslant u_{\lambda_2}.$$

Set

$$\mathscr{K} := \{x \in \overline{\Omega} : v_\lambda(x) = u_{\lambda_2}(x)\}.$$

Using (3.24), we have that $\mathscr{K} \neq \overline{\Omega}$ and so by the Proposition 2.1 in Guedda–Veron [11], we obtain that $0 < v_\lambda < u_{\lambda_2}$. Therefore, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$,

$$u_{\lambda_1}(x) + \varepsilon\delta(x) \leqslant v_\lambda \leqslant u_{\lambda_2}(x) - \varepsilon\delta(x),$$

where $\delta(x) = \inf\{|x-y|;\, y \in \partial\Omega\}$. Moreover, it is easy to see that the function $\hat{F}(u) := \tilde{F}(u) - F(u)$ on the interval of functions $[u_{\lambda_1}, u_{\lambda_2}]$ is independent of u, so $\widetilde{J}_\lambda - J_\lambda$ is constant in $\mathscr{C}^1$-ball, $\{u \in C^1(\Omega) \cap W^{1,p}(\Omega) : \|u - v_\lambda\|_{1,0} \leqslant \varepsilon\}$, which means that v_λ is a local minimum of J_λ in the $\mathscr{C}^1$-topology. Finally, using the same argument as in the proof of Theorem 1.1 in [4] (see also [8]) we obtain that v_λ is also a local minimum of functional J_λ in the space $W^{1,p}(\Omega)$. $\square$

3.2. The perturbed functional

Here, we are denoting by v_λ the local minimum obtained in Proposition 3.1. Next we are going to prove that the perturbed functional $I_\lambda(u) := J_\lambda(u + v_\lambda)$ has the mountain pass geometry.

Lemma 3.2 (*Mountain pass geometry*). *The functional J_λ satisfies the following*:

(i) *there exist $\alpha \in \mathbb{R}$ and $\rho > 0$ such that*

$$J_\lambda(u) \geqslant \alpha \quad for\ u \in W^{1,p}(\Omega) \quad with\ \|u - v_\lambda\|_{1,p} = \rho;$$

(ii) *there exists $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that $\|\tilde{u}_\lambda\|_{1,p} > \rho$ and $J_\lambda(\tilde{u}_\lambda) < \alpha$.*

Proof. (i) follows from the fact that v_λ is local minimum of J_λ. To prove (ii) it is enough to observe that

$$\begin{aligned} J_\lambda(v_\lambda + t v_\lambda) &= \frac{(1+t)^p}{p}\, \|v_\lambda\|_{1,p}^p - \frac{(1+t)^{q+1}}{q+1}\, \|v_\lambda\|_{L^q(\Omega)}^{q+1} - (1+t)\int_{\partial\Omega} v_\lambda \varphi \, \mathrm{d}\sigma_y \\ &\leqslant \frac{(1+t)^p}{p}\, \|v_\lambda\|_{1,p}^p - \frac{(1+t)^{q+1}}{q+1}\, \|v_\lambda\|_{L^q(\Omega)}^{q+1} \end{aligned}$$

and $q + 1 > p$. $\square$

Therefore, we can conclude that the set

$$\Gamma = \{\gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = v_\lambda \text{ and } J_\lambda(\gamma(1)) < J_\lambda(v_\lambda)\},$$

is nonempty and the *mountain pass level*

$$c(v_\lambda) := \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} J_\lambda(\gamma(t)),$$

is well defined. Moreover, following [10] we have the following characterization to the minimax level $c(v_\lambda)$,

$$c(v_\lambda) = \inf_{v \in W^{1,p}(\Omega)\backslash\{0\}} \max_{t \geqslant 0} I_\lambda(tv) = \inf_{v \in W^{1,p}(\Omega)\backslash\{0\}} \max_{t \geqslant 0} J_\lambda(v_\lambda + tv). \tag{3.25}$$

Next, using this characterization we can state

Proposition 3.3. *If* $q = p^* - 1$, *then the following estimate is true*:

$$c(v_\lambda) < J_\lambda(v_\lambda) + \frac{1}{2n} S^{n/p}.$$

Proof. By (3.25) we have

$$c(v_\lambda) \leqslant \max_{t \geqslant 0} J_\lambda(v_\lambda + tv), \quad \text{for all } v \in W^{1,p}(\Omega)\backslash\{0\}. \tag{3.26}$$

Since equation (1_λ) is equivariant with respect to rotations and translations in $\mathbb{R}^n$, we can assume without lost of generality that $x_0 = 0 \in \partial\Omega$ and $\Omega \subset \{x_n > 0\}$. For each $x \in \mathbb{R}^n$ we write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In the following, we assume that in some neighborhood of origin the boundary of Ω is given by

$$x_n = h(x') = g(x') + o(|x'|^2), \quad \text{for } x' = (x_1, \dots, x_{n-1}) \in D(0,\delta), \tag{3.27}$$

where

$$D(0,\delta) = B(0,\delta) \cap \{x_n = 0\}, \quad g(x') := \frac{1}{2} \sum_{i=1}^{n-1} \alpha_i x_i^2$$

and $\alpha_i > 0$ are the principal curvatures of $\partial\Omega$ in $x_0 = 0$.

Next, we are going to estimate

$$\begin{aligned} J(v_\lambda + tu_\varepsilon) = {} & \frac{1}{p} \int_\Omega [|\nabla(v_\lambda + tu_\varepsilon)|^p + \lambda|v_\lambda + tu_\varepsilon|^p]\,\mathrm{d}x \\ & - \frac{1}{p^*} \int_\Omega |v_\lambda + tu_\varepsilon|^{p^*}\,\mathrm{d}x - \int_{\partial\Omega} \varphi(v_\lambda + tu_\varepsilon)\,\mathrm{d}\sigma_y. \end{aligned}$$

For the sake of clarity we estimate separately the gradient term, critical and subcritical term. We are going to use the following notations:

$$K_{1,s}(\varepsilon) := \int_\Omega |\nabla u_\varepsilon|^s\,\mathrm{d}x, \quad K_{2,r}(\varepsilon) := \int_\Omega u_\varepsilon^r\,\mathrm{d}x.$$

(i) *Estimate of the gradient term*: Let $t \in [0, \infty)$, $p \in [2, 3)$, $\alpha \in [0, 2\pi]$ and $\gamma \in [p-1, 2]$. The following elementary inequality holds:

$$(1 + t^2 + 2t \cos\alpha)^{p/2} \leqslant 1 + t^p + pt \cos\alpha + Ct^\gamma. \tag{3.28}$$

Since

$$\int_\Omega |\nabla(v_\lambda + tu_\varepsilon)|^p \,dx = \int_\Omega |\nabla v_\lambda|^p \left(1 + 2t \frac{\nabla v_\lambda \nabla u_\varepsilon}{|\nabla v_\lambda|^2} + t^2 \frac{|\nabla u_\varepsilon|^2}{|\nabla v_\lambda|^2}\right)^{p/2} dx$$

from (3.28) we obtain

$$\begin{aligned}&\int_\Omega |\nabla(v_\lambda + tu_\varepsilon)|^p \,dx \\ &\quad\leqslant \int_\Omega (|\nabla v_\lambda|^p + t^p |\nabla u_\varepsilon|^p + pt|\nabla v_\lambda|^{p-2}\langle \nabla v_\lambda \nabla u_\varepsilon\rangle + t^\gamma |\nabla u_\varepsilon|^\gamma)\,dx,\end{aligned}$$

which together with L^∞ estimate due to Libermann [12] and Cauchy–Schwarz's inequality implies

$$\int_\Omega |\nabla(v_\lambda + tu_\varepsilon)|^p \,dx \leqslant \int_\Omega |\nabla v_\lambda|^p \,dx + t^p \int_\Omega |\nabla u_\varepsilon|^p \,dx + t^\gamma K_{1,\gamma}(\varepsilon). \tag{3.29}$$

(ii) *Estimate of the critical power term*: In order to estimate the critical power term we consider the elementary inequality

$$(1 + s)^{p^*} \geqslant 1 + s^{p^*} + p^* s + p^* s^{p^*-1} + Cs^\gamma, \quad s \geqslant 0, \tag{3.30}$$

where $\gamma \in (1, p^* - 1]$ (see [3] for more details). Thus, from (3.30),

$$\int_\Omega (v_\lambda + tu_\varepsilon)^{p^*} \,dx \geqslant \int_\Omega v_\lambda^{p^*} \,dx + t^{p^*} \int_\Omega u_\varepsilon^{p^*} \,dx + p^* t^{p^*-1} \int_\Omega u_\varepsilon^{p^*-1} v_\lambda \,dx. \tag{3.31}$$

(iii) *Estimate of the subcritical power term*: Firstly, we notice that for each $a, b \geqslant 0$ and $1 < p < n$ we have

$$(a + b)^p \leqslant a^p +^p + C \max\{ab^{p-1}, ba^{p-1}\},$$

which implies that

$$\begin{aligned}\int_\Omega |v_\lambda + tu_\varepsilon|^p \,dx \leqslant{}& \int_\Omega v_\lambda^p \,dx + t^p \int_\Omega u_\varepsilon^p \,dx + C_1 t^{p-1} \int_\Omega v_\lambda u_\varepsilon^{p-1} \,dx \\ &+ C_2 t \int_\Omega v_\lambda^{p-1} u_\varepsilon \,dx.\end{aligned}$$

Since $v_\lambda \in L^\infty(\overline{\Omega})$, we get

$$\begin{aligned}\int_\Omega |v_\lambda + tu_\varepsilon|^p \,dx \leqslant{}& \int_\Omega v_\lambda^p \,dx + t^p \int_\Omega u_\varepsilon^p \,dx \\ &+ C_3 t^{p-1} \int_\Omega u_\varepsilon^{p-1} \,dx + C_4 t \int_\Omega u_\varepsilon \,dx.\end{aligned} \tag{3.32}$$

Using estimates (3.29), (3.31) and (3.32) we obtain

$$J_\lambda(v_\lambda + tu_\varepsilon) \leqslant J_\lambda(v_\lambda) + F_\lambda(t,\varepsilon) + G_\lambda(t,\varepsilon), \tag{3.33}$$

where

$$F_\lambda(t,\varepsilon) = \frac{t^p}{p}\,(K_{1,p} + \lambda K_{2,p}) - \frac{t^{p^*}}{p^*} K_{2,p^*}$$

and

$$\begin{aligned} G_\lambda(t,\varepsilon) = {} & C_1 t^\gamma K_{1,\gamma}(\varepsilon) + C_2 t^{p-1} K_{2,p-1}(\varepsilon) + C_3 t K_{2,1}(\varepsilon) \\ & - t^{p^*-1} \int_\Omega u_\varepsilon^{p^*-1} v_\lambda \,\mathrm{d}x. \end{aligned}$$

To finish the proof of Proposition 3.3, we need the following result.

Lemma 3.4. *For each $\lambda > 0$ and $\varepsilon > 0$ sufficiently small we have*

$$\max_{t>0} F_\lambda(t,\varepsilon) < \frac{1}{2n} S^{n/p} \tag{3.34}$$

and

$$G(t,\varepsilon) = t^\gamma O(\varepsilon^\alpha) + t^{p-1} O(\varepsilon^\beta) + t\,O(\varepsilon^\delta) - t^{p^*-1} O(\varepsilon^\eta), \tag{3.35}$$

where

$$\begin{aligned} \alpha &= \frac{n-p}{p^2}\,\gamma + \frac{\gamma}{p} - \frac{n\gamma}{p} + \frac{p-1}{p} n, \\ \beta &= \frac{n-p}{p^2}\,(p-1) - \frac{(n-p)}{p}\,(p-1) + \frac{p-1}{p}\,n, \\ \delta &= \frac{n-p}{p^2} - \frac{(n-p)}{p} + \frac{p-1}{p}\,n, \\ \eta &= \frac{n-p}{p^2}\,(p^*-1) - \frac{(n-p)}{p}\,(p^*-1) + \frac{p-1}{p}\,n. \end{aligned}$$

Proof. We begin by proving estimate (3.34). For this purpose, we consider two cases: $p^2 \leqslant n$ and $p^2 > n$.

Case: $p^2 \leqslant n$. Notice that

$$K_{1,p}(\varepsilon) = \int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p \,\mathrm{d}x - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p}), \tag{3.36}$$

because

$$- \int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p \,\mathrm{d}x + \int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x + \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n = O(\varepsilon^{(n-p)/p}).$$

Since

$$\begin{aligned}
&\left|\int_\Omega |\nabla u_\varepsilon|^p\,\mathrm{d}x - \int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\,\mathrm{d}x + \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n\right| \\
&\quad = \left|-\int_{\mathbb{R}^n_+\setminus\Omega} |\nabla u_\varepsilon|^p\,\mathrm{d}x + \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n\right| \\
&\quad \leqslant \int_{\mathbb{R}^n_+\setminus B_+(0,\delta)} |\nabla u_\varepsilon|^p\,\mathrm{d}x = C(n,\,p)\varepsilon^{(n-p)/p} \int_{\mathbb{R}^n_+\setminus B_+(0,\delta)} \frac{|x|^{p/(p-1)}}{(\varepsilon+|x|^{p/(p-1)})^n}\,\mathrm{d}x \\
&\quad \leqslant C(n,\,p)\varepsilon^{(n-p)/p} \int_\delta^\infty \frac{r^{p/(p-1)+n-1}}{r^{p(n-1)/(p-1)}}\,\mathrm{d}r \\
&\quad = C(n,\,p)\varepsilon^{(n-p)/p} \int_\delta^\infty \frac{1}{r^{(n-1)/(p-1)}}\,\mathrm{d}r < \infty,
\end{aligned}$$

because $1 < p^2 \leqslant n$ implies $2p-1 < p^2 \leqslant n$ and consequently $(n-p)/(p-1) > 1$. Now, notice that

$$\begin{aligned}
K_1 &= 2\int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\,\mathrm{d}x = \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^p\,\mathrm{d}x \\
&= \left(\frac{n-p}{p-1}\right)^n \int_{\mathbb{R}^n} \frac{|x|^{p/(p-1)}}{(1+|x|^{p/(p-1)})^n}\,\mathrm{d}x.
\end{aligned} \tag{3.37}$$

Thus, K_1 does not depend on ε.

From (3.36)–(3.37) it follows that

$$\begin{aligned}
K_{1,p}(\varepsilon) = \frac{1}{2}K_1 &- \int_{D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n \\
&- \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p}) \\
= \frac{1}{2}K_1 &- \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n - \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n \\
&+ \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n \\
&- \int_{D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p}).
\end{aligned}$$

Thus,

$$\begin{aligned}
K_{1,p}(\varepsilon) = \frac{1}{2}K_1 &- \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n \\
&- \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p}),
\end{aligned} \tag{3.38}$$

where in the last inequality we have used the following estimate:

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n \\
&\quad = \int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n \\
&\quad = C(n,p)\varepsilon^{(n-p)/p} \int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} \frac{|x|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})^n} \,\mathrm{d}x_n \\
&\quad \leqslant C(n,p)\varepsilon^{(n-p)/p} \int_{\mathbb{R}^{n-1}\setminus D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} \frac{1}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \,\mathrm{d}x_n.
\end{aligned}
$$

Using radial variable we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n \\
&\quad \leqslant C_1(n,p)\varepsilon^{(n-p)/p} \int_\delta^\infty \frac{r^2 r^{n-2}}{r^{p(n-1)/(p-1)}} \,\mathrm{d}r \\
&\quad \leqslant C_2(n,p)\varepsilon^{(n-p)/p} \int_\delta^\infty \frac{1}{r^{(n-p)/(p-1)}} \,\mathrm{d}r < \infty.
\end{aligned}
$$

Now, notice that

$$
\begin{aligned}
I(\varepsilon) := &\int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |\nabla u_\varepsilon|^p \,\mathrm{d}x_n \\
&= \left(\frac{n-p}{p-1}\right)^p \varepsilon^{(n-p)/p} \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} \frac{|x|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})^n} \,\mathrm{d}x_n \\
&= \left(\frac{n-p}{p-1}\right)^p \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{\varepsilon^{(p-1)/p} g(x')} \frac{|x|^{p/(p-1)}}{(1 + |x|^{p/(p-1)})^n} \mathrm{d}x_n.
\end{aligned} \tag{3.39}
$$

Thus,

$$
\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{\varepsilon^{(p-1)/p}} = \left(\frac{n-p}{p-1}\right)^p \int_{\mathbb{R}^{n-1}} \frac{|x'|^{p/(p-1)} g(x')}{(1 + |x'|^{p/(p-1)})^n} \,\mathrm{d}x'
$$

which implies that

$$
I(\varepsilon) = O(\varepsilon^{(p-1)/p}).
$$

Moreover,

$$
\begin{aligned}
|I_1(\varepsilon):| &= \left| \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |\nabla u_\varepsilon|^p \, \mathrm{d}x_n \right| \\
&= C(n,p)\varepsilon^{(n-p)/p} \left| \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} \frac{|x|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})^n} \, \mathrm{d}x_n \right| \\
&= C(n,p)\varepsilon^{(n-p)/p} \left| \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} \right. \\
&\quad \left. \times \frac{|x|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})(\varepsilon + |x|^{p/(p-1)})^{n-1}} \, \mathrm{d}x_n \right| \\
&\leqslant C(n,p)\varepsilon^{(n-p)/p} \left| \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} \frac{1}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \, \mathrm{d}x_n \right| \\
&\leqslant C(n,p)\varepsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{|h(x') - g(x')|}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \, \mathrm{d}x'.
\end{aligned}
$$

Since $h(x') = g(x') + o(|x'|^2)$, it follows that for all $\sigma > 0$, there exists $C(\varepsilon) > 0$ such that $|h(x') - g(x')| \leqslant \sigma |x'|^2 + C(\sigma)|x'|^\alpha$ for all $x' \in D(0,\delta)$, where $2 < \alpha < (n-1)/(p-1)$. Thus,

$$
I_1(\varepsilon) \leqslant C(n,p)\varepsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{\sigma|x'|^2 + C(\sigma)|x'|^\alpha}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \, \mathrm{d}x'.
$$

Now, observing that

$$
\varepsilon^{(n-p)/p} / \varepsilon^{(p-1)/p} \int_{D(0,\delta)} \frac{|x'|^2}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \, \mathrm{d}x' \leqslant C
$$

and

$$
\varepsilon^{(n-p)/p} / \varepsilon^{(p-1)/p} \int_{D(0,\delta)} \frac{|x'|^\alpha}{(\varepsilon + |x'|^{p/(p-1)})^{n-1}} \, \mathrm{d}x' \leqslant C(n,p)\varepsilon^{(p-1)(\alpha-2)/p},
$$

we obtain

$$
I_1(\varepsilon) \leqslant C(n,p)\varepsilon^{(p-1)/p}(\sigma + C(\sigma)\varepsilon^{(p-1)(\alpha-2)/p}).
$$

Since σ is arbitrary and $\alpha > 2$, we conclude that $I_1(\varepsilon) = o(\varepsilon^{(p-1)/p})$. Therefore,

$$
\begin{aligned}
K_{1,p}(\varepsilon) &= \frac{1}{2} K_1 - I(\varepsilon) - I_1(\varepsilon) + O(\varepsilon^{(n-p)/p}) \\
&= \frac{1}{2} K_1 - I(\varepsilon) + o(\varepsilon^{(p-1)/p}).
\end{aligned}
\tag{3.40}
$$

Now, let us obtain a more refined estimate of $K_{2,p^*}(\varepsilon)$. To this end, firstly notice that

$$\begin{aligned}
K_{2,p^*}(\varepsilon) &= \int_{\mathbb{R}^n_+} |u_\varepsilon|^{p^*}\,\mathrm{d}x - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} |u_\varepsilon|^{p^*}\,\mathrm{d}x_n + O(\varepsilon^{n/p}) \\
&= \frac{1}{2}K_2 - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{g(x')} |u_\varepsilon|^{p^*}\,\mathrm{d}x_n \\
&\quad - \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |u_\varepsilon|^{p^*}\,\mathrm{d}x_n + O(\varepsilon^{n/p}) \\
&= \frac{1}{2}K_2 - \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} |u_\varepsilon|^{p^*}\,\mathrm{d}x_n \\
&\quad - \int_{D(0,\delta)} \mathrm{d}x' \int_{g(x')}^{h(x')} |u_\varepsilon|^{p^*}\,\mathrm{d}x_n + O(\varepsilon^{n/p}) \\
&= \frac{1}{2}K_2 - II(\varepsilon) - III(\varepsilon) + O(\varepsilon^{n/p}).
\end{aligned}$$

Since

$$\begin{aligned}
II(\varepsilon) &:= \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} u_\varepsilon^{p^*}\,\mathrm{d}x_n \\
&= \varepsilon^{n/p} \int_{\mathbb{R}^{n-1}} \mathrm{d}x' \int_0^{g(x')} \frac{1}{(\varepsilon + |x|^{p/(p-1)})^n}\,\mathrm{d}x_n \\
&= \int_{\mathbb{R}^{n-1}} \mathrm{d}y' \int_0^{\varepsilon^{(p-1)/p} g(y')} \frac{1}{(1 + |y|^{p/(p-1)})^n}\,\mathrm{d}y_n,
\end{aligned} \tag{3.41}$$

we have $II(\varepsilon) = O(\varepsilon^{(p-1)/p})$. Using the same estimate as in $I_1(\varepsilon)$ we have $III(\varepsilon) = o(\varepsilon^{(p-1)/p})$. Thus, for $1 < p^2 \leqslant n$ we have

$$K_{2,p^*}(\varepsilon) = \frac{1}{2}K_2 - II(\varepsilon) + o(\varepsilon^{(p-1)/p}). \tag{3.42}$$

We can now proceed analogously to obtain a refined estimate for $K_{2,p}(\varepsilon)$. To this end, we consider two cases $p^2 < n$ and $p^2 = n$ separately.

Case 1: $p^2 < n$. In this case we have

$$\begin{aligned}
K_{2,p}(\varepsilon) &= \int_\Omega u_\varepsilon^p\,\mathrm{d}x \leqslant \int_{\mathbb{R}^n} u_\varepsilon^p\,\mathrm{d}x = \varepsilon^{(n-p)/p} \int_{\mathbb{R}^n} \frac{1}{(\varepsilon + |x|^{p/(p-1)})^{n-p}}\,\mathrm{d}x \\
&= w_n \varepsilon^{(n-p)/p} \left(\int_0^1 \frac{r^{n-1}}{(\varepsilon + |r|^{p/(p-1)})^{n-p}}\mathrm{d}r + \int_1^\infty \frac{r^{n-1}}{(\varepsilon + |r|^{p/(p-1)})^{n-p}}\,\mathrm{d}r \right) \\
&= O(\varepsilon^{(n-p)/p}) \\
&= o(\varepsilon^{(p-1)/p}).
\end{aligned}$$

Case 2: $p^2 = n$. Let $R > 0$ such that $\Omega \subset B(0, R)$. Notice that

$$\begin{aligned}
K_{2,p}(\varepsilon) &= \int_\Omega u_\varepsilon^p \, dx \leqslant \int_{B(0,R)} u_\varepsilon^p \, dx \\
&= \varepsilon^{(n-p)/p} \int_{B(0,R)} \frac{1}{(\varepsilon + |x|^{p/(p-1)})^{n-p}} \, dx \\
&= w_n \varepsilon^{(n-p)/p} \int_0^R \frac{r^{n-1}}{(\varepsilon + |r|^{p(p-1)})^{n-p}} \, dr \\
&= w_n \varepsilon^{p-1} \int_0^{R/\varepsilon^{(p-1)/p}} \frac{s^{n-1}}{(1 + |s|^{p/(p-1)})^{n-p}} \, ds \\
&= C\varepsilon^{p-1}(1 - \log(\varepsilon^{(p-1)/p})) \\
&= \varepsilon^{(p-1)/p}(\varepsilon^{(p-1)^2/p} - \varepsilon^{(p-1)^2/p} \log(\varepsilon^{(p-1)/p})) \\
&= o(\varepsilon^{(p-1)/p}).
\end{aligned}$$

Hence, for $1 < p^2 \leqslant n$ we have

$$K_{2,p}(\varepsilon) = o(\varepsilon^{(p-1)/p}). \tag{3.43}$$

Since $p^* > p$, there exists $t_\varepsilon > 0$ such that

$$J_\lambda(t_\varepsilon u_\varepsilon) = \max_{t>0} \left\{ \frac{1}{p}(K_{1,p}(\varepsilon) + \lambda K_{2,p}(\varepsilon))t^p - \frac{K_{2,p}(\varepsilon)}{p^*} t^{p^*} \right\}. \tag{3.44}$$

It follows from estimates (3.40), (3.42) and (3.43) that there exists $\varepsilon_0 > 0$, $K' > 0$ and $K'' > 0$ such that

$$K_{2,p^*}(\varepsilon) \geqslant K' \quad \text{and} \qquad K_{1,p}(\varepsilon) + K_{2,p}(\varepsilon) \leqslant K'', \quad \forall\ \varepsilon \in (0, \varepsilon_0). \tag{3.45}$$

Consequently, t_ε is uniformly bounded in $(0, \varepsilon_0)$. Since $K_3(\varepsilon) = o(\varepsilon^{(p-1)/p})$ for $p^2 \leqslant n$, we get

$$\begin{aligned}
J_\lambda(t_\varepsilon) &= \sup_{t>0} \left\{ \frac{1}{p} K_1(\varepsilon) t^p - \frac{K_2(\varepsilon)}{p^*} t^{p^*} \right\} + o(\varepsilon^{(p-1)/p}) \\
&= \frac{1}{p} K_1(\varepsilon) \frac{K_2(\varepsilon)}{K_1(\varepsilon)} t_1^{p^*} - \frac{1}{p^*} K_2(\varepsilon) t_1^{p^*} + o(\varepsilon^{(p-1)/p}) \\
&= \frac{1}{n} K_2(\varepsilon) t_1^{p^*} + o(\varepsilon^{(p-1)/p}) \\
&= \frac{1}{n} K_2(\varepsilon) \left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)} \right)^{n/p} + o(\varepsilon^{(p-1)/p}) \\
&= \frac{1}{n} \left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{(n-p)/n}} \right)^{n/p} + o(\varepsilon^{(p-1)/p}).
\end{aligned}$$

Finally, we observe that statement (3.34) will be proved once we have proved the following claim

Claim 3.1. *The following estimate holds*

$$\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{(n-p)/n}} < 2^{-p/n} S + o(\varepsilon^{(p-1)/p}). \tag{3.46}$$

F. rom (1.2), inequality (3.46) is equivalent to

$$\begin{aligned}\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{(n-p)/n}} &< 2^{-p/n} \frac{K_1}{K_2^{(n-p)/n}} + o(\varepsilon^{(p-1)/p}) \\ &= \frac{K_1}{2} \frac{1}{\left(\frac{K_2}{2}\right)^{(n-p)/n}} + o(\varepsilon^{(p-1)/p}),\end{aligned}$$

that is,

$$K_1(\varepsilon)\left(\frac{K_2}{2}\right)^{(n-p)/n} < \frac{K_1}{2} K_2(\varepsilon)^{(n-p)/n} + o(\varepsilon^{(p-1)/p}).$$

From (3.40)–(3.42) we have

$$\begin{aligned}\left(\frac{K_1}{2} - I(\varepsilon)\right)\left(\frac{K_2}{2}\right)^{(n-p)/n} &< \frac{K_1}{2}\left(\frac{K_2}{2} - II(\varepsilon) + o(\varepsilon^{(p-1)/p})\right)^{(n-p)/n} \\ &\quad + o(\varepsilon^{(p-1)/p}).\end{aligned} \tag{3.47}$$

Now, notice that for $a\alpha > 0$, we have

$$(1-t)^{\alpha} = 1 - \alpha t + o(t), \quad \text{as} \quad t \to 0.$$

In particular, taking

$$t = \frac{II(\varepsilon) + o(\varepsilon^{(p-1)/p})}{\frac{K_2}{2}},$$

we obtain

$$\begin{aligned}&\left(\frac{K_2}{2} - II(\varepsilon) + o(\varepsilon^{(p-1)/p})\right)^{(n-p)/n} \\ &\quad = \left(\frac{K_2}{2}\right)^{(n-p)/n} - \left(\frac{n-p}{n}\right)\left(\frac{K_2}{2}\right)^{-p/n} II(\varepsilon) + o(\varepsilon^{(p-1)/p}).\end{aligned}$$

Thus, (3.47) is equivalent to

$$-I(\varepsilon)\left(\frac{K_2}{2}\right)^{(n-p)/n} < -\frac{K_1}{2}\left(\frac{K_2}{2}\right)^{-p/n}\left(\frac{n-p}{n}\right) II(\varepsilon) + o(\varepsilon^{(p-1)/p}).$$

Since, $II(\varepsilon) = O(\varepsilon^{(p-1)/p})$ we get

$$\frac{I(\varepsilon)}{II(\varepsilon)} > \left(\frac{n-p}{n}\right)\frac{K_1}{K_2} + o(1),$$

which implies that (3.46) is equivalent to

$$\lim_{\varepsilon\to 0} \frac{I(\varepsilon)}{II(\varepsilon)} > \left(\frac{(n-p)}{n}\right)\frac{K_1}{K_2}. \tag{3.48}$$

From (3.39) and (3.41) we get

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{I(\varepsilon)}{II(\varepsilon)} &= ((n-p)/(p-1))^p \lim_{\varepsilon\to 0} \frac{\int_{\mathbb{R}^{n-1}} \mathrm{d}y' \int_0^{\varepsilon^{(p-1)/p} g(y')} \frac{|y|^{p/(p-1)}}{(1+|y|^{p/(p-1)})^n}\,\mathrm{d}y_n}{\int_{\mathbb{R}^{n-1}} \mathrm{d}y' \int_0^{\varepsilon^{(p-1)/p} g(y')} \frac{1}{(1+|y|^{p/(p-1)})^n}\,\mathrm{d}y_n} \\
&= \left(\frac{n-p}{p-1}\right)^p \frac{\int_{\mathbb{R}^{n-1}} \frac{|y'|^{p/(p-1)}}{(1+|y'|^{p/(p-1)})^n}\,\mathrm{d}y'}{\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y'|^{p/(p-1)})^n}\,\mathrm{d}y'} \\
&= \left(\frac{n-p}{p-1}\right)^p \frac{\int_0^\infty \frac{r^{n+p/(p-1)}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r}{\int_0^\infty \frac{r^n}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r}.
\end{aligned} \tag{3.49}$$

Now we calculate the last term in (3.49). If $p/(p-1) \leqslant \beta \leqslant p(n-1) + 1/(p-1)$, integrating by parts we have

$$\int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}\,\mathrm{d}r = \frac{p(n-1)}{(p-1)\beta-1}\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r. \tag{3.50}$$

Observing that

$$\frac{r^\beta}{(1+r^{p/(p-1)})^n} = \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}\left(1-\frac{1}{1+r^{p/(p-1)}}\right),$$

we obtain

$$\begin{aligned}
\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r &= \int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^{n-1}}\,\mathrm{d}r \\
&\quad - \int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r.
\end{aligned} \tag{3.51}$$

From (3.50) and (3.51) we get

$$\left(1-\frac{(n-1)p}{(p-1)\beta-1}\right)\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r = -\int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r,$$

that is,

$$\int_0^\infty \frac{r^\beta}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r = \frac{(p-1)\beta-1}{(n-1)p-(p-1)\beta+1} \times \int_0^\infty \frac{r^{\beta-p/(p-1)}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r. \tag{3.52}$$

From (3.49) and (3.52) with $\beta = n + p/(p-1)$ we obtain

$$\lim_{\varepsilon\to 0}\frac{I(\varepsilon)}{II(\varepsilon)} = \left(\frac{n-p}{p-1}\right)^p \frac{(p-1)(n+1)}{n-2p+1} = \frac{(n-p)^p}{(p-1)^{p-1}}\,\frac{n+1}{n-2p+1}. \tag{3.53}$$

By (1.2), we have

$$\left(\frac{n-p}{n}\right)\frac{K_1}{K_2} = \left(\frac{n-p}{n}\right)\frac{\int_0^\infty \frac{r^{n+p/(p-1)-1}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r}{\int_0^\infty \frac{r^{n-1}}{(1+r^{p/(p-1)})^n}\,\mathrm{d}r}\left(\frac{n-p}{p-1}\right)^p.$$

Taking $\beta = n + p/(p-1) - 1$ in (3.52) we have

$$\begin{aligned}\frac{n-p}{n}\left(\frac{K_1}{K_2}\right) &= \frac{n-p}{n}\,\frac{(p-1)((n-1)+p/(p-1))}{(n-1)p-(p-1)((n-1)+p/(p-1))}\left(\frac{n-p}{p-1}\right)^p\\ &= \frac{(n-p)^p}{(p-1)^{p-1}}.\end{aligned} \tag{3.54}$$

Since $n+1 > n-2p+1$, (3.53)–(3.54) yields that (3.48) is true. Therefore, the claim was proved in the case $1 < p^2 \leqslant n$.

Case 2: $p^2 > n$. Let $R > 0$ such that $\Omega \subset B(0, R)$. Notice that

$$K_3(\varepsilon) = \int_\Omega u_\varepsilon^p\,\mathrm{d}x \leqslant c\varepsilon^{(n-p)/p}\int_0^R \frac{r^{n-1}}{(\varepsilon + r^{p/(p-1)})^{n-p}}\,\mathrm{d}r.$$

Consequently,

$$K_3(\varepsilon) = O(\varepsilon^{(n-p)/p}). \tag{3.55}$$

Choosing $0 < a \leqslant A < \infty$ such that $a|x'|^2 \leqslant h(x') \leqslant A|x'|^2$ for $x' \in D(0,\delta)$, we have

$$\begin{aligned}K_1(\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^p\,\mathrm{d}x &= \int_{\mathbb{R}^n_+} |\nabla u_\varepsilon|^p\,\mathrm{d}x\\ &\quad - \int_{D(0,\delta)}\mathrm{d}x'\int_0^{h(x')} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p})\\ &= \frac{K_1}{2} - \int_{D(0,\delta)}\mathrm{d}x'\int_0^{h|x'|^2} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p})\\ &\leqslant \frac{K_1}{2} - \int_{D(0,\delta)}\mathrm{d}x'\int_0^{a|x'|^2} |\nabla u_\varepsilon|^p\,\mathrm{d}x_n + O(\varepsilon^{(n-p)/p}).\end{aligned}$$

Using $|x|^{p/(p-1)} \geqslant |x'|^{p/(p-1)}$, we have

$$\int_{D(0,\delta)} \mathrm{d}x' \int_0^{a|x'|^2} |\nabla u_\varepsilon|^p \, \mathrm{d}x_n \\ \geqslant \varepsilon^{(n-p)/p} \int_{D(0,\delta)} \mathrm{d}x' \int_0^{a|x'|^2} \frac{|x'|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})^n} \, \mathrm{d}x_n. \tag{3.56}$$

For $\delta \in (0, 1)$, we have $\varepsilon + |x|^{p/(p-1)} \leqslant C(\varepsilon + |x'|^{p/(p-1)})$. Consequently

$$\varepsilon^{(n-p)/p} \int_{D(0,\delta)} \mathrm{d}x' \int_0^{a|x'|^2} \frac{|x'|^{p/(p-1)}}{(\varepsilon + |x|^{p/(p-1)})^n} \, \mathrm{d}x_n \\ \geqslant c_1 \varepsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{a|x'|^2 |x'|^{p/(p-1)}}{(\varepsilon + |x'|^{p/(p-1)})^n} \mathrm{d}x'. \tag{3.57}$$

Now, observe that

$$\varepsilon^{(n-p)/p} \int_{D(0,\delta)} \frac{a|x'|^2 |x'|^{p/(p-1)}}{(\varepsilon + |x'|^{p/(p-1)})^n} \, \mathrm{d}x' = \varepsilon^{(n-p)/p} \int_0^\delta \frac{r^2 r^{p/(p-1)} r^{n-1}}{(\varepsilon + r^{p/(p-1)})^n} \, \mathrm{d}r \\ = \varepsilon^{(n-p)/p} \varepsilon^{(2p-n-1)/p} \int_0^{\delta/\varepsilon^{(p-1)/p}} \frac{s^{p/(p-1)+n}}{(1 + s^{p/(p-1)})^n} \, \mathrm{d}s \\ \geqslant \varepsilon^{(n-p)/p} \varepsilon^{(2p-n-1)/p} \int_1^{\delta/\varepsilon^{(p-1)/p}} \frac{s^{p/(p-1)+n}}{(1 + s^{p/(p-1)})^n} \, \mathrm{d}s \\ \geqslant \varepsilon^{(n-p)/p} \varepsilon^{(2p-n-1)/p} \int_1^{\delta/\varepsilon^{(p-1)/p}} \frac{1}{(1 + s^{p/(p-1)})^n} \, \mathrm{d}s \\ \geqslant c_2 \varepsilon^{(n-p)/p} \varepsilon^{(2p-n-1)/p} \int_1^{\delta/\varepsilon^{(p-1)/p}} \frac{1}{s^{pn/p-1}} \, \mathrm{d}s,$$

where in the last inequality above we have used the fact that $1 + s^{p/(p-1)} \leqslant s^{p/(p-1)} + s^{p/(p-1)}$. Setting

$$f(\varepsilon) := \varepsilon^{(2p-n-1)/p} \int_1^{\delta/\varepsilon^{(p-1)/p}} \frac{1}{s^{pn/p-1}} \, \mathrm{d}s,$$

we have

$$K_1(\varepsilon) \leqslant \frac{1}{2} K_1 - c_2 \varepsilon^{(n-p)/p} f(\varepsilon) + O(\varepsilon^{(n-p)/p}). \tag{3.58}$$

To estimate $K_2(\varepsilon)$, notice that

$$K_2(\varepsilon) = \frac{1}{2} K_2 - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{h(x')} u_\varepsilon^{p^*} \, \mathrm{d}x_n + O(\varepsilon^{n/p}) \\ \geqslant \frac{1}{2} K_2 - \int_{D(0,\delta)} \mathrm{d}x' \int_0^{A|x'|^2} u_\varepsilon^{p^*} \, \mathrm{d}x_n + O(\varepsilon^{n/p})$$

and

$$\int_{D(0,\delta)} \mathrm{d}x' \int_0^{A|x'|^2} u_\varepsilon^{p^*}\,\mathrm{d}x \leqslant A\varepsilon^{n/p} \int_{D(0,\delta)} \frac{|x'|^2}{(\varepsilon+|x'|^{p/(p-1)})^n}\,\mathrm{d}x'$$
$$= A\varepsilon^{n/p}\int_0^\delta$$
$$= O(\varepsilon^{n/p}).$$

Thus,

$$K_2(\varepsilon) \geqslant \frac{1}{2} K_2 - O(\varepsilon^{(n-p)/p}). \tag{3.59}$$

Let t_ε be such that

$$\max_{t>0} J_\lambda(tu_\varepsilon) = J_\lambda(t_\varepsilon u_\varepsilon).$$

From (3.55)–(3.59) we conclude that t_ε is uniformly bounded for $\varepsilon \in (0, \varepsilon_o)$. Thus,

$$J_\lambda(t_\varepsilon u_\varepsilon) \leqslant \sup_{t>0}\left\{\frac{1}{p} t^p K_1(\varepsilon) t^p - \frac{1}{p^*} t^{p^*} K_2(\varepsilon)\right\} + O(\varepsilon^{p/(p-1)})$$
$$= \frac{1}{n}\left(\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{p/(p-1)}}\right)^{n/p} + O(\varepsilon^{p/(p-1)}).$$

Now we claim that

$$\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{p/(p-1)}} < 2^{-p/n} S - O(\varepsilon^{p/(p-1)}) \tag{3.60}$$

for ε small (that is sufficiently to show (3.46)). Indeed by (3.58)–(3.59), we see that (3.60) is equivalent to

$$\frac{K_1}{2} - c_o \varepsilon^{p/(p-1)} f(\varepsilon) < 2^{-p/n} S\left(\frac{1}{2} K_2 - O(\varepsilon^{p/(p-1)})\right)^{n/p} + O(\varepsilon^{p/(p-1)})$$
$$= \frac{1}{2} S K_2^{(n-p)/n} + O(\varepsilon^{p/(p-1)}).$$

Since $S = K_1/K_2^{n/p}$ we have that

$$\frac{K_1}{2} - c_o \varepsilon^{p/(p-1)} f(\varepsilon) < \frac{1}{2} K_1 + O(\varepsilon^{p/(p-1)}), \tag{3.61}$$

because $\lim_{\varepsilon\to 0} f(\varepsilon) = \infty$. Therefore, (3.60) is true. Thus, (3.46) holds in the case $p^2 > n$.

Finally we are going to prove (3.35). To this end, notice that

$$\begin{aligned}K_{1,\gamma}(\varepsilon) &= \int_\Omega |\nabla u_\varepsilon|^\gamma \,\mathrm{d}x \\ &= \left(\frac{n-p}{p-1}\right)^\gamma \varepsilon^{(n-p)\gamma/p^2} \int_\Omega \frac{|x|^{\gamma/p-1}}{(\varepsilon+|x|^{p/(p-1)})^{n\gamma/p}}\,\mathrm{d}x \\ &= C\varepsilon^\alpha \int_\Omega \frac{|x|^{\gamma/p-1}}{(1+|x|^{p/(p-1)})^{n\gamma/p}}\,\mathrm{d}x \\ &= O(\varepsilon^\alpha),\end{aligned}$$

where $\alpha = (n-p)\gamma/p^2 + \gamma p - n\gamma/p + (p-1)n/p$. On the other hand, if $r > 1$ we have

$$\begin{aligned}K_{2,r}(\varepsilon) &:= \int_\Omega u_\varepsilon^r \,\mathrm{d}x \\ &= \varepsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p} \int_\Omega \frac{1}{(1+|x|^{p/(p-1)})^{r(n-p)/p}}\,\mathrm{d}x \\ &= O(\varepsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p}).\end{aligned}$$

Taking $r = p-1$ and $r = 1$, we obtain, respectively, β and δ. Since $v_\lambda \in L^\infty(\overline{\Omega})$, we have

$$\left|\int_\Omega u_\varepsilon^{p^*-1} v_\lambda\right| \mathrm{d}x \leqslant \int_\Omega u_\varepsilon^{p^*-1}\,\mathrm{d}x = O(\varepsilon^{(n-p)r/p^2 - r(n-p)/p + (p-1)n/p}),$$

with $r = p^* - 1$. Thus, we obtain η. □

Proof of Proposition 3.3 (*conclusion*). If $p \in [2,3)$, fix $\varepsilon_0 > 0$ and consider the function $h : [0,+\infty) \times [0,\varepsilon_0) \to \mathbb{R}$ defined by $h_\lambda(t,\varepsilon) = F_\lambda(t,\varepsilon) + G_\lambda(t,\varepsilon)$.

From (3.45) and (3.34), there exists $C_1 > 0$ and $C_2 > 0$ such that

$$h_\lambda(t,\varepsilon) \leqslant C_1(t^p + t^\gamma + t^{p-1} + t) - C_2 t^{p^*-1}.$$

Since $\gamma < p^*$, there exists $t_0 > 0$ such that $t_\varepsilon \leqslant t_0$ for all $0 < \varepsilon \leqslant \varepsilon_0$, where $h(t_\varepsilon,\varepsilon) = \max_{t\geqslant 0} h(t,\varepsilon)$. Thus,

$$h_\lambda(t,\varepsilon) \leqslant h_\lambda(t_0,\varepsilon) = F_\lambda(t_0,\varepsilon) + G_\lambda(t_0,\varepsilon) \leqslant \max_{t\geqslant 0} F_\lambda(t,\varepsilon) + G_\lambda(t_0,\varepsilon).$$

From (3.34) we obtain $G(t_0,\varepsilon) = O(\varepsilon^\theta)$ for some $\theta > 0$. Thus, we obtain from (3.34) that

$$h_\lambda(t,\varepsilon) < \frac{1}{2n} S^{n/p}.$$

Noting that $u \in C^{1,\alpha}(\overline{\Omega})$ (see [12]) we obtain $u, \nabla u \in L^\infty$. Thus, the cases $1 < p < 2$ and $3 \leqslant p$ follow using the same argument as in Azorero–Peral [3]. This completes the proof of Proposition 3.3. □

3.3. Proof of Theorem 1.2

Until this moment we have proved the existence of a local minimum v_λ of energy functional J_λ and we are ready to prove the existence of a second critical point of J_λ, which is of the mountain pass type. Indeed, in view of Lemma 3.2 we can apply the Mountain-Pass Theorem to obtain a sequence (w_n) in $W^{1,p}(\Omega)$ such that $J_\lambda(w_n) \to c(v_\lambda)$ and $J'_\lambda(w_n) \to 0$ in $W^{-1,p'}(\Omega)$. Now, we consider two cases.

Subcritical case: $p-1<q<p^*-1$. In this case, since the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact the result follows easily.

Critical case: $q=p^*-1$. Here we are going to prove that J_λ satisfies the $(PS)_{c(v_\lambda)}$ condition, or there exists one solution w_λ such that

$$J_\lambda(w_\lambda) < J_\lambda(v_\lambda).$$

Since

$$\frac{1}{p}\int_\Omega |\nabla w_n|^p + \lambda |w_n|^p - \frac{1}{p^*}\int_\Omega w_n^{p^*} - \int_{\partial\Omega} w_n \varphi = o_n(1) + c(v_\lambda)$$

and

$$\int_\Omega |\nabla w_n|^p + \lambda |w_n|^p - \int_\Omega w_n^{p^*} - \int_{\partial\Omega} w_n \varphi = o_n(1)\|w_n\|_{1,p},$$

by Sobolev embedding and Holder's inequality, we obtain

$$\left(\frac{1}{p}-\frac{1}{p^*}\right)\|w_n\|_{1,p}^p \leqslant c(v_\lambda) + (o_n(1) + C_1\|\varphi\|_{L^{p'}(\partial\Omega)})\|w_n\|_{1,p}.$$

Consequently, (w_n) is bounded in $W^{1,p}(\Omega)$. Thus, we may extract a subsequence still denoted by (w_n) such that

$$\begin{aligned} w_n &\rightharpoonup w, && \text{weakly in } W^{1,p}(\Omega);\\ w_n &\to w, && \text{strongly in } L^p(\Omega);\\ w_n &\to w, && \text{a.e. on } \Omega. \end{aligned}$$

By a convergence result due to Lucio–Bocardo (see [5, Theorem 2.1]) we have $\nabla w_n \to \nabla w$ almost everywhere in Ω. Using this and standard argument it yields that *w* must be a critical point of J_λ. We observe that $w \neq 0$. In fact, by the definition of the weak solution, we obtain

$$\lambda\int_\Omega w_n^{p-1}\,\mathrm{d}x = \int_\Omega w_n^q\,\mathrm{d}x + \int_{\partial\Omega}\varphi\,\mathrm{d}\sigma_y.$$

Making $n \to +\infty$, we get a contradiction.

We shall have established the Theorem 1.2 if we prove the following:

Claim 3.2. $J_\lambda(w) = c(v_\lambda)$, *or* $J_\lambda(w) < J_\lambda(v_\lambda)$.

Applying the Brezis–Lieb , we obtain

$$\|\nabla w_n\|^p = \|\nabla w\|^p + \|\nabla(w_n - w)\|^p + o_n(1) \tag{3.62}$$

and

$$\|w_n\|_{L^{p^*}}^{p^*} = \|w\|_{L^{p^*}}^{p^*} + \|w_n - w\|_{L^{p^*}}^{p^*} + o_n(1). \tag{3.63}$$

From (3.62) and (3.63) we have

$$\frac{1}{p}\|w_n - w\|^p - \frac{1}{p^*}\|w_n - w\|^{p^*} + J_\lambda(w_n) = c(v_\lambda) + o_n(1). \tag{3.64}$$

$$\|w_n - w\|^p - \|w_n - w\|_{L^{p^*}}^{q+1} + J_\lambda'(w)w = J_\lambda'(w_n)w_n + o_n(1). \tag{3.65}$$

Substituting (3.65) in (3.64) we obtain that

$$o_n(1) + c(v_\lambda) = J_\lambda(w) + \left(\frac{1}{p} - \frac{1}{p^*}\right)\|w_n - w\|_{L^{p^*}}^{p^*}, \tag{3.66}$$

or let, $\|w_n - w\|_{L^{p^*}}^{p^*} \to l \geqslant 0$. If $l = 0$, the proof is finished. If not, $l > 0$. By Sobolev inequality, we get

$$\|w_n - w\|_{L^{p^*}} \leqslant S\|w_n - w\|_{1,p}.$$

Thus,

$$l \geqslant S^{n/p}.$$

Returning to (3.66) we obtain

$$\begin{aligned} c(v_\lambda) &= J_\lambda(w) + \left(\frac{1}{p} - \frac{1}{p^*}\right)\|w_n - w\|_{L^{p^*}}^{p^*} - o_n(1) \\ &= J_\lambda(w) + \left(\frac{1}{p} - \frac{1}{p^*}\right) l \\ &\geqslant J_\lambda(w) + \frac{1}{n} S^{n/p} \\ &> J_\lambda(w) + \frac{1}{2n} S^{n/p}. \end{aligned} \tag{3.67}$$

Since

$$c(v_\lambda) < J_\lambda(v_\lambda) + \frac{1}{2n} S^{n/p},$$

we conclude Claim 3.2. This finishes the proof of Theorem 1.2.

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