Universidade Federal da Paraíba
Centro de Ciências Exatas e da Natureza
Departamento de Matemática
Mestrado Profissional em Matemática em Rede Nacional

PROFMAT

1^a **Prova: MA23 - Geometria Analítica**

João Pessoa, 28 de setembro de 2018

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Nome: ______________________________ Matrícula: ____________________

1 (**2 pts.**) *Sejam* $A = (1, 2)$ *e* $\overrightarrow{BC} = (3, 4)$, *determine os vértices* B *e* C *do triângulo* ΔABC *sabendo que o seu baricentro encontra-se na origem.*

2 (**2 pts.**) *Determine a equação cartesiana da circunferência inscrita ao triângulo* ΔABC, *onde* $A = (3, 4)$, $B = (6, -2)$ *e* $C = (4, 6)$.

3 (**3 pts.**) *Dada a equação* $7x^2 + 24xy - 256x - 192y + 1456 = 0$, *identifique a cônica que ela representa, encontre seus principais elementos e faça um esboço do seu gráfico.*

4 (**3 pts.**) *Em cada caso determine a equação da cônica pedida.*

(a) *Elipse com centro no ponto* $C = (1, 2)$, *um vértice, na reta focal, no ponto* $V = (3, 2)$ *e excentricidade* $e = \frac{1}{2}$;

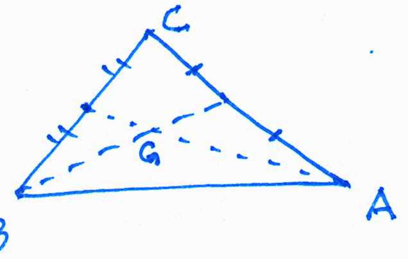
(b) *Hipérbole com vértices nos pontos* $V_1 = (-6, 0)$ *e* $V_2 = (6, 0)$ *e assintotas* $y = \pm\frac{7}{6}x$;

(c) *Parábola com vértice sobre a reta* $7x + 3y = 4$, *reta focal paralela ao eixo* X *e passa pelos pontos* $A = (3, -5)$ *e* $B = (\frac{3}{2}, 1)$.

Boa Prova.

MA23 Prova 1 - 2018

(1)



$A = (1,2)$, $\vec{BC} = [3,4]$

Determinar B e C, sabendo que o baricentro G do ΔABC encontra-se na origem. ($G = (0,0)$)

Solução

Suponha $B = (b_1, b_2)$ e $C = (c_1, c_2)$

$$\vec{BC} = [3,4] \Rightarrow \begin{cases} c_1 - b_1 = 3 \\ c_2 - b_2 = 4 \end{cases} \Rightarrow \begin{cases} c_1 = 3 + b_1 \\ c_2 = 4 + b_2 \end{cases} \quad \ldots (*)$$

As coord. do baricentro (encontro das medianas) são

$$G = \left(\frac{1 + b_1 + c_1}{3} , \frac{2 + b_2 + c_2}{3} \right)$$

usando (*) temos $G = \left(\frac{4 + 2b_1}{3} , \frac{6 + 2b_2}{3} \right)$

e como $G = (0,0)$ obtemos

$$\frac{4 + 2b_1}{3} = 0 \Rightarrow \boxed{b_1 = -2} \quad e \quad \frac{6 + 2b_2}{3} = 0 \Rightarrow \boxed{b_2 = -3}$$

$\therefore$ $\boxed{B = (-2, -3)}$

$\left. \begin{array}{l} b_1 = -2 \\ c_1 = 3 + b_1 \end{array} \right\} \Rightarrow \boxed{c_1 = 1}$

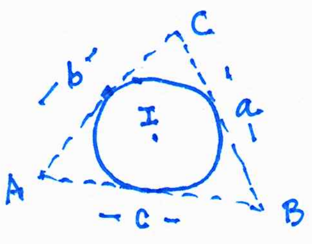
$\left. \begin{array}{l} b_2 = -3 \\ c_2 = 4 + b_2 \end{array} \right\} \Rightarrow \boxed{c_2 = 1}$

$\therefore$ $\boxed{C = (1, 1)}$

(2) Eq. da circunf. inscrita ao ΔABC.

$A = (3, 4)$, $B = (6, -2)$ e $C = (4, 6)$

Solução



I = incentro, centro da circunf. inscrita

As coord. do incentro são:

$$I = \frac{aA + bB + cC}{a+b+c}$$

onde a, b e c são as medidas dos lados opostos aos vértices A, B e C respect.

$$a = d(B,C) = \sqrt{(4-6)^2+(6+2)^2} = \sqrt{4+64} = \sqrt{68}$$
$$= 2\sqrt{17} \quad \therefore \quad \boxed{a = 2\sqrt{17}}$$

$$b = d(A,C) = \sqrt{(4-3)^2+(6-4)^2} = \sqrt{1+4} = \sqrt{5}$$
$$\boxed{b = \sqrt{5}}$$

$$c = d(A,B) = \sqrt{(6-3)^2+(-2-4)^2} = \sqrt{9+36} = \sqrt{45} = \boxed{3\sqrt{5}}$$

$$I = \left(\underbrace{\frac{2\sqrt{17}\cdot 3 + \sqrt{5}\cdot 6 + 3\sqrt{5}\cdot 4}{2\sqrt{17}+\sqrt{5}+3\sqrt{5}}}_{I_x} , \underbrace{\frac{2\sqrt{17}\cdot 4 + \sqrt{5}\cdot(-2) + 3\sqrt{5}\cdot 6}{2\sqrt{17}+\sqrt{5}+3\sqrt{5}}}_{I_y} \right)$$

$$I_x = \frac{6\sqrt{17}+18\sqrt{5}}{2\sqrt{17}+4\sqrt{5}} = \frac{3(\sqrt{17}+3\sqrt{5})}{\sqrt{17}+2\sqrt{5}}$$

$$I_y = \frac{8\sqrt{17}+16\sqrt{5}}{2\sqrt{17}+4\sqrt{5}} = \frac{4(\sqrt{17}+2\sqrt{5})}{\sqrt{17}+2\sqrt{5}} = 4$$

$$I = \left(\frac{3(\sqrt{17}+3\sqrt{5})}{\sqrt{17}+2\sqrt{5}}, 4 \right)$$

$$\text{raio} = d(I, \ell_{AB}) \qquad \ell_{AB}: \begin{cases} x = 3+t \\ y = 4-2t \end{cases}$$

ou $\boxed{2x+y=10}$

$$d(I, \ell_{AB}) = \left| \frac{6(\sqrt{17}+3\sqrt{5})}{\sqrt{17}+2\sqrt{5}} + 4 - 10 \right| \cdot \frac{1}{\sqrt{5}}$$

$$= 6 \left| \frac{\cancel{\sqrt{17}} + 3\sqrt{5} - \cancel{\sqrt{17}} - 2\sqrt{5}}{\sqrt{17}+2\sqrt{5}} \right| \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{6 \cdot \cancel{\sqrt{5}}}{\sqrt{17}+2\sqrt{5}} \cdot \frac{1}{\cancel{\sqrt{5}}} = \frac{6}{\sqrt{17}+2\sqrt{5}}$$

$$d(I, \ell_{AC}) = ? \quad \left(= d(I, \ell_{AB}) \right)$$

$$\ell_{AC} : \begin{cases} x = 3 + t \\ y = 4 + 2t \end{cases} \quad \text{ou} \quad 2x - y = 2$$

$$d(I, \ell_{AC}) = \frac{1}{\sqrt{5}} \cdot \left| \frac{6(\sqrt{17}+3\sqrt{5})}{\sqrt{17}+2\sqrt{5}} - 4 - 2 \right|$$

$$= \frac{6}{\sqrt{5}} \left| \frac{\sqrt{17}+3\sqrt{5}}{\sqrt{17}+2\sqrt{5}} - 1 \right|$$

$$= \ldots = \frac{6}{\sqrt{17}+2\sqrt{5}} = d(I, \ell_{AB})$$

Eq. da circunf.

$$\left(x - \frac{3(\sqrt{17}+3\sqrt{5})}{\sqrt{17}+2\sqrt{5}} \right)^2 + (y-4)^2 = \left(\frac{6}{\sqrt{17}+2\sqrt{5}} \right)^2$$

③ $7x^2 + 24xy - 256x - 192y + 1456 = 0 \dots (*)$

Identificar a cônica, encontrar seus principais elementos e fazer o seu gráfico.

Solução:

$$A = \begin{pmatrix} 7 & 12 \\ 12 & 0 \end{pmatrix}. \quad \det(A - \lambda I) = 0 \Rightarrow -\lambda(7-\lambda) - 144 = 0$$

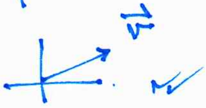
$$\Rightarrow \lambda^2 - 7\lambda - 144 = 0$$

$$\Rightarrow (\lambda - 16)(\lambda + 9) \quad \therefore \boxed{\lambda = 16 \text{ ou } \lambda = -9}$$

$\lambda = 16$

$$\begin{pmatrix} -9 & 12 \\ 12 & -16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -9x + 12y = 0 \\ 12x - 16y = 0 \end{cases} \Rightarrow \boxed{3x - 4y = 0}$$

$$\vec{v} = (1, 3/4)$$



$$\therefore \boxed{\lambda_1 = 16}$$

$\lambda_1 = 16, \quad \vec{v}_1 = (4/5\,;\,3/5)$ $\qquad B' = \{\vec{v}_1, \vec{v}_2\}$

$\lambda_2 = -9, \quad \vec{v}_2 = (-3/5, 4/5)$ $\qquad B = \{\vec{i}, \vec{j}\}$

$$[I]_B^{B'} = \begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix} \qquad \boxed{[I]_B^{B'} \cdot \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\begin{cases} x = \frac{1}{5}(4\bar{x} - 3\bar{y}) \\ y = \frac{1}{5}(3\bar{x} + 4\bar{y}) \end{cases}$$

A cônica (*) se transforma em:

$$16\bar{x}^2 - 9\bar{y}^2 - 256 \cdot \frac{1}{5}(4\bar{x} - 3\bar{y}) - 192 \cdot \frac{1}{5}(3\bar{x} + 4\bar{y}) + 1456 = 0$$

$$16\bar{x}^2 - 9\bar{y}^2 - \frac{4 \cdot 256 + 3 \cdot 192}{5}\bar{x} + \cancel{\frac{3 \cdot 256 - 4 \cdot 192}{5}}^{\,0}\,\bar{y} + 1456 = 0$$

$$16\bar{x}^2 - 9\bar{y}^2 - 320\bar{x} + 1456 = 0$$

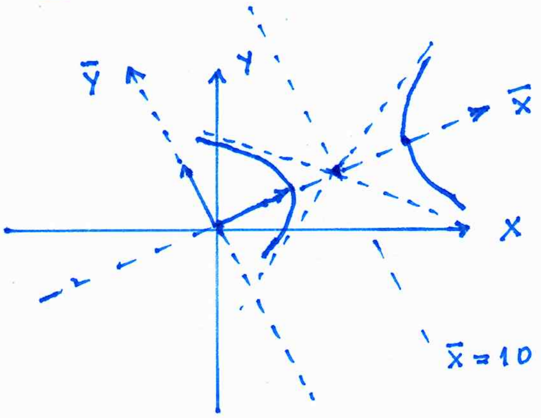
$$16(\bar{x}^2 - 20\bar{x}) - 9\bar{y}^2 + 1456 = 0$$

$$16(\bar{x}-10)^2 - 1600 - 9\bar{y}^2 + 1456 = 0$$

$$16(\bar{x}-10)^2 - 9\bar{y}^2 = 144$$

$$\boxed{\frac{(\bar{x}-10)^2}{9} - \frac{\bar{y}^2}{16} = 1}$$

<u>Hipérbole</u>



Centro: $\boxed{\bar{x}, \bar{y}}$

$(10,0)$

Focos: $(5,0)$, $(15,0)$

Vértices $(7,0)$, $(13,0)$

$$\left.\begin{array}{l} a = 3 \\ b = 4 \end{array}\right\} \Rightarrow c = 5 \quad (c^2 = a^2 + b^2)$$

etc

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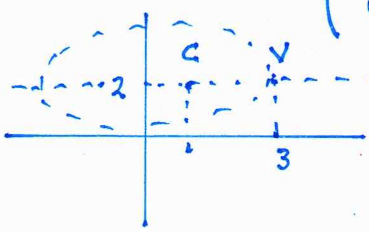
4 Eq. da cônica

(a) <u>Elipse</u>

$$\begin{cases} \text{Centro } C = (1,2) \\ \text{Um vértice na reta focal, } V = (3,2) \\ e = 1/2 \quad (\text{excentricidade}) \end{cases}$$



A eq. tem a forma: $\dfrac{(x-1)^2}{a^2} + \dfrac{(y-2)^2}{b^2} = 1$

$$d(V,C) = \sqrt{(3-1)^2 + (2-2)^2} = \sqrt{4} = 2 \quad \boxed{a = 2}$$

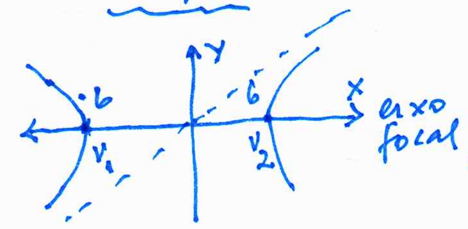
$$\frac{1}{2} = e = c/a = \frac{c}{2} \Longrightarrow \boxed{c = 1}$$

$$b^2 = a^2 - c^2 \Longrightarrow b^2 = 4 - 1 = 3 \quad \therefore \quad \boxed{b = \sqrt{3}}$$

Eq.: $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{3} = 1$ //

(b) Hipérbole: $\begin{cases} \text{Vértices: } V_1 = (-6,0), \quad V_2 = (6,0) \\ \text{Assíntotas: } y = \pm 7/6\, x \end{cases}$

Solução



$y = \frac{7}{6} x = \frac{b}{a} x$

centro: $(0,0)$

Eq. do tipo $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\boxed{\frac{b}{a} = 7/6}$

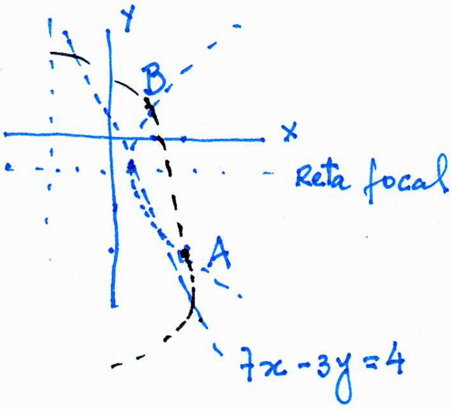
$a = d(\text{Centro}, \text{Vértice}) = 6$

$a = 6 \qquad \frac{b}{a} = \frac{7}{6} \Rightarrow \boxed{b = 7}$

Eq.: $\boxed{\frac{x^2}{36} - \frac{y^2}{49} = 1}$

(c) Parábola: $\begin{cases} \text{Vértice na reta } 7x + 3y = 4 \\ \text{reta focal paralela ao eixo } X \\ \text{passa nos ptos } A = (3,-5) \text{ e } B = (3/2, 1) \end{cases}$

Solução



Eq. da forma

$$(y - y_0)^2 = 4p(x - x_0)$$

com $7x_0 + 3y_0 = 4$ e

$$\begin{cases} (-5 - y_0)^2 = 4p(3 - x_0) \\ (1 - y_0)^2 = 4p(3/2 - x_0) \end{cases}$$

$$\left.\begin{aligned} (5 + y_0)^2 &= 12p - 4px_0 \\ (1 - y_0)^2 &= 6p - 4px_0 \end{aligned}\right\} \Rightarrow (5 + y_0)^2 - (1 - y_0)^2 = 6p$$

$$25 + 10y_0 + \cancel{y_0^2} - (1 - 2y_0 + \cancel{y_0^2}) = 6p$$

$$12y_0 + 24 = 6p \qquad \boxed{p = 2y_0 + 4}$$

$$\therefore \quad \boxed{y_0 = \frac{1}{2}p - 2}$$

$$7x_0 + 3y_0 = 4 \Rightarrow 7x_0 + \frac{3}{2}p - 6 = 4$$

$$\Rightarrow x_0 = \frac{10 - 3/2\, p}{7} = \frac{10}{7} - \frac{3}{14}p$$

$$\boxed{x_0 = \frac{10}{7} - \frac{3}{14}p}$$

$$(1 - y_0)^2 = 4p\left(\frac{3}{2} - x_0\right) \Rightarrow \left(1 - \frac{1}{2}p + 2\right)^2 = 4p\left(3/2 - \frac{10}{7} + \frac{3}{14}p\right)$$

$$\Rightarrow \left(3 - \frac{1}{2}p\right)^2 = 4p\left(\frac{1}{14} + \frac{3}{14}p\right) = \frac{2}{7}p(1 + 3p)$$

$$\Rightarrow \frac{1}{4}(6 - p)^2 = \frac{2}{7}p(1 + 3p)$$

$$\Rightarrow (6 - p)^2 = \frac{8}{7}p + \frac{24}{7}p^2$$

$$\Rightarrow 36 - 12p + p^2 - \frac{24}{7}p^2 - \frac{8}{7}p = 0$$

$$\Rightarrow -\frac{17}{7}p^2 - \frac{92p}{7} + 36 = 0$$

$$\Rightarrow 17p^2 + 92p - 252 = 0$$

$$\Rightarrow (p - 2)(17p + 126) = 0 \quad \therefore \quad p = 2 \text{ ou } p = \frac{-126}{17}$$

$p = 2$

$$y_0 = \frac{1}{2}p - 2 \Rightarrow \boxed{y_0 = -1}$$

$$x_0 = \frac{10}{7} - \frac{3}{14}p \Rightarrow \boxed{x_0 = 1}$$

Neste caso, a eq. da parábola é $(y+1)^2 = 8(x-1)$ //

$$\boxed{p = \frac{-126}{17}}$$

$$x_0 = \frac{10}{7} - \frac{3}{\cancel{14}}\left(\frac{-\cancel{126}^{\,9}}{17}\right) = \frac{10}{7} + \frac{27}{17} = \frac{359}{119}$$

$$\boxed{x_0 = \frac{359}{119}}$$

$$y_0 = \frac{1}{2}\left(\frac{-126}{17}\right) - 2 = \frac{-126}{34} - \frac{68}{34} = -\frac{\cancel{194}^{\,97}}{\cancel{34}_{\,17}}$$

$$\boxed{y_0 = \frac{-97}{17}}$$

Eq. da Parábola

$$\left(y + \frac{97}{17}\right)^2 = \frac{-126}{17} \cdot 4\left(x - \frac{359}{119}\right)$$

etc.