Universidade Federal da Paraíba

Centro de Ciências Exatas e da Natureza

Departamento de Matemática

Mestrado Profissional em Matemática em Rede Nacional

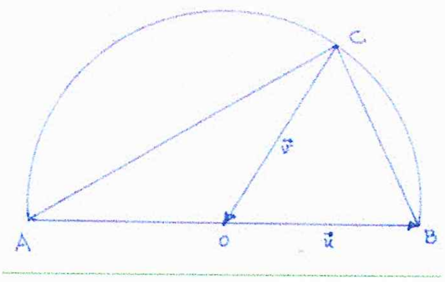
PROFMAT

2014.2

MA23 - Geometria Analítica - Avaliação Local 1

Nome: ______________________ Matrícula: ____________

1 *Na figura abaixo o triângulo ABC está inscrito no semi-círculo de centro O. Usando os vetores*
$\vec{u} = \overrightarrow{OB}$ *e* $\vec{v} = \overrightarrow{CO}$ *mostre que tal triângulo é retângulo.*



2 *Sejam* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *a transformação de* $\mathbb{R}^2$ *dada por* $T(x,y) = (2x - y + 3, 2x + y - 1)$ *e* C *o círculo de equação* $x^2 + y^2 = 1$. *Determine a imagem de* C *por* T.

3 *As retas de equações* $r_1 : x + y = -1$, $r_2 : \begin{cases} x = \frac{9}{4} + t \\ y = \frac{1}{2} - 2t \end{cases}$ *e* $r_3 : \begin{cases} x = \frac{9}{4} + 3s \\ y = \frac{1}{2} - 2s \end{cases}$ *se intersetam duas a duas formando um triângulo. Determine a equação da circunferência circunscrita a esse triângulo.*

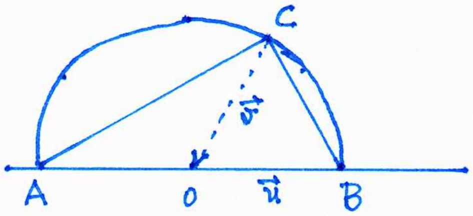
4 *Classifique a cônica de equação* $8x^2 - 2xy + 8y^2 - 46x - 10y + 11 = 0$ *indicando centro, focos, eixos, etc.*

5 *A região* $\mathcal{R}$ *contém o ponto* $(1,0)$ *e é limitada pelo círculo* $x^2 + (y-1)^2 = 1$ *e pelas retas* $y = x$, $y = -x$ *e* $x = 2$. *Descreva a região* $\mathcal{R}$ *como uma reunião de regiões da forma*
$\begin{cases} \rho_1(\theta) \le \rho \le \rho_2(\theta) \\ \theta_1 \le \theta \le \theta_2 \end{cases}$

Boa Prova !!

MA23 Prova 1 - 2014.2

(1)



$\vec{u} = \overrightarrow{OB}$
$\vec{v} = \overrightarrow{CO}$

ΔABC inscrito no semi-círculo de centro O

usando os vetores $\vec{u}$ e $\vec{v}$ provar que ΔABC é retângulo

Solução:

Basta mostrar que $\overrightarrow{CA} \perp \overrightarrow{CB}$, ou seja, $\langle \overrightarrow{CA}, \overrightarrow{CB} \rangle = 0$.

$\overrightarrow{CA} = \vec{v} + \overrightarrow{OA} = \vec{v} - \vec{u} \quad (\overrightarrow{OA} = -\vec{u})$
$\overrightarrow{CB} = \vec{v} + \vec{u}$.

Além disso, $\|\vec{v}\| = \|\vec{u}\|$.

Daí

$$\begin{aligned}\langle \overrightarrow{CA}, \overrightarrow{CB} \rangle &= \langle \vec{v} - \vec{u}, \vec{v} + \vec{u} \rangle \\ &= |\vec{v}|^2 + \langle \vec{v}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - |u|^2 \\ &= 0\end{aligned}$$

$\therefore$ $\overrightarrow{CA} \perp \overrightarrow{CB}$ e ΔABC é retângulo.

(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (2x - y + 3, 2x + y - 1)$

$\mathcal{C}: x^2 + y^2 = 1$.

Determinar $T(\mathcal{C})$.

Solução

$T(x,y) = (x', y')$

$$\left.\begin{aligned} x' &= 2x - y + 3 \\ y' &= 2x + y - 1 \end{aligned}\right\} \Rightarrow \begin{cases} x' - 3 = 2x - y \\ y' + 1 = 2x + y \end{cases}$$

$$\begin{cases} x = \frac{1}{4}(x'+y'-2) \\ y = \frac{1}{2}(y'-x'+4) \end{cases}$$

$$x^2+y^2=1 \Rightarrow \frac{1}{16}(x'+y'-2)^2+\frac{1}{4}(y'-x'+4)^2=1$$

$$\Rightarrow (x'+y'-2)^2+4(y'-x'+4)^2=16$$

$$\Rightarrow x'^2+y'^2+4+2x'y'-4x'-4y'+4x'^2+4y'^2+64-8x'y'$$
$$+32y'-32x'+\cancel{64}=16$$

$$\Rightarrow \underline{5x'^2+5y'^2-6x'y'-36x'+28y'=-52}$$

$$\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

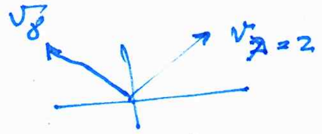
$$(5-\lambda)^2-9=0 \qquad \lambda = 5\pm 3$$
$$5-\lambda=\pm 3 \qquad \underline{\lambda=8 \text{ ou } \lambda=2}$$

$\underline{\lambda=8}$

$$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3x+3y=0$$
$$\Rightarrow \underline{y=-x}$$

$$\underline{v_8=(1,-1)}$$

$\underline{\lambda=2}$ $\quad v_{\lambda=2}=(-1,-1) \parallel (1,1)$



$$\lambda_1=2 \qquad v_1=(1/\sqrt{2},\,1/\sqrt{2})$$
$$\lambda_2=8 \qquad v_2=(-1/\sqrt{2},\,1/\sqrt{2})$$

$$\begin{cases} x' = 1/\sqrt{2}\,(\bar{x}-\bar{y}) \\ y' = 1/\sqrt{2}\,(\bar{x}+\bar{y}) \end{cases}$$

$$2\bar{x}^2 + 8\bar{y}^2 - 36/\sqrt{2}\,(\bar{x}-\bar{y}) + 28/\sqrt{2}\,(\bar{x}+\bar{y}) + 52 = 0$$

$$2\bar{x}^2 + 8\bar{y}^2 - 4\sqrt{2}\,\bar{x} + 32\sqrt{2}\,\bar{y} + 52 = 0$$

$$2(\bar{x}-\sqrt{2})^2 - 4 + 8(\bar{y}+2\sqrt{2})^2 - 64 + 52 = 0$$

$$2(\bar{x}-\sqrt{2})^2 + 8(\bar{y}+2\sqrt{2})^2 - 16 = 0$$

$$\underbrace{\frac{(\bar{x}-\sqrt{2})^2}{8} + \frac{(\bar{y}+2\sqrt{2})^2}{2} = 1}$$

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(3) $r_1: x+y=-1$, $r_2: \begin{cases} x = 9/4 + t \\ y = 1/2 - 2t \end{cases}$, $r_3: \begin{cases} x = 9/4 + 3s \\ y = 1/2 - 2s \end{cases}$

r_1, r_2 e r_3 se cortam formando um triângulo. Determinar a eq. da circunferência circunscrita a esse Δ.

Solução

$r_1 \cap r_2$ $\quad \frac{9}{4} + t + \frac{1}{2} - 2t = -1$

$\Rightarrow \frac{11}{4} - t = -1$ $\quad \underbrace{t = \frac{15}{4}}$

$x = \frac{9}{4} + \frac{15}{4} = \frac{24}{4} = 6$

$y = \frac{1}{2} - \frac{30}{4} = \frac{-28}{4} = -7$

$$\underbrace{r_1 \cap r_2 = \{(6,-7)\}}$$

$r_2 \cap r_3$ $\quad \begin{aligned} 9/4 + t &= 9/4 + 3s \\ 1/2 - 2t &= 1/2 - 2s \end{aligned}$! $\quad \{(9/4, 1/2)\} = r_2 \cap r_3$

$r_1 \cap r_3$ $\quad \frac{9}{4} + 3s + \frac{1}{2} - 2s = -1 \Rightarrow s + \frac{11}{4} = -1$

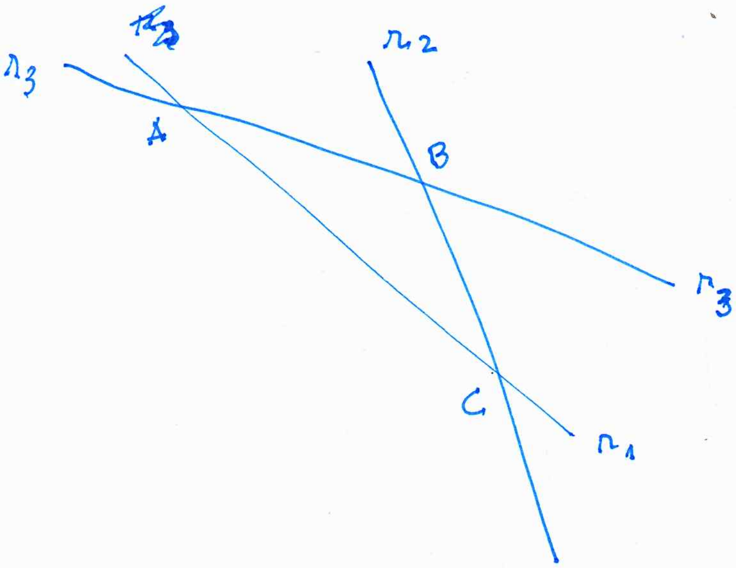
$\Rightarrow \underbrace{s = -\frac{15}{4}}$

$x = 9/4 + 3\left(\frac{-15}{4}\right) = \frac{9}{4} - \frac{45}{4} = \frac{-36}{4} = -9$

$\boxed{x=-9}$

$y = \frac{1}{2} - 2\left(\frac{-15}{4}\right) = \frac{1}{2} + \frac{15}{2} = \underline{8}$

$\boxed{y = 8}$ $\quad \underbrace{r_1 \cap r_3 = \{(-9, 8)\}}$



$A = (-9, 8) \;:\; r_1 \cap r_3$

$B = (9/4, 1/2) \;:\; r_2 \cap r_3$

$C = (6, -7) \;:\; r_1 \cap r_2$

Seja $O = (x, y)$ o centro da circunf. Temos

$$(x+9)^2 + (y-8)^2 \underset{(1)}{=} (x - 9/4)^2 + (y - 1/2)^2 \underset{(2)}{=} (x-6)^2 + (y+7)^2$$

$$\Rightarrow (1) \quad 18x + 81 \cancel{+} 16y + 64 = -\frac{9}{2}x + \frac{81}{16} - y + 1/4$$

$$\Rightarrow \frac{45}{2}x - 15y = \frac{-60}{16} = \frac{-15}{4} \qquad \boxed{\frac{45}{2}x - 15y = \frac{-15}{4}}$$

$$(2) \quad 18x + 81 - 16y + 64 = -12x + 36 + 14y + 49$$

$$30x - 30y = 85 - 145 = 60 \qquad \boxed{x - y = 2}$$

$$\frac{45}{2}x - 15(x-2) = \frac{-15}{4} \Rightarrow \frac{45-30}{2}x + 30 = \frac{-15}{4}$$

$$\frac{15}{2}x + 30 = \frac{-15}{4} \qquad \frac{15}{2}x = -\frac{15}{4} - 30 = \frac{-135}{4}$$

$$\Rightarrow x = -\frac{\cancel{2}}{\cancel{15}} \cdot \frac{\overset{9}{\cancel{135}}}{\cancel{4}_2} = -9/2 \qquad \boxed{x = -9/2}$$

$$y = x - 2 = -\frac{9}{2} - 2 = \frac{-13}{2}$$

$$\boxed{y = \frac{-13}{2}}$$

$$O = \left(-9/2, -13/2\right)$$

$$r^2 = \left(-9/2 - 9/4\right)^2 + \left(\frac{-13}{2} - \frac{1}{2}\right)^2 = \left(\frac{27}{4}\right)^2 + \left(\frac{14}{4}\right)^2$$

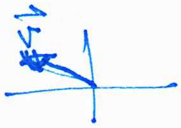
$$= \frac{729 + 196}{16} = \frac{955}{16} = \qquad \text{etc}$$

(4) $$8x^2-2xy+8y^2-46x-10y+11=0$$

$$A=\begin{pmatrix}8 & -1\\ -1 & 8\end{pmatrix}, \quad \det(A-\lambda I)=0 \Rightarrow \cancel{(8-\lambda)^2-1=0}$$

$$\Rightarrow (8-\lambda)^2-1=0 \Rightarrow \underline{\lambda=9 \text{ ou } \lambda=7}$$

$\underline{\lambda=9}$ $$\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow x+y=0 \quad v=(1,-1)$$



$\underline{\lambda_2=9}$ $$\vec{v}_2=\frac{1}{\sqrt{2}}(-1,1)$$

$\lambda_1=7$ $$\vec{v}_1=\frac{1}{\sqrt{2}}(1,1)$$

$B'=\{\vec{v}_1,\vec{v}_2\}$, B a base canônica do $\mathbb{R}^2$

$$[I]_B^{B'}=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \qquad [I]_B^{B'}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{cases} x=\frac{1}{\sqrt{2}}(\bar{x}-\bar{y})\\ y=\frac{1}{\sqrt{2}}(\bar{x}+\bar{y})\end{cases}$$

$$7\bar{x}^2+9\bar{y}^2-\frac{46}{\sqrt{2}}(\bar{x}-\bar{y})-\frac{10}{\sqrt{2}}(\bar{x}+\bar{y})+11=0$$

$$7\bar{x}^2+9\bar{y}^2-\frac{56}{\sqrt{2}}\bar{x}-\frac{36}{\sqrt{2}}\bar{y}+11=0$$

$$7\left(\bar{x}^2-\frac{8}{\sqrt{2}}\bar{x}\right)+9\left(\bar{y}^2-\frac{4}{\sqrt{2}}\bar{y}\right)+11=0$$

$$8/\sqrt{2}=4\sqrt{2} \qquad 4/\sqrt{2}=2\sqrt{2}$$

$$7(\bar{x}-2\sqrt{2})^2-56+9(\bar{y}-\sqrt{2})^2-18+11=0$$

$$7(\bar{x}-2\sqrt{2})^2+9(\bar{y}-\sqrt{2})^2=63$$

$$\frac{(\bar{x}-2\sqrt{2})^2}{9}+\frac{(\bar{y}-\sqrt{2})^2}{7}=1$$

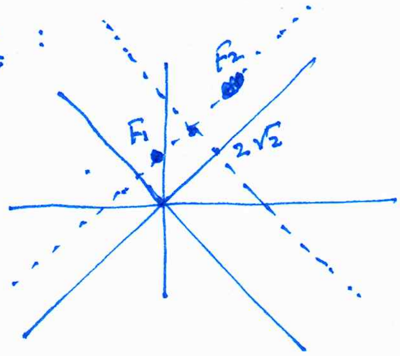
centro (coord. $(\bar{x},\bar{y})$)

$(2\sqrt{2},\sqrt{2})$

$\bar{a}=3$, $\bar{b}=\sqrt{7}$ $\qquad \bar{a}^2=\bar{b}^2+\bar{c}^2 \Rightarrow 9=7+\bar{c}^2$

$\boxed{\bar{c}=\sqrt{2}}$ $\qquad = \boxed{\bar{c}^2=2}$

Focos :



etc

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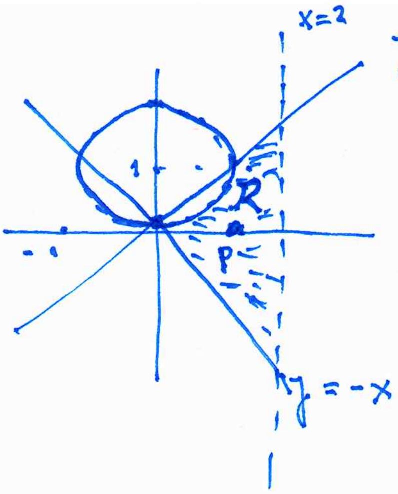

$P=(1,0)$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}$$

$y = x \Rightarrow \theta = \pi/4$

$y = -x \Rightarrow \theta = -\pi/4$

$$x^2+(y-1)^2 \geq 1 \qquad (P\in R)$$

$$\rho^2\cos^2\theta + (\rho\,\mathrm{sen}\,\theta - 1)^2 \geq 1$$

$$\Rightarrow \rho^2\cos^2\theta + \rho^2\mathrm{sen}^2\theta - 2\rho\,\mathrm{sen}\,\theta + \cancel{1} \geq \cancel{1}$$

$$1 - 2\rho\,\mathrm{sen}\,\theta \geq 0$$

$0 \leq \theta \leq \pi/4$

$$\rho \geq \frac{1}{2\,\mathrm{sen}\,\theta}$$

$\cancel{0\leq\theta}$ $-\pi/4 \leq \theta \leq 0$ $(\mathrm{sen}\,\theta < 0)$

$$\rho \leq \frac{1}{2\,\mathrm{sen}\,\theta}$$

$x \leq 2$

$$\rho\cos\theta \leq 2 \qquad \rho \leq 2/\cos\theta.$$

$0 \leq \theta \leq \pi/4$ $\qquad$ $-\pi/4 \leq \theta \leq 0$

$\frac{1}{2\,\mathrm{sen}\,\theta} \leq \rho \leq \frac{2}{\cos\theta}$ $\qquad$ $0 \leq \rho \leq 2/\cos\theta$