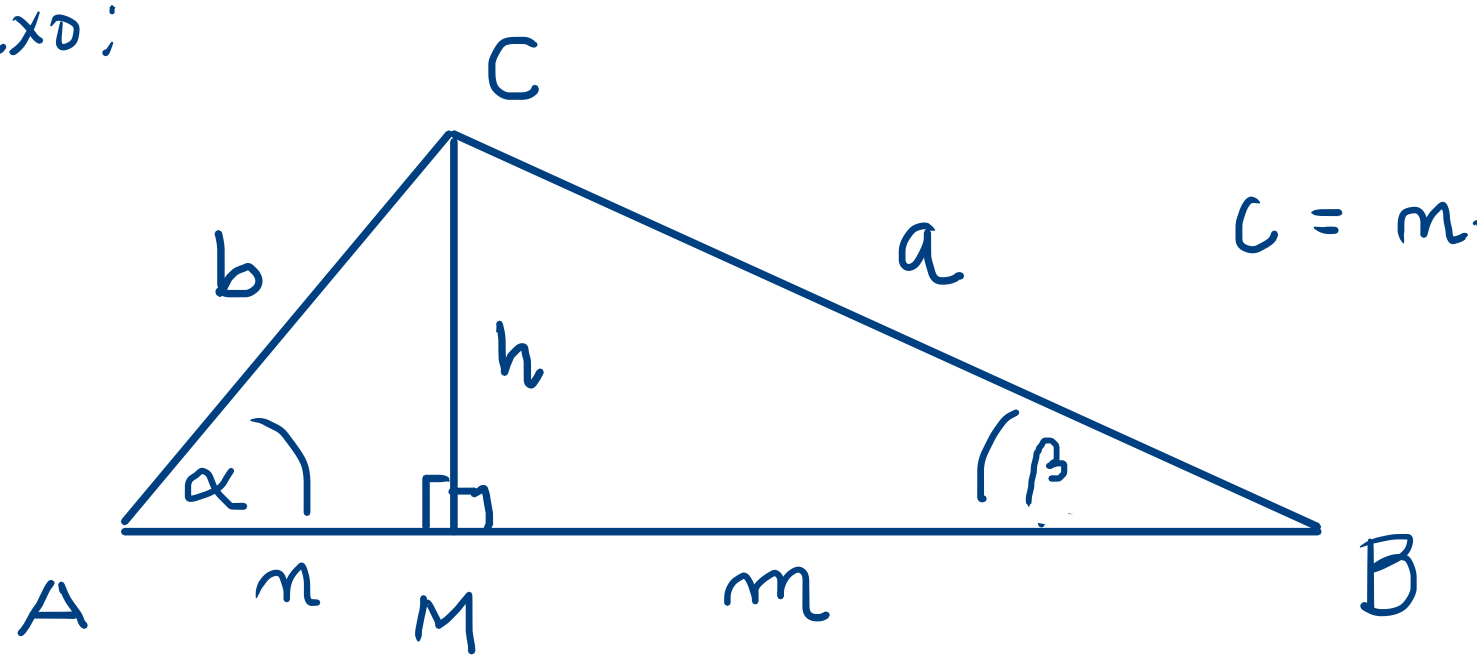
A lei dos senos

Consideremos um ΔABC qualquer, como na figura abaixo:



$c = n + m$

CM = altura baixada de C sobre o lado AB

Temos:

$\operatorname{sen}\alpha = h/b \quad \therefore \{h = b\operatorname{sen}\alpha\}$

$\operatorname{sen}\beta = h/a \quad \therefore \{h = a\operatorname{sen}\beta\}$

Daí,

$$b\operatorname{sen}\alpha = a\operatorname{sen}\beta$$

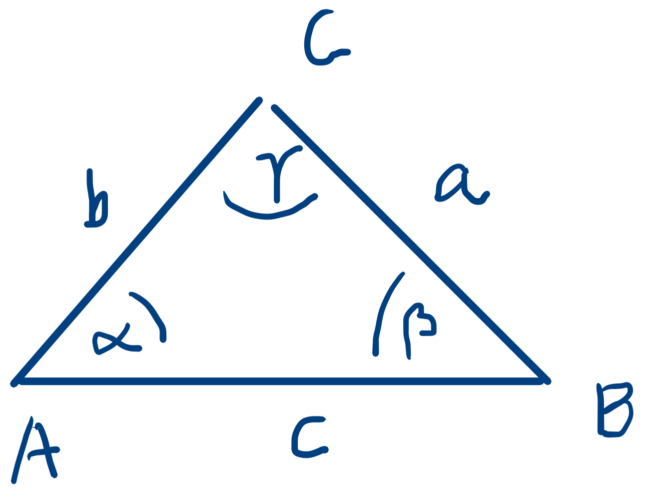
donde

$$\frac{a}{\operatorname{sen}\alpha} = \frac{b}{\operatorname{sen}\beta}$$

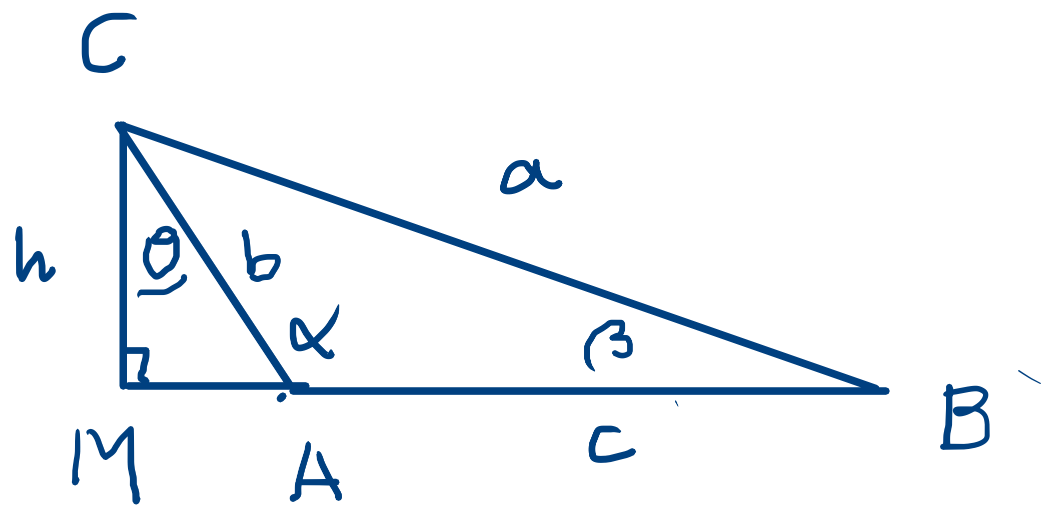
Analogamente: $\dfrac{a}{\operatorname{sen}\alpha} = \dfrac{c}{\operatorname{sen}\gamma}$ $\quad (\gamma = \measuredangle ACB)$

Assim obtemos:

$$\left\{ \frac{a}{\operatorname{sen}\alpha} = \frac{b}{\operatorname{sen}\beta} = \frac{c}{\operatorname{sen}\gamma} \right.$$



No caso de um Δ obtusângulo a situação é a "mesma" (basta lembrar que $\operatorname{sen}(90+\theta) = \cos\theta$)



observe que $\{\alpha = 90° + \theta\}$

$\operatorname{sen}\alpha = \operatorname{sen}(90° + \theta) = \cos\theta = h/b$

$\therefore \{h = b\operatorname{sen}\alpha\}$

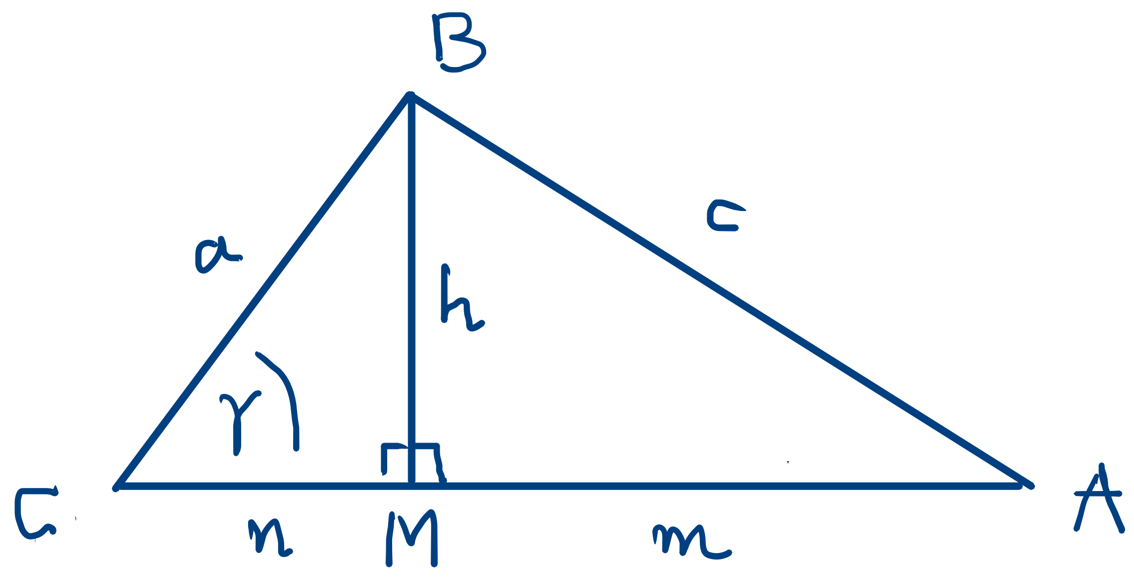
$\operatorname{sen}\beta = h/a, \quad \therefore \{h = a\operatorname{sen}\beta\}$

Daí $\left\{ \dfrac{a}{\operatorname{sen}\alpha} = \dfrac{b}{\operatorname{sen}\beta} \right\}$

etc...

Lei dos cossenos

Como antes, consideremos um ΔABC qualquer.



$b = n + m$

$\overline{BM}$ altura

$\cos\gamma = n/a \quad \therefore \quad \{n = a\cos\gamma\}$

Pitágoras: $h^2 = a^2 - n^2$ e $h^2 = c^2 - m^2$

Daí temos: $a^2 - n^2 = c^2 - m^2$ e como $m = b - n$,

obtemos $a^2 - n^2 = c^2 - (b-n)^2 = c^2 - b^2 + 2bn - n^2$

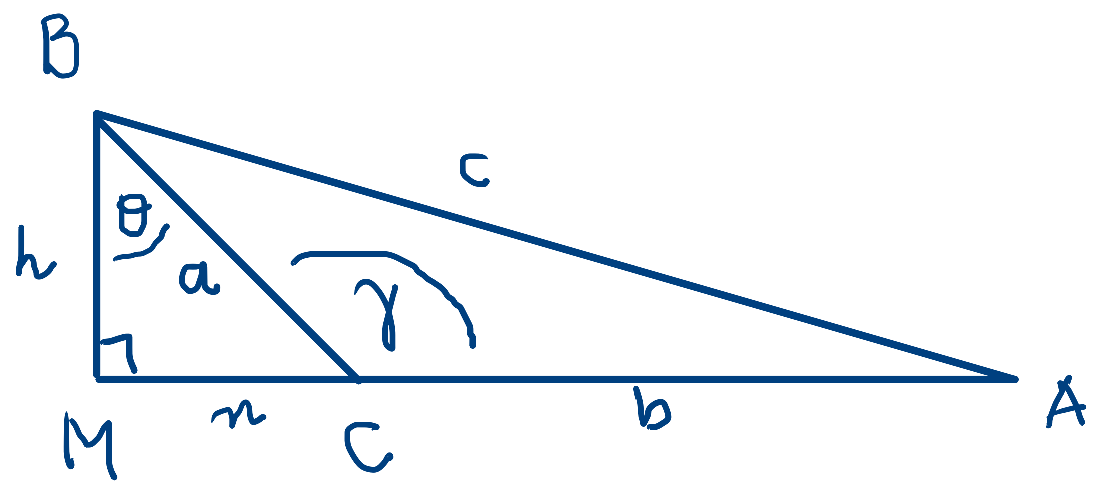
Logo, $a^2 = c^2 - b^2 + 2bn$, mas $n = a\cos\gamma$

portanto $a^2 = c^2 - b^2 + 2ab\cos\gamma$.

Daí:

$$\{c^2 = a^2 + b^2 - 2ab\cos\gamma\}$$

Se o ΔABC é obtusângulo, a conclusão é a mesma.
Basta lembrar que $\cos(90^\circ + \theta) = -\operatorname{sen}\theta$.



$\gamma = 90^\circ + \theta$ $\quad \cos\gamma = -\operatorname{sen}\theta$ e $\operatorname{sen}\theta = \frac{n}{a}$

$\therefore \cos\gamma = -\frac{n}{a}$ donde, $\{n = -a\cos\gamma\}$

De novo usando Pitágoras temos

$h^2 = a^2 - n^2$ e $h^2 = c^2 - (n+b)^2$

Logo $a^2 - n^2 = c^2 - (n+b)^2$

$= c^2 - n^2 - 2nb - b^2$

$\Rightarrow c^2 = a^2 + b^2 + 2nb$, mas $n = -a\cos\gamma$

Logo,

$$\{c^2 = a^2 + b^2 - 2ab\cos\gamma\}$$