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Prova 2 - Introdução à Álgebra Linear - 03/11/2022

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1 (2,0 pts.) *Considere a aplicação* $T : \mathbb{R}^3 \to \mathbb{R}^2,\ T(x, y, z) = (x + 2y - z, 2x - y + 3z)$.

(a) Mostre que T *é linear;*

(b) Determine $[T]$, *a matriz de* T, *com relação a base canônica de* $\mathbb{R}^3$.

2 (2,0 pts.) *Dê exemplo de uma aplicação linear como a que se pede em cada caso.*

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ *injetiva;*

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ *sobrejetiva.*

3 (2,0 pts.) *Seja* $T : \mathbb{R}^3 \to \mathbb{R}^3$, *a aplicação linear que satisfaz:* $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (-1, 1, 2)$ *e* $T(0, 0, 1) = (2, 0, 1)$.

(a) Determine $T(x, y, z)$;

(b) Encontre $Ker(T)$ *e* $Im(T)$.

4 (2,0 pts.) *Dadas as bases* $B_1 = \{(1, 1), (-1, 1)\}$ *de* $\mathbb{R}^2$ *e* $B_2 = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}$ *de* $\mathbb{R}^3$. *Encontrar a transformação linear* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *tal que* $[T]^{B_1}_{B_2} = \begin{pmatrix} 1 & 1 \\ -3 & 2 \\ 0 & -1 \end{pmatrix}$

5 (3,0 pts.) *Considere a aplicação linear*

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ T(x, y, z) = (x + 2y - 2z, 2x + y - 2z, 2x + 2y - 3z)\,.$$

Determine:

(a) O polinômio Característico de T;

(a) Os autovalores de T;

(a) Uma base para cada subespaço próprio;

Boa Prova !!

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① $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (x+2y-z,\ 2x-y+3z)$

(a) T é linear.

Sejam $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ e $\lambda \in \mathbb{R}$

Vamos provar que

$$T(\lambda\vec{u} + \vec{v}) = \lambda T(\vec{u}) + T(\vec{v}).$$

$$\begin{aligned}
T(\lambda\vec{u}+\vec{v}) &= T\big(\lambda(u_1,u_2,u_3) + (v_1,v_2,v_3)\big) \\
&= T\left(\lambda u_1 + v_1,\ \lambda u_2 + v_2,\ \lambda u_3 + v_3\right) \\
&= \Big((\lambda u_1+v_1) + 2(\lambda u_2+v_2) - (\lambda u_3+v_3),\ 2(\lambda u_1+v_1) - (\lambda u_2+v_2) + \\
&\qquad + 3(\lambda u_3+v_3)\Big) \\
&= \Big(\lambda(u_1+2u_2-u_3) + (v_1+2v_2-v_3),\ \lambda(2u_1-u_2+3u_3) + (2v_1-v_2+3v_3)\Big) \\
&= \lambda\left(u_1+2u_2-u_3,\ 2u_1-u_2+3u_3\right) + \left(v_1+2v_2-v_3,\ 2v_1-v_2+3v_3\right) \\
&= \lambda T(u_1,u_2,u_3) + T(v_1,v_2,v_3) \\
&= \lambda T(\vec{u}) + T(\vec{v})
\end{aligned}$$

$\therefore$ T é linear

(b) Determinar $[T]$, a matriz de T, com relação às bases canônicas de $\mathbb{R}^3$ e $\mathbb{R}^2$.

$$\begin{aligned}
T(1,0,0) &= (1,2) = 1(1,0) + 2(0,1) \\
T(0,1,0) &= (2,-1) = 2(1,0) - 1(0,1) \\
T(0,0,1) &= (-1,3) = -1(1,0) + 3(0,1)
\end{aligned}$$

$$[T] = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

② Exemplos de Aplic. lineares:

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ injetiva

$$T(1,0) = (1,0,0)$$
$$T(0,1) = (0,1,0)$$

$$T(x,y) = x\,T(1,0) + y\,T(0,1) = (x,y,0)$$

$$T(x,y) = (x,y,0)$$

T é injetiva. De fato $(x,y) \in \ker T \Rightarrow T(x,y) = (0,0,0)$

$$\Rightarrow (x,y,0) = (0,0,0)$$
$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

(b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ sobrejetiva

$$T(1,0,0) = (1,0)$$
$$T(0,1,0) = (0,1)$$
$$T(0,0,1) = (0,0)$$

$$Im(T) = span\{T(1,0,0), T(0,1,0), T(0,0,1)\}$$
$$= span\{(1,0),(0,1)\} = \mathbb{R}^2$$

$\therefore$ T é sobrejetiva

$$T(x,y,z) = (x,y)$$

③ $T: \mathbb{R}^3 \to \mathbb{R}^3$ t.q. $\begin{cases} T(1,0,0) = (1,1,0) \\ T(0,1,0) = (-1,1,2) \\ T(0,0,1) = (2,0,1) \end{cases}$

(a) Calcular $T(x,y,z)$.

$$T(x,y,z) = T\big(x(1,0,0) + y(0,1,0) + z(0,0,1)\big)$$
$$= x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$$
$$= x(1,1,0) + y(-1,1,2) + z(2,0,1)$$

$T(x,y,z) = (x-y+2z,\ x+y,\ 2y+z)$

(b) Encontre $\ker(T)$ e $\mathrm{Im}(T)$.

$\ker(T)$

$(x,y,z) \in \ker(T) \iff T(x,y,z) = (0,0,0)$

$$\iff (*) \begin{cases} x - y + 2z = 0 \\ x + y = 0 \\ 2y + z = 0 \end{cases}$$

$$\det \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = 1+4+0-(0+0-1) = 6 \neq 0.$$

A matriz dos coef. do sist $(*)$ tem determinante não nulo, logo o sist $(*)$ tem solução única $x=0,\ y=0,\ z=0$

$\therefore\ \ker(T) = \{(0,0,0)\}.$

$\dim \ker(T) = 0.$

$$\underbrace{\dim \ker(T)}_{0} + \dim \mathrm{Im}(T) = \dim \mathbb{R}^3 = 3$$

$\therefore\ \dim \mathrm{Im}(T) = 3$

Como $\mathrm{Im}(T)$ é subesp. de $\mathbb{R}^3$, temos

$\mathrm{Im}(T) = \mathbb{R}^3$

4) $B_1 = \{(1,1),(-1,1)\}$ base de $\mathbb{R}^2$

$B_2 = \{(1,0,1),(0,1,0),(0,1,1)\}$ base de $\mathbb{R}^3$

Encontrar $T: \mathbb{R}^2 \to \mathbb{R}^3$ linear t.q. $[T]_{B_2}^{B_1} = \begin{pmatrix} 1 & 1 \\ -3 & 2 \\ 0 & -1 \end{pmatrix}$

$$T(1,1) = 1\cdot(1,0,1) - 3(0,1,0) + 0(0,1,1)$$

$$\underbrace{T(1,1) = (1,-3,1)}$$

$$T(-1,1) = 1(1,0,1) + 2(0,1,0) - 1(0,1,1)$$

$$\underbrace{T(-1,1) = (1,1,0)}$$

$$(x,y) = a(1,1) + b(-1,1) \Rightarrow \begin{cases} a-b = x \\ a+b = y \end{cases}$$

$$\therefore\ a = \frac{x+y}{2}\ ,\ b = \frac{y-x}{2}$$

$$\therefore\ (x,y) = \frac{x+y}{2}(1,1) + \frac{y-x}{2}(-1,1)$$

Daí,

$$T(x,y) = \frac{x+y}{2}T(1,1) + \frac{y-x}{2}T(-1,1)$$

$$= \frac{x+y}{2}(1,-3,1) + \frac{y-x}{2}(1,1,0)$$

$$\underbrace{T(x,y) = \left(y, -2x+y, \frac{x+y}{2}\right)}$$

(5) $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x+2y-2z,\ 2x+y-2z,\ 2x+2y-3z)$

Determinar:

(a) O polinômio característico de T

$$p(\lambda) = \det([T] - \lambda I)$$

$$[T] = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$$

$T(1,0,0) = (1,2,2)$
$T(0,1,0) = (2,1,2)$
$T(0,0,1) = (-2,-2,-3)$

$$\det([T]-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 & -2 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -3-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2(-3-\lambda) - 8 - 8 - \left(-4(1-\lambda) - 4(1-\lambda) + 4(-3-\lambda)\right)$$

$$= -(1-\lambda)^2(3+\lambda) - 16 + 8(1-\lambda) + 4(3+\lambda)$$

$$= -(1-\lambda)^2(3+\lambda) - 16 + 8 + 12 - 8\lambda + 4\lambda$$

$$= -(1-\lambda)^2(3+\lambda) + 4 - 4\lambda$$

$$= -(1-\lambda)^2(3+\lambda) + 4(1-\lambda)$$

$$= (1-\lambda)\left(-(1-\lambda)(3+\lambda) + 4\right)$$

$$= (1-\lambda)(\lambda^2 + 2\lambda + 1)$$

$$\boxed{p(\lambda) = (1-\lambda)(1+\lambda)^2}$$

(b) Autovalores de T

Os autovalores de T são raizes da eq. $p(\lambda) = 0$

$\therefore \quad (1-\lambda)(1+\lambda)^2 = 0 \implies \lambda = 1$ ou $\lambda = -1$

(c) Uma base para cada subespaço próprio.

$\lambda = 1$

$S_1 = \{\vec{v} \in \mathbb{R}^3 : T\vec{v} = \vec{v}\}$ $\quad (T\vec{v} = \vec{v} \iff (T - 1\cdot I)\vec{v} = \vec{0})$

$$\begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} 2y - 2z &= 0 \\ 2x - 2z &= 0 \\ 2x + 2y - 4z &= 0 \end{aligned}$$

$$\begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} x - z = 0 \\ y - z = 0 \end{cases}$$

ou $\;\forall z \begin{cases} x = z \\ y = z \end{cases}$

$S_1 = \text{span}\{(1,1,1)\}$

Uma base para S_1 é $\{(1,1,1)\}$

$\lambda = -1$

$$\begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad 2x + 2y - 2z = 0$$

ou

$x + y - z = 0$

$S_{-1}: \; x + y - z = 0$

Uma base p/ S_{-1}: $\{(1,0,1), (0,1,1)\}$