Cálculo III 1ª Prova 2015.1 18/05/2015

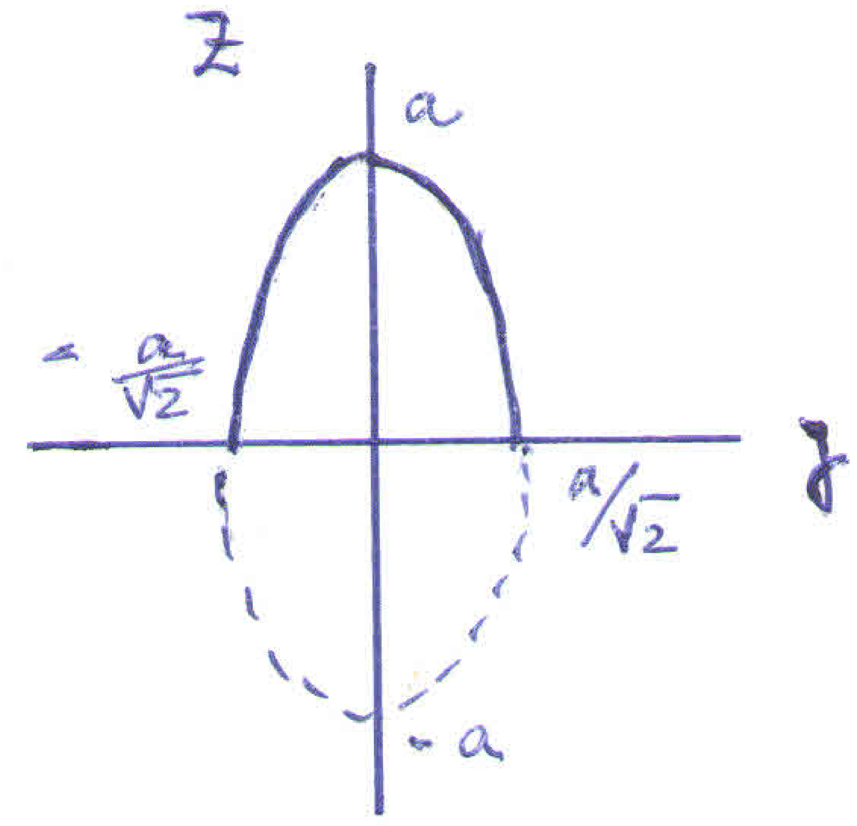
① $C: \begin{cases} x^2+y^2+z^2 = a^2 \\ y = x \\ z \geq 0 \end{cases}$, $a > 0$

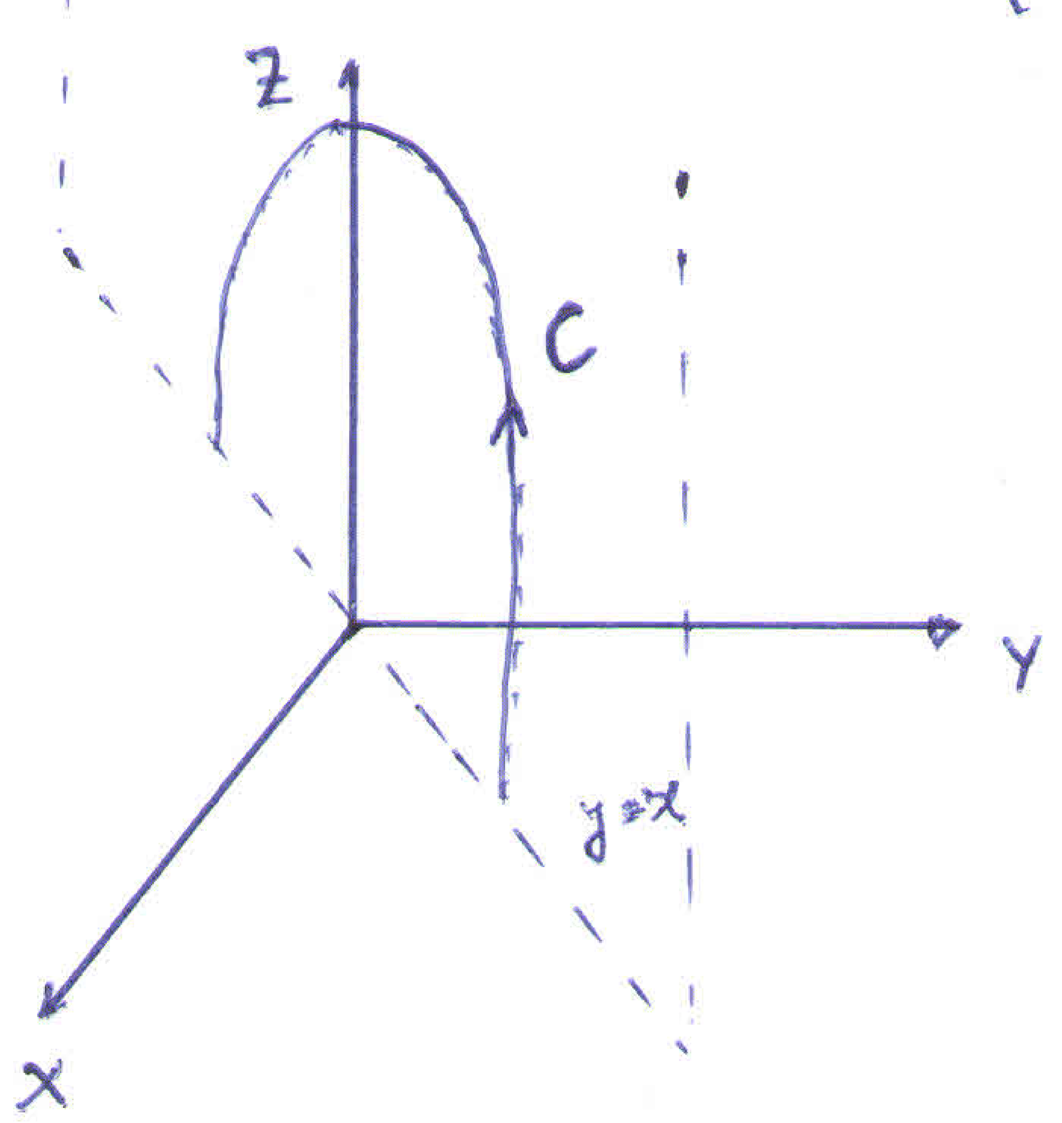
$\int_C xyz\, ds = 8 \Rightarrow \underline{a = ??}$

Solução

$\left.\begin{array}{l} x^2+y^2+z^2 = a^2 \\ y = x \end{array}\right\} \Rightarrow 2y^2 + z^2 = a^2$

$\Rightarrow \left(\dfrac{y}{a/\sqrt{2}}\right)^2 + \left(\dfrac{z}{a}\right)^2 = 1$





Uma parametrização para C é

$\begin{cases} y = \dfrac{a}{\sqrt{2}} \cos\theta \\ z = a \operatorname{sen}\theta \\ x = y = \dfrac{a}{\sqrt{2}} \cos\theta \end{cases}$ $\theta \in [0, \pi]$

$\gamma : [0,\pi] \longrightarrow \mathbb{R}^3, \; \gamma(\theta) = \left(\dfrac{a}{\sqrt{2}}\cos\theta, \dfrac{a}{\sqrt{2}}\cos\theta, a\operatorname{sen}\theta\right)$

$\gamma'(\theta) = \left(-\dfrac{a}{\sqrt{2}}\operatorname{sen}\theta, -\dfrac{a}{\sqrt{2}}\operatorname{sen}\theta, a\cos\theta\right)$

$|\gamma'(\theta)|^2 = \dfrac{a^2}{2}\operatorname{sen}^2\theta + \dfrac{a^2}{2}\operatorname{sen}^2\theta + a^2\cos^2\theta = a^2(\operatorname{sen}^2\theta + \cos^2\theta)$

$= a^2 \quad \therefore \quad \underline{|\gamma'(\theta)| = a}$

$xyz = \dfrac{a^3}{2}\cos^2\theta \operatorname{sen}\theta .$

$$\int_C xyz\,ds = \int_0^{\pi} \frac{a^3}{2}\cos^2\theta \,\text{sen}\,\theta \cdot a\,d\theta = \frac{a^4}{2}\int_0^{\pi} \cos^2\theta \cdot \text{sen}\,\theta\,d\theta$$

$$= -\frac{a^4}{2}\,\frac{\cos^3\theta}{3}\Big|_0^{\pi} = -\frac{a^4}{6}(-1-1) = \frac{a^4}{3}$$

$$\int_C xyz\,ds = 8 \Rightarrow \frac{a^4}{3} = 8 \quad \therefore \quad a^4 = 24$$

$$a = \sqrt[4]{24}$$

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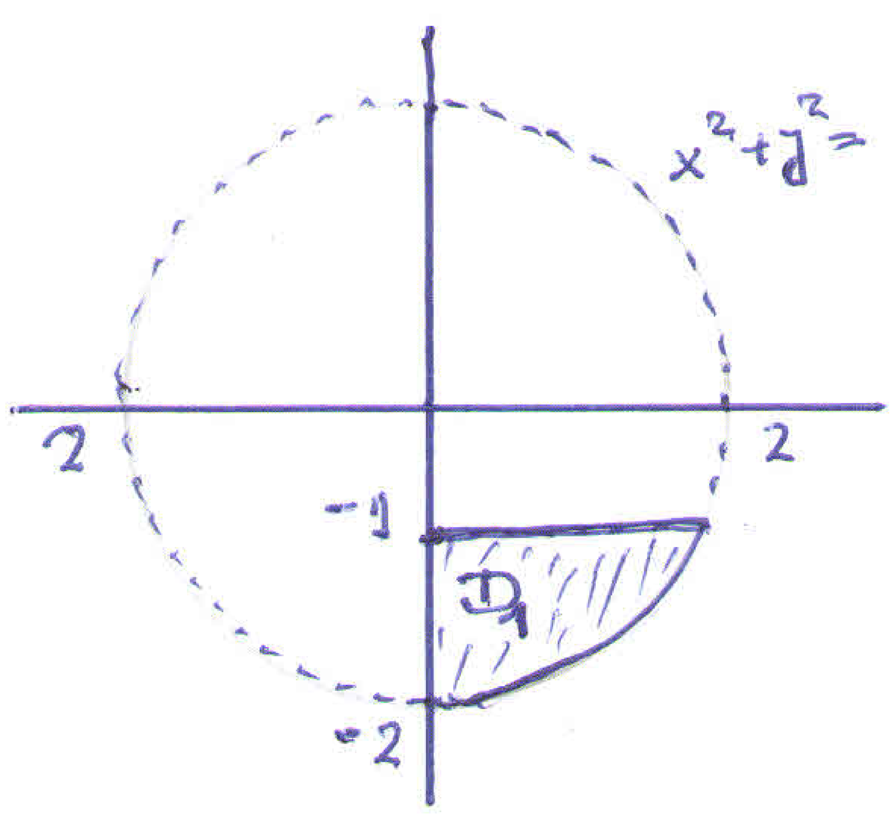
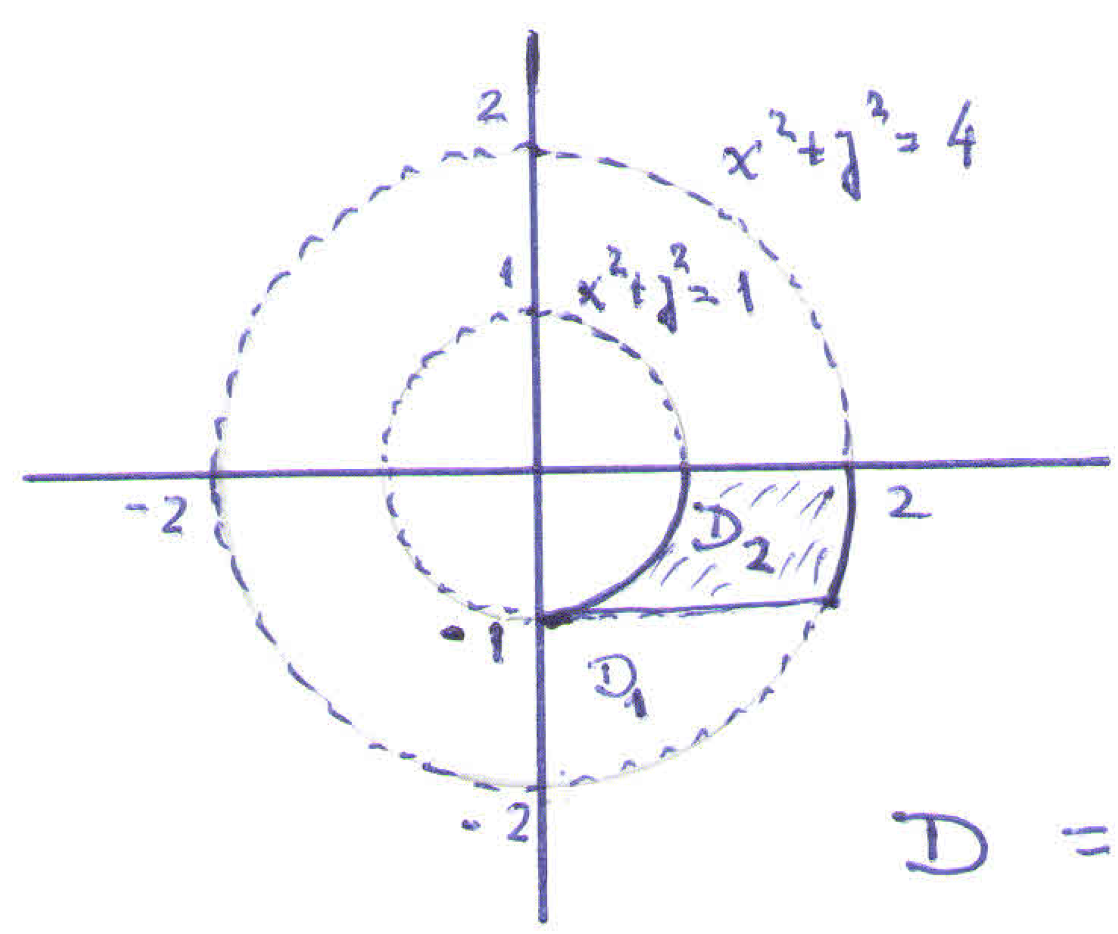
(2) W sólido acima de uma região D do plano XY

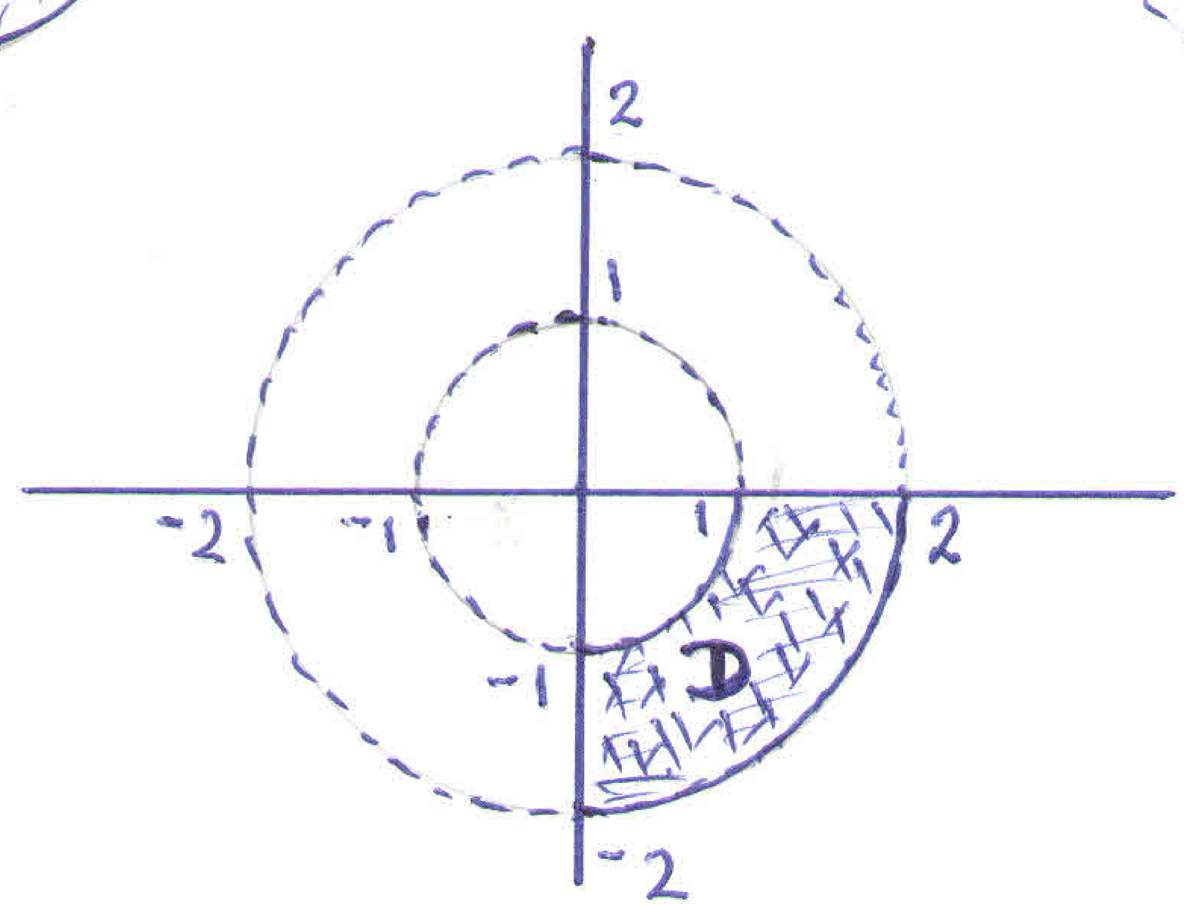
$$\text{vol}(W) = \int_{-2}^{-1}\int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy + \int_{-1}^{0}\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$

a) Esboce a região D



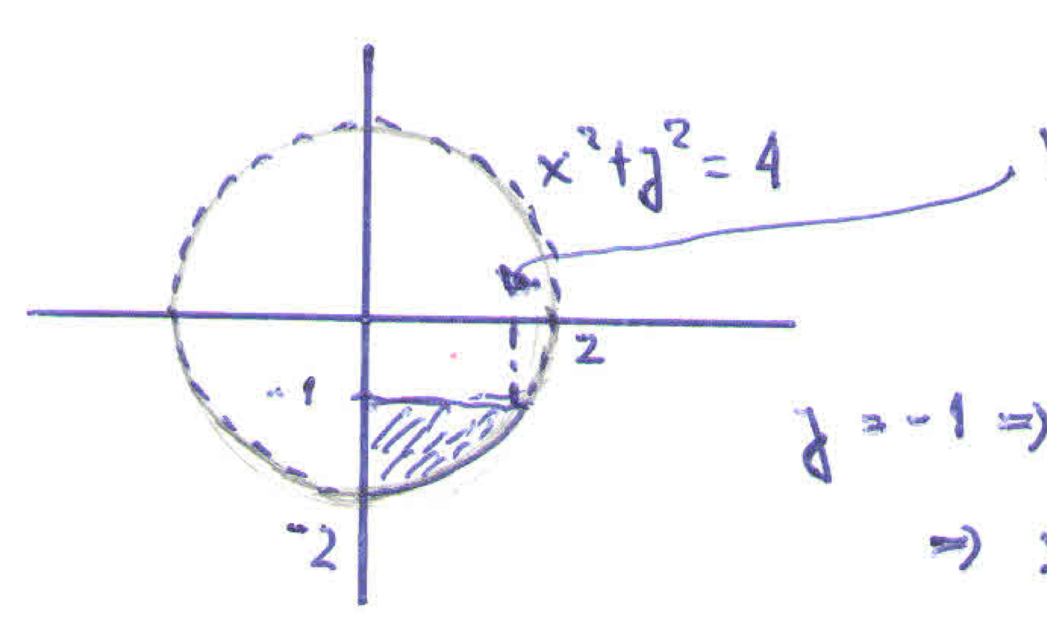


$$D = D_1 \cup D_2$$



b) Mude a ordem de integração

$$I_1 = \int_{-2}^{-1}\int_0^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$



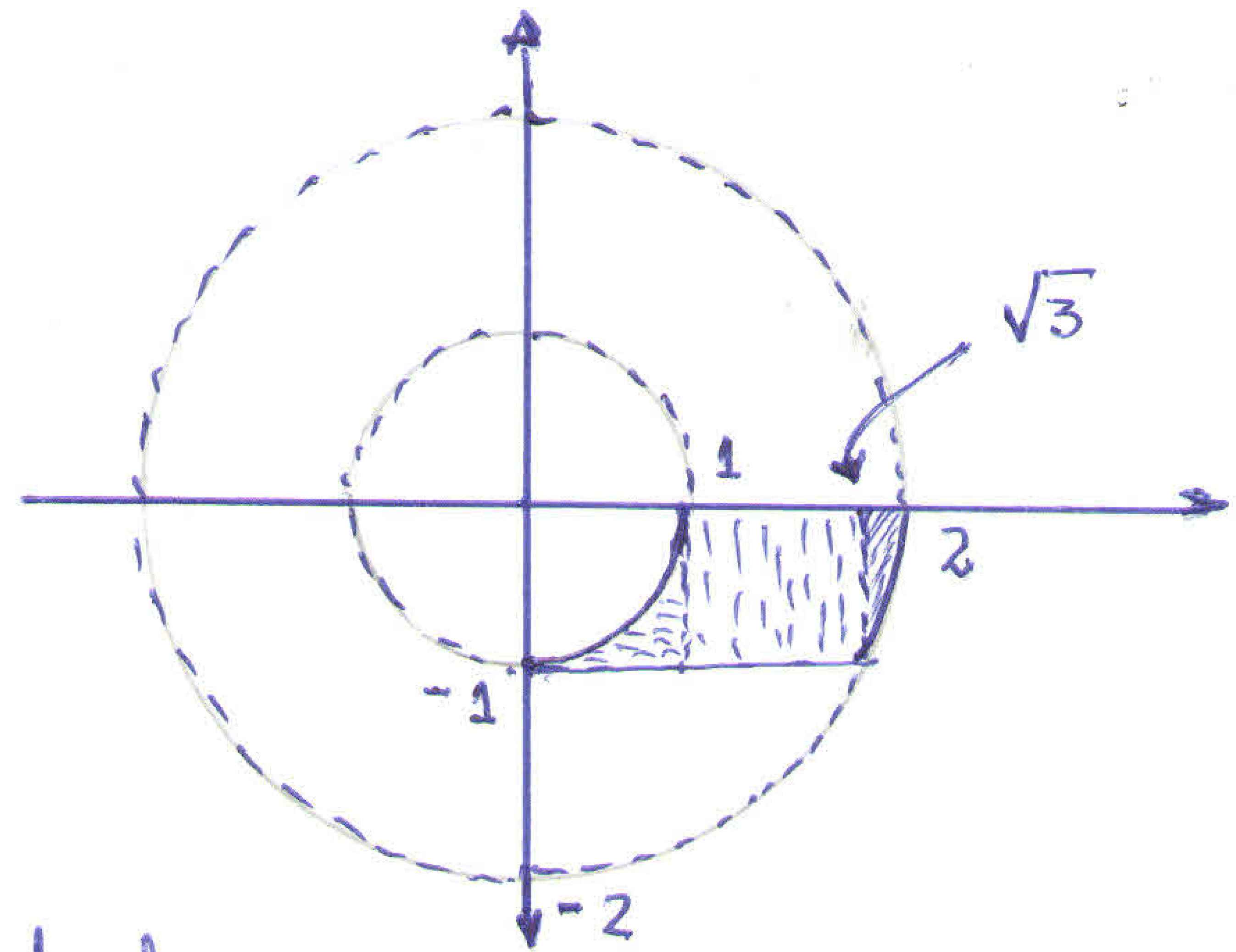
$y=-1 \Rightarrow x^2+1=4$

$\Rightarrow x^2=3$

$\Rightarrow x=\pm\sqrt{3}$

$$I_1 = \int_0^{\sqrt{3}} \int_{-\sqrt{4-x^2}}^{-1} (x^2+y^2)\,dy\,dx.$$

$$I_2 = \int_{-1}^{0} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$



$$I_2 = \int_0^1 \int_{-1}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx + \int_1^{\sqrt{3}} \int_{-1}^{0} (x^2+y^2)\,dy\,dx$$

$$+ \int_{\sqrt{3}}^{2} \int_{\sqrt{4-x^2}}^{0} (x^2+y^2)\,dy\,dx$$

c) calcular vol(W).

$$\mathrm{vol}(W) = I_1 + I_2$$

$$= \int_{-2}^{0} \int_{-\sqrt{1-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)\,dx\,dy$$

$x = r\cos\theta$
$y = r\,\mathrm{sen}\,\theta$

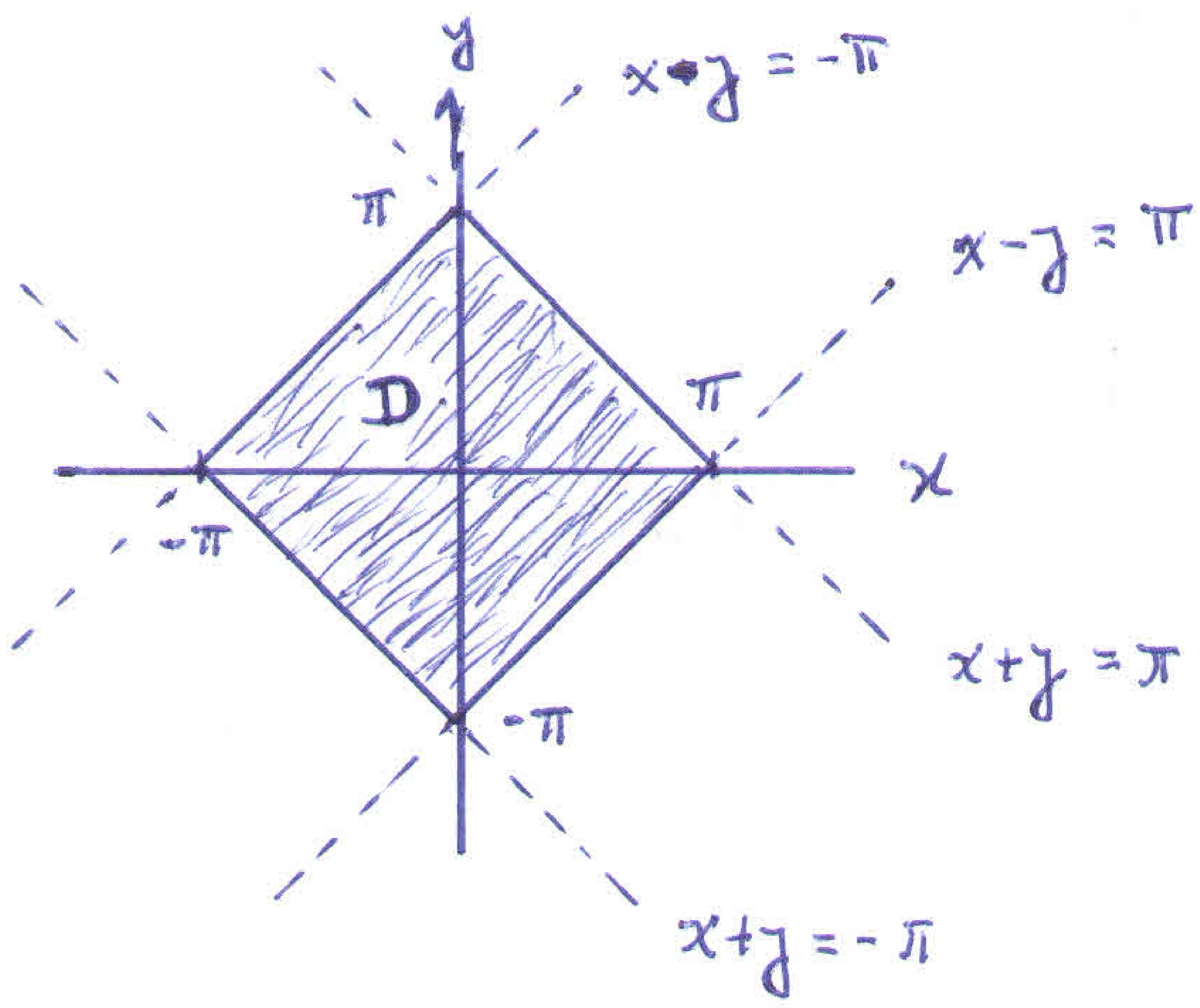
$$D_{r,\theta} : \begin{cases} 1 \le r \le 2 \\ -\frac{\pi}{2} \le \theta \le 0 \end{cases}$$

$$\mathrm{vol}(W) = \int_1^2 \int_{-\pi/2}^{0} r^2 \cdot r\,d\theta\,dr = \frac{\pi}{2} \int_1^2 r^3\,dr = \frac{\pi}{2} \cdot \frac{r^4}{4} \Big|_1^2$$

$$= \frac{\pi}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}\pi$$

③ Calcular $\int_D f \, dA$.

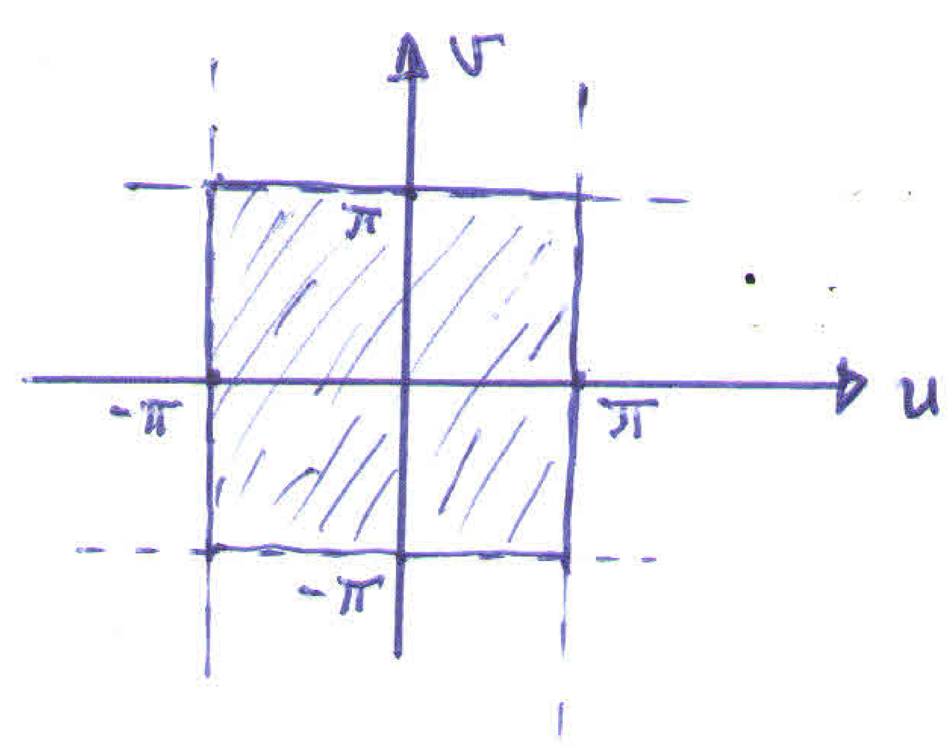
(a) $f(x,y) = (x+y+1)(x-y)$, $D: |x| + |y| \leq \pi$.



$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$$

$$\therefore \quad J = 1/2 \, , \quad dA = \frac{1}{2} du dv$$

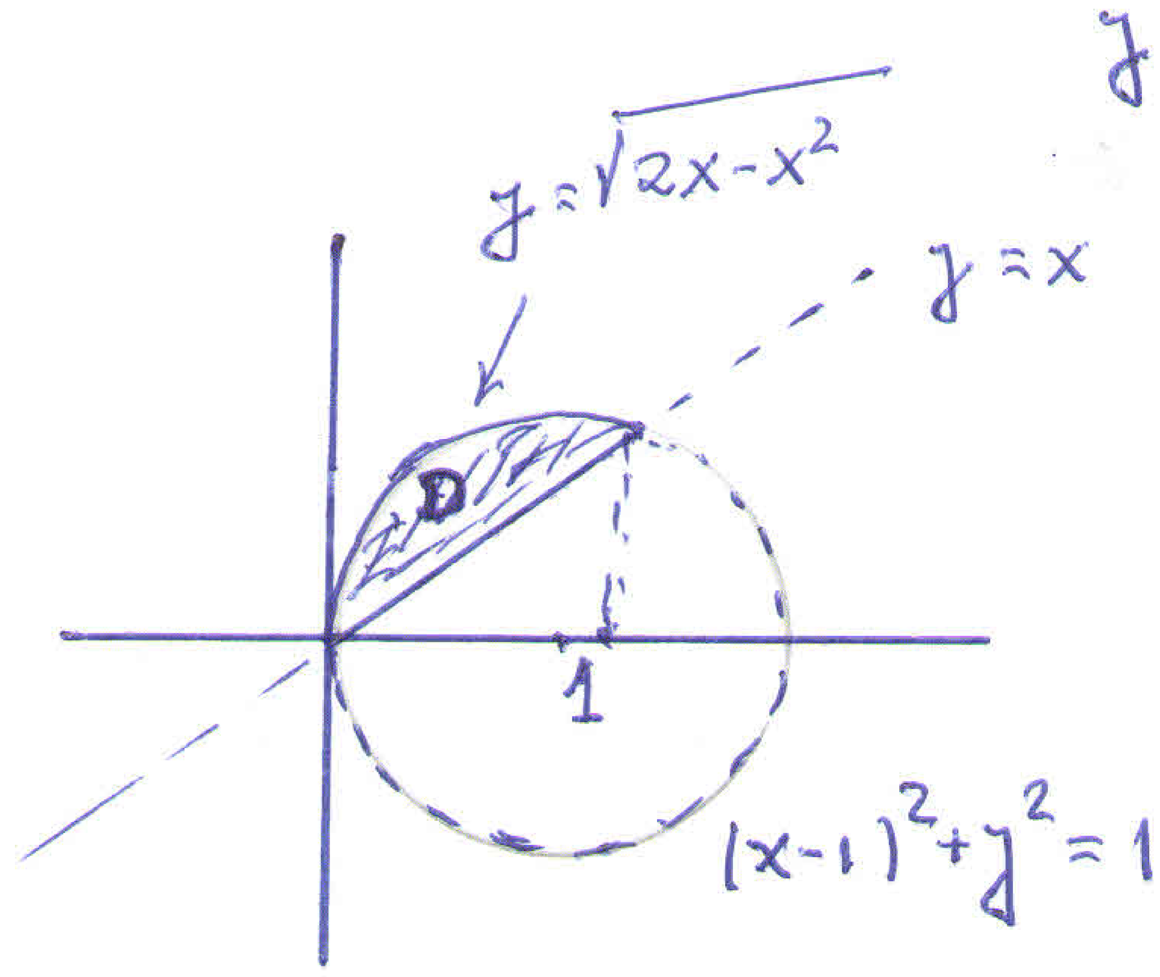


$$\int_D f dA = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} (u+1) v \cdot \frac{1}{2} du dv$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} v \left(\frac{u^2}{2} + u\right)\Bigg|_{u=-\pi}^{u=\pi} dv$$

$$= \pi \int_{-\pi}^{\pi} v \, dv = 0$$

(b) $f(x,y) = \sqrt{x^2+y^2}$, D é a região limitada pelas curvas $y = \sqrt{2x - x^2}$ e $y = x$.



D em coord. polares

$$\begin{cases} \pi/4 \leq \theta \leq \pi/2 \\ 0 \leq r \leq 2\cos\theta \end{cases}$$

$$\left(\begin{array}{l} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{array} \quad y = \sqrt{2x - x^2} \Rightarrow x^2 + y^2 = 2x \right.$$

$$\Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 2\cos\theta \Big)$$

$$\int_D f\,dA = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} \sqrt{r^2}\cdot r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \frac{8}{3}\int_{\pi/4}^{\pi/2} \cos^3\theta\,d\theta$$

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2} (1-\text{sen}^2\theta)\cos\theta\,d\theta$$

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2} \cos\theta\,d\theta - \frac{8}{3}\int_{\pi/4}^{\pi/2} \text{sen}^2\theta\cos\theta\,d\theta$$

$$= -\frac{8}{3}\,\text{sen}\,\theta\Big|_{\pi/4}^{\pi/2} - \frac{8}{3}\cdot\frac{\text{sen}^3\theta}{3}\Big|_{\pi/4}^{\pi/2}$$

$$= -\frac{8}{3}\left(1-\frac{\sqrt{2}}{2}\right) - \frac{8}{9}\left(1-\left(\frac{\sqrt{2}}{2}\right)^3\right)$$

$$= \ldots\ \text{etc.}$$

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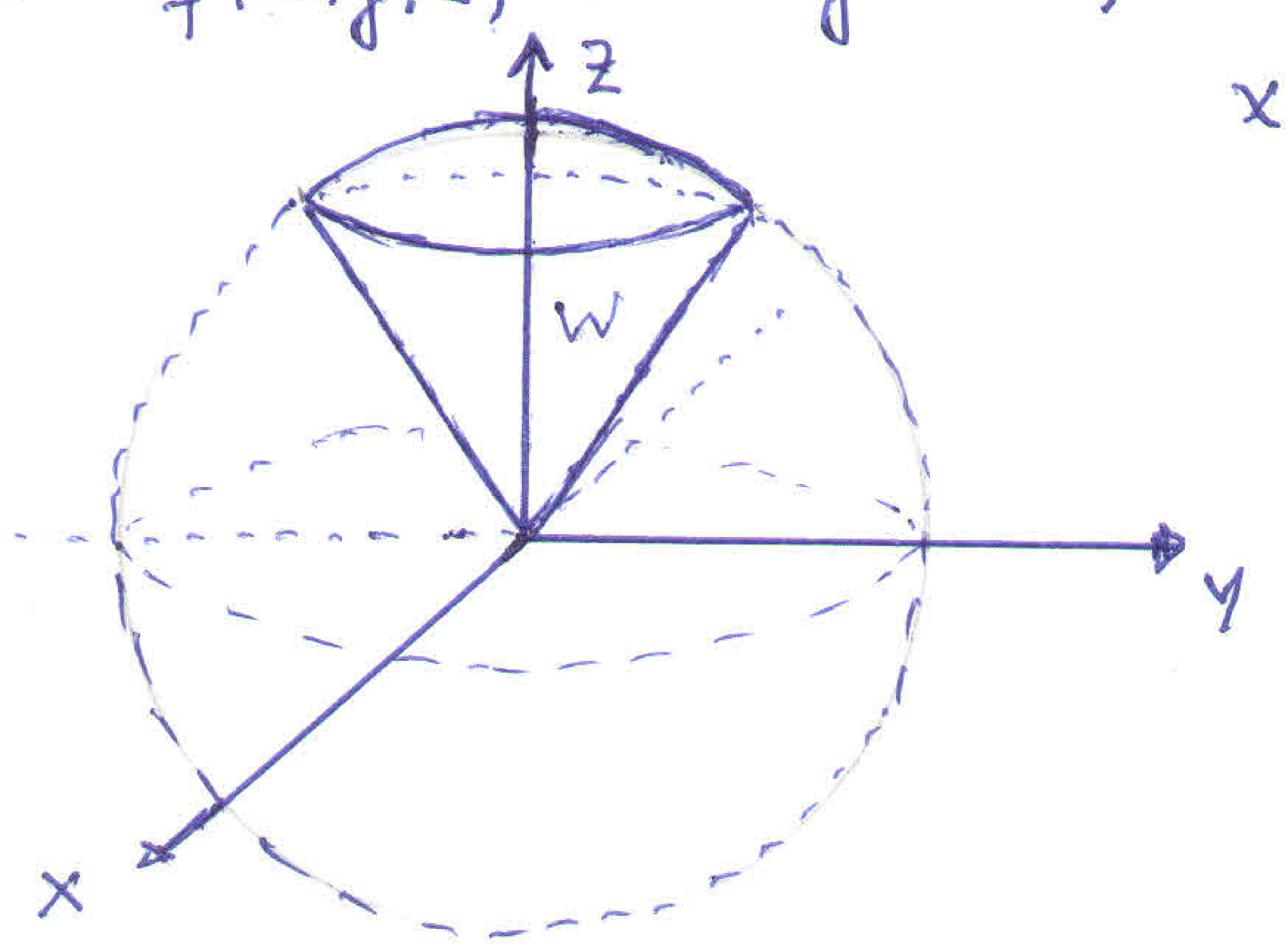
(4) Calcule $\int_W f\,dV$.

(a) $f(x,y,z) = x^2+y^2+z^2$, W é a região limitada acima pela esfera $x^2+y^2+z^2=16$ e abaixo pelo cone $z=\sqrt{x^2+y^2}$



W em coord esféricas

$$W: \begin{cases} 0 \le \rho \le 4 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/4 \end{cases}$$

$$x^2+y^2+z^2=\rho^2$$

$$J=\rho^2 \operatorname{sen}\phi$$

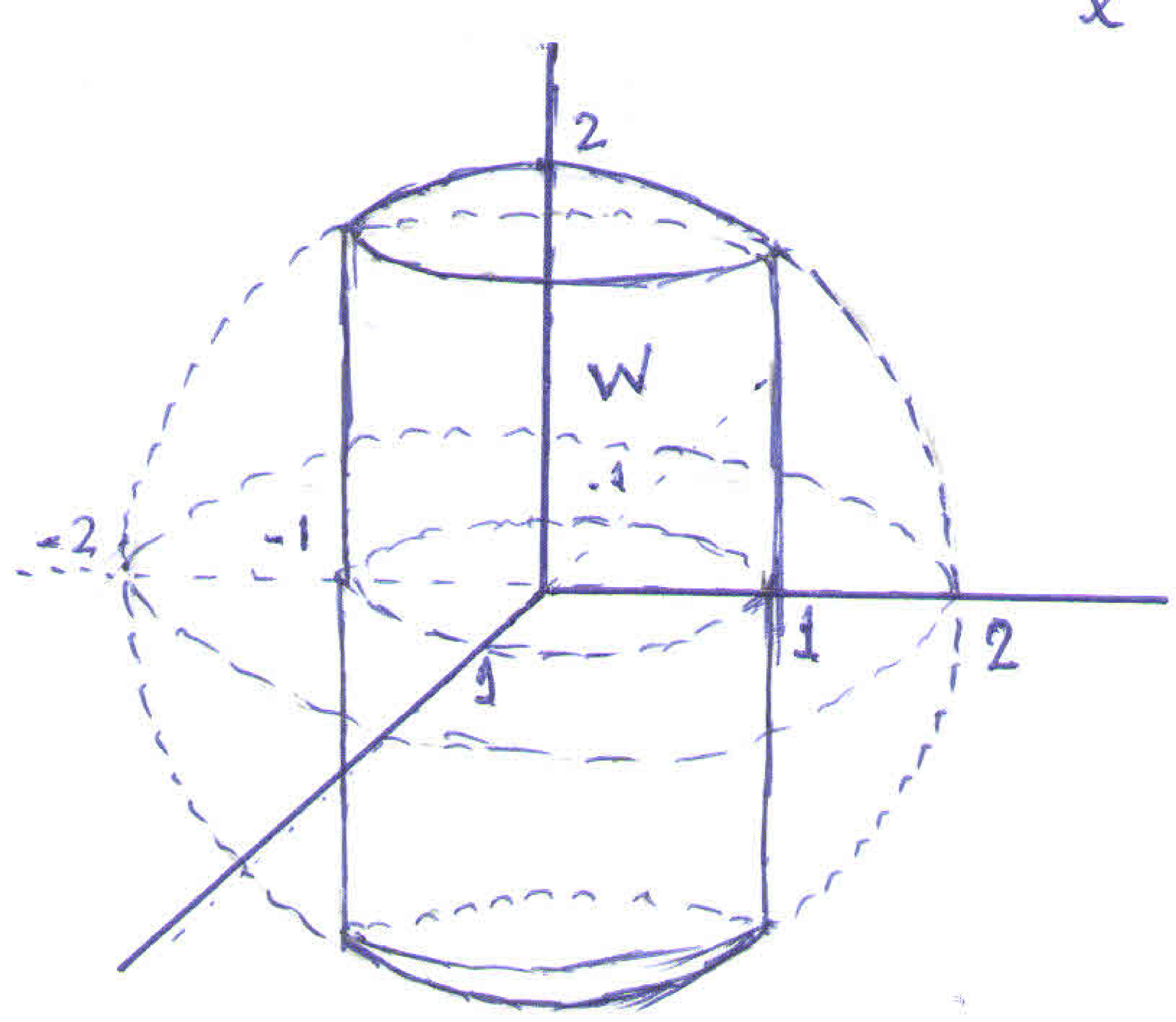
$$\int_W f\,dV = \int_0^4\int_0^{2\pi}\int_0^{\pi/4} \rho^2\cdot\rho^2 \operatorname{sen}\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^4\int_0^{2\pi} -\rho^4 \cos\phi \Big|_0^{\pi/4} d\theta\, d\rho$$

$$= \left(1-\frac{\sqrt2}{2}\right)\cdot 2\pi \int_0^4 \rho^4\, d\rho = 2\left(1-\frac{\sqrt2}{2}\right)\pi \, \frac{\rho^5}{5}\Big|_0^4$$

$$= \frac{2\cdot 4^5}{5}\left(1-\sqrt2/2\right)\pi$$

(b) $f(x,y,z)=x^2+y^2$, W é a região interior ao cilindro $x^2+y^2=1$ e à esfera $x^2+y^2+z^2=4$



W em coord. cilíndricas

$$\begin{cases} 0\le r\le 1\\ 0\le\theta\le 2\pi\\ -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}\end{cases}$$

$$J=r$$

$$\int_W f\,dV = \int_0^1\int_0^{2\pi}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2\cdot r\, dz\, d\theta\, dr$$

$$= 2\pi\int_0^1 2\sqrt{4-r^2}\cdot r^3\, dr = -2\pi\int_0^1 r^2\sqrt{4-r^2}\,(-2r)\,dr$$

$u = 4 - r^2 \Rightarrow r^2 = 4 - u$ $\qquad$ $r = 0 \Rightarrow u = 4$

$du = -2r\,dr$ $\qquad$ $r = 1 \Rightarrow u = 3$

$$\int_W f\,dV = -2\pi \int_4^3 (4-u)\sqrt{u}\,du = 2\pi \int_3^4 \left(4\sqrt{u} - u^{3/2}\right) du$$

$$= 2\pi \left(4 \cdot \frac{2u^{3/2}}{3} - \frac{2}{5} u^{5/2} \right) \Bigg|_3^4$$