C3 Lista 7 Ex 6

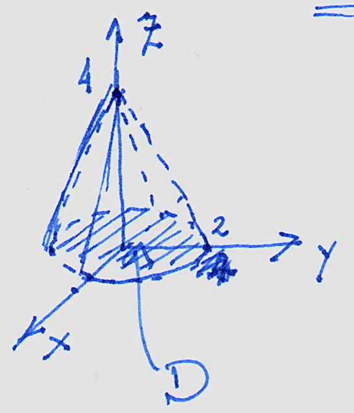
$$S: \begin{cases} z = 4 - 2\sqrt{x^2+y^2} \\ z \geq 0 \end{cases}$$

densidade: $f = K\sqrt{x^2+y^2}$

$$I_z = \frac{12}{5} M, \quad M = \text{massa}(S)$$

$$I_z = \int_S (x^2+y^2) f \, dS, \quad M = \int_S f \, dS$$

S:



Parametrizamos S como gráfico de $z = 4 - 2\sqrt{x^2+y^2}$

$\varphi: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\varphi(x,y) = (x, y, 4 - 2\sqrt{x^2+y^2})$

$$D: \begin{cases} x^2+y^2 \leq 4 \\ (z = 0) \end{cases}$$

$$\varphi_x = \left(1, 0, \frac{-2x}{\sqrt{x^2+y^2}}\right)$$

$$\varphi_y = \left(0, 1, \frac{-2y}{\sqrt{x^2+y^2}}\right)$$

$$\varphi_x \wedge \varphi_y = \left(\frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}, 1\right)$$

$$\|\varphi_x \wedge \varphi_y\| = \sqrt{\left(\frac{2x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{2y}{\sqrt{x^2+y^2}}\right)^2 + 1} = \sqrt{5}$$

(Vc. já sabia disso!!)

(Neste caso, $\|\varphi_x \wedge \varphi_y\| = \sqrt{1 + |\nabla z|^2} = \sqrt{1 + z_x^2 + z_y^2}$)

$$\therefore \quad dS = \sqrt{5} \, dx \, dy$$

$$M = \int_S f\,dS = \int_D K\sqrt{x^2+y^2}\,\sqrt{5}\,dx\,dy$$

$D_{r\theta}$, D em coord. polares $D_{r\theta}: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \end{cases}$

$$M = \int_0^2 \int_0^{2\pi} K\sqrt{r^2}\,.\sqrt{5}\,.\,r\,d\theta\,dr$$

$$= \sqrt{5}\,K \cdot 2\pi \int_0^2 r^2\,dr = 2\sqrt{5}\,K\pi \cdot \frac{1}{3} r^3 \Big|_0^2$$

$$= \frac{16}{3}\sqrt{5}\,K\,\pi \qquad \boxed{M = \frac{16}{3}\sqrt{5}\,K\,\pi}$$

$$I_z = \int_S (x^2+y^2) f\,dS = K\int_S (x^2+y^2)\sqrt{x^2+y^2}\,dS$$

$$= \sqrt{5}\,K \int_S (x^2+y^2)^{3/2}\,dx\,dy = \sqrt{5}\,K \int_0^2 \int_0^{2\pi} (r^2)^{3/2} \cdot r\,d\theta\,dr$$

$$= \sqrt{5} \cdot 2\pi \cdot K \int_0^2 r^3 \cdot r\,dr = 2\sqrt{5}\,K\pi \int_0^2 r^4\,dr$$

$$= \frac{2\sqrt{5}\,K\pi}{5} \cdot 2^5 = \frac{64\sqrt{5}\,K\pi}{5}$$

$$= \frac{4}{5} \cdot 16\sqrt{5}\,K\pi = \frac{4 \cdot 3}{5} \cdot \underbrace{\frac{16\sqrt{5}\,K\pi}{3}}_{M}$$

$$= \frac{12}{5} M$$